STABILITY OF DISPERSION MANAGED SOLITONS FOR VANISHING AVERAGE DISPERSION

DIRK HUNDERTMARK, PEER KUNSTMANN, AND ROLAND SCHNAUBELT

Abstract. We study dynamical properties of the dispersion management equation with vanishing average dispersion. Our main result establishes the stability of the set of ground states.

1. Introduction

In optical fiber cables information is transmitted by means of localized pulses, which exist thanks to an interplay between dispersion and Kerr-type nonlinear material properties. Depending on the sign of the dispersion, the pulses tend to concentrate (the focusing case) or to smear out (defocusing case). Both effects are undesirable: The first one leads to high concentration of energy, the second one to interference between different signals. In the dispersion management technique one uses fibers whose dispersion changes its sign periodically. The average dispersion $d_{\rm av} \geq 0$ has to be small or, ideally, to vanish. This idea has turned out to be enormously successful, see [3], [4], [9], [11] and the references therein. The existence of such localized pulses has rigorously been shown in [12] and [8] for $d_{\rm av} > 0$ and $d_{\rm av} = 0$, respectively. Their stability is known so far only for the case $d_{\rm av} > 0$, [12], where additional regularity and compactness properties are available. In this note we establish the stability of the set of these dispersion management solitions also for vanishing average dispersion $d_{\rm av} = 0$.

The signal in an optical fiber is transmitted via amplitude modulation of a carrier wave. The amplitude w is then approximately determined by the one-dimensional cubic nonlinear Schrödinger equation

$$\mathrm{i}\partial_t w(t) + d(t)\partial_{xx} w(t) + \varepsilon\,|w(t)|^2\,w(t) = 0, \qquad t \in \mathbb{R},\ x \in \mathbb{R}, \tag{1.1}$$

see [9], where we have normalized some constants. This equation is formulated in a reference frame moving with the group velocity of the pulse, where $t \in \mathbb{R}$ actually denotes the position at the cable and $x \in \mathbb{R}$ is the retarded time. Nevertheless we keep the letters t and x that are familiar from the Schrödinger equation in quantum mechanics and other evolution equations. The dispersion is given by $d(t) = \varepsilon d_{\rm av} + d_0(t)$ for the average dispersion $d_{\rm av} \geq 0$, the dispersion profile $d_0 : \mathbb{R} \to \mathbb{R}$ with period $L > 0$ and with mean zero, and a small parameter $\varepsilon > 0$. We thus work in the regime of *strong* dispersion management in which amplitude and non-linearity are small compared to the varying dispersion d_0, but the nonlinear effects are still significant, see e.g. [11].

2000 *Mathematics Subject Classification.* 35B35, 35Q55, 35Q60.

Key words and phrases. Dispersion managed NLS, stability, ground states, compactness.

D.H. is supported by the Alfried Krupp von Bohlen and Halbach Foundation.

One is mainly interested in localized standing wave solutions $u(t,x) = \mathrm{e}^{\mathrm{i}\omega t}v(x)$ of (1.1) and their stability properties, since such solitary waves are the building blocks of signal transmission. These topics are hard to tackle in non-autonomous problems such as (1.1). For this reason Gabitov and Turitsyn proposed in [3] and [4] to study the averaged equation

$$\mathrm{i}\partial_t u(t) + d_{\mathrm{av}}\partial_{xx}u(t) + \frac{1}{L}\int_0^L T_{D(s)}^{-1}[T_{D(s)}u(t)\,\overline{T_{D(s)}u(t)}\,T_{D(s)}u(t)]\,\mathrm{d}s = 0, \quad (1.2)$$

where $t,x \in \mathbb{R}$, $T_\tau = \mathrm{e}^{\mathrm{i}\tau\partial_x^2}$ is the free Schrödinger group, $D(s) = \int_0^s d_0(r)\,\mathrm{d}r$, and we have removed the parameter $\varepsilon > 0$ by scaling. It was shown in [12] that for a solution u of (1.2) the function $T_{D(t)}u(\varepsilon^{-1}t,\cdot)$ is close to the solution of (1.1) with initial value $u(0,\cdot)$ if $\varepsilon > 0$ is sufficiently small. We point out that in order to resort to the autonomous equation (1.2) one has to pay the considerable price of a nonlocal nonlinearity with a highly oscillating kernel.

For non-vanishing mean dispersion $d_{\mathrm{av}} > 0$, standing wave solutions for (1.2) have been constructed in [12]. They arise as ground states v of a constrained minimization problem which is solved in $H^1(\mathbb{R})$ using the concentration–compactness result Theorem 6.1 of [12]. These functions v are called dispersion management solitons. In [12], it was also indicated that the set of ground states is stable in $H^1(\mathbb{R})$ under the flow of (1.2), see p. 798. This fact follows from the concentration–compactness result and arguments introduced in [1]. Finally, standard boot–strapping implies the smoothness of the ground state.

The situation for the (in the applications most interesting) case of vanishing mean dispersion $d_{\mathrm{av}} = 0$ is quite different since here the natural state space is $L^2(\mathbb{R})$. It is not clear how to use concentration–compactness arguments here, and there is no apparent second derivative available which would give regularity for free. Nevertheless for piecewise constant profiles d_0 and $d_{\mathrm{av}} = 0$, it was possible to show existence and smoothness of dispersion management solitons in [8] and [10] with considerable efforts. By completely different methods, the existence of ground states for $d_{\mathrm{av}} = 0$ and a very large class of profiles d_0 was established in [7], co–authored by one of us. We use certain compactness results of [7] in the present paper. Moreover, for $d_{\mathrm{av}} \geq 0$ every dispersion management soliton v and its Fourier transform $\mathcal{F}v$ decay exponentially, see [5] and also [2] for a special case.

We work in the setting of [7] and write the nonlinearity in (1.2) as the integral

$$q(u(t),u(t),u(t)) := \int_{\mathbb{R}} T_{-\tau}\left[T_\tau u(t)\,\overline{T_\tau u(t)}\,T_\tau u(t)\right]\psi(\tau)\mathrm{d}\tau$$

for weights $0 \leq \psi \in L^2(\mathbb{R})\cap L^4(\mathbb{R})\cap L^4(\mathbb{R}, t^2\,\mathrm{d}t)$. The above integral is obtained via the transformation $\tau = D(s)$ from that of (1.2). As explained in Lemma 1.4 and Remarks 1.5 of [7], locally integrable dispersion profiles d_0 yield a density ψ with the above stated integrability properties if d_0 has finitely many sign changes and $d_0^{-1} \in L^3(0,L)$. The latter is clearly satisfied if d_0 is strictly separated from 0.

In this paper, we study the dispersion management equation with $d_{\mathrm{av}} = 0$ and the above general nonlinearity, i.e.,

$$\begin{aligned} \mathrm{i}\partial_t u(t) + q(u(t), u(t), u(t)) = 0, \qquad & t \in \mathbb{R}, \\ u(0) = u_0, & \end{aligned} \tag{1.3}$$

At first, Proposition 2.2 yields the global wellposedness of (1.3) in $L^2(\mathbb{R})$. This result is based on regularity properties of the map q shown in [7] and the preservation of the L^2–norm under the flow. In Theorem 2.3 we then establish that the set of ground states of (1.3) is stable in $L^2(\mathbb{R})$. As in [1] this fact is shown by a contradiction argument which involves conserved quantities and a compactness property of minimizing sequences. In contrast to the concentration–compactness principle used in [1] or [12], we employ a compactness criterion from [7] which says that a weakly converging series $(f_n)_n$ in $L^2(\mathbb{R}^d)$ converges strongly if $(f_n)_n$ and $(\mathcal{F} f_n)_n$ satisfy a certain tightness property. In the next section we further recall the necessary background mainly from [7].

2. Results

We investigate the dispersion management equation (1.3) in $L^2(\mathbb{R})$ with mean average 0 and the non-local nonlinearity

$$q(v_1, v_2, v_3) = \int_{\mathbb{R}} T_{-\tau}\big[T_\tau v_1\, \overline{T_\tau v_2}\, T_\tau v_3\big]\, \psi(\tau)\mathrm{d}\tau,$$

where $T_\tau = \mathrm{e}^{\mathrm{i}\tau\partial_x^2}$ is the free Schrödinger group. We mostly assume that the density ψ satisfies

$$0 \le \psi \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}) \cap L^4(\mathbb{R}, t^2\, \mathrm{d}t). \tag{2.1}$$

We further need the functionals

$$\begin{aligned} Q(v_1, v_2, v_3, v_4) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{T_\tau v_1}\, T_\tau v_2\, \overline{T_\tau v_3}\, T_\tau v_4\, \mathrm{d}x\, \psi(\tau)\mathrm{d}\tau = \langle v_1, q(v_2, v_3, v_4)\rangle_2, \\ Q[v] := Q(v, v, v, v) &= \int_{\mathbb{R}}\int_{\mathbb{R}} |T_\tau v|^4\, \mathrm{d}x\, \psi(\tau)\mathrm{d}\tau, \end{aligned}$$

where $\langle u, v\rangle_2 = \int_{\mathbb{R}} \overline{u} v\, \mathrm{d}x$. Lemma B.1 of [7] yields the following basic property, see also Lemma 1 of [6] and Remark 3.5 of [5]. (Abusing notation, we use the term multi-linear even if the map is anti-linear in some variables.)

Proposition 2.1. *Let $0 \le \psi \in L^2(\mathbb{R})$. Then $q : L^2(\mathbb{R})^3 \to L^2(\mathbb{R})$ and $Q : L^2(\mathbb{R})^4 \to \mathbb{C}$ are bounded trilinear and four-linear maps, respectively.*

This continuity result directly implies local wellposedness of (1.3), whereas the global existence of solutions is a consequence of the preservation of the norm in $L^2(\mathbb{R})$ as stated in the next proposition.

Proposition 2.2. *Let $0 \le \psi \in L^2(\mathbb{R})$. Then for each $u_0 \in L^2(\mathbb{R})$ there is a unique global solution $u = u(\cdot; u_0)$ of (1.3) in $C^1(\mathbb{R}, L^2(\mathbb{R}))$. It satisfies $\|u(t)\|_2 = \|u_0\|_2$ and $Q[u(t)] = Q[u_0]$ for all $t \in \mathbb{R}$. The map $L^2(\mathbb{R}) \to C([-t_0, t_0], L^2(\mathbb{R}))$; $u_0 \mapsto u(\cdot; u_0)$, is Lipschitz on bounded sets for each $t_0 > 0$.*

Proof. The nonlinearity $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ given by $F(v) = q(v,v,v)$ is Lipschitz on bounded sets due to Proposition 2.1. Hence, there exists a unique, maximally defined solution of $u \in C^1((-t_1,t_2), L^2(\mathbb{R}))$ of (1.3). If $t_1 > 0$ or $t_2 > 0$ is finite, then $\|u(t)\|_2$ becomes unbounded as $t \to t_2$ or $t \to -t_1$, respectively. ¿From (1.3) we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_2^2 = 2\,\mathrm{Re}\langle \partial_t u(t), u(t)\rangle_2 = -2\,\mathrm{Re}(\mathrm{i}\langle q(u(t),u(t),u(t)), u(t)\rangle_2). \\ = -2\,\mathrm{Re}(\mathrm{i}Q[u(t)]) = 0.$$

This means that $\|u(t)\|_2 = \|u_0\|_2$ for all t, and so solutions are global in time.

Observe that the derivative of $v \mapsto Q[v]$ on $L^2(\mathbb{R})$ is given by $\langle h, Q'[v]\rangle_2 = 4\,\mathrm{Re}\,Q(h,v,v,v)$, see Lemma 2.7 of [7]. Equation (1.3) thus implies

$$\frac{\mathrm{d}}{\mathrm{d}t}Q[u(t)] = 4\,\mathrm{Re}\,Q(\partial_t u(t), u(t), u(t), u(t)) \\ = 4\,\mathrm{Re}\langle iq(u(t),u(t),u(t)), q(u(t),u(t),u(t))\rangle_2 = 0$$

for all $t \in \mathbb{R}$; i.e., solutions preserve $Q[\cdot]$.

Finally, the continuous dependence on initial data follows from the boundednes of q by integrating (1.3) in t and using Gronwall's inequality. □

If (2.1) holds, then Theorem 1.1 of [7] shows that for each $\lambda > 0$ there is a function $v \in L^2(\mathbb{R})$ maximizing the functional $Q[v]$ under the condition $\|v\|_2^2 = \lambda$. We call such maximizers *ground states.* Every ground state also solves the stationary problem

$$\omega v = q(v,v,v) \tag{2.2}$$

for some $\omega > 0$, see p.21 and p.22 in [7]. In particular, the problem (1.3) admits the standing wave solution $u(t,x) = \mathrm{e}^{\mathrm{i}\omega t}v(x)$. The functions v are called dispersion management solitons.

It is easy to check that phase factors, translations and boosts of v do not change $Q[v]$ so that also the function given by $\tilde{v}(x) = \mathrm{e}^{\mathrm{i}(\theta+\xi_0 x)}v(x-x_0)$ is a ground state for all $x_0, \xi_0, \theta \in \mathbb{R}$. Hence, $\tilde{v}$ solves (2.2) and $w(t,x) = \mathrm{e}^{\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}(\theta+\xi_0 x)}v(x-x_0)$ satisfies (1.3). We thus obtain sets of ground states

$$S_\lambda = \{v \in L^2(\mathbb{R}) \,|\, \|v\|_2^2 = \lambda,\ Q[v] = P_\lambda\}, \quad \text{where } P_\lambda := \sup\{Q[f] \,|\, \|f\|_2^2 = \lambda\}$$

for each $\lambda > 0$. Observe that S_λ is invariant under the flow of (1.3) in view of the above observations.

Setting $\mathrm{d}(\varphi, S_\lambda) = \inf_{v\in S_\lambda}\|\varphi - v\|_2$, we now state our main stability result.

Theorem 2.3. *If* (2.1) *holds and* $\lambda > 0$*, then* S_λ *is stable in* $L^2(\mathbb{R})$*; i.e., for each* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that for each initial value* $u_0 \in L^2(\mathbb{R})$ *with* $\mathrm{d}(u_0, S_\lambda) < \delta$ *we obtain* $\mathrm{d}(u(t;u_0), S_\lambda) < \varepsilon$ *for all* $t \in \mathbb{R}$.

Proof. If the assertion was wrong, there would exist initial values $\varphi_n \in L^2(\mathbb{R})$, times $t_n \in \mathbb{R}$ and a number $\eta > 0$ such that $\mathrm{d}(\varphi_n, S_\lambda) \to 0$ as $n \to \infty$, but $\|u(t_n;\varphi_n) - v\|_2 \geq \eta$ for all $n \in \mathbb{N}$ and $v \in S_\lambda$. We set $u_n = u(t_n;\varphi_n)$ and note

$$\|u_n - v\|_2 \geq \eta \qquad \text{for all } n \in \mathbb{N} \text{ and } v \in S_\lambda. \tag{2.3}$$

For each $\varepsilon > 0$ we obtain $n_0 \in \mathbb{N}$ and $v_n \in S_\lambda$ such that $\|\varphi_n - v_n\|_2 \le \varepsilon$ if $n \ge n_0$. It follows

$$\big| \|\varphi_n\|_2 - \lambda^{1/2} \big| = \big| \|\varphi_n\|_2 - \|v_n\|_2 \big| \le \varepsilon$$

for $n \ge n_0$; i.e., $\alpha_n := \lambda^{1/2} \|\varphi_n\|_2^{-1}$ tends to 1 as $n \to \infty$ and the sequence (φ_n) is bounded in $L^2(\mathbb{R})$. Here and below, we may asume that $\varphi_n \neq 0$ for every $n \in \mathbb{N}$. Using Proposition 2.1, we also deduce

$$\big| Q[\varphi_n] - P_\lambda \big| = \big| Q[\varphi_n] - Q[v_n] \big| \le c\, \|\varphi_n - v_n\|_2 \le c\varepsilon$$

for $n \ge n_0$, so that $Q[\varphi_n] \to P_\lambda$ as $n \to \infty$.

The conservation laws in Proposition 2.2 next imply that $\|u_n\|_2 = \|\varphi_n\|_2$ for all $n \in \mathbb{N}$ and that $Q[u_n] = Q[\varphi_n]$ tends to P_λ as $n \to \infty$. Setting $w_n = \alpha_n u_n$, we then derive $\|w_n\|_2^2 = \lambda$ for all $n \in \mathbb{N}$ and $Q[w_n] \to P_\lambda$ as $n \to \infty$, also employing Proposition 2.1.

Since (w_n) is a maximizing sequence for $Q[\cdot]$ with $\|w_n\|_2^2 = \lambda$, Proposition 2.4 of [7] yields shifts $x_n, \xi_n \in \mathbb{R}$ such that the functions given by $\tilde{w}_n(x) = \mathrm{e}^{\mathrm{i}\xi_n x} w_n(x - x_n)$ satisfy

$$\lim_{R\to\infty} \sup_{n\in\mathbb{N}} \int_{|x|\ge R} |\tilde{w}_n|^2 \,\mathrm{d}x = 0 \quad \text{and} \quad \lim_{R\to\infty} \sup_{n\in\mathbb{N}} \int_{|\xi|\ge R} |\mathcal{F}\tilde{w}_n|^2 \,\mathrm{d}\xi = 0,$$

where $\mathcal{F}$ is the Fourier transform. Because of $\|\tilde{w}_n\|_2^2 = \lambda$, there is a subsequence (denoted by same index) which converges weakly in $L^2(\mathbb{R})$ to some $\tilde{w}$. Due to the above tightness property, Lemma A.1 in [7] now shows that $\tilde{w}_n$ tends to $\tilde{w}$ even in $L^2(\mathbb{R})$. Using $Q[\tilde{w}_n] = Q[w_n]$ and the continuity of Q, we infer that $Q[\tilde{w}] = P_\lambda$. Since $\|\tilde{w}\|_2^2 = \lambda$, the function $\tilde{w}$ thus belongs to S_λ.

But, then also the maps $\tilde{v}_n = \mathrm{e}^{-\mathrm{i}\xi_n(\cdot + x_n)} \tilde{w}(\cdot + x_n)$ are contained in S_λ and we estimate

$$\|u_n - \tilde{v}_n\|_2 \le |1 - \alpha_n|\, \|u_n\|_2 + \|w_n - \tilde{v}_n\|_2 = |1 - \alpha_n|\, \|u_n\|_2 + \|\tilde{w}_n - \tilde{w}\|_2$$

for $n \in \mathbb{N}$. In view of the above results, the right hand side vanishes as $n \to \infty$, which contradicts (2.3). □

References

[1] T. Cazenave and P.L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations.* Comm. Math. Phys. **85** (1982), 549–561.

[2] M.B. Erdoğan, D. Hundertmark and Y.-R. Lee, *Exponential decay of dispersion management solitons.* Math. Res. Lett. **18** (2011), 11–24.

[3] L. Gabitov and S.K. Turitsyn, *Averaged pulse dynamics in a cascaded transmission system with passive dispersion compensation.* Opt. Lett. **21** (1996), 327–329.

[4] L. Gabitov and S.K. Turitsyn, *Breathing solitons in optical fiber links.* JETP Lett. **63** (1996), 861.

[5] W.R. Green and D. Hundertmark, *Exponential Decay for dispersion managed solitons for general dispersion profiles.* Preprint, see http://arxiv.org/abs/1212.4004.

[6] D. Hundertmark and Y.-R. Lee, *Decay estimates and smoothness for solutions of the dispersion managed non-linear Schrödinger equation.* Comm. Math. Phys. **286** (2009), 851–873.

[7] D. Hundertmark and Y.-R. Lee, *On non-local variational problems with lack of compactness related to non-linear optics.* J. Nonlinear Sci. **22** (2012), 1–38.

[8] M. Kunze, *On a variational problem with lack of compactness related to the Strichartz inequality.* Calc. Var. Partial Differential Equations **19** (2004), 307–336.

[9] J. Moloney and A. Newell, *Nonlinear Optics.* Westview Press, Boulder (Co), 2004.

[10] M. Stanislalova, *Regularity of ground state solutions of dispersion managed nonlinear Schrödinger equations.* J. Differential Equations **210** (2005), 87–105.

[11] S. K. Turitsyn, E.G. Shapiro, S.B. Medvedev, M.P. Fedoruk and V.K. Mezentsev, *Physics and mathematics of dispersion-managed optical solitons.* Comptes Rendus Physique **4** (2003), 145–161.

[12] V. Zharnitsky, E. Grenier, C.K.R.T. Jones and S.K. Turitsyn, *Stabilizing effects of dispersion management.* Phys. D **152-153**(2001), 794–817.

Department of Mathematics, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany.

E-mail address: dirk.hundertmark@kit.edu

E-mail address: peer.kunstmann@kit.edu

E-mail address: schnaubelt@kit.edu