CENTER MANIFOLDS AND ATTRACTIVITY FOR QUASILINEAR PARABOLIC PROBLEMS WITH FULLY NONLINEAR DYNAMICAL BOUNDARY CONDITIONS

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ABSTRACT. We construct and investigate local invariant manifolds for a large class of quasilinear parabolic problems with fully nonlinear dynamical boundary conditions and study their attractivity properties. In a companion paper we have developed the corresponding solution theory. Examples for the class of systems considered are reaction–diffusion systems or phase field models with dynamical boundary conditions and to the two–phase Stefan problem with surface tension.

1. INTRODUCTION

Quasilinear parabolic evolution equations have been studied intensively in the past decades. In recent years problems with dynamical boundary conditions have attracted a lot interest in this context. Moreover, after a transformation to a fixed domain problems such as the Stefan problem with surface tension yield a quasilinear problem with a nonlinear dynamical boundary condition, see e.g. [7] and [17]. In the companion paper [20] we have identified a general class of systems comprising these examples (see (1.1)), developed a solution theory for such systems, and treated stable and unstable manifolds. In the introduction of [20] we have given further references to papers dealing with the Stefan problem and with reaction–diffusion systems or phase field models with dynamical boundary conditions.

A crucial step in the study of nonlinear equations is the investigation of the long–time behavior of solutions near a given equilibrium w_*. Typically, the structure of the flow in a neighborhood of a w_* is largely determined by the spectrum of the linearization at w_*, see e.g. [3], [6], [11], [12], [19], [21], [23], [25] and [26]. In this paper we construct center–like invariant manifolds for quasilinear parabolic problems with fully nonlinear dynamical boundary conditions and show that the center manifold $\mathcal{M}_{\mathrm{c}}$ attracts the center–stable manifold with a tracking solution if the flow on $\mathcal{M}_{\mathrm{c}}$ is stable, see Theorem 6.5. Our results are applied to the Stefan problem with surface tension in Examples 5.4 and 6.6.

In earlier work [8], [9] and [10] we have studied the case of quasilinear problems with fully nonlinear boundary conditions. These problems have led to

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various new problems, e.g. one had to parametrize the nonlinear solution manifold and the resulting chart enters in the fixed point problems. The necessary spectral information is carried by the semigroup governing of the linearized problem with 0 boundary conditions. To combine it with the nonlinear boundary conditions, we had to use extrapolation theory for the semigroup. Another difficult came from the solution spaces for such problems which involve time regularity. This forced us to introduce nonautonomous and nonlocal cutoff functions in the contruction of local center–like manifolds.

In this paper we also add a dynamical boundary condition and consider the system

$$\begin{aligned}
\partial_t u(t) + \mathcal{A}(u(t),\rho(t))u(t) &= \mathcal{R}(u(t),\rho(t),\dot\rho(t)), && \text{on } \Omega,\ \ t>0,\\
\partial_t \rho(t) + \mathcal{D}_0(u(t),\rho(t)) &= 0, && \text{on } \Omega,\ \ t>0,\\
\mathcal{D}_j(u(t),\rho(t)) &= 0, && \text{on } \partial\Omega,\ \ t\ge 0,\ \ j=1,\cdots,m,\\
u(0) &= u_0, && \text{on } \Omega,\\
\rho(0) &= \rho_0, && \text{on } \Sigma,
\end{aligned} \tag{1.1}$$

on a spatial domain Ω which either has the smooth boundary Σ (one phase setting) or is the disjoint union of two domains whose boundary consists of the common part Σ and possibly of further disjoint 'outer parts' (two phase setting). On these outer parts we impose linear boundary conditions not shown in (1.1). The solutions u and ρ take values in finite dimensional vector spaces.

In Ω act the main quasilinear diffusion type operator $\mathcal{A}$ of (differential) order $2m$ and the lower order perturbation $\mathcal{R}$. On the boundary we have a dynamical boundary condition governed by the nonlinear term $\mathcal{D}_0$ and static boundary condition governed by $\mathcal{D}_1,\cdots,\mathcal{D}_m$. One can also consider this system as an evolution equation for the function $w=(u,\rho)$, where u and ρ are directly coupled via the nonlinearities and also via the static boundary condition. In the operators $\mathcal{D}_j$ the orders with respect to u are stricly less than $2m$. However, the orders in ρ are not bounded apriori. The solution space for ρ has to be adapted to the degree of unboundedness of these operators. We will assume that the nonlinearities are C^1 on the solution spaces of the linear theory and that the resulting linearized boundary value problems are normally elliptic and satisfy Lopatinsky–Shapiro conditions. (See Section 2.)

Our approach is based on results about maximal regularity of type L_p for inhomogeneous linear boundary value problems from [5]. In our work the equations in (1.1) at the boundary are understood classically and the evolution equation in Ω holds in L_p sense. This setting was proposed for the Stefan problem with surface tension in [7] and has proved to be very successful, see e.g. [17]. In other approaches boundary conditions are understood only in a weak sense on the state space of the resulting flow, see e.g. [1], [2], [22], [23]. Another possibility is the treatment in the framework of higher regularity which also covers fully nonlinear problems, but requires more compatibility conditions and does not give smoothing effects, see e.g. [11], [12].

One main difficulty in (1.1) is the occurence of a time derivative of the second component ρ in the evolution equation for u. Such terms arise if one transforms a problem with moving boundaries to a fixed domain, cf. Example 2.2 in [20] or [7], [17]. In the solution of the nonlinear problem this term can be treated as a perturbation, which requires the solution space of ρ to provide extra time regularity of $\partial_t \rho$.

In [20] we have established the local wellposedness of (1.1) and showed a smoothing effect of the solution with corresponding estimates which give extra regularity of some of the invariant manifolds, see e.g. Theorem 4.6(f). This property is crucial for the convergence analysis in Section 6. The solution manifold still incorporates the static boundary conditions (as in [8], [9], [10]), but now also a 'dynamical' regularity constraint coming from the dynamical boundary condition. The latter arises because $\partial_t \rho(t)$ possesses extra space regularity which must also be fulfilled by $\mathcal{D}_0(u(t), \rho(t))$. We found a suitable parametrization for the solution manifold which allowed to handle this difficulty. However, these nonlinear compatibility conditions play an important role all over the paper.

Moreover, the semigroup generated by the linearization lives on a smaller state space than the nonlinear problem which makes it diffucult to combine the maximal regularity theory for the linearized problems (from [5]) with the spectral decompositions of this semigroup. For instance, the spectral projections do not leave invariant the solution spaces, which causes trouble in Section 6.

In our main results we then develop a rather complete theory for the center–like manifolds in Theorems 4.6, 5.1, 5.2 and Corollary 5.3. If the flow on the center manifold is stable, we can show that small solutions and those starting on the center–stable manifold converge to a solution on the center manifold, see Theorems 6.3 and 6.5. In our arguments we use the implicit function theorem and methods knwon from dynamical systems, but these tools have to be combined with the sophisticated technical devices needed for the system (1.1).

We apply our results to the Stefan problem with surface tension in Examples 5.4 and 6.6, using the description of equilibria and the spectrum of the linearization provided by [15]. Except for a degenerate case, the center manifold here only consists of equilibria, and depending on the parameters there is a one dimensional unstable manifold or none. As noted in these examples in the recent paper [17] the asmptotic stability of the center manifold of a closely related system was shown if there is no unstable spectrum, and also global properties were established there. Our results provide here additional information, as described in Examples 5.4 and 6.6. Moreover, the approach of [17] only works for center manifolds consisting of equlibria only (see [16] and [18] for related results for linear boundary conditions). Our apporach is more flexible and also suited for bifurcation arguments as in [22], for instance.

In Sections 2 and 3 we recall the setting and the solution theory used and established in [20] to make this paper readable independently of [20] (though it makes it also longer). In Section 4 we treat the center manifold and recall the necessary facts about a nonautonomous cut off introduced in [10]. The following section is devoted to the center–stable and center–unstable manifolds. The last section treats the convergence analysis.

2. Setting and function spaces

We first describe our setting and introduce the relevant spaces. For more details, references and proofs, we refer to Section 2 of [20]. We denote by c a generic constant and by $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ a generic nondecreasing function with $\varepsilon(r) \to 0$ as $r \to 0$. Moreover, $J \subset \mathbb{R}$ is an interval with a nonempty interior. Other notation is listed in the introduction of [20].

We fix numbers $m \in \mathbb{N}$, $m_j \in \{0, 1, 2m-1\}$, and $k_j \in \mathbb{N}_0 \cup \{-\infty\}$ for $j \in \{0, 1, \dots, m\}$, describing the order of the differentiable operators appearing in (1.1), where $k_j = -\infty$ if $\mathcal{D}_j$ does not depend on ρ, see (R) and (2.15) below. We have $m_j < m$, but k_j is not restricted apriori. We consider two different types of domains.

In the *one phase setting*, let $\Omega \subset \mathbb{R}^n$ be an open connected set with a compact boundary $\partial\Omega$ of class $C^{2m+\ell-m_0}$ and outer unit normal $\nu(x)$, where $\ell \in \{m_0, m_0+1, \cdots\}$ is given by (2.5) below.

In the *two phase setting*, let $\Omega = \Omega_1 \dot\cup \Omega_2$ for two open subsets $\Omega_j \subseteq \mathbb{R}^n$ having compact boundaries of class $C^{2m+\ell-m_0}$, where $\partial\Omega_j = \Sigma \dot\cup \Gamma_j$ for $j = 1, 2$, $\partial\Omega_1 \cap \partial\Omega_2 = \Sigma$, and Γ_j may be empty. In this case, $\nu(x)$ is the outer normal of the interface Σ with respect to Ω_1.

We set $\Sigma := \partial\Omega$ and $\Gamma_1 = \Gamma_2 := \emptyset$ in the one phase case. Since we will impose fixed linear homogeneous boundary conditions on Γ_j, in both settings Σ is the important part of the boundary.

Throughout this paper, we fix a finite exponent $p \in (n+2m, \infty)$. Let V_u and V_ρ be finite dimensional Banach spaces with norms $|\cdot|$, being the range spaces of the solutions to (1.1). As function spaces on Ω we use

$$
\begin{aligned}
&X = L_p(\Omega; V_u) \qquad \text{in the one and in the two phase case;}\\
&X_\gamma = W_p^{2m(1-1/p)}(\Omega; V_u), \quad X_1 = W_p^{2m}(\Omega; V_u) \qquad \text{in the one phase case;}\\
&X_1 = \{v \in W_p^{2m}(\Omega; V_u) \,|\, B^0 v = 0\}, \quad X_\gamma = (X, X_1)_{1-\frac{1}{p},p} \quad \text{in the two phase case,}
\end{aligned}
$$

where B^0 is an m-tupel of fixed linear boundary operators on $\Gamma_1 \cup \Gamma_2$ which are given by (2.16) below. We have $X_\gamma \subseteq \{v \in W_p^{2m(1-1/p)}(\Omega; V_u) \,|\, B^0 v = 0\}$ in the two phase case. At the boundary we employ the spaces

$$
\begin{aligned}
&Y_u = L_p(\Sigma; V_u), \quad Y_{j\gamma} = W_p^{2m\kappa_j - 2m/p}(\Sigma; V_u), \quad Y_{j1} = W_p^{2m\kappa_j}(\Sigma; V_u),\\
&Y_\rho = L_p(\Sigma; V_\rho), \quad Y_{0\gamma} = W_p^{2m\kappa_0 - 2m/p}(\Sigma; V_\rho), \quad Y_{01} = W_p^{2m\kappa_0}(\Sigma; V_\rho),\\
&Y_k = Y_{0k} \times \cdots \times Y_{mk}, \qquad \widehat{Y}_k = Y_{1k} \times \cdots \times Y_{mk},\\
&Z = W_p^{2m\kappa_0}(\Sigma; V_\rho) = Y_{01}, \qquad Z_1 = W_p^{\ell+2m\kappa_0}(\Sigma; V_\rho),
\end{aligned}
$$

for $j \in \{1, \cdots, m\}$, $k \in \{\gamma, 1\}$, and the numbers

$$
\kappa_j = 1 - \frac{m_j}{2m} - \frac{1}{2mp}, \qquad j = 0, 1, \dots, m. \tag{2.1}
$$

We observe that $X_1 \hookrightarrow X_\gamma \hookrightarrow X$, $Y_{j1} \hookrightarrow Y_{j\gamma} \hookrightarrow Y_u$, $Y_{01} \hookrightarrow Y_{0\gamma} \hookrightarrow Y_\rho$,

$$\begin{aligned} &X_\gamma \hookrightarrow C_0^{2m-1}(\overline{\Omega}; V_u) \qquad \text{in the one phase case,} \\ &X_\gamma \hookrightarrow C_0^{2m-1}(\overline{\Omega_1}; V_u) \times C_0^{2m-1}(\overline{\Omega_2}; V_u) \qquad \text{in the two phase case;} \\ &Y_{j\gamma} \hookrightarrow C^{2m-1-m_j}(\Sigma; V_u), \quad \text{and} \quad Y_{0\gamma} \hookrightarrow C^{2m-1-m_0}(\Sigma; V_\rho) \end{aligned} \tag{2.2}$$

for $j = 1, \ldots, m$. The base space and solution space for u in (1.1) are

$$\begin{aligned} &\mathbb{E}(J) = L_p(J; L_p(\Omega; V_u)) = L_p(J; X) \qquad \text{and} \\ &\mathbb{E}_u(J) = W_p^1(J; X) \cap L_p(J; X_1) \ \subseteq \ W_p^1(J; L_p(\Omega; V_u)) \cap L_p(J; W_p^{2m}(\Omega; V_u)), \end{aligned}$$

respectively, where the last inclusion is an equality in the one phase case. If J not compact, we write $\mathbb{E}_{\mathrm{loc}}(J)$ for the space of functions whose their restrictions to each interval $[a, b] \subseteq J$ belong to $\mathbb{E}([a, b])$. Analogous notations are used for $\mathbb{E}_u$ and the other function spaces introduced below.

We denote by $\gamma_t : u \mapsto u(t)$ the trace operator at $t \in \overline{J}$ (if defined). It holds

$$\begin{aligned} &\mathbb{E}_u(J) \hookrightarrow C_{ub}(J; X_\gamma) \hookrightarrow C_{ub}(J; C_0^{2m-1}(\overline{\Omega}; V_u)) \quad \text{(one phase),} \\ &\mathbb{E}_u(J) \hookrightarrow C_{ub}(J; X_\gamma) \hookrightarrow C_{ub}(J; C_0^{2m-1}(\overline{\Omega_1}; V_u) \times C_0^{2m-1}(\overline{\Omega_2}; V_u)) \quad \text{(two phase);} \\ &\gamma_t : \mathbb{E}_u(J) \to X_\gamma \quad \text{is continuous and has a bounded right inverse} \end{aligned} \tag{2.3}$$

for all $t \in \overline{J}$. The norms of the first embeddings in (2.3) are uniform for J of length greater than a fixed $d_0 > 0$. For functions vanishing at $t = \inf J$, this constant can be chosen independent of J.

In view of e.g. Section 3 of [4], the natural trace spaces of the solution space $\mathbb{E}_u$ are given by

$$\begin{aligned} \mathbb{F}_j(J) &= W_p^{\kappa_j}(J; L_p(\Sigma; V_u)) \cap L_p(J; W_p^{2m\kappa_j}(\Sigma; V_u)) = W_p^{\kappa_j}(J; Y_u) \cap L_p(J; Y_{j1}), \\ \mathbb{F}_0(J) &= W_p^{\kappa_0}(J; L_p(\Sigma; V_\rho)) \cap L_p(J; W_p^{2m\kappa_0}(\Sigma; V_\rho)) = W_p^{\kappa_0}(J; Y_\rho) \cap L_p(J; Y_{01}) \end{aligned}$$

for $j \in \{1, \cdots, m\}$ endowed with their canonical norms, where we put

$$\mathbb{F}(J) = \mathbb{F}_0(J) \times \cdots \times \mathbb{F}_m(J) \quad \text{and} \quad \widehat{\mathbb{F}}(J) = \mathbb{F}_1(J) \times \cdots \times \mathbb{F}_m(J).$$

We further have

$$\begin{aligned} &\mathbb{F}_j(J) \hookrightarrow C_{ub}(J; Y_{j\gamma}) \hookrightarrow C_{ub}(J \times \Sigma; V) \\ &\gamma_t : \mathbb{F}_j(J) \to Y_{j\gamma} \quad \text{is continuous and has a bounded right inverse} \end{aligned} \tag{2.4}$$

for all $t \in \overline{J}$, where $j = 0, 1, \ldots, m$ and we write $V = V_u$ if $j \geq 1$ and $V = V_\rho$ if $j = 0$. The same remarks as after (2.3) apply.

The solution space $\mathbb{E}_\rho$ for ρ in (1.1) is more sophisticated. It is chosen such that the operators $\mathcal{D}_j$ in (1.1) map $\mathbb{E}_\rho$ into the trace spaces $\mathbb{F}_j$ of the solutions u. We follow [5] and put $\widetilde{\mathcal{J}} = \big\{ j \in \{0, 1, \ldots, m\} \,\big|\, k_j \neq -\infty \big\}$ as well as

$$\ell_j = k_j - m_j + m_0, \qquad \ell = \max_{j=0,1,\ldots,m} \ell_j \geq m_0. \tag{2.5}$$

We then define

$$\begin{aligned} \mathbb{E}_\rho(J) = \ &W_p^{1+\kappa_0}\big(J; L_p(\Sigma; V_\rho)\big) \cap L_p\big(J; W_p^{\ell+2m\kappa_0}(\Sigma; V_\rho)\big) \\ &\cap W_p^1\big(J; W_p^{2m\kappa_0}(\Sigma; V_\rho)\big) \cap \bigcap\nolimits_{j \in \widetilde{\mathcal{J}}} W_p^{\kappa_j}\big(J; W_p^{k_j}(\Sigma; V_\rho)\big). \end{aligned} \tag{2.6}$$

Observe that ρ has extra space and time regularity compared to u. This is needed in important applications and for the underlying linear theory, see e.g. Example 5.4 and Proposition 3.5. We further need the embeddings

$$\begin{aligned}
&\partial^\beta \in \mathcal{B}(\mathbb{E}_\rho(J), \mathbb{F}_j(J)), \quad \mathbb{E}_\rho(J) \hookrightarrow C_{ub}(J; Z_\gamma), \quad \partial_t \in \mathcal{L}(\mathbb{E}_\rho(J), C_{ub}(J; Z_\gamma^1)) \\
&(\gamma_t, \gamma_t \partial_t) \in \mathcal{L}(\mathbb{E}_\rho(J), Z_\gamma \times Z_\gamma^1) \quad \text{has a bounded right inverse,} \qquad (2.7) \\
&\gamma_t \in \mathcal{L}(\mathbb{E}_\rho(J), Z_\gamma) \quad \text{has a bounded right inverse,}
\end{aligned}$$

if $|\beta| \le k_j$ and $t \in \overline{J}$. The spaces Z_γ and Z_γ^1 are Slobodeckii spaces on Σ. Their order depends on the cases $\ell < 2m$, $\ell = 2m$ and $\ell > 2m$. Similarly, depending on these cases one obtains a simpler description of the space $\mathbb{E}_\rho$, see [5] and also [13] and [20]. These results are omitted here since we do not need them, but we note that $Z_1 \hookrightarrow Z_\gamma \hookrightarrow Z$. To formulate (1.1) on product spaces, we set

$$E = X \times Z, \quad E_1 = X_1 \times Z_1, \quad E_\gamma = X_\gamma \times Z_\gamma, \quad \mathbb{E}_1(J) = \mathbb{E}_u(J) \times \mathbb{E}_\rho(J).$$

We note that the index 1 refers to the basic domain of the respective operators, 0 to the range space and γ to the spaces given by time traces, where one has control uniform in time.

Throughout, W_γ denotes a nonempty convex open subset of E_γ on which the operators in (1.1) will be defined. We set

$$W_1 = \{w_0 \in E_1 \,\big|\, w_0 \in W_\gamma\}, \quad \mathbb{W}_1(J) = \{w \in \mathbb{E}_1(J) \,\big|\, w(t) \in W_\gamma \ (\forall\, t \in J)\} \qquad (2.8)$$

The nonlinear maps in (1.1) shall satisfy

(R) $\mathcal{A} \in C^1(W_\gamma; \mathcal{L}(X_1, X))$, $\mathcal{R} \in C^1(W_\gamma \times Y_{0\gamma}; X)$, and $\mathcal{D} = (\mathcal{D}_0, \dots, \mathcal{D}_m) \in C^1(W_1; Y_1)$ induces a map $\mathcal{D} \in C^1(\mathbb{W}_1(J); \mathbb{F}(J))$ for any compact J. The first derivatives of these maps are bounded and uniformly continuous on all closed balls.

We consider $\mathcal{A}'(w)$ as bilinear map from $E_\gamma \times X_1$ to X and $\mathcal{A}'(w)v$ as a bounded linear map from E_γ to X, where $w \in W_\gamma$ and $v \in X_1$. The embeddings (2.3), (2.4), (2.7) and (2.13) then imply that these operators also induce maps

$$\begin{aligned}
&\mathcal{A} \in C^1(\mathbb{W}_1(J); C_b(J; \mathcal{L}(X_1, X))) \cap C^1(\mathbb{W}_1(J) \times L_p(J; X_1); \mathbb{E}(J)), \\
&\mathcal{R} \in C^1(\mathbb{W}_1(J); C_b(J; X)), \qquad \mathcal{D} \in C^1(W_\gamma; Y_\gamma),
\end{aligned}$$

respectively. We set $\widehat{\mathcal{D}} = (\mathcal{D}_1, \dots, \mathcal{D}_m)$. Some results require one more degree of smoothness than (R), namely

(RR) Condition (R) holds and the maps $\mathcal{A}' : W_\gamma \to \mathcal{L}_2(E_\gamma \times X_1, X)$, $\mathcal{R}' : W_\gamma \times Y_{0\gamma} \to \mathcal{L}(E_\gamma \times Y_{0\gamma}, X)$, $\mathcal{D}' : \mathbb{W}_1(J) \to \mathcal{L}(\mathbb{E}_1(J), \mathbb{F}(J))$ are Lipschitz on closed balls.

We further impose ellipticity conditions on the linearizations of our nonlinear maps $\mathcal{A}$, $\mathcal{R}$ and $\mathcal{D}_j$ at $w_* \in \mathbb{W}_1(J)$, given by

$$\begin{aligned}
B_j(t) &= \partial_1 \mathcal{D}_j(u_*(t), \rho_*(t)) \in \mathcal{L}(X_1, Y_{j1}) \cap \mathcal{L}(X_\gamma, Y_{j\gamma}), \\
C_j(t) &= \partial_2 \mathcal{D}_j(u_*(t), \rho_*(t)) \in \mathcal{L}(Z_1, Y_{j1}) \cap \mathcal{L}(Z_\gamma, Y_{j\gamma}), \\
A(t) &= \mathcal{A}(w_*(t)) \in \mathcal{L}(X_1, X), \qquad (2.9) \\
A_{*u}(t) &= \mathcal{A}(w_*(t)) + \partial_1 \mathcal{A}(u_*(t), \rho_*(t)) u_*(t) - \partial_1 \mathcal{R}(u_*(t), \rho_*(t), \dot\rho_*(t)) \in \mathcal{L}(X_1, X), \\
A_{*\rho}(t) &= \partial_2 \mathcal{A}(u_*(t), \rho_*(t)) u_*(t) - \partial_2 \mathcal{R}(u_*(t), \rho_*(t), \dot\rho_*(t)) \in \mathcal{L}(Z_\gamma, X),
\end{aligned}$$

$A_{*\dot\rho}(t) = -\partial_3\mathcal{R}(u_*(t),\rho_*(t),\dot\rho_*(t)) \in \mathcal{L}(Y_{0\gamma},X),$

$A_*(t) = (A_{*u}(t), A_{*\rho}(t), A_{*\dot\rho}(t)) \in \mathcal{L}(X_1 \times Z_\gamma \times Y_{0\gamma}, X).$

$j \in \{0,1,\dots,m\}$ and $t \in J$. For a time independent $w_0 = (u_0,\rho_0) \in W_\gamma$, we take some $(u_*,\rho_*) \in \mathbb{W}_1([0,1])$ with $u_*(0) = u_0$ and $\rho_*(0) = \rho_0$ (e.g. with $\dot\rho_*(0) = 0$) and write $A = A(0)$, $B_j = B_j(0)$ and $C_j = C_j(0)$, cf. (2.3), (2.4), (2.7). For $(w_0, y_0) \in W_1 \times Y_{0\gamma}$ we define A_* by inserting (w_0,y_0) instead of $(w_*(t),\dot\rho_*(t))$. For an equilibrium w_0 we will always take $y_0 = 0$. We abbreviate

$$B = (B_0,\dots,B_m),\quad \widehat{B} = (B_1,\dots,B_m),\quad C = (C_0,\dots,C_m),\quad \widehat{C} = (C_1,\dots,C_m).$$

We also make use of the operator matrices

$$\Lambda = \begin{pmatrix} A & 0 \\ B_0 & C_0 \end{pmatrix}, \qquad \Lambda_* = \begin{pmatrix} A_{*u} - A_{*\dot\rho}B_0 & A_{*\rho} - A_{*\dot\rho}C_0 \\ B_0 & C_0 \end{pmatrix} \tag{2.10}$$

acting from E_1 to E, see (2.9). We see below that $-\Lambda_*$ induces the generator of an analytic semigroup which is crucuial for our analysis. For $\ell \le 2m$ this semigroup acts in E, but for $\ell > 2m$ it lives in the smaller space E_0 defined by

$$Z_0 = B^{\varsigma}_{pp}(\Sigma; V_\rho), \qquad E_0 = X \times Z_0. \tag{2.11}$$

Here, $\varsigma = 2m\kappa_0$ and thus $Z_0 = Z$ if $\ell \le 2m$, but $\varsigma > 2m\kappa_0$ and thus $Z_0 \hookrightarrow Z$ if $\ell > 2m$, see [5] or [20]. The space Z_0 occurs naturally in view of the embedding

$$\mathbb{E}_\rho \hookrightarrow W^1_p(J; Z_0) \tag{2.12}$$

The trace spaces are ordered as

$$Z_0 \hookrightarrow Z, \qquad Z_1 \hookrightarrow Z_\gamma \hookrightarrow Z_0 \hookrightarrow Z^1_\gamma \hookrightarrow Y_{0\gamma}. \tag{2.13}$$

The domain of the generators will contain compatibility conditions expressed by the spaces

$$\begin{aligned} \widetilde{E}_\gamma &= \{(v,\sigma) \in E_\gamma \,\big|\, B_0 v + C_0\sigma \in Z^1_\gamma\}, \\ E^0_\gamma &= \{(v,\sigma) \in \widetilde{E}_\gamma \,\big|\, \widehat{B}v + \widehat{C}\sigma = 0\}, \\ E^0_1 &= \{(v,\sigma) \in E_1 \,\big|\, B_0 v + C_0\sigma \in Z_0,\ \widehat{B}v + \widehat{C}\sigma = 0\}, \end{aligned} \tag{2.14}$$

which are Banach spaces endowed with the canonical norms $|(v,\sigma)|_{E_\gamma} + |B_0 v + C_0\sigma|_{Z^1_\gamma}$ and $|(v,\sigma)|_{E_1} + |B_0 v + C_0\sigma|_{Z_0}$, respectively, due to (2.9) and (2.13).

We equip Λ with the domain $D(\Lambda) = E^0_1$ and denote by Λ_0 the restriction of Λ_* to $D(\Lambda_0) = E^0_1$.

To apply [5], the operators in (2.9) have to be differential operators of the following form, where we insert $(u_*,\rho_*) \in \mathbb{W}_1([0,T])$ and any $T > 0$:

$$A(t)v(x) = \sum_{|\alpha|\le 2m} a_\alpha(t,x)D^\alpha v(x), \qquad B_j(t)v(y) = \sum_{|\beta|\le m_j} b_{j\beta}(t,y)\gamma_\Omega D^\beta v(y)$$

$$C_j(t)\sigma\circ \mathrm{g}(z) = \sum_{|\gamma|\le k_j} c^{\mathrm{g}}_{j\gamma}(t,z)D^\gamma_{n-1}(\sigma\circ \mathrm{g})(z) \tag{2.15}$$

for $(v,\sigma) \in E_\gamma$, $j \in \{0,1,\dots,m\}$, $x \in \Omega$, $y \in \Sigma$, $t \in [0,T]$, local coordinates g for Σ and z belonging to the domain of g in $\mathbb{R}^{n-1}$. Usually we omit the trace operator γ_Ω on Ω here. In the two phase case the term $b_{j\beta}(t,y)\gamma_\Omega D^\beta v(y)$ is understood as $b^1_{j\beta}(t,y)\gamma^1_\Omega D^\beta v(y) - b^2_{j\beta}(t,y)\gamma^2_\Omega D^\beta v(y)$ where γ^i_Ω gives the trace of functions on Ω_i to the interface Σ. Still in the two phase case, on the (possibly empty) outside boundaries Γ_1 and Γ_2 we consider boundary operators

$$B_j^0 v(y) = \sum_{|\beta|\le m_j^0} b_{j\beta}^0(y)\gamma_\Omega D^\beta v(y), \quad B^0 := (B_1^0, \dots, B_m^0), \tag{2.16}$$

of order $m_j^0 \in \{0, \cdots, 2m-1\}$ for $y \in \Gamma_1 \dot\cup \Gamma_2$ and $j = \{1, \dots, m\}$.

In view of (2.2) and the representation of Z_γ given in [5] (and recalled in Section 2 of [20]), one can see that the derivatives in the above operators are well defined. In Section 2 of [20] we stated the regularity assumption (S), the ellpiticity assumption (E) and the Lopatinskii-Shapito conditions(LS) and ($LS_\infty^\pm$) on these operators (which are taken from [5]). Here (S) and (E) are fairly standard, but the conditions (LS) and ($LS_\infty^\pm$) on the boundary operators is more involved than usual in particular if $\ell \neq 2m$. Below we only need the consequences of these conditions so that we do not recall them in this paper. We summarize our hypotheses for the wellposedness theory.

Hypothesis 2.1. *Let (R) and (S) from* [20] *be true, and (E), (LS) and (if $\ell \gtreqless 2m$) ($LS_\infty^\pm$) from* [20] *hold for every $w_0 = (u,\rho_0) \in W_\gamma$.*

We also recall the simple Lemma 2.8 from [20].

Lemma 2.2. *Let $a < b < d$, $q \in (1, \infty)$, $\kappa > 1/q$, and V be a Banach space. If $u \in W_q^\kappa((a, b); V)$ and $v \in W_q^\kappa((b, d); V)$ satisfy $u(b) = v(b)$ (where the trace exists by Sobolev's embedding), then the function w given by $w = u$ on $(a, b]$ and $w = v$ on $[b, d)$ belongs to $W_q^\kappa((a, d); V)$ with $\|w\|_{W_q^\kappa} \le c_W (\|u\|_{W_q^\kappa} + \|v\|_{W_q^\kappa})$.*

3. Solution theory

Our main results rely the following linearization setup, where we use the operators from (2.9) for $w_* = (u_*, \rho_*) \in \mathbb{W}_1 = \mathbb{W}^1(J)$. We put $\mathbb{W}_1^* = \mathbb{W}_1 - w_*$ and define the nonlinear maps

$$\begin{aligned} &F \in C^1(\mathbb{W}_1^*; \mathbb{E}) \quad \text{and} \quad G \in C^1(\mathbb{W}_1^*; \mathbb{F}) \quad \text{with} \\ &\text{loc. bdd. derivative,} \quad F(0) = 0,\ G(0) = 0 \quad \text{and} \quad F'(0) = 0,\ G'(0) = 0, \end{aligned} \tag{3.1}$$

by setting

$$\begin{aligned} F(v, \sigma) &= \big(\mathcal{A}(w_*)v - \mathcal{A}(w_* + (v, \sigma))v\big) \\ &\quad - \big(\mathcal{A}(w_* + (v, \sigma))u_* - \mathcal{A}(w_*)u_* - [\mathcal{A}'(w_*)u_*](v, \sigma)\big) \\ &\quad + \big(\mathcal{R}(w_* + (v, \sigma), \dot\rho_* + \dot\sigma) - \mathcal{R}(w_*, \dot\rho_*) - \mathcal{R}'(w_*, \dot\rho_*)(v, \sigma, \dot\sigma)\big), \\ G(v, \sigma) &= \mathcal{D}'(w_*)(v, \sigma) + \mathcal{D}(w_*) - \mathcal{D}(w_* + (v, \sigma)), \end{aligned} \tag{3.2}$$

for $(v, \sigma) \in \mathbb{W}_1^*$. We put $\widehat{G} = (G_1, \dots, G_m)$. It holds

$$\begin{aligned} F'(\varphi)(u, \rho) &= [\mathcal{A}(w_*) - \mathcal{A}(w_* + \varphi)]u + [\mathcal{A}'(w_*)u_* - \mathcal{A}'(w_* + \varphi)(u_* + v)](u, \rho) \\ &\quad + [\mathcal{R}'(u_* + v, \rho_* + \sigma, \dot\rho_* + \dot\sigma) - \mathcal{R}'(u_*, \rho_*, \dot\rho_*)](u, \rho, \dot\rho), \\ G'(\varphi)(u, \rho) &= [\mathcal{D}'(w_*) - \mathcal{D}'(w_* + \varphi)](u, \rho) \end{aligned} \tag{3.3}$$

for $\varphi = (v,\sigma) \in \mathbb{W}_1^*$ and $(u,\rho) \in \mathbb{E}_1$. The asserted mapping properties easily follow from (R) and the embeddings (2.3), (2.7), (2.13). Observe that $\mathcal{D}(w_*) = 0$ if w_* is an equilibrium of (1.1).

In order to treat the nonlinear compatibility conditions related to (1.1), we need an 'almost right inverse' of the map (B, C). It is given by the next lemma, which is Corollary 2.7 of [20].

Lemma 3.1. *Assume that Hypothesis 2.1 holds. Given $(u_0, \rho_0) \in W_\gamma$, take some $(u_*, \rho_*) \in \mathbb{W}_1([0,T])$ and $T > 0$ with $u_*(0) = u_0$ and $\rho_*(0) = \rho_0$. In (2.9) put $A = A(0)$, $B = B(0)$ and $C = C(0)$. Then there is a map $\mathcal{N}_\gamma \in \mathcal{L}(Y_\gamma, E_\gamma)$ such that $(\widehat{B}, \widehat{C})\mathcal{N}_\gamma = I_1$, $(B_0, C_0)\mathcal{N}_\gamma - I_0 \in \mathcal{L}(Y_\gamma, Z_\gamma^1)$, where $I_0(\psi_0, \dots, \psi_m) = \psi_0$ and $I_1(\psi_0, \dots, \psi_m) = (\psi_1, \dots, \psi_m) =: \widehat{\psi}$.*

Let $w_* = (u_*, \rho_*) \in \mathbb{W}_1(J)$ be a solution of (1.1) for some J with $\min J = 0$ and initial values (u_{0*}, ρ_{0*}). (In later sections w_* will be an equilibrium.) At each time t the solution belongs to the solution manifold

$$\mathcal{M} = \{w_0 = (u_0, \rho_0) \in W_\gamma \,\big|\, \widehat{\mathcal{D}}(w_0) = 0,\ \mathcal{D}_0(w_0) \in Z_\gamma^1\}. \tag{3.4}$$

For $(u_0, \rho_0) \in W_\gamma$ and $w = (u, \rho) \in \mathbb{E}_1(J)$, we put $(v_0, \sigma_0) = (u_0 - u_{0*}, \rho_0 - \rho_{0*})$ and $(v, \sigma) = (u - u_*, \rho - \rho_*)$. Using the linearization described above and (2.9), we see that $(u_0, \rho_0) \in \mathcal{M}$ if and only if (v_0, σ_0) belongs to

$$\begin{aligned}\mathcal{M}^* = \mathcal{M} - (u_{*0}, \rho_{*0}) = \{(v_0, \sigma_0) \in W_\gamma - (u_*, \rho_*) \,\big|\, (\widehat{B}, \widehat{C})(v_0, \sigma_0) = \widehat{G}(v_0, \sigma_0),\\ B_0 v_0 + C_0 \sigma_0 - G_0(v_0, \sigma_0) \in Z_\gamma^1\}.\end{aligned} \tag{3.5}$$

Moreover, $(u, \rho) \in \mathbb{W}_1$ solves (1.1) if and only if $(v, \sigma) \in \mathbb{W}_1^*$ solves

$$\begin{aligned}
\partial_t v(t) + A_*(t)(v(t), \sigma(t), \dot\sigma(t)) &= F(v, \sigma)(t), &&\text{on } \Omega,\ t \in (0, T],\\
\partial_t \sigma(t) + B_0(t)v(t) + C_0(t)\sigma(t) &= G_0(v, \sigma)(t), &&\text{on } \Sigma,\ t \in [0, T],\\
\widehat{B}(t)v(t) + \widehat{C}(t)\sigma(t) &= \widehat{G}(v, \sigma)(t), &&\text{on } \Sigma,\ t \in [0, T],\\
B^0 v(t) &= 0, &&\text{on } \Gamma_1 \cup \Gamma_2,\ t \in [0, T],\\
(v(0), \sigma(0)) &= (v_0, \sigma_0), &&\text{on } \Omega \times \Sigma.
\end{aligned} \tag{3.6}$$

Here drop the equation $B^0 u(t) = 0$ in the one phase setting. This equation is mostly omitted in the following since it is already contained in the domain of $A_*(t)$ and in the solution space.

For $w_{0*} \in \mathcal{M}$ and $\psi = (v_0, \sigma_0) \in \widetilde{E}_\gamma$ (see (2.14)), we further define

$$\begin{aligned}
\langle\psi\rangle_\gamma &= |\psi|_{E_\gamma} + [\psi]_\gamma, \quad [\psi]_\gamma = |\mathcal{D}_0(\psi + w_{0*}) - \mathcal{D}_0(w_*)|_{Z_\gamma^1} = |(B_0, C_0)\psi - G_0(\psi)|_{Z_\gamma^1},\\
\langle\psi\rangle_1 &= |\psi|_{E_1} + [\psi]_1, \quad [\psi]_1 = |\mathcal{D}_0(\psi + w_{0*} - \mathcal{D}_0(w_*)|_{Z_\gamma} = |(B_0, C_0)\psi - G_0(\psi)|_{Z_\gamma}
\end{aligned} \tag{3.7}$$

For a solution $\psi(t) = (v(t), \sigma(t))$ of (3.6), the above quantities simplify to

$$\langle\psi(t)\rangle_\gamma = |\psi(t)|_{E_\gamma} + |\dot\sigma(t)|_{Z_\gamma^1}, \qquad \langle\psi(t)\rangle_1 = |\psi(t)|_{E_1} + |\dot\sigma(t)|_{Z_\gamma}. \tag{3.8}$$

We note that $[\psi]_\gamma \le c\,|\psi|_{E_\gamma}$ if $\ell \le 2m$ since then $Z_\gamma^1 = Y_{0\gamma}$ as observed in Section 2 of [20], and thus $|\psi|_{E_\gamma}$ and $\langle\psi\rangle_\gamma$ are locally equivalent in this case. Given $r > 0$, we further introduce

$$\mathcal{M}^*(r) := \{\psi \in \mathcal{M}^* \,\big|\, \langle\psi\rangle_\gamma < r\}.$$

We recall Lemma 3.2 of [20] which gives a local chart for the above set.

Lemma 3.2. *In the setting of Lemma 3.1, we define G by (3.2) for some $w_{0*} = (u_{0*}, \rho_{0*}) \in \mathcal{M}$. Then the map $\mathcal{Q} = I - \mathcal{N}_\gamma G$ belongs to $C^1(W_\gamma - w_{0*}; E_\gamma)$ with a locally bounded derivative, $\mathcal{Q}(0) = 0$ and $\mathcal{Q}'(0) = I$. It maps $\mathcal{M}^*$ into E_γ^0 (see (2.14)) with $|\psi - \mathcal{N}_\gamma G(\psi)|_{E_\gamma^0} \leq c\,\langle\psi\rangle_\gamma$ for $\psi \in \mathcal{M}^*$. We can invert $I - \mathcal{N}_\gamma G$ on some ball $B_{E_\gamma}(0, r_0) \subseteq W_\gamma - w_*$ and set $h = \mathcal{N}_\gamma G(I - \mathcal{N}_\gamma G)^{-1}$. There is a radius $r > 0$ such that $\mathcal{M}^*(r)$ is the graph of h, i.e.,*

$$\mathcal{M}^*(r) = \{\psi = \xi + h(\xi) \,\big|\, \xi \in B_{E_\gamma^0}(0, r_0),\ \langle\psi\rangle_\gamma < r\}.$$

In particular, $w_{0*} + E_\gamma^0$ is the tangent plane of $\mathcal{M}$ at w_{0*} and $\mathcal{Q}$ is a local chart.

We next summarize Theorem 3.3 and Propositions 3.1, 3.4 and 3.5 from [20], omitting some details. They yield the local well–posedness and smoothing properties of (1.1). We write tw for the function $t \mapsto tw(t)$.

Theorem 3.3. *Let Hypothesis 2.1 hold. Let $w_{0*} = (u_{0*}, \rho_{0*}) \in \mathcal{M}$. Take $T \in (0, t^+(w_{0*}))$ and set $J = [0, T]$ and $J^+ = [0, t^+(w_{0*}))$, cf. (a). Then the following assertions are true.*

(a) There is a number $t^+(w_{0*}) > 0$ such that the problem (1.1) has a unique solution $w_* = w(\cdot; w_{0*}) = (u_*, \rho_*) \in \mathbb{W}_1([0, T]) \hookrightarrow C([0, T]; W_\gamma)$.

(b) There is a radius $r > 0$ such that for each $\varphi_0 = (v_0, \sigma_0) \in \mathcal{M}^*(r)$ there exists a solution $w = (u, \rho) \in \mathbb{W}_1(J)$ of (1.1) with $w(0) = w_0 = w_{0*} + \varphi_0$. Moreover, the map $\varphi_0 \mapsto w - w_*$ from $\mathcal{M}^*(r)$ to $\mathbb{W}_1(J)$ is C_b^1. It holds

$$\|w - w_*\|_{\mathbb{E}_1(J)} \leq c\,\langle w_0 - w_{0*}\rangle_\gamma = c\,|w_0 - w_{0*}|_{E_\gamma} + c\,|\mathcal{D}_0(w_0 + w_{0*}) - \mathcal{D}_0(w_{0*})|_{Z_\gamma^1}$$

We further have $t\partial_t w \in \mathbb{E}_1(J)$.

(c) In the setting of (b), assume also that $w_* \in E_1$ is an equilibrium of (1.1). Then there is an $r_1 \in (0, r]$ such that for $w_0 \in w_* + \mathcal{M}^*(r_1)$ and $T_0 \in (0, T)$ the solution $w = (u, \rho) = w(\cdot; w_0) \in \mathbb{W}_1([0, T])$ satisfies

$$|w(t) - w_*|_{E_1} + |\dot{w}(t)|_{E_\gamma} \leq c\,\langle w_0 - w_*\rangle_\gamma, \quad \|t\,\partial_t(w - w_*)\|_{\mathbb{E}_1([0,T])} \leq c\,\langle w_0 - w_*\rangle_\gamma,$$

for $t \in [T_0, T]$ and constants independet of t and w_0.

(d) In the setting of (b), assume also that (RR) holds and that $w_* = (u_*, \rho_*) \in \mathbb{W}_1([0, T_*])$ solves (1.1) with $w_*(0) = w_{0*} \in \mathcal{M}$. Take $T \in (0, T_*)$ and $T_0 \in (0, T)$. Then there is an $r_2 \in (0, r]$ such that for every $w_0 \in w_{0*} + \mathcal{M}^*(r_2)$ the solution $w = (u, \rho) \in \mathbb{W}_1([0, T])$ of (1.1) satisfies

$$\langle w(t) - w_*(t)\rangle_1 \leq c\,\langle w_0 - w_{*0}\rangle_\gamma\,, \quad \|t\,\partial_t(w - w_*)\|_{\mathbb{E}_1([0,T])} \leq c\,\langle w_0 - w_{*0}\rangle_\gamma,$$

for $t \in [T_0, T]$ and constants independent of t and w_0.

The next hypothesis will be assumed in the rest of the paper.

Hypothesis 3.4. *Let (R) and (S) of [20] be true, and (E), (LS) and (if $\ell \gtrless 2m$) ($LS_\infty^\pm$) of [20] hold for every $w_0 \in W_\gamma$. Let $w_* = (u_*, \rho_*) \in W_1$ be an equilibrium of (1.1) and define the maps A_*, B, C, F, G, Λ_* and $\Lambda_0 = \Lambda_*|E_1^0$ as well as the expressions $\langle\psi\rangle_\gamma$ and $\langle\psi\rangle_1$ for this w_* as in (2.9), (3.2), (2.10), (2.14), (3.7).*

Our main results are based on linearization at the equilibrium w_*. We collect the relevant results from [20], starting with the corrresponding the inhomogeneous problem. This proposition is a special case of Corollary 2.6 of [20] which in turn follows from results in [5] by perturbation. We look at the problem

$$\begin{aligned} \partial_t u(t) + A_{*u}u(t) + A_{*\rho}\rho(t) + A_{*\dot\rho}\dot\rho(t) &= f(t), && \text{on } \Omega,\ t\in(0,T],\\ \partial_t\rho(t) + B_0u(t) + C_0\rho(t) &= g_0(t), && \text{on } \Sigma,\ t\in[0,T],\\ \widehat{B}u(t) + \widehat{C}\rho(t) &= \widehat{g}(t), && \text{on } \Sigma,\ t\in[0,T],\\ B^0u(t) &= 0, && \text{on } \Gamma_1\cup\Gamma_2,\ t\in[0,T],\\ (u(0),\rho(0)) &= (u_0,\rho_0), && \text{on } \Omega\times\Sigma, \end{aligned} \tag{3.9}$$

(where we drop the equation $B^0u(t)=0$ in the one phase setting).

Proposition 3.5. *Assume that Hypothesis 3.4 holds. Then the following assertions are true.*

(a) There is a unique solution $(u,\rho)\in\mathbb{E}_1(J)$ of the problem (3.9) if and only if f, g, u_0 and ρ_0 belong to the data space

$$\mathbb{D}(J) := \big\{(u_0,\rho_0,f,g)\in X_\gamma\times Z_\gamma\times\mathbb{E}(J)\times\mathbb{F}(J)\,\big|\, B_ju_0 + C_j\rho_0 = g_j(0) \\ \textit{for } j=1,\dots,m;\quad g_0(0) - B_0u_0 - C_0\rho_0 \in Z^1_\gamma\big\}.$$

The corresponding solution operator $\mathcal{S}:\mathbb{D}(J)\to\mathbb{E}_1(J)$ is continuous. The norm of $\mathcal{S}$ is bounded uniformly in $T'\in(0,T]$ if we restrict it to the subspace $\mathbb{D}_0([0,T'])$ containing g with $g(0)=0$.

(b) The operator $\Lambda_0=\Lambda_*|E^0_1$ generates an analytic C_0–semigroup $T(\cdot)$ in E_0.

(c) There is a $\mu_0\ge 0$ larger than the growth bound of $-\Lambda_0$ such that for each $(u_0,\rho_0,f,g)\in\mathbb{D}(\mathbb{R}_+)$ there is a unique solution $(u,\rho)\in\mathbb{E}_1(\mathbb{R}_+)$ of

$$\begin{aligned} &\partial_t u(t) + (A_{*u}(t) + \mu - A_{*\dot\rho}(t)B_0(t))u(t)\\ &\qquad + (A_{*\rho}(t) - A_{*\dot\rho}(t)C_0(t))\rho(t) = f(t), && \textit{on } \Omega,\ t\in(0,T],\\ &\partial_t\rho(t) + B_0(t)u(t) + (C_0(t)+\mu)\rho(t) = g_0(t), && \textit{on } \Sigma,\ t\in[0,T],\\ &\widehat{B}(t)u(t) + \widehat{C}(t)\rho(t) = \widehat{g}(t), && \textit{on } \Sigma,\ t\in[0,T],\\ &B^0u(t) = 0, && \textit{on } \Gamma_1\cup\Gamma_2,\ t\in[0,T],\\ &(u(0),\rho(0)) = (u_0,\rho_0), && \textit{on } \Omega\times\Sigma, \end{aligned} \tag{3.10}$$

(where we drop the equation $B^0u(t)=0$ in the one phase setting) for each $\mu\ge\mu_0$, and it holds $\|(u,\rho)\|_{\mathbb{E}_1(\mathbb{R}_+)}\le c\,\|(u_0,\rho_0,f,g)\|_{\mathbb{D}(\mathbb{R}_+)}$.

The data space $\mathbb{D}(J)$ is endowed with the norm

$$\|f\|_{\mathbb{E}(J)} + \|g\|_{\mathbb{F}(J)} + |(u_0,\rho_0)|_{E_\gamma} + |g_0(0) - B_0u_0 - C_0\rho_0|_{Z^1_\gamma}.$$

It is a Banach space and $\mathbb{D}_0(J)$ is a closed subspace.

We need the *extrapolation space* E_{-1} which is the completion of E_0 with respect to the norm $|(\mu+\Lambda_0)^{-1}w|_{E_0}$ for any $\mu\ge\mu_0$. There is a bounded extension $-\Lambda_{-1}:E_0\to E_{-1}$ of $-\Lambda_0$ which is similar to $-\Lambda_0$ and generates the extension $T_{-1}(\cdot)$ of $T(\cdot)$ on E_{-1}. It further holds $T_{-1}(t)\in\mathcal{L}(E_{-1},E^0_1)$ for $t>0$.

A *solution* of the problem (1.1), (3.6), or (3.9) (or of some lines of them) on an (unbounded) interval J is a function $w\in\mathbb{E}^{\mathrm{loc}}_1(J)$ satisfying the respective problem. Let $\alpha, \beta \in \mathbb{R}$. To study our equations on unbounded time intervals we set $\mathrm{e}_\alpha(t) = \mathrm{e}^{\alpha t}$ for $t \in \mathbb{R}$, denoting restrictions of this function by the same symbol. Moreover, on $J = \mathbb{R}$ we fix a smooth, strictly positive function $\mathrm{e}_{\alpha,\beta}$ satisfying $\mathrm{e}_{\alpha,\beta}(t) = \mathrm{e}_\alpha(t)$ for $t \le -1$ and $\mathrm{e}_{\alpha,\beta}(t) = e_\beta(t)$ for $t \ge 1$. We then introduce the weighted spaces

$$\mathbb{E}_1(\mathbb{R}_\pm, \alpha) = \{w \,|\, \mathrm{e}_\alpha w \in \mathbb{E}_1(\mathbb{R}_\pm)\}, \qquad \mathbb{E}_1(\alpha, \beta) = \{w \,|\, \mathrm{e}_{\alpha,\beta} w \in \mathbb{E}_1(\mathbb{R})\}, \tag{3.11}$$

and their analogues for $\mathbb{E}$, $\mathbb{F}$ and $\mathbb{D}$, which are complete if endowed with the canonical norms $\|w\|_{\mathbb{E}_1(\mathbb{R}_+,\alpha)} = \|\mathrm{e}_\alpha w\|_{\mathbb{E}_1(\mathbb{R}_+)}$ etc. We also use the corresponding norms on compact intervals J. The embeddings (2.3) and (2.7) imply that

$$|\varphi(t)|_{E_\gamma} + |\dot{\sigma}(t)|_{Z^1_\gamma} \le |\mathrm{e}^{\delta t}\varphi(t)|_{E_\gamma} + |\mathrm{e}^{\delta t}\dot{\sigma}(t)|_{Z^1_\gamma} \le c\, \|\varphi\|_{\mathbb{E}_1(J,\delta)} \tag{3.12}$$

for $t \in J = \mathbb{R}_\pm$ and $\delta \ge 0$.

Let $w = (u, \rho)$ be the solution of (3.9) and $\widetilde{f} = f - A_{*\dot{\rho}} g_0$. We insert $\dot{\rho} = g_0 - B_0 u - C_0 \rho$ into the term $A_{*\dot{\rho}}\dot{\rho}$ in (3.9), obtaining

$$\partial_t w(t) + \Lambda_* w(t) = (f(t) - A_{*\dot{\rho}} g_0(t), g_0(t)) = (\widetilde{f}(t), g_0(t)), \qquad t \in [0, T]. \tag{3.13}$$

The next result allows to use the asymptotic behavior of $T(\cdot)$ (determined by $\sigma(\Lambda_0)$) in the investigation of the longterm behavior of the nonlinear problem (1.1), by means of the ‘mild formula’ in (d). Observe that part (c) decribes the difference between Λ_{-1} and Λ_* which expresses the impact of the boundary conditions. We define

$$\Pi = (\mu + \Lambda_{-1})\mathcal{N}_1.$$

Proposition 3.6. *Under Hypothesis 3.4, the following assertions hold.*

(a) There are operators $\mathcal{N}_1 \in \mathcal{L}(\widehat{Y}_1, E_1)$ and $R \in \mathcal{L}(E, E_1)$ such that $(\mu + \Lambda_*)\mathcal{N}_1 = 0$ and $(\widehat{B}, \widehat{C})\mathcal{N}_1 = I_{\widehat{Y}_1}$, as well as $(\mu + \Lambda_*)R = I_E$ and $(\widehat{B}, \widehat{C})R = 0$.

(b) We have $E \hookrightarrow E_{-1}$ and $\Lambda_{-1} w = \Lambda_* w$ for all $w \in E_1$ with $(\widehat{B}, \widehat{C})w = 0$.

(c) It holds $\Pi \in \mathcal{L}(\widehat{Y}, E_{-1})$ and $\Lambda_* w = \Lambda_{-1} w - \Pi(\widehat{B}, \widehat{C})w$ for all $w \in E_1$.

(d) Let $J = [0, T]$, $(w_0, f, g) \in \mathbb{D}(J)$, and put $\widetilde{f} := f - A_{*\dot{\rho}} g_0 \in \mathbb{E}(J)$. Then the solution $w \in \mathbb{E}_1(J)$ of (3.9) is given by

$$w(t) = T(t)w_0 + \int_0^t T_{-1}(t-\tau)[(\widetilde{f}(\tau), g_0(\tau)) + \Pi\widehat{g}(\tau))\,\mathrm{d}\tau, \quad t \in J. \tag{3.14}$$

Moreover, w is the solution of (3.10) with data $(w_0, \widetilde{f}, g)$ and $\mu = 0$, where we have $\|\widetilde{f}\|_{\mathbb{E}(J)} \le c\,(\|f\|_{\mathbb{E}(J)} + \|g_0\|_{L_p(J;Y_{0\gamma})}) \le c\,(\|f\|_{\mathbb{E}(J)} + \|g_0\|_{\mathbb{F}(J)})$.

In the following we rewrite the solutions of (3.9) on unbounded time intervals $J \in \{\mathbb{R}_\pm, \mathbb{R}\}$ as in (3.14). We first recall some results from [20] for the case that the (rescaled) semigroup $\{\mathrm{e}^{\delta t}T(t)\}_{t\ge 0}$ has an exponential dichotomy for $\delta \in [\delta_1, \delta_2]$. Let $P \in \mathcal{L}(E_0)$ be the corresponding (stable) spectral projection for $-\Lambda_0 + \delta$ and set $Q = I - P$. Then, $P \in \mathcal{L}(E_1^0)$, P commutes with $T(t)$ and Λ_0, $Q \in \mathcal{L}(E_0, E_1^0)$, $T(t)$ is invertible on QE_0 with the inverse $T_Q(-t)Q$, and $\|\mathrm{e}^{t\delta}T(t)P\|_{\mathcal{L}(E_0)}$, $\|\mathrm{e}^{-t\delta}T_Q(-t)Q\|_{\mathcal{L}(E_0)} \le c\mathrm{e}^{-\epsilon t}$ for $t \ge 0$ and some $\epsilon > 0$. Further, there are extensions $P_{-1} \in \mathcal{L}(E_{-1})$ of P and $Q_{-1} \in \mathcal{L}(E_{-1}, E_1^0)$ of Q

such that $T_{-1}(t)$ has an exponential dichotomy on E_{-1} with the same constants. From $P = I - Q$, we deduce

$$P \in \mathcal{L}(E_1) \cap \mathcal{L}(E_\gamma) \cap \mathcal{L}(E_\gamma^0) \quad \text{and} \quad P_{-1} \in \mathcal{L}(E). \tag{3.15}$$

We partly omit the subscript -1. (Compare e.g. §2 of [9] for these facts.) It further holds:

$$\text{If} \quad (w_0, f, g) \in \mathbb{D}(J), \quad \text{then} \quad (Pw_0, f, g) \in \mathbb{D}(J). \tag{3.16}$$

Let $\mathrm{e}_\delta T(\cdot)$ have an exponential dichotomy. Given $(\varphi_0, f, g) \in E_\gamma \times \mathbb{E}(\mathbb{R}_+, \delta) \times \mathbb{F}(\mathbb{R}_+, \delta)$, resp. $(\varphi_0, f, g) \in E_{-1} \times \mathbb{E}(\mathbb{R}_-, \delta) \times \mathbb{F}(\mathbb{R}_-, \delta)$, we can then define

$$\begin{aligned} L^+_{P,\Lambda_0}(\varphi_0, f, g)(t) &= T(t)\varphi_0 + \int_0^t T_{-1}(t-\tau)P_{-1}[(\widetilde{f}(\tau), g_0(\tau)) + \Pi\widehat{g}(\tau)]\,\mathrm{d}\tau \\ &\quad - \int_t^\infty T_Q(t-\tau)Q[(\widetilde{f}(\tau), g_0(\tau)) + \Pi\widehat{g}(\tau)]\,\mathrm{d}\tau, \qquad t \geq 0, \end{aligned} \tag{3.17}$$

$$\phi_0^+ = -\int_0^\infty T_Q(-\tau)Q[(\widetilde{f}(\tau), g_0(\tau)) + \Pi\widehat{g}(\tau)]\,\mathrm{d}\tau, \tag{3.18}$$

$$\begin{aligned} L^-_{P,\Lambda_0}(\varphi_0, f, g)(t) &= T_Q(t)Q\varphi_0 + \int_{-\infty}^t T_{-1}(t-\tau)P_{-1}[(\widetilde{f}(\tau), g_0(\tau)) + \Pi\widehat{g}(\tau)]\,\mathrm{d}\tau \\ &\quad - \int_t^0 T_Q(t-\tau)Q[(\widetilde{f}(\tau), g_0(\tau)) + \Pi\widehat{g}(\tau)]\,\mathrm{d}\tau, \quad t \leq 0, \end{aligned} \tag{3.19}$$

$$\phi_0^- = \int_{-\infty}^0 T_{-1}(-\tau)P_{-1}[(\widetilde{f}(\tau), g_0(\tau)) + \Pi\widehat{g}(\tau)]\,\mathrm{d}\tau, \tag{3.20}$$

where $\widetilde{f} := f - A_{*\dot{\rho}}g_0 \in \mathbb{E}(J, \delta)$. We recall Propositions 4.5 and 4.6 of [20].

Proposition 3.7. *Assume that Hypothesis 3.4 holds and that for $\delta \in [\delta_1, \delta_2] \subset \mathbb{R}$ the semigroup $\mathrm{e}_\delta T(\cdot)$ has an exponential dichotomy with the stable projection P, and let $Q = I - P$. Given $(w_0, f, g) \in \mathbb{D}(\mathbb{R}_+, \delta)$, the following assertions are equivalent.*

(a) $\mathcal{S}_{\Lambda_0}(w_0, f, g) \in \mathbb{E}(\mathbb{R}_+, \delta)$.
(b) $L^+_{P,\Lambda_0}(w_0 - \phi_0^+, f, g) \in \mathbb{E}(\mathbb{R}_+, \delta)$.
(c) $\phi_0^+ = Qw_0$.

If these assertions hold, then $(u, \rho) := \mathcal{S}_{\Lambda_0}(w_0, f, g) = L^+_{P,\Lambda_0}(Pw_0, f, g)$ belongs to $\mathbb{E}_1(\mathbb{R}_+, \delta)$ and solves (3.9), and we have

$$\begin{aligned} \|\mathcal{S}_{\Lambda_0}(w_0, f, g)\|_{\mathbb{E}_1(\mathbb{R}_+,\delta)} \leq c\,(&|w_0|_{E_\gamma} + |(B_0, C_0)w_0 - g_0(0)|_{Z^1_\gamma} \\ &+ \|f\|_{\mathbb{E}(\mathbb{R}_+,\delta)} + \|g\|_{\mathbb{F}(\mathbb{R}_+,\delta)}), \end{aligned}$$

where c does not depend on w_0, f, g or δ. (Note that $\dot{\rho}(0) = g_0(0) - (B_0, C_0)w_0$.)

Proposition 3.8. *Assume that Hypothesis 3.4 holds and that for $\delta \in [\delta_1, \delta_2] \subset \mathbb{R}$ the semigroup $\mathrm{e}_\delta T(\cdot)$ has an exponential dichotomy with the stable projection P, and let $Q = I - P$. Given $(w_0, f, g) \in E_{-1} \times \mathbb{E}(\mathbb{R}_-, \delta) \times \mathbb{F}(\mathbb{R}_-, \delta)$, there is a* *solution* $w = \mathcal{S}_{\Lambda_0}(w_0, f, g)$ *of* (3.9) *in* $\mathbb{E}(\mathbb{R}_-, \delta)$ *if and only if* $P_{-1}w_0 = \phi_0^-$. *In this case, this solution is unique,* $w = L^-_{P,\Lambda_0}(w_0, f, g) \in \mathbb{E}_1(\mathbb{R}_-, \delta)$, *and*

$$\|\mathcal{S}_{\Lambda_0}(w_0, f, g)\|_{\mathbb{E}_1(\mathbb{R}_-,\delta)} \le c\,(|Qw_0|_E + \|f\|_{\mathbb{E}(\mathbb{R}_-,\delta)} + \|g\|_{\mathbb{F}(\mathbb{R}_-,\delta)}),$$

where c *does not depend on* w_0, f, g *or* δ.

In order to treat the interval $J = \mathbb{R}$, we assume that $T(\cdot)$ has an *exponential trichotomy*; i.e., there is a splitting

$$\sigma(-\Lambda_0) = \sigma_s \cup \sigma_c \cup \sigma_u \qquad \text{with} \tag{3.21}$$
$$\max \operatorname{Re} \sigma_s < -\omega_s < -\underline{\omega}_c < \min \operatorname{Re} \sigma_c \le 0 \le \max \operatorname{Re} \sigma_c < \overline{\omega}_c < \omega_u < \min \operatorname{Re} \sigma_u.$$

(If Ω is bounded, $\sigma(-\Lambda_0)$ is discrete and thus (3.21) automatically holds with $\sigma_u \subset i\mathbb{R}$ and arbitrarily small $\underline{\omega}_c = \overline{\omega}_c > 0$.) We take numbers $\alpha \in [\underline{\omega}_c, \omega_s]$ and $\beta \in [\overline{\omega}_c, \omega_u]$ and denote by P_k the spectral projections for $-\Lambda_0$ corresponding to σ_k with $k = s, c, u$. We set $P_{\rm cs} = P_{\rm s} + P_{\rm c}$, $P_{\rm cu} = P_{\rm c} + P_{\rm u}$, and $P_{\rm su} = P_{\rm s} + P_{\rm u}$. Then the rescaled semigroups $\mathrm{e}_\alpha T(\cdot)$ and $\mathrm{e}_{-\beta} T(\cdot)$ have an exponential dichotomy on E_0 with stable projections $P_{\rm s}$ and $P_{\rm cs}$, respectively. The restriction of $T(t)$ to $P_k E_0$ yields a group denoted by $T_k(t)$, $t \in \mathbb{R}$, where $k = c, u, cu$. For $f \in \mathbb{E}(\alpha, -\beta)$, $g \in \mathbb{F}(\alpha, -\beta)$ and $w_0 \in E_{-1}$, we can then define

$$\begin{aligned} L_{\Lambda_0}(w_0, f, g)(t) = T_c(t)P_{\rm c}w_0 &+ \int_0^t T_c(t-\tau)P_{\rm c}[(\widetilde{f}(\tau), g_0(\tau)) + \Pi\widehat{g}(\tau)]\,\mathrm{d}\tau \\ &+ \int_{-\infty}^t T_{-1}(t-\tau)P_{{\rm s},-1}[(\widetilde{f}(\tau), g_0(\tau)) + \Pi\widehat{g}(\tau)]\,\mathrm{d}\tau \\ &- \int_t^\infty T_u(t-\tau)P_{\rm u}[(\widetilde{f}(\tau), g_0(\tau)) + \Pi\widehat{g}(\tau)]\,\mathrm{d}\tau, \qquad t \in \mathbb{R}, \end{aligned} \tag{3.22}$$

$$\begin{aligned} \phi_0 = &\int_{-\infty}^0 T_{-1}(-\tau)P_{{\rm s},-1}[(\widetilde{f}(\tau), g_0(\tau)) + \Pi\widehat{g}(\tau)]\,\mathrm{d}\tau \\ &- \int_0^\infty T_u(-\tau)P_{\rm u}[(\widetilde{f}(\tau), g_0(\tau)) + \Pi\widehat{g}(\tau)]\,\mathrm{d}\tau, \end{aligned} \tag{3.23}$$

where again $\widetilde{f} = f - A_{*\dot\rho}g_0$. The trichotomy and the assumptions on the data imply that the integrals exist in E_{-1}.

In the next result the equivalence and the formula for the solution follow from Propositions 3.7 and 3.8 combined with Lemma 2.2 and the fact that we can glue together the solutions on $\mathbb{R}_\pm$ as noted before Lemma 3.2 of [20]. For the asserted inequality, we can treat the function $w_c = T_c(\cdot)P_{\rm c}w_0$ separately since $P_c \in \mathcal{L}(E_{-1}, D(\Lambda_0^n))$ for all $n \in \mathbb{N}_0$. The difference $w_1 = w - w_c$ has the initial value $(P_{\rm su})_{-1}w_0 = \phi_0$. The two parts of it can be controlled in E_γ via f and g using (3.23), Proposition 3.8 and P_u. Finally, $|\dot\rho(0)|_{Z^1_\gamma} \le c\,\|\rho\|_{\mathbb{E}_\rho(\mathbb{R}_-,\alpha)}$ due to (3.12), which can also be bounded by means of Proposition 3.8.

Proposition 3.9. *Assume that Hypothesis 3.4 holds and that* $T(\cdot)$ *has a trichotomy as in* (3.21). *Take* $\alpha \in [\underline{\omega}_c, \omega_s]$ *and* $\beta \in [\overline{\omega}_c, \omega_u]$. *Given* $(w_0, f, g) \in E_{-1} \times \mathbb{E}(\alpha, -\beta) \times \mathbb{F}(\alpha, -\beta)$, *there is a solution* $w = \mathcal{S}_{\Lambda_0}(w_0, f, g)$ *of* (3.9) *in* $\mathbb{E}(\alpha, -\beta)$ *if and only if* $(P_{\rm su})_{-1}w_0 = \phi_0$. *In this case, this solution is unique,* *we have* $w = L_{\Lambda_0}(w_0, f, g) \in \mathbb{E}_1(\alpha, -\beta)$ *and*

$$\|\mathcal{S}_{\Lambda_0}(w_0, f, g)\|_{\mathbb{E}_1(\alpha,-\beta)} \le c\,(|P_c w_0|_E + \|f\|_{\mathbb{E}(\alpha,-\beta)} + \|g\|_{\mathbb{F}(\alpha,-\beta)}),$$

where c does not depend on w_0, f, g, α *or* β.

The next result (Proposition 4.9 of [20]) describes the properties of F and G on $\mathbb{R}_\pm$ with weights larger than 0. For $\delta \ge 0$, we set

$$\mathbb{W}^1_*(\mathbb{R}_\pm, \pm\delta) = \{w \in \mathbb{E}_1(\mathbb{R}_\pm, \pm\delta) \,|\, w(t) \in W_\gamma - w_* \quad \text{for all} \quad t \in \mathbb{R}_\pm\}.$$

It is straightforward to check that this set is open in $\mathbb{E}_1(\mathbb{R}_\pm, \pm\delta)$ if $\delta > 0$ using (3.12). Moreover, 0 belongs to the interior of $\mathbb{W}^1_*(\mathbb{R}_\pm) := \mathbb{W}^*_1(\mathbb{R}_\pm, 0)$.

Proposition 3.10. *Let (R) hold,* $\delta \in (0, d]$ *and define* F *and* G *as in* (3.2) *for an equilibrium* $w_* = (u_*, \rho_*) \in W_1$. *We then have*

$$F \in C^1(\mathbb{W}^*_1(\mathbb{R}_\pm, \pm\delta), \mathbb{E}(\mathbb{R}_\pm, \pm\delta)) \qquad \textit{and} \qquad G \in C^1(\mathbb{W}^*_1(\mathbb{R}_\pm, \pm\delta), \mathbb{F}(\mathbb{R}_\pm, \pm\delta))$$

and $F(0) = 0$, $G(0) = 0$, $F'(0) = 0$, $G'(0) = 0$. *Moreover, the derivatives are bounded and uniformly continuous on closed balls. If* $\delta = 0$, *the above results hold on sufficiently small balls in* $\mathbb{E}_1(\mathbb{R}_\pm)$ *with center 0.*

4. The cutoff problem and a center manifold

In this section we want to extend the construction of invariant manifolds from the setting of Theorem 5.1 of [20] to the case of an exponential trichotomy as in (3.21). Under this assumptions we encounter unbounded semigroup orbits on the center part so that we must deal with spaces of (say, exponentially) growing functions on $\mathbb{R}$. For such functions our substitution operators are not locally Lipschitz (unless they are globally Lipschitz). It is well known that one has to introduce cutoff functions in the nonlinearities to deal with this problem. As in [10] for static nonlinear boundary conditions, we have to control the full $\mathbb{E}_1$ norm of solutions so that we need a nonlocal cuoff $\Gamma(t, w)$ introduced below. Since the arguments are parallel to those in [10], we omit most of the proofs. We assume throughout that Hypothesis 3.4 holds with the equilibrium $w_* = (u_*, \rho_*) \in W_1$.

The cutoff depends on a parameter $\eta > 0$ to be fixed in the following theorems. We first introduce

$$J(t) = [t - \tfrac{3}{2}, t + \tfrac{3}{2}] \quad \text{and} \quad N(t, \varphi) = \|\varphi\|_{\mathbb{E}_1(J(t))} \quad \text{for} \quad \varphi \in \mathbb{E}^{\text{loc}}_1(\mathbb{R}),\ t \in \mathbb{R}.$$

Given an $\eta > 0$, we take even functions $\chi, \gamma \in C^\infty(\mathbb{R})$ such that $0 \le \chi \le 1$, $\chi(t) = 1$ for $t \in [-\eta, \eta]$, $\operatorname{supp} \chi \subset (-2\eta, 2\eta)$, $\|\chi^{(k)}\|_\infty \le c/\eta^k$ for $k = 1, 2$ and such that $\gamma \ge 0$, $\int_{\mathbb{R}} \gamma(t)\, dt = 1$, $\operatorname{supp} \gamma \subseteq (-1/4, 1/4)$. We now define the cutoff

$$\begin{aligned} \Gamma_{\mathbb{R}}(t, \varphi) = \Gamma(t, \varphi) &:= (\gamma * \chi(N(\cdot\,, \varphi)))(t) \\ &= \int_{\mathbb{R}} \gamma(t - \tau)\, \chi(\|\varphi\|_{\mathbb{E}_1([\tau-3/2, \tau+3/2])})\, \mathrm{d}\tau \end{aligned} \tag{4.1}$$

for $t \in \mathbb{R}$ and $\varphi \in \mathbb{E}^{\text{loc}}_1(\mathbb{R})$. Observe that the integrand is continuous in τ and that $\Gamma(t, \varphi)$ depends on the restriction of φ to $(t - 7/4, t + 7/4)$. For functions $\varphi \in \mathbb{E}^{\text{loc}}_1(J)$, we define $\Gamma(t, \varphi)$ as in (4.1) for $t \in [\frac{7}{4} + \inf J, -\frac{7}{4} + \sup J]$, where J is an interval of length greater than 7/2.

To treat $\varphi \in \mathbb{E}_1^{\mathrm{loc}}(\mathbb{R}_\pm)$, we fix continuous extension operators $R_\pm : \mathbb{E}_1^{\mathrm{loc}}(\mathbb{R}_\pm) \to \mathbb{E}_1^{\mathrm{loc}}(\mathbb{R})$ satisfying $\operatorname{supp} R_+\varphi \subset (-1,\infty)$, $\operatorname{supp} R_-\varphi \subset (-\infty,1)$,

$$\|R_+\varphi\|_{\mathbb{E}_1([-1,1])} \le c_R \|\varphi\|_{\mathbb{E}_1([0,1])}, \qquad \|R_-\varphi\|_{\mathbb{E}_1([-1,1])} \le c_R \|\varphi\|_{\mathbb{E}_1([-1,0])} \tag{4.2}$$

for $\varphi \in \mathbb{E}_1$ and a constant $c_R > 0$. In this context we further observe that

$$\begin{aligned} &\|\varphi\|_{\mathbb{E}_1([0,T])} \le c_E \|\varphi\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)}, \qquad \|\varphi\|_{\mathbb{E}_1([-T,0])} \le c_E \|\varphi\|_{\mathbb{E}_1(\mathbb{R}_-,\alpha)} \\ &\|\varphi\|_{\mathbb{E}_1([-T,T])} \le c_E \|\varphi\|_{\mathbb{E}_1(\alpha,-\beta)} \end{aligned} \tag{4.3}$$

for a constant $c_E > 0$ depending on $T > 0$ and being uniform in $\alpha, \beta \ge 0$ in compact intervals. We then define the cutoffs

$$\Gamma_{\mathbb{R}_\pm}(t,\varphi) = \Gamma_\pm(t,\varphi) := \Gamma(t, R_\pm\varphi) = (\gamma * \chi(N(\cdot\,, R_\pm\varphi)))(t) \tag{4.4}$$

for $t \in \mathbb{R}$ and $\varphi \in \mathbb{E}_1^{\mathrm{loc}}(\mathbb{R}_\pm)$. We now collect several properties of the cutoffs in (4.1) and (4.4) for $J \in \{\mathbb{R}, \mathbb{R}_+, \mathbb{R}_-\}$, which follow easily from the above definitions and observations, cf. §3 in [10].

Remark 4.1. If $\varphi \in \mathbb{E}_1^{\mathrm{loc}}(J)$ satisfies $\|\varphi\|_{\mathbb{E}_1([t-2,t+2])} \le \eta$ for some $t \in J$ (where $|t| \ge 2$ if $J = \mathbb{R}_\pm$), then $\Gamma_J(t,\varphi) = 1$. If $J = \mathbb{R}_\pm$ and $t \in J \cap [-2,2]$, then $\|\varphi\|_{\mathbb{E}_1([t-2,t+2]\cap J)} \le c_W^{-1}(1+c_R)^{-1}\eta$ implies $\Gamma_\pm(t,\varphi) = 1$, where $c_W \ge 1$ is the constant given by Lemma 2.2.

Remark 4.2. Let $\varphi \in \mathbb{E}_1^{\mathrm{loc}}(\mathbb{R})$ and $J = \mathbb{R}$. By its definition, the cutoff is translation invariant in the sense that $\Gamma_\mathbb{R}(t+t_0,\varphi) = \Gamma_\mathbb{R}(t,\varphi(\cdot + t_0))$ for all $t, t_0 \in \mathbb{R}$. We point out that for $J = \mathbb{R}_\pm$ the cutoff is *not* translation invariant. However, for $\varphi \in \mathbb{E}_1^{\mathrm{loc}}(\mathbb{R}_\pm)$ we have $\Gamma_+(t,\varphi) = \Gamma_\mathbb{R}(t,\varphi)$ for $t \ge 7/4$ and $\Gamma_-(t,\varphi) = \Gamma_\mathbb{R}(t,\varphi)$ for $t \le -7/4$, respectively. As a result, $\Gamma_+(t+t_0,\varphi) = \Gamma_\mathbb{R}(t,\varphi(\cdot + t_0))$ holds for $t+t_0 \ge 7/4$ and $\Gamma_-(t+t_0,\varphi) = \Gamma_-(t,\varphi(\cdot+t_0))$ holds for $t + t_0 \le -7/4$, where $t, t_0 \in \mathbb{R}$. (Here $\varphi(\cdot + t_0)$ is defined on $[-t_0,\infty)$, resp. on $(-\infty,-t_0]$.)

Remark 4.3. Let $\varphi \in \mathbb{E}_1^{\mathrm{loc}}(J)$, $t \in J$, and $J \in \{\mathbb{R}, \mathbb{R}_\pm\}$. We put $\varphi_\mathbb{R} = R_\pm\varphi$ if $J = \mathbb{R}_\pm$ and $\varphi_\mathbb{R} = \varphi$ if $J = \mathbb{R}$. Assume that $\Gamma_J(t,\varphi) \ne 0$. Then there exists an $s \in [t-1/4, t+1/4]$ such that $\chi(N(s,\varphi_\mathbb{R})) \ne 0$, and hence $\|\varphi_\mathbb{R}\|_{\mathbb{E}_1(J(s))} \le 2\eta$. The embeddings (2.3) and (2.7) now imply that $|\varphi(t)|_{E_\gamma} \le c_0 \|\varphi_\mathbb{R}\|_{\mathbb{E}_1(J(s))} \le 2c_0\eta$ for $t \in J$ if $\Gamma_J(t,\varphi) \ne 0$, where $c_0 > 0$ is a constant. We can thus fix a number $\eta_0 > 0$ such that $\Gamma_J(t,\varphi) \ne 0$ for some $t \in J$ implies that $\varphi(t) + w_* \in W_\gamma$, provided that $\eta \in (0,\eta_0]$.

From now on we always assume that $\eta \in (0,\eta_0]$.

We add a related observation needed in the proofs of the following two propositions, cf. Remark 3.3 in [10]. Suppose that $t \in [n-1/8, n+9/8] \cap J =: J_n^*$ satisfies $\Gamma_J(t,\varphi) \ne 0$ for some $\varphi \in \mathbb{E}_1^{\mathrm{loc}}(J)$ and $n \in \mathbb{Z}$. Take s as in Remark 4.3. We then have $J_n^* \subset J(s)$, and hence $\|\varphi_\mathbb{R}\|_{\mathbb{E}_1(J_n^*)} \le 2\eta$. (The same estimate holds if $\Gamma_J'(t,\varphi) \ne 0$, where the derivative is given by (4.7).) Moreover, $|\varphi_\mathbb{R}(\tau)|_{E_\gamma} \le c_0 \|\varphi_\mathbb{R}\|_{\mathbb{E}_1(J(s))} \le 2c_0\eta$ implying $\varphi_\mathbb{R}(\tau) + w_* \in W_\gamma$ for all $\tau \in J_n^*$.

Finally, for $\varphi \in \mathbb{E}_1^{\mathrm{loc}}(J)$ and $J \in \{\mathbb{R}, \mathbb{R}_+, \mathbb{R}_-\}$, we define the cutoff versions

$$F_{\Gamma J}(\varphi)(t) = \Gamma_J(t,\varphi)F(\varphi(t)), \quad G_{\Gamma J}(\varphi)(t) = \Gamma_J(t,\varphi)G(\varphi(t)), \quad t \in J, \tag{4.5}$$

of the nonlinear maps F and G defined in (3.2) for the equilibrium w_*. In this definition, we set $F(t,\varphi(t)) = 0$ and $G(t,\varphi(t)) = 0$ if $\varphi(t) + w_* \notin W_\gamma$. We also abbreviate $F_\Gamma = F_{\Gamma\mathbb{R}}$, $F_{\Gamma\pm} = F_{\Gamma\mathbb{R}_\pm}$, $G_\Gamma = G_{\Gamma\mathbb{R}}$, and $G_{\Gamma\pm} = G_{\Gamma\mathbb{R}_\pm}$. The cutoff version of the initial-boundary value problem (3.6) is then given by

$$
\begin{aligned}
\partial_t v(t) + A_*(t)(v(t),\sigma(t),\dot\sigma(t)) &= F_{\Gamma J}(v,\sigma)(t), && \text{on } \Omega,\ t \in J,\\
\partial_t \sigma(t) + B_0(t)v(t) + C_0(t)\sigma(t) &= G_{0,\Gamma J}(v,\sigma)(t), && \text{on } \Sigma,\ t \in J,\\
\widehat{B}(t)v(t) + \widehat{C}(t)\sigma(t) &= \widehat{G}_{\Gamma J}(v,\sigma)(t) && \text{on } \Sigma,\ t \in J,\\
v(0) &= v_0, && \text{on } \Omega,\\
\sigma(0) &= \sigma_0, && \text{on } \Sigma.
\end{aligned}
\tag{4.6}
$$

where $J \in \{\mathbb{R},\mathbb{R}_+,\mathbb{R}_-\}$ and $\varphi = (v,\sigma) \in \mathbb{E}_1^{\mathrm{loc}}(J)$. Thanks to Remark 4.1, a solution $\varphi \in \mathbb{E}_1^{\mathrm{loc}}(J)$ of (4.6) actually satisfies (3.6) on $[a,b] \subset J$ if $\|\varphi\|_{\mathbb{E}_1([t-2,t+2]\cap J)}$ is sufficiently small for each $t \in [a,b]$. We stress that the cutoff problem (4.6) is not local in time. In particular, even for $J = \mathbb{R}_+$ it is not a well–posed Cauchy problem. Nevertheless, based on our results for the linear problem on J we can solve (4.6) globally in function spaces on J.

We now consider the maps $F_{\Gamma J}$ and $G_{\Gamma J}$ on the spaces $\mathbb{E}_1(\mathbb{R}_\pm,\mp\alpha)$ and $\mathbb{E}_1(\alpha,-\beta)$, where $\alpha,\beta \ge 0$, see (4.5) and (3.11). (These values of α,β were not treated in Proposition 3.10.) We start with the Lipschitz properties.

Proposition 4.4. *Assume that Hypothesis 3.4 holds. Let $\eta \in (0,\eta_0]$ be the parameter for the cutoff and $\alpha,\beta \in [0,d]$ for some $d > 0$. Then the maps $F_{\Gamma\pm} : \mathbb{E}_1(\mathbb{R}_\pm,\mp\alpha) \to \mathbb{E}(\mathbb{R}_\pm,\mp\alpha)$, $F_\Gamma : \mathbb{E}_1(\alpha,-\beta) \to \mathbb{E}(\alpha,-\beta)$, $G_{\Gamma\pm} : \mathbb{E}_1(\mathbb{R}_\pm,\mp\alpha) \to \mathbb{F}(\mathbb{R}_\pm,\mp\alpha)$, and $G_\Gamma : \mathbb{E}_1(\alpha,-\beta) \to \mathbb{F}(\alpha,-\beta)$ are (globally) Lipschitz with a Lipschitz constant $\varepsilon(\eta)$ for a nondecreasing function ε converging to 0 as $\eta \to 0$, independent of α or β. Moreover, $F_{\Gamma J}(0) = 0$ and $G_{\Gamma J}(0) = 0$ for $J = \mathbb{R}_\pm,\mathbb{R}$.*

We omit the (technical) proof of this result since it is very similar to that of Proposition 3.6 in [10]. It uses the above listed properties of the cutoff, Lemma 4.7 in [20], (R), and a straightforward extension of Remark 3.4 of [10] to the present situation.

Next, we want to establish the continuous differentiability of F_Γ and G_Γ in certain spaces. We first observe that the map $\varphi \mapsto N(t,\varphi) = \|\varphi\|_{\mathbb{E}_1(J(t))}$ is continuously differentiable on $\mathbb{E}_1(J(t)) \setminus \{0\}$ and that its derivative is uniformly bounded (cf. §3 of [10] and Theorem 2.3.2(a) in [24] for the Slobodeckii spaces). As in §3 of [10] one verifies that the map $\mathbb{E}_1([t-2,t+2]) \ni \varphi \mapsto \Gamma(t,\varphi)$ is C^1 with the derivative

$$\Gamma'(t,\varphi) = [\gamma * \chi'(N(\cdot,\varphi))N'(\cdot,\varphi)](t). \tag{4.7}$$

Here we set $N'(t,0) = 0$ and note that $\Gamma(t,\varphi) = 1$, and thus $\Gamma'(t,\varphi) = 0$, if $\|\varphi\|_{\mathbb{E}_1([t-2,t+2])} < \eta$. The cutoffs $\Gamma_\pm(t,\varphi) = \Gamma(t,R_\pm\varphi)$ on $\mathbb{R}_\pm$ have the analogous differentiability properties.

Given $\alpha,\beta \ge 0$ and $w \in \mathbb{E}_1(\alpha,-\beta)$, we introduce the linear operators F'_Γ and G'_Γ acting on $w \in \mathbb{E}_1(\alpha,-\beta)$ by the formulas

$$
\begin{aligned}
[F'_\Gamma(w)\varphi](t) &= \langle \varphi, \Gamma'(t,w)\rangle\, F(w(t)) + \Gamma(t,w)F'(w(t))\varphi(t),\\
[G'_\Gamma(w)\varphi](t) &= \langle \varphi, \Gamma'(t,w)\rangle\, G(w(t)) + \Gamma(t,w)G'(w(t))\varphi(t),
\end{aligned}
\tag{4.8}
$$

see (3.2) and (3.3) for the definitions of F, G, F' and G' We also set

$$[F'_{\Gamma\pm}(w)\varphi](t) = [F'_\Gamma(R_\pm w)R_\pm\varphi](t), \quad [G'_{\Gamma\pm}(w)\varphi](t) = [G'_\Gamma(R_\pm w)R_\pm\varphi](t) \quad (4.9)$$

for $t \in \mathbb{R}_\pm$ and $w, \varphi \in \mathbb{E}_1(\mathbb{R}_+, -\alpha)$ in the case $J = \mathbb{R}_+$, respectively, $w, \varphi \in \mathbb{E}_1(\mathbb{R}_-, \alpha)$ in the case $J = \mathbb{R}_-$.

We stress that the maps F_Γ and G_Γ are not differentiable if the domain and range spaces have the *same* weight function. However, the next proposition shows that they are C^1 maps with the derivatives F'_Γ and G'_Γ introduced above if we take a *smaller* weight function in the range space, cf. [25].

Proposition 4.5. *Assume that Hypothesis 3.4 holds. Let $\eta \in (0, \eta_0]$ be the parameter for the cutoff, $0 \le \alpha \le \beta \le d$ and $0 \le \alpha' \le \beta' \le d$ for some $d > 0$. Define the operators F'_Γ, $F'_{\Gamma\pm}$ G'_Γ, and $G'_{\Gamma\pm}$ by (4.8) and (4.9). Then the following assertions hold.*

(a) The operators $F'_\Gamma(w) : \mathbb{E}_1(\alpha, -\alpha') \to \mathbb{E}(\beta, -\beta')$, $G'_\Gamma(w) : \mathbb{E}_1(\alpha, -\alpha') \to \mathbb{F}(\beta, -\beta')$, $F'_{\Gamma\pm}(w) : \mathbb{E}_1(\mathbb{R}_\pm, \mp\alpha) \to \mathbb{E}(\mathbb{R}_\pm, \mp\beta)$, and $G'_{\Gamma\pm}(w) : \mathbb{E}_1(\mathbb{R}_\pm, \mp\alpha) \to \mathbb{F}(\mathbb{R}_\pm, \mp\beta)$ are all bounded with the norms $\varepsilon(\eta)$, where ε is a nondecreasing function converging to 0 as $\eta \to 0$ which does not depend on $w, \alpha, \alpha', \beta, \beta'$.

(b) If $\beta > \alpha$ and $\beta' > \alpha'$, then the maps $F_\Gamma : \mathbb{E}_1(\alpha, -\alpha') \to \mathbb{E}(\beta, -\beta')$, $G_\Gamma : \mathbb{E}_1(\alpha, -\alpha') \to \mathbb{F}(\beta, -\beta')$, $F_{\Gamma\pm} : \mathbb{E}_1(\mathbb{R}_\pm, \mp\alpha) \to \mathbb{E}(\mathbb{R}_\pm, \mp\beta)$, and $G_{\Gamma\pm} : \mathbb{E}_1(\mathbb{R}_\pm, \mp\alpha) \to \mathbb{F}(\mathbb{R}_\pm, \mp\beta)$ are continuously differentiable with derivatives F'_Γ G'_Γ, $F'_{\Gamma\pm}$, and $G'_{\Gamma\pm}$, respectively. Moreover, $F'_{\Gamma J}(0) = 0$ and $G'_{\Gamma J}(0) = 0$ for $J \in \{\mathbb{R}_+, \mathbb{R}_-, \mathbb{R}\}$.

We again omit the lengthy proof which it is analogous to that of Proposition 3.8 in [10], using the properties of the cutoff stated above, (R), Lemma 4.7 of [20], Proposition 4.4, as well as straightforward extensions of the statements (3.14), (3.15) and (3.16) in [10] to the present situation.

We now establish one of the main result of this paper where we construct a *local center manifold* $\mathcal{M}_c$ and show some of its basic properties. In particular, $\mathcal{M}_c$ is a C^1–submanifold of $\mathcal{M}$ being tangent to P_cE at w_*. Further properties of $\mathcal{M}_c$ are stated in Corollary 5.3 and Theorem 6.5. We assume that the spectrum of $-\Lambda_0$ has the decomposition given by (3.21), i.e., $\sigma(-\Lambda_0)$ has spectral gaps in the left and in the right open halfplane. Recall that this assumption automatically holds if the spatial domain Ω is bounded.

There is no description of $\mathcal{M}_c$ directly in terms of (1.1) as provided by Theorem 5.1 of [20] for the stable and unstable manifold, respectively. In fact, there are simple ODEs in dimension two admitting infinitely many locally invariant manifolds which are tangent to P_cE at w_* and satisfy $\mathcal{M}_c \cap \mathcal{M}_s = \mathcal{M}_c \cap \mathcal{M}_u = \{u_*\}$ (cf. Corollary 5.3). However, if w_* is stable in forward and backward time, then Theorem 4.6(e) implies that our $\mathcal{M}_c$ is the only submanifold of $\mathcal{M}$ near w_* with these properties.

Theorem 4.6. *Assume that Hypothesis 3.4 and (3.21) hold. Let the projections P_k and the numbers ω_k be given by (3.21). Take any $\alpha \in (\underline{\omega}_c, \omega_s)$ and $\beta \in (\overline{\omega}_c, \omega_u)$. Then there is a number $\eta_c \in (0, \eta_0]$ such that for each $\eta \in (0, \eta_c]$ there exists a radius $r_c = r_c(\eta) > 0$ such that the following assertions hold, where the cutoff Γ is defined in (4.1) for the chosen $\eta \in (0, \eta_c]$.*

(a) There exists a map $\phi_c \in C^1(P_{\mathrm{c}}E; P_{\mathrm{su}}E_\gamma)$ with a bounded derivative such that $\phi_c(0) = 0$, $\phi_c'(0) = 0$, and

$$\begin{aligned}\widetilde{\mathcal{M}}_c &:= \big\{w_0 = w_* + \xi + \phi_c(\xi) \in E_\gamma \,\big|\, \xi \in P_{\mathrm{c}}E\big\} \\ &= \big\{w_0 = w_* + \varphi(0) \in E_\gamma \,\big|\, \exists \text{ solution } \varphi \in \mathbb{E}_1(\alpha, -\beta) \text{ of (4.6) on } J = \mathbb{R}\big\}.\end{aligned} \tag{4.10}$$

If $w_0 \in \widetilde{\mathcal{M}}_c$, then $w_* + \varphi(t) \in \widetilde{\mathcal{M}}_c$ for each $t \in \mathbb{R}$ and $\varphi = \Phi_c(P_{\mathrm{c}}(w_0 - w_*)) = P_{\mathrm{c}}\varphi + \phi_c(P_{\mathrm{c}}\varphi)$ for a map $\Phi_c \in C^1(P_{\mathrm{c}}E; \mathbb{E}_1(\alpha, -\beta))$ having a bounded derivative, where φ is given by (4.10).

(b) We define $\mathcal{M}_{\mathrm{c}} = \{w_0 \in \widetilde{\mathcal{M}}_c \,|\, \langle w_0 - w_*\rangle_\gamma < r_c\}$. Let $w_0 \in \mathcal{M}_{\mathrm{c}}$ and φ be given by (4.10) with $w_0 = \varphi(0) + w_*$. Then $\Gamma(t, \varphi) = 1$ and φ solves the (original) equation (3.6) for $t \in [-3, 3]$, at least, so that $\mathcal{M}_{\mathrm{c}} \subset \mathcal{M}$. The dimension of $\mathcal{M}_{\mathrm{c}}$ is equal to that of $P_{\mathrm{c}}E$. Moreover, $w_0 = w_* + \xi + \phi_c(\xi) \in \mathcal{M}_{\mathrm{c}}$ and $\langle w_0 - w_*\rangle_\gamma \le c\,|\xi|_E$ for $\xi = P_{\mathrm{c}}(w_0 - w_*)$.

(c) Let $w_0 \in \mathcal{M}_{\mathrm{c}}$ and φ be given by (4.10). If the forward solution w of (1.1) exists and satisfies $\langle w - w_*\rangle_\gamma < r_c$ on $[0, t]$ for some $t > 0$, then $w(t) \in \mathcal{M}_{\mathrm{c}}$. If the function $\widetilde{w} = \varphi + w_*$ satisfies $\langle \widetilde{w} - w_*\rangle_\gamma < r_c$ on $[t, 0]$ for some $t < 0$, then $\widetilde{w}(t) \in \mathcal{M}_{\mathrm{c}}$, and $\widetilde{w}$ solves (1.1) on $[t, 0]$.

(d) Let $w_0 = w_* + \varphi_0 \in \mathcal{M}_{\mathrm{c}}$ and let φ be given by (4.10). Assume that $\varphi(t) + w_* \in \mathcal{M}_{\mathrm{c}}$ for all $t \in (a, b)$ and some $a < 0 < b$. Then $z = P_{\mathrm{c}}\varphi$ satisfies the equations

$$\begin{aligned}\dot{z}(t) &= -\Lambda_0 P_{\mathrm{c}} z(t) + P_{\mathrm{c}}\Pi\widehat{G}(z(t) + \phi_c(z(t))) \\ &\qquad + P_{\mathrm{c}}[\widetilde{F}(z(t) + \phi_c(z(t))), G_0(z(t) + \phi_c(z(t)))], \\ z(0) &= P_{\mathrm{c}}(w_0 - w_*),\end{aligned} \tag{4.11}$$

on $P_{\mathrm{c}}E$ for $t \in (a, b)$, where $\widetilde{F} = F - A_{*\dot{\rho}_*}G_0$ and $\Pi \in \mathcal{L}(\widehat{Y}, E_{-1})$ is given by Proposition 3.6. Moreover, $\varphi \in C((a, b); E_1)$ and

$$(B_0, C_0)\phi_c(P_{\mathrm{c}}\varphi_0) - G_0(\varphi_0) \in Z_\gamma^1, \qquad (\widehat{B}, \widehat{C})\phi_c(P_{\mathrm{c}}\varphi_0) = \widehat{G}(\varphi_0), \tag{4.12}$$

$$P_{\mathrm{su}}[\Lambda_*\varphi_0 - (\widetilde{F}(\varphi_0), G_0(\varphi_0)))] = \phi_c'(P_{\mathrm{c}}\varphi_0)P_{\mathrm{c}}[\Lambda_*\varphi_0 - (\widetilde{F}(\varphi_0), G_0(\varphi_0))]. \tag{4.13}$$

(e) If w solves (1.1) on $\mathbb{R}$ with $\langle w(t) - w_*\rangle_\gamma < r_c$ for all $t \in \mathbb{R}$, then $w(t) \in \mathcal{M}_{\mathrm{c}}$ for all $t \in \mathbb{R}$.

(f) If also (RR) holds, then there is a $r_0 > 0$ such that the map $\phi_c : \widehat{D}_c := P_{\mathrm{c}}E \cap B_E(0, r_0) \to P_{\mathrm{su}}E_1$ is Lipschitz, and $\phi_c'(\xi)$ is uniformly bounded in $\mathcal{L}(P_{\mathrm{c}}E, P_{\mathrm{su}}E_1)$ for $\xi \in \widehat{D}_c$.

Proof. We first construct a manifold $\widetilde{\mathcal{M}}_c$ consisting of solutions to (4.6) on $\mathbb{R}$ on the space $\mathbb{E}_1(\alpha, -\beta)$. The desired center manifold $\mathcal{M}_{\mathrm{c}}$ is then obtained by restriction to small balls.

(a) Let $\alpha' \in (\underline{\omega}_c, \alpha)$ and $\beta' \in (\overline{\omega}_c, \beta)$. We define the map

$$\mathcal{T}_c : P_{\mathrm{c}}E \times \mathbb{E}_1(\alpha', -\beta') \to \mathbb{E}_1(\alpha, -\beta); \quad \mathcal{T}_c(\xi, \varphi) = L_{\Lambda_0}(\xi, F_\Gamma(\varphi), G_\Gamma(\varphi))$$

for the operators given in (3.22) and (4.5). Let ϕ_0 be given as in (3.23) and set $w_0 = \xi + \phi_0$. Due to Propositions 3.9 and 4.5, the map $\mathcal{T}_c^0 : w \mapsto \mathcal{T}_c(\xi, w)$ is C^1 from $\mathbb{E}_1(\alpha', -\beta')$ to $\mathbb{E}_1(\alpha, -\beta)$ and the derivative of $\mathcal{T}_c^0$ is bounded by $c_1\varepsilon(\eta)$ in the norm of both $\mathcal{L}(\mathbb{E}_1(\alpha', -\beta'))$ and $\mathcal{L}(\mathbb{E}_1(\alpha, -\beta))$, independent of $\xi \in P_{\mathrm{c}}E$.

Moreover, $\mathcal{T}_c^0$ is Lipschitz in $\mathbb{E}_1(\alpha', -\beta')$ with constant $c_1\varepsilon(\eta)$ independent of $\xi \in P_c E$ by Propositions 3.9 and 4.4. Finally, the map $\xi \mapsto \mathcal{T}_c(\xi, w)$ is affine from $P_c E$ to $\mathbb{E}_1(\alpha', -\beta')$ with the derivative $T(\cdot)P_c$.

We now fix $\eta = \eta_c > 0$ such that $c_1\varepsilon(\eta) \le 1/2$. (Note that this estimate also holds for every $\eta' \in (0, \eta)$ and that it is independent of the choice of $\alpha, \alpha', \beta, \beta'$ as above.) Then Theorem 3 of [25] (with $Y_0 = Y = \mathbb{E}_1(\alpha', -\beta')$ and $Y_1 = \mathbb{E}_1(\alpha, -\beta)$) shows that for each $\xi \in P_c E$ there exists a unique solution $\varphi = \Phi_c(\xi) \in \mathbb{E}_1(\alpha', -\beta')$ of the equation $\varphi = \mathcal{T}_c(\xi, \varphi)$, where $\Phi_c \in C^1(P_c E; \mathbb{E}_1(\alpha, -\beta))$ and $\Phi_c(0) = 0$. Moreover, (4.4) in [25] implies that $\Phi_c'(\xi) \in \mathcal{L}(P_c E, \mathbb{E}_1(\alpha, -\beta))$ is bounded uniformly in ξ. Observe that φ is also a unique fix point in $\mathbb{E}_1(\alpha, -\beta)$ since we can vary $\alpha' < \alpha$ and $\beta' < \beta$ in the gaps. We now introduce

$$\begin{aligned}\phi_c(\xi) &:= \gamma_0 P_{\mathrm{su}} \Phi_c(\xi) \qquad (4.14)\\ &= \int_{-\infty}^0 T_{-1}(-\tau) P_{\mathrm{s},-1}[(\widetilde{F}_\Gamma(\Phi_c(\xi))(\tau), G_{0\Gamma}(\Phi_c(\xi))(\tau)) + \Pi \widehat{G}_\Gamma(\Phi_c(\xi))(\tau)]\,\mathrm{d}\tau\\ &\quad - \int_0^\infty T_u(-\tau) P_{\mathrm{u}}[(\widetilde{F}_\Gamma(\Phi_c(\xi))(\tau), G_{0\Gamma}(\Phi_c(\xi))(\tau)) + \Pi \widehat{G}_\Gamma(\Phi_c(\xi))(\tau)]\,\mathrm{d}\tau\end{aligned}$$

for $\xi \in P_c E$, where we have $\widetilde{F}_\Gamma(\psi) = F_\Gamma(\psi) - A_{*\hat{\rho}} G_{0\Gamma}(\psi)$ and recall (3.22). Using the embeddings (2.3) and (2.7), we see that $\phi_c \in C^1(P_c E; P_{\mathrm{su}} E_\gamma)$, ϕ_c' is bounded and $\phi_c(0) = 0$. The properties of $T(\cdot)$ and Propositions 3.6 and 4.5 further yield that $\phi_c'(0) = 0$. In the formula (4.10) the inclusion '$\subset$' follows from the construction. If $\varphi \in \mathbb{E}_1(\alpha, -\beta)$ is given by the second description of $\widetilde{\mathcal{M}}_c$, then Proposition 3.9 implies that $\varphi = L_{\Lambda_0}(P_c\varphi(0), F_\Gamma(\varphi), G_\Gamma(\varphi))$. The other inclusion in (4.10) thus is a consequence of the uniqueness of the equation $\varphi = \mathcal{T}_c(\xi, \varphi)$ with $\xi = P_c\varphi(0)$. If $w_0 = w_* + \varphi_0 \in \widetilde{\mathcal{M}}_c$ with the corresponding solution φ of (4.6) and $t \in \mathbb{R}$, then $\psi = \varphi(\cdot + t)$ solves (4.6) with the initial condition $\psi(0) = \varphi(t)$ thanks to Remark 4.2. This means that $w_* + \varphi(t) \in \widetilde{\mathcal{M}}_c$, and thus $\varphi(t) = P_c\varphi(t) + \phi_c(P_c\varphi(t))$.

(b) Take $w_0 = w_* + \varphi_0 \in \widetilde{\mathcal{M}}_c$ with $\langle \varphi_0 \rangle_\gamma < r$ for some $r > 0$. Set $\xi = P_c\varphi_0$ and $\varphi = \Phi_c(\xi)$. Estimate (4.3) and assertion (a) imply that

$$\begin{aligned}\|\varphi\|_{\mathbb{E}_1([-5,5])} &\le c\,\|\varphi\|_{\mathbb{E}_1(\alpha,-\beta)} = c\,\|\Phi_c(\xi) - \Phi_c(0)\|_{\mathbb{E}_1(\alpha,-\beta)}\\ &\le c\,|\xi|_E \le c\,|\varphi_0|_E < c' r. \qquad (4.15)\end{aligned}$$

If we take $r \le r_c := \eta/c'$, Remark 4.1 implies that $\Gamma(t, \varphi) = 1$ for $t \in [-3, 3]$; i.e., φ solves (3.6) on $[-3, 3]$ in this case. The last assertion in (b) now follows from (4.15), using also (3.8) and the embeddings (2.3) and (2.7). The other claims in (b) are clear. We will decrease $r_c > 0$ below, if necessary.

(c) Take $w_0 \in \mathcal{M}_c$ and let $w = w_* + \psi$ be the forward solution of (1.1) satisfying $\langle \psi \rangle_\gamma < r_c$. Since the function φ from (4.10) solves (3.6) on $[0, 3]$ by assertion (b), the uniqueness of (3.6) implies that $\psi = \varphi$ on $[0, 3]$. We thus deduce $w(\tau) = \varphi(\tau) + w_* \in \widetilde{\mathcal{M}}_c$ for $\tau \in [0, 3]$ from part (a). Due the assumption, we have $w(t) \in \mathcal{M}_c$ if $t \le 3$. If $t > 3$, we can iterate the argument using the translation invariance of (4.6) and (3.6). The asserted backward invariance of $\mathcal{M}_c$ is a direct consequence of parts (a) and (b).

(d) Let $w_0 = w_* + \varphi_0 \in \mathcal{M}_{\rm c}$ and let φ be given by (4.10) so that $\varphi(t) + w_* \in \mathcal{M}_{\rm c}$ for $t \in (a,b)$. Set $z = P_{\rm c}\varphi$. Due to (a)–(c), we have $\varphi = z + \phi_c(z)$ and φ solves (3.6) on (a,b). In particular, it holds $(\widehat{B},\widetilde{C})\varphi(t) = \widehat{G}(\varphi(t))$ on (a,b). Theorem 3.3 further shows that φ is continuous in E_1. Using (3.13) and Propsition 3.6, we compute

$$\begin{aligned}\dot z(t) = P_{\rm c}\dot\varphi(t) &= P_{\rm c}[-\Lambda_*\varphi(t) + (\widetilde{F}(\varphi(t)), G_0(\varphi(t)))]\\ &= -P_{\rm c}\Lambda_{-1}z(t) + P_{\rm c}\Pi\widehat{G}(z(t)+\phi_c(z(t)))\\ &\quad + P_{\rm c}[\widetilde{F}(z(t)+\phi_c(z(t))), G_0(z(t)+\phi_c(z(t)))]\end{aligned}$$

for $t \in (a,b)$. Since $P_{\rm c}\Lambda_{-1} = \Lambda_0 P_{\rm c}$, the equation (4.11) is shown. The assertion (4.12) follows from $\varphi_0 = P_{\rm c}\varphi_0 + \phi_c(P_{\rm c}\varphi_0) \in \mathcal{M}$ and $P_{\rm c}E \subset E_1^0$. Differentiating $\varphi = P_{\rm c}\varphi + \phi_c(P_{\rm c}\varphi)$, we deduce

$$\begin{aligned}\dot\varphi(t) &= -\Lambda_*\varphi(t) + (\widetilde{F}(\varphi(t)), G_0(\varphi(t))),\\ \dot\varphi(t) &= P_{\rm c}[-\Lambda_*\varphi(t) + (\widetilde{F}(\varphi(t)), G_0(\varphi(t)))]\\ &\quad + \phi_c'(P_{\rm c}\varphi(t))P_{\rm c}[-\Lambda_*\varphi(t) + (\widetilde{F}(\varphi(t)), G_0(\varphi(t)))]\end{aligned}$$

for $t \in (a,b)$, so that (4.13) follows by taking $t = 0$.

(e) Let w be a solution of (1.1) on $\mathbb{R}$ staying such that $\langle w(t) - w_*\rangle_\gamma < r_c$ for $t \in \mathbb{R}$. The estimate in Theorem 3.3(b) yields that $\|w\|_{\mathbb{E}_1([t-2,t+2])} \le c_* r_c$ for each $t \in \mathbb{R}$ and a constant $c_* > 0$ (possibly after decreasing $r_c > 0$). In particular, $w \in \mathbb{E}_1(\alpha, -\beta)$ due to Lemma 4.7 in [20]. Fixing $r_c \le \eta/c_*$, we deduce from Remark 4.1 that $\varphi = w - w_*$ solves (4.6) on $J = \mathbb{R}$. Assertion (e) thus follows from the definition of $\mathcal{M}_{\rm c}$.

(f) If $\xi \in P_{\rm c}E \cap B_E(0, r_0)$ for a sufficiently small $r_0 > 0$, as in the proof of assertion (b) we can deduce from (4.15) that $w = w_* + \varphi = w_* + \Phi_c(\xi)$ solves the original problem (3.6) on $[-3,3]$. We can now show assertion (f) as Theorem 5.1(e) in [20]. □

5. A center-stable and a center-unstable manifold

For a better understanding of the center manifold, cf. Corollary 5.3, it is useful to relate it with local stable, unstable, center-stable and center-unstable manifolds. To construct them, we assume the existence of numbers $\omega_s, \omega_u, \omega_{cu}, \omega_{cs} > 0$ such that at least one of the following assertions holds:

$$\sigma(-\Lambda_0) = \sigma_s \cup \sigma_{cu} \quad \text{with} \quad \max \operatorname{Re}\sigma_s < -\omega_s < -\omega_{cu} < \min\operatorname{Re}\sigma_{cu}\,, \tag{5.1}$$

$$\sigma(-\Lambda_0) = \sigma_{cs} \cup \sigma_u \quad \text{with} \quad \max \operatorname{Re}\sigma_{cs} < \omega_{cs} < \omega_u < \min\operatorname{Re}\sigma_u\,. \tag{5.2}$$

In other words, $-\Lambda_0$ has spectral gaps in the open left or the open right half plane. We denote again by P_k the spectral projections for $-A_0$ corresponding to σ_k, $k \in \{s, cs, cu, u\}$. The map $\mathcal{Q} = I - \mathcal{N}_\gamma G$ was introduced in Lemma 3.2.

In the next theorem we construct and study the stable and center-stable manifolds, whereas the center-stable and unstable manifolds are treated afterwards.

Theorem 5.1. *Let Hypothesis 3.4 and (5.2) hold. Take any $\beta \in (\omega_{cs}, \omega_u)$. Then there are numbers $r_u' \ge r_u > 0$, $r_0^u > 0$ and $\eta_{cs} > 0$ such that for each $\eta \in$*

$(0,\eta_{cs}]$ *there exists a radius* $r_{cs}=r_{cs}(\eta)>0$ *such that the following assertions hold, where the cutoff* Γ_+ *is defined in* (4.4) *for the chosen* $\eta\in(0,\eta_{cs}]$.

(a) There is a C_b^1 *map*

$$\phi_u : D_u := \{\xi\in P_{\rm u}E \,\big|\, |\xi|_E<r_0^u\} \longrightarrow P_{\rm cs}E_\gamma$$

such that $\phi_u(0)=0$, $\phi_u'(0)=0$ *and*

$$\begin{aligned}\mathcal{M}_{\rm u} &:= \{w_0=w_*+\xi+\phi_u(\xi)\,\big|\,\xi\in D_u,\ \langle w_0-w_*\rangle_\gamma<r_u\}\\ &= \{w_0\in\mathcal{M}\,\big|\,\langle w_0-w_*\rangle_\gamma<r_u,\ \exists \text{ solution } w=(u,\rho) \text{ of } (1.1) \text{ on } \mathbb{R}_- \text{ with}\\ &\qquad \langle w(t)-w_*\rangle_\gamma\le r_u',\quad \langle w(t)-w_*\rangle_1\le c\mathrm{e}^{\beta t}\ \ (\forall\, t\le 0)\}\end{aligned} \tag{5.3}$$

In (5.3) *we can take* $c=\overline{c}\,|w(0)-w_*|_E$ *for a constant* $\overline{c}$ *independent of* w_0,t,β, *and we have* $w=w_*+\Phi_u(P_{\rm u}(w_0-w_*))$ *for a map* $\Phi_u\in C_b^1(D_u;\mathbb{E}_1(\mathbb{R}_-,-\beta))$ *with* $\Phi_u(0)=0$. *It holds* $\mathcal{M}_{\rm u}\subset\mathcal{M}$.

(b) If $w_0\in\mathcal{M}_{\rm u}$ *and the forward solution* w *of* (1.1) *fulfills* $\langle w-w_*\rangle_\gamma<r_u$ *on* $[0,t]$ *for some* $t>0$, *then* $w(t)\in\mathcal{M}_{\rm u}$. *If* $w_0\in\mathcal{M}_{\rm u}$ *and the solution* w *from* (5.3) *satisfies* $\langle w-w_*\rangle_\gamma<r_u$ *on* $[t,0]$ *for some* $t<0$, *then* $w(t)\in\mathcal{M}_{\rm u}$.

The dimension of $\mathcal{M}_{\rm u}$ *is equal to the dimension of* $P_{\rm u}E$. *If* $\sigma_u\neq\emptyset$, *then* w_* *is (Lyapunov) unstable in* $E_\gamma\times Z_\gamma^1$ *for* (1.1).

If also (RR) holds, then there is a $\widehat{r}_0^u\in(0,r_0^u]$ *such that the map* ϕ_u *is Lipschitz from* $\widehat{D}_u:=\{\xi\in P_{\rm u}E\,\big|\,|\xi|_E<\widehat{r}_0^u\}$ *to* $P_{\rm cs}E_1$, *and* $\phi_u'(\xi)$ *is uniformly bounded in* $\mathcal{L}(P_{\rm u}E,P_{\rm cs}E_1)$ *for* $\xi\in\widehat{D}_u$.

(c) There exist maps $\phi_{cs}\in C^1(P_{\rm cs}E_\gamma^0;P_{\rm u}E)$ *and* $\vartheta_{cs}\in C^1(P_{\rm cs}E_\gamma^0;P_{\rm cs}E_\gamma)$ *with bounded derivatives such that* $\phi_{cs}(0)=\vartheta_{cs}(0)=0$, $\phi_{cs}'(0)=\vartheta_{cs}'(0)=0$, *and*

$$\begin{aligned}\widetilde{\mathcal{M}}_{cs} &:= \big\{w_0=w_*+\xi+\vartheta_{cs}(\xi)+\phi_{cs}(\xi)\,\big|\,\xi\in P_{\rm cs}E_\gamma^0\big\} \\ &= \{w_0=w_*+\varphi(0)\,\big|\,\exists \text{ solution } \varphi\in\mathbb{E}_1(\mathbb{R}_+,-\beta) \text{ of } (4.6) \text{ on } J=\mathbb{R}_+\}.\end{aligned} \tag{5.4}$$

Moreover, the function φ *in* (5.4) *is given by* $\varphi=\Phi_{cs}(\xi)$ *for a map* $\Phi_{cs}\in C_b^1(P_{\rm cs}E_\gamma^0;\mathbb{E}_1(\mathbb{R}_+,-\beta))$.

(d) We define $\mathcal{M}_{\rm cs}=\{w_0\in\widetilde{\mathcal{M}}_{cs}\,\big|\,|w_0-w_*|_{E_\gamma}+|\dot\sigma(0)|_{Z_\gamma^1}<r_{cs}\}$, *where* $\varphi=(v,\sigma)$ *is given by* (5.4). *Then* $\Gamma_+(t,\varphi)=1$ *and* φ *solves the (original) equation* (3.6) *for* $t\in[0,4]$, *at least, and thus* $\mathcal{M}_{\rm cs}\subset\mathcal{M}$. *In particular, we have* $|w_0-w_*|_{E_\gamma}+|\dot\sigma(0)|_{Z_\gamma^1}=\langle w_0-w_*\rangle_\gamma$.

(e) Let $w_0\in\mathcal{M}_{\rm cs}$ *and* φ *be given by* (5.4). *Assume that a forward or a backward solution* w *of* (1.1) *exists and satisfies* $\langle w-w_*\rangle_\gamma<r_{cs}$ *on* $[0,t_0]$ *or on* $[-t_0,0]$ *for some* $t_0>0$, *respectively. Set* $\varphi(t)=w(t)-w_*$ *for* $-t_0\le t\le 0$ *in the second case. Then*

$$\begin{aligned}w(t)=w_*+\varphi(t) &= w_*+P_{\rm cs}\mathcal{Q}(\varphi(t))+\phi_{cs}(P_{\rm cs}\mathcal{Q}\varphi(t)))+\vartheta_{cs}(P_{\rm cs}\mathcal{Q}(\varphi(t)))\\ &= w_*+P_{\rm cs}\varphi(t)+\phi_{cs}(P_{\rm cs}\mathcal{Q}\varphi(t)))\end{aligned} \tag{5.5}$$

belongs to $\mathcal{M}_{\rm cs}$ *for* $0\le t\le t_0$ *or* $-t_0\le t\le 0$, *respectively.*

(f) We have $\mathcal{M}_{\rm cs}\cap\mathcal{M}_{\rm u}=\{w_*\}$.

(g) If w *solves* (1.1) *on* $\mathbb{R}_+$ *with* $\langle w(t)-w_*\rangle_\gamma<r_{cs}$ *for all* $t\ge0$, *then* $w(t)\in\mathcal{M}_{\rm cs}$ *for all* $t\ge0$.

Proof. The proof of assertions (a) and (b) is similar to the corresponding parts of Theorem 5.1 of [20] (and actually a bit simpler), so that we omit it, cf. Theorem 4.1 in [10]. For the center-stable menifold, we proceed as in the case of the stable manifold in Theorem 5.1 of [20], but now we must work in the space $\mathbb{E}_1(\mathbb{R}_+, -\beta)$ containing exponentially growing functions. Hence, as in Theorem 4.6, we have to involve the cutoff Γ_+ which leads to various technical difficulties.

(c) We define the map $\mathcal{T}_{cs} : P_{\rm cs}E_\gamma^0 \times \mathbb{E}_1(\mathbb{R}_+, -\beta) \to \mathbb{E}_1(\mathbb{R}_+, -\beta)$ by setting

$$\mathcal{T}_{cs}(\xi, \varphi) = L^+_{P_{\rm cs},\Lambda_0}(\xi + P_{\rm cs}\mathcal{N}_\gamma\gamma_0 G_{\Gamma+}(\varphi), F_{\Gamma+}(\varphi), G_{\Gamma+}(\varphi)),$$

where the operators $L^+_{P_{\rm cs},\Lambda_0}$, $F_{\Gamma+}$ and $G_{\Gamma+}$ are given by (3.17) and (4.5). Observe that the semigroup $\mathrm{e}_{-\beta}T(\cdot)$ has an exponential dichotomy with the stable projection $P_{\rm cs}$. Due to Corollary 3.1, (2.4) and (3.16), the operator $L^+_{P_{\rm cs},\Lambda_0}$ above is applied to elements of $\mathbb{D}(\mathbb{R}_+, -\beta)$. Propositions 3.7 and 4.5 thus show that the map $\mathcal{T}^0_{cs} : \varphi \mapsto \mathcal{T}_{cs}(\xi, \varphi)$ is C^1 from $\mathbb{E}_1(\mathbb{R}_+, -\beta')$ to $\mathbb{E}_1(\mathbb{R}_+, -\beta)$ for any $\beta' \in (\omega_{cs}, \beta)$ and the derivative of $\mathcal{T}^0_{cs}$ is bounded by $c_1\varepsilon(\eta)$ in the norm of both $\mathcal{L}(\mathbb{E}_1(\mathbb{R}_+, -\beta'))$ and $\mathcal{L}(\mathbb{E}_1(\mathbb{R}_+, -\beta))$, independent of $\xi \in P_{\rm cs}E_\gamma^0$. Moreover, $\mathcal{T}^0_{cs}$ is Lipschitz in $\mathbb{E}_1(\mathbb{R}_+, -\beta')$ with constant $c_1\varepsilon(\eta)$ independent of $\xi \in P_{\rm cs}E^0_\gamma$ by Proposition 4.4. Finally, the map $\xi \mapsto \mathcal{T}_{cs}(\xi, \varphi)$ is affine from $P_{\rm cs}E_\gamma^0$ to $\mathbb{E}_1(\mathbb{R}_+, -\beta')$ with the derivative $T(\cdot)P_{\rm cs}$.

We find an $\eta_{cs} > 0$ such that $c_1\varepsilon(\eta) \le 1/2$ for all $\eta \in (0, \eta_{cs}]$. For any fixed $\eta \in (0, \eta_{cs}]$, Theorem 3 of [25] (with $Y_0 = Y = \mathbb{E}_1(\mathbb{R}_+, -\beta')$ and $Y_1 = \mathbb{E}_1(\mathbb{R}_+, -\beta)$) then shows that for each $\xi \in P_{\rm cs}E^0_\gamma$ there exists a unique solution $\varphi = \Phi_{cs}(\xi) \in \mathbb{E}_1(\mathbb{R}_+, -\beta')$ of the equation $\varphi = \mathcal{T}_{cs}(\xi, \varphi)$, where $\Phi_{cs} \in C^1(P_{\rm cs}E^0_\gamma; \mathbb{E}_1(\mathbb{R}_+, -\beta))$ and $\Phi_{cs}(0) = 0$. Therefore, φ solves (4.6) on $J = \mathbb{R}_+$. Due to (4.4) in [25], the derivatives $\Phi'_{cs}(\xi) \in \mathcal{L}(P_{\rm cs}E^0_\gamma, \mathbb{E}_1(\mathbb{R}_+, -\beta))$ are bounded uniformly in ξ. (We note that φ is also the unique fix point in $\mathbb{E}_1(\mathbb{R}_+, -\beta)$.) We then introduce

$$\vartheta_{cs}(\xi) = P_{\rm cs}\mathcal{N}_\gamma\gamma_0 G_{\Gamma+}(\Phi_{cs}(\xi)) \quad \text{and} \quad \phi_{cs}(\xi) = P_{\rm u}\gamma_0\Phi_{cs}(\xi), \quad \text{i.e.,} \tag{5.6}$$

$$\phi_{cs}(\xi) = -\int_0^\infty T_u(-\tau)P_{\rm u}\Big(\Pi\widehat{G}_{\Gamma+}(\Phi_{cs}(\xi))(\tau) \\ + \big[F_{\Gamma+}(\Phi_{cs}(\xi))(\tau) - A_{*,\dot{\rho}}G_{0\Gamma+}(\Phi_{cs}(\xi))(\tau), G_{0\Gamma+}(\Phi_{cs}(\xi))(\tau)\big]\Big)\,\mathrm{d}\tau,$$

for $\xi \in P_{\rm cs}E^0_\gamma$. Hence, Corollary 3.1, (3.15), (2.4), Propositions 3.6 and 4.5 imply that $\phi_{cs} \in C^1(P_{\rm cs}E^0_\gamma; P_{\rm u}E_\gamma)$ and $\vartheta_{sc} \in C^1(P_{\rm cs}E^0_\gamma; P_{\rm cs}E_\gamma)$ with bounded derivatives, as well as $\phi_{cs}(0) = \vartheta_{cs}(0) = 0$ and $\phi'_{cs}(0) = \vartheta'_{cs}(0) = 0$. The inclusion '$\subset$' in (5.4) now follows from our construction.

Conversely, any solution $\varphi = (v, \sigma) \in \mathbb{E}_1(\mathbb{R}_+, -\beta)$ of (4.6) on $\mathbb{R}_+$ is given by

$$\varphi = L^+_{P_{\rm cs},\Lambda_0}(P_{\rm cs}\varphi(0), F_{\Gamma+}(\varphi), G_{\Gamma+}(\varphi))$$

due to Proposition 3.7. We set $\xi = P_{\rm cs}[\varphi(0) - \mathcal{N}_\gamma\gamma_0 G_{\Gamma+}(\varphi))]$. Combined with the embeddings (2.3), (2.4), (2.7) and the equations (4.6), Corollary 3.1 shows that $\varphi(0) - \mathcal{N}_\gamma\gamma_0 G_{\Gamma+}(\varphi) \in E_\gamma$,

$$(\widehat{B}, \widehat{C})\mathcal{N}_\gamma\gamma_0 G_{\Gamma+}(\varphi) = \gamma_0\widehat{G}_{\Gamma+}(\varphi) = (\widehat{B}, \widehat{C})\varphi(0), \qquad \text{and}$$

$$(B_0, C_0)\mathcal{N}_\gamma\gamma_0 G_{\Gamma+}(\varphi) - (B_0, C_0)\varphi(0) = (B_0, C_0)\mathcal{N}_\gamma\gamma_0 G_{\Gamma+}(\varphi) - \gamma_0 G_{\Gamma+}(\varphi) + \dot{\sigma}(0)$$

belongs to Z^1_γ. Hence, $\xi \in P_{\mathrm{cs}} E^0_\gamma$ and $\varphi = \mathcal{T}_{cs}(\xi, \varphi)$. Because of the uniqueness of the latter equation, we arrive at $\varphi = \Phi_{cs}(\xi)$, and thus $\varphi(0) = \xi + \vartheta_{cs}(\xi) + \phi_{cs}(\xi) \in \widetilde{\mathcal{M}}_{cs} - w_*$.

(d) Take $w_0 \in \widetilde{\mathcal{M}}_{cs}$ with the corresponding $\xi \in P_{\mathrm{cs}} E^0_\gamma$ and the solution $\varphi = (v, \sigma) \in \mathbb{E}_1(\mathbb{R}_+, -\beta)$ of (4.6) with $\varphi(0) = w_0 - w_* =: \varphi_0$. Assume that $|w_0 - w_*|_{E_\gamma} + |\dot\sigma(0)|_{Z^1_\gamma} < r$ for an $r > 0$ to be fixed below. Assertion (c) and its proof, (3.15), Corollary 3.1, (4.6), (2.4) and Proposition 4.4 yield

$$\begin{aligned}
\|\varphi\|_{\mathbb{E}_1(\mathbb{R}_+,-\beta)} &= \|\Phi_{cs}(\xi) - \Phi_{cs}(0)\|_{\mathbb{E}_1(\mathbb{R}_+,-\beta)} \le c\,|\xi|_{E^0_\gamma} \le c\,|\varphi(0) - \mathcal{N}_\gamma \gamma_0 G_{\Gamma+}(\varphi)|_{E^0_\gamma} \\
&\le c\,\big(|\varphi_0 - \mathcal{N}_\gamma \gamma_0 G_{\Gamma+}(\varphi)|_{E_\gamma} + |\dot\sigma(0)|_{Z^1_\gamma} \\
&\qquad + |\gamma_0 G_{\Gamma+}(\varphi) - (B_0, C_0)\mathcal{N}_\gamma \gamma_0 G_{\Gamma+}(\varphi)|_{Z^1_\gamma}\big) \\
&\le c\,\big(|\varphi_0|_{E_\gamma} + |\dot\sigma(0)|_{Z^1_\gamma} + |\gamma_0 G_{\Gamma+}(\varphi)|_{Y_\gamma}\big) \\
&\le c\,\big(|\varphi_0|_{E_\gamma} + |\dot\sigma(0)|_{Z^1_\gamma} + \|G_{\Gamma+}(\varphi) - G_{\Gamma+}(0)\|_{\mathbb{E}_1(\mathbb{R}_+,-\beta)}\big) \\
&\le c\,\big(|\varphi_0|_{E_\gamma} + |\dot\sigma(0)|_{Z^1_\gamma}\big) + \varepsilon(\eta)\,\|\varphi\|_{\mathbb{E}_1(\mathbb{R}_+,-\beta)}.
\end{aligned}$$

Fixing a sufficiently small $\eta_{cs} > 0$ and using also (4.3), we thus obtain

$$\|\varphi\|_{\mathbb{E}_1([0,6])} \le c\,\|\varphi\|_{\mathbb{E}_1(\mathbb{R}_+,-\beta)} \le \widetilde{c}\,\big(|\varphi_0|_{E_\gamma} + |\dot\sigma(0)|_{Z^1_\gamma}\big) \tag{5.7}$$

for a constant $\widetilde{c} > 0$ and all $\eta \in (0, \eta_{cs}]$. We take an $r^0_{cs} = r^0_{cs}(\eta) > 0$ such that $r^0_{cs}\widetilde{c} \le c_W^{-1}(1 + c_R)^{-1}\eta \le c_W^{-1}(1 + c_R)^{-1}\eta_{cs}$. For every $r \in (0, r^0_{cs}]$, Remark 4.1 and (5.7) thus imply that $\Gamma_+(t, \varphi) = 1$ for $0 \le t \le 4$. As a result,

$$\xi = P_{\mathrm{cs}}[\varphi(0) - \mathcal{N}_\gamma G(\varphi_0)] = P_{\mathrm{cs}} \mathcal{Q}(\varphi_0) \tag{5.8}$$

and φ solves the original problem (3.6) on $[0, 4]$, so that $|\dot\sigma(0)|_{Z^1_\gamma} = [w_0 - w_*]_\gamma$.

(e.i) Let $w_0 \in \widetilde{\mathcal{M}}_{cs}$ and $w = w_* + \psi$ be a solution of (1.1) on $[0, t_0]$ with $w(0) = w_0$ for some $t_0 > 0$. Let $\varphi = (v, \sigma) \in \mathbb{E}_1(\mathbb{R}_+, -\beta)$ be the solution of (4.6) with $\varphi(0) = w_0 - w_*$. We assume that $\langle \psi(t)\rangle_\gamma < r \le r^0_{cs}$ for $0 \le t \le t_0$, and want to derive that $\psi(t) = \varphi(t)$ and $w(t) \in \widetilde{\mathcal{M}}_{cs}$ for $0 \le t \le t_0$. Part (d) of the proof implies that $\Gamma_+(t, \varphi) = 1$ and that φ solves (3.6) for $t \in [0, 4]$. The uniqueness of (3.6) thus gives $\psi(t) = \varphi(t)$ for $t \in [0, 4] \cap [0, t_0]$.

First, let $t_0 \le 2$ and set $\widetilde{\varphi}(t) = \varphi(t + t_0)$ for $t \ge 0$. Remark 4.2 yields that $\Gamma_+(t, \widetilde{\varphi}) = \Gamma_+(t + t_0, \varphi)$ for $t \ge 2$. From (5.7) we further deduce that $\|\widetilde{\varphi}\|_{\mathbb{E}_1([t-2,t+2]\cap\mathbb{R}_+)} \le \|\varphi\|_{\mathbb{E}_1([0,6])} < c'r \le c'r^0_{cs}$ for $0 \le t \le 2$. Decreasing $r^0_{cs} > 0$, we arrive at $\Gamma_+(t, \widetilde{\varphi}) = 1$ for $0 \le t \le 2$ thanks to Remark 4.1. Since $2 + t_0 \le 4$, we also have $\Gamma_+(t + t_0, \varphi) = 1$ for $0 \le t \le 2$. Summing up, it holds $\Gamma_+(t, \widetilde{\varphi}) = \Gamma_+(t + t_0, \varphi)$ for all $t \ge 0$ so that $\widetilde{\varphi} \in \mathbb{E}_1(\mathbb{R}_+, -\beta)$ solves (4.6) on $J = \mathbb{R}_+$ with $\widetilde{\varphi}(0) = \varphi(t_0)$. This means that $\psi(t_0) = \varphi(t_0) \in (\widetilde{\mathcal{M}}_{cs} - w_*) \cap \mathcal{M}^*(r)$. Since we can replace here t_0 by $t \in [0, t_0]$, part (c) and formula (5.8) show

$$w(t) = w_* + \varphi(t) = w_* + P_{\mathrm{cs}}\mathcal{Q}(\varphi(t)) + \phi_{cs}(P_{\mathrm{cs}}\mathcal{Q}(\varphi(t))) + \vartheta_{cs}(P_{\mathrm{cs}}\mathcal{Q}(\varphi(t))) \tag{5.9}$$

belongs to $\widetilde{\mathcal{M}}_{cs}$ for $0 \le t \le t_0$. If $t_0 > 2$, we obtain this result by a finite iteration of the above argument.

(e.ii) Let $w_0 \in \widetilde{\mathcal{M}}_{cs}$ possess a solution w of (1.1) on $[-t_0, 0]$ with $w(0) = w_0$ for some $t_0 > 0$. We set $z(t) = w(t) - w_* = (u(t), \rho(t)) - w_*$ and assume that $\langle z(t)\rangle_\gamma < r \le r^0_{cs}$ for $-t_0 \le t \le 0$. Let $\varphi = (v, \sigma) \in \mathbb{E}_1(\mathbb{R}_+, -\beta)$ be the solution of (4.6) with $\varphi(0) = w_0 - w_*$ given by (5.4). To show that $w(t) \in \widetilde{\mathcal{M}}_{cs}$ for $-t_0 \le t \le 0$, we set $z(t) = \varphi(t)$ and $\psi(t) = z(t - t_0)$ for $t \ge 0$. Clearly, $\psi \in \mathbb{E}_1(\mathbb{R}_+, -\beta)$, $\psi(0) = w(-t_0) - w_*$, and ψ satisfies (3.6) on $[0, t_0 + 2]$ since z and φ solve (3.6) on $[-t_0, 0]$ and $[0, 2]$, respectively, and we have $z(0) = \varphi(0)$ and thus $\dot\rho(0) = \dot\sigma(0)$. (Here we also use Lemma 2.2.) Take $t \in [0, t_0 + 2]$ and s with $|t - s| \le 1/4$. Noting that $[0, 1] \subset J(s)$ if $J(s) \cap \mathbb{R}_- \ne \emptyset$, we deduce from Lemma 2.2 and (4.2) that

$$\begin{aligned}\|R_+\psi\|_{\mathbb{E}_1(J(s))} &\le c\left(\|\psi\|_{\mathbb{E}_1(J(s)\cap[0,t_0])} + \|\psi\|_{\mathbb{E}_1(J(s)\cap[t_0,\infty))} + \|R_+\psi\|_{\mathbb{E}_1(J(s)\cap[-1,0])}\right)\\ &\le c_*\left(\|z(\cdot - t_0)\|_{\mathbb{E}_1(J(s)\cap[0,t_0])} + \|\varphi(\cdot - t_0)\|_{\mathbb{E}_1(J(s)\cap[t_0,\infty))}\right)\\ &= c_*\left(\|z\|_{\mathbb{E}_1(J(s-t_0)\cap[-t_0,0])} + \|\varphi\|_{\mathbb{E}_1(J(s-t_0)\cap\mathbb{R}_+)}\right)\end{aligned}$$

for a constant $c_* > 0$ given by these lemmas. Since z solves (3.6) on $J(s - t_0) \cap [-t_0, 0] =: [a, b]$, the estimate in Theorem 3.3(b) yields $\|z\|_{\mathbb{E}_1([a,b])} \le \widehat{c}\,\langle z(a)\rangle_\gamma < \widehat{c}r \le \widehat{c}r^0_{cs}$ for a constant $\widehat{c} > 0$, where we possibly decrease $r^0_{cs} > 0$ to apply the theorem. Moreover, from inquality (5.7) we infer that $\|\varphi\|_{\mathbb{E}_1(J(s-t_0)\cap\mathbb{R}_+)} \le \widetilde{c}\,\langle\varphi(0)\rangle_\gamma < \widetilde{c}r$ using that $J(s - t_0) \cap \mathbb{R}_+ \subset [0, 4]$. As a consequence,

$$\|R_+\psi\|_{\mathbb{E}_1(J(s))} < c_*(\widehat{c} + \widetilde{c})r \le c_*(\widehat{c} + \widetilde{c})r^0_{cs}.$$

Decreasing r^0_{cs} once more if necessary, we obtain $\|R_+\psi\|_{\mathbb{E}_1(J(s))} \le \eta_{cs}/(c_W(1 + c_R))$ so that $\Gamma_+(t, \psi) = 1$ for $0 \le t \le t_0 + 2$ due to Remark 4.1.

The function ψ thus satisfies (4.6) for $0 \le t \le t_0{+}2$. For $t \ge t_0{+}2$, Remark 4.2 yields $\Gamma_+(t, \psi) = \Gamma_+(t{-}t_0, \varphi)$ so that ψ fulfills (4.6) also on $[t_0{+}2, \infty)$. Summing up, $\psi \in \mathbb{E}_1(\mathbb{R}_+, -\beta)$ solves (4.6) on $\mathbb{R}_+$ and so $w(-t_0) - w_* = \psi(0) \in (\widetilde{\mathcal{M}}_{cs} - w_*) \cap \mathcal{M}^*(r)$. Replacing here $-t_0$ by $t \in [-t_0, 0]$, and writing $\varphi(t) = z(t)$, we arrive at (5.9) for $-t_0 \le t \le 0$. Since $P_{\rm cs}\varphi(t) = P_{\rm cs}\mathcal{Q}(\varphi(t)) + \vartheta_{cs}(P_{\rm cs}\mathcal{Q}(\varphi(t)))$, formula (5.5) follows from (5.9).

(f) Assume that $w_0 = w_* + \varphi_0 \in \widetilde{M}_{cs} \cap \mathcal{M}_{\rm u}$ with $\langle\varphi_0\rangle_\gamma < r \le \min\{r^0_{cs}, r_u\}$, and let $\varphi = (v, \sigma) \in \mathbb{E}_1(\mathbb{R}_+, -\beta)$ be the solution of (4.6) with $\varphi(0) = \varphi_0$ given by (5.4). Take $\beta + \epsilon \in (\beta, \omega_u)$. Due to assertion (a), there is a solution z of (3.6) on $\mathbb{R}_-$ with $z(0) = \varphi_0$ satisfying

$$\langle z(t)\rangle_\gamma \le \bar{c}\mathrm{e}^{(\beta+\epsilon)t}\,\langle\varphi_0\rangle_\gamma < \bar{c}r \tag{5.10}$$

for all $t \le 0$. We choose a sufficiently small $r =: r_{cs} > 0$ such that $\bar{c}r \le r^0_{cs}$, and take $t \le 0$. We define $\mathcal{M}_{\rm cs}$ for this $r_{cs} \le r^0_{cs}$. Part (e.ii) of the proof implies that $w_* + z(t) \in \mathcal{M}_{\rm cs}$ and that the function $\psi_t \in \mathbb{E}_1(\mathbb{R}_+, -\beta)$ given by $\psi_t(\tau) = z(t + \tau)$ for $\tau \in [0, -t]$ and $\psi_t(\tau) = \varphi(t + \tau)$ for $\tau \ge -t$ solves (4.6) on $J = \mathbb{R}_+$. Estimate (5.7) thus yields

$$\|\psi_t\|_{\mathbb{E}_1(\mathbb{R}_+,-\beta)} \le c\,\langle\psi_t(0)\rangle_\gamma = c\,\langle z(t)\rangle_\gamma\,, \tag{5.11}$$

where the constant c does not depend on $t \le 0$. Observe that

$$\langle\varphi_0\rangle_\gamma = \mathrm{e}^{-\beta t}\mathrm{e}^{-\beta(-t)}\left(|\varphi(0)|_{E_\gamma} + |\dot\sigma(0)|_{Z^1_\gamma}\right) \le c\mathrm{e}^{-\beta t}\langle(\mathrm{e}_{-\beta}\psi_t)(-t)\rangle_\gamma$$

due to (3.8) and (2.13). Using the embeddings (2.3), (2.7) and the inequalities (5.11) and (5.10), we then estimate

$$\langle\varphi_0\rangle_\gamma \le c\mathrm{e}^{-\beta t}\|\psi_t\|_{\mathbb{E}_1(\mathbb{R}_+,-\beta)} \le c\mathrm{e}^{-\beta t}\langle z(t)\rangle_\gamma \le c\mathrm{e}^{\epsilon t}\langle\varphi_0\rangle_\gamma,$$

where the constants do not depend on t. Letting $t \to -\infty$, we conclude that $w_0 - w_* = \varphi_0 = 0$.

(g) The last assertion can be shown as part (f) of Theorem 4.6. $\square$

Theorem 5.2. *Let Hypothesis 3.4 and (5.1) hold. Take any $\alpha \in (\omega_{cu}, \omega_s)$. Then there are numbers $r_s' \geq r_s > 0$, $r_0^s > 0$ and $\eta_{cu} > 0$ such that for each $\eta \in (0, \eta_{cu}]$ there exists a radius $r_{cu} = r_{cu}(\eta) > 0$ such that the following assertions hold, where the cutoff Γ_- is defined in (4.4) for the chosen $\eta \in (0, \eta_{cu}]$.*

(a) There are C_b^1 maps

$$\phi_s : D_s := \{\xi \in P_\mathrm{s} E_\gamma^0 \,\big|\, |\xi|_{E_\gamma^0} < r_0^s\} \longrightarrow P_\mathrm{cu} E, \qquad \vartheta_s : D_s \longrightarrow P_\mathrm{s} E_\gamma\,,$$

such that $\phi_s(0) = \vartheta_s(0) = 0$, $\phi_s'(0) = \vartheta_s'(0) = 0$ and

$$\begin{aligned} \mathcal{M}_\mathrm{s} := \{w_0 = w_* + \xi + \vartheta_s(\xi) + \phi_s(\xi) \,\big|\, \xi \in D_s,\ \langle w_0 - w_*\rangle_\gamma < r_s\} \\ = \{w_0 \in \mathcal{M} \,\big|\, \langle w_0 - w_*\rangle_\gamma < r_s,\ \exists \text{ solution } w = (u, \rho) \text{ of } (1.1) \text{ on } \mathbb{R}_+ \text{ with} \\ \langle w(t) - w_*\rangle_\gamma \leq r_s',\ \ \langle w(t) - w_*\rangle_1 \leq c\mathrm{e}^{-\alpha t}\ \ (\forall\, t \geq 0)\}. \end{aligned} \tag{5.12}$$

In (5.12) we can take $c = \overline{c}\,\langle w(0) - w_*\rangle_\gamma$ for a constant $\overline{c}$ independent of w_0, t, α, and we have $w = w_* + \Phi_s(P_\mathrm{s}\mathcal{Q}(w_0 - w_*))$ for a map $\Phi_s \in C_b^1(D_s; \mathbb{E}_1(\mathbb{R}_+, \alpha))$ with $\Phi_s(0) = 0$.

(b) If $w_0 \in \mathcal{M}_\mathrm{s}$ and the forward (resp., a backward) solution w of (1.1) satisfies $\langle w - w_*\rangle_\gamma < r_s$ on $[0, t]$ for some $t > 0$ (resp., on $[t, 0]$ for some $t < 0$), then $w(t) \in \mathcal{M}_\mathrm{s}$.

(c) There exists a map $\phi_{cu} \in C^1(P_\mathrm{cu}E; P_\mathrm{s}E_\gamma)$ with a bounded derivative such that $\phi_{cu}(0) = 0$, $\phi_{cu}'(0) = 0$, and

$$\begin{aligned} \widetilde{\mathcal{M}}_{cu} := \big\{w_0 = w_* + \xi + \phi_{cu}(\xi) \,\big|\, \xi \in P_\mathrm{cu} E\big\} \\ = \{w_0 = w_* + \varphi(0) \,\big|\, \exists \text{ solution } \varphi \in \mathbb{E}_1(\mathbb{R}_-, \alpha) \text{ of } (4.6) \text{ on } J = \mathbb{R}_-\}. \end{aligned} \tag{5.13}$$

Moreover, the function φ in (5.13) is given by $\varphi = \Phi_{cu}(P_\mathrm{cu}(w_0 - w_*))$ for a map $\Phi_{cu} \in C^1(P_\mathrm{cu}E; \mathbb{E}_1(\mathbb{R}_-, \alpha))$ having a bounded derivative.

(d) We define $\mathcal{M}_\mathrm{cu} = \{w_0 \in \widetilde{\mathcal{M}}_{cu} \,\big|\, \langle w_0 - w_*\rangle_\gamma < r_{cu}\}$. For $w_0 \in \mathcal{M}_\mathrm{cu}$, let φ be given by (5.13). Then $\Gamma_-(t, \varphi) = 1$ and φ solves the (original) equation (3.6) for $t \in [0, 4]$, at least. The dimension of $\mathcal{M}_\mathrm{cu}$ is equal to $\dim P_\mathrm{cu}E$. We have $w_0 = w_* + \xi + \phi_{cu}(\xi) \in \mathcal{M}_\mathrm{cu}$ and $\langle w_0 - w_*\rangle_\gamma \leq c\,|\xi|_E$ if $|\xi|_E$ for $\xi = P_\mathrm{cu}(w_0 - w_*)$.

(e) Let $w_0 \in \mathcal{M}_\mathrm{cu}$ and φ be given by (5.13). If the forward solution w of (1.1) exists and satisfies $\langle w - w_*\rangle_\gamma < r_{cu}$ on $[0, t_0]$ for some $t_0 > 0$, then $w(t) =: w_* + \varphi(t) \in \mathcal{M}_\mathrm{cu}$ for $0 \leq t \leq t_0$. If the function $\widetilde{w} = w_* + \varphi$ satisfies $\langle \widetilde{w} - w_*\rangle_\gamma < r_{cu}$ on $[t_0, 0]$ for some $t_0 < 0$, then $\widetilde{w}(t) = w_* + \varphi(t) \in \mathcal{M}_\mathrm{cu}$ and $\widetilde{w}$ solves (1.1) for $t_0 \leq t \leq 0$. In particular, $\varphi(t) = P_\mathrm{cu}\varphi(t) + \phi_{cu}(P_\mathrm{cu}\varphi(t))$ for $t \in [0, t_0]$, resp. $t \in [t_0, 0]$.

(f) We have $\mathcal{M}_\mathrm{cu} \cap \mathcal{M}_\mathrm{s} = \{w_*\}$.

(g) Let $w_0 = w_* + \varphi_0 \in \mathcal{M}_\mathrm{cu}$ and let φ be given by (5.13). Assume that $\varphi(t) + w_* \in \mathcal{M}_\mathrm{cu}$ for all $t \in (a, b)$ and some $a < 0 < b$. Then $z = P_\mathrm{cu}\varphi$ satisfies

the equations

$$\begin{aligned}\dot z(t) &= -\Lambda_0 P_{\rm cu} z(t) + P_{\rm cu}\Pi\widehat G(z(t)+\phi_{cu}(z(t)))\\ &\qquad + P_{\rm cu}[\widetilde F(z(t)+\phi_{cu}(z(t))), G_0(z(t)+\phi_{cu}(z(t)))],\\ z(0) &= P_{\rm cu}(w_0-w_*),\end{aligned}\tag{5.14}$$

on $P_{\rm cu}E$ *for* $t\in(a,b)$*, where* $\widetilde F = F - A_{*\dot\rho_*}G_0$*. Also,* $\varphi\in C((a,b);E_1)$ *and*

$$(B_0,C_0)\phi_{cu}(P_{\rm cu}\varphi_0) - G_0(\varphi_0)\in Z^1_\gamma, \qquad (\widehat B,\widehat C)\phi_{cu}(P_{\rm cu}\varphi_0) = \widehat G(\varphi_0), \tag{5.15}$$

$$P_{\rm s}[\Lambda_*\varphi_0 - (\widetilde F(\varphi_0), G_0(\varphi_0)))] = \phi'_{cu}(P_{\rm cu}\varphi_0)P_{\rm cu}[\Lambda_*\varphi_0 - (\widetilde F(\varphi_0), G_0(\varphi_0))]. \tag{5.16}$$

(h) If w *solves* (1.1) *on* $\mathbb{R}_-$ *with* $\langle w(t)-w_*\rangle_\gamma < r_{cu}$ *for all* $t\le 0$*, then* $w(t)\in\mathcal{M}_{\rm cu}$ *for all* $t\le 0$.

(i) Assume that (RR) holds, too. Then there is a $r_0>0$ *such that the map* ϕ_{cu} *is Lipschitz from* $\widehat D_{cu} := \{\xi\in P_{\rm cu}E \,\big|\, |\xi|_E < r_0\}$ *to* $P_{\rm s}E_1$*, and* $\phi'_{cu}(\xi)$ *is uniformly bounded in* $\mathcal{L}(P_{\rm cu}E, P_{\rm s}E_1)$ *for* $\xi\in\widehat D_{cu}$.

Proof. As in Theorem 5.1 we do not give the proof of (a) and (b), cf. Theorem 5.1 in [20] and also Theorem 4.1 in [10]. The proof of assertions (c)–(i) is similar the corresponding parts in the previous theorem, so that we can omit some details and focus on the differences.

(c) We define the map $\mathcal{T}_{cu}: P_{\rm cu}E\times\mathbb{E}_1(\mathbb{R}_-,\alpha)\to\mathbb{E}_1(\mathbb{R}_-,\alpha)$ by setting

$$\mathcal{T}_{cu}(\xi,\varphi) = L^-_{P_{\rm s},\Lambda_0}(\xi, F_{\Gamma-}(\varphi), G_{\Gamma-}(\varphi)),$$

where the operators $L^-_{P_{\rm s},\Lambda_0}$, $F_{\Gamma-}$ and $G_{\Gamma-}$ are given by (3.19) and (4.5). Using Propositions 3.8, 4.4 and 4.5, we find an $\eta_{cu}>0$ such that the assumptions of Theorem 3 of [25] hold for the cutoff Γ_- for any parameter $\eta\in(0,\eta_{cu}]$. Let $\alpha'\in(\omega_{cu},\alpha)$. As in the proof of Theorem 4.6, for each $\xi\in P_{\rm cu}E$ there exists a unique solution $\varphi=\Phi_{cu}(\xi)\in\mathbb{E}_1(\mathbb{R}_-,\alpha')$ of the equation $\varphi=\mathcal{T}_{cu}(\xi,\varphi)$, where $\Phi_{cu}\in C^1(P_{\rm cu}E;\mathbb{E}_1(\mathbb{R}_-,\alpha))$, $\Phi_{cu}(0)=0$, and the derivatives $\Phi'_{cu}(\xi)\in\mathcal{L}(P_{\rm cu}E,\mathbb{E}_1(\mathbb{R}_-,\alpha))$ are bounded uniformly in ξ. We then introduce the map

$$\begin{aligned}\phi_{cu}(\xi) &= \gamma_0 P_{\rm s}\Phi_{cu}(\xi)\\ &= \int_{-\infty}^0 T_{-1}(-\tau)P_{\rm s}\Big(\big[F_{\Gamma+}(\Phi_{cs}(\xi))(\tau) - A_{*\dot\rho}G_{0\Gamma+}(\Phi_{cs}(\xi))(\tau), G_{0\Gamma+}(\Phi_{cs}(\xi))(\tau)\big]\\ &\qquad\qquad + \Pi\widehat G_{\Gamma+}(\Phi_{cs}(\xi))(\tau)\Big)\,\mathrm{d}\tau,\end{aligned}$$

for $\xi\in P_{\rm cu}E$. As before, we obtain that $\phi_{cu}\in C^1(P_{\rm cu}E;P_{\rm s}E_\gamma)$ with a bounded derivative and that $\phi_{cu}(0)=0$ and $\phi'_{cu}(0)=0$. Moreover, the identity (5.13) follows from Proposition 3.8, where $\varphi=\Phi_{cu}(\xi)$ and $\xi=P_{\rm cu}(w_0-w_*)$.

(d) Take $w_0\in\widetilde{\mathcal{M}}_{cu}$ with $\langle w_0-w_*\rangle_\gamma < r\le r^0_{cu}$ where $r^0_{cu}>0$ is fixed below. Let φ be the corresponding solution of (4.6) on $\mathbb{R}_-$ given by (5.13). From (4.3) and part (c) we deduce

$$\|\varphi\|_{\mathbb{E}_1([-6,0])} \le c\,\|\varphi\|_{\mathbb{E}_1(\mathbb{R}_-,\alpha)} \le c\,|\xi|_E \le c\,|\varphi(0)|_E \le c'r \le c'r^0_{cu}, \tag{5.17}$$

where the constants do not depend on r and α. For any fixed $\eta\in(0,\eta_{cu}]$, we can choose a sufficiently small $r^0_{cu}>0$ so that $c'r^0_{cu}\le(c_W(1+c_R))^{-1}\eta$. Remark 4.1 thus yields $\Gamma_-(t,\varphi)=1$ for $-4\le t\le 0$. As a result, φ solves the original problem (3.6) on $[-4,0]$. The last assertion now follows from (5.17), using also the embeddings (2.3) and (2.7).

(e.i) Take $w_0 \in \widetilde{\mathcal{M}}_{cu}$ and $t_0 > 0$ such that the solution $w = w_* + \psi$ of (1.1) on $[0,t_0]$ with $w(0) = w_0$ satisfies $\langle w(t) - w_* \rangle_\gamma < r \le r_{cu}^0$ for $t \in [0,t_0]$. Let $\varphi \in \mathbb{E}_1(\mathbb{R}_-,\alpha)$ be the solution of (4.6) on $J = \mathbb{R}_-$ with $\varphi(0) = w_0 - w_*$ given by (5.13). We further define $\psi(t) = \varphi(t)$ and $z(t) = \psi(t+t_0)$ for $t \le 0$. As before, $z \in \mathbb{E}_1(\mathbb{R}_-,\alpha)$, $z(0) = \varphi(t_0)$, and z satisfies (3.6) on $[-t_0-2,0]$ since ψ and φ solve (3.6) on $[0,t_0]$ and $[-2,0]$, respectively. Take $t \in [-t_0-2,0]$ and s with $|t-s| \le 1/4$. As in part (e.ii) of the proof of Theorem 5.2, we deduce from Lemma 2.2 and (4.2) that

$$\begin{aligned}\|R_- z\|_{\mathbb{E}_1(J(s))} &\le c\left(\|\psi(\cdot + t_0)\|_{\mathbb{E}_1(J(s)\cap[-t_0,0])} + \|\varphi(\cdot + t_0)\|_{\mathbb{E}_1(J(s)\cap(-\infty,-t_0])}\right)\\ &= c\left(\|\psi\|_{\mathbb{E}_1(J(s+t_0)\cap[0,t_0])} + \|\varphi\|_{\mathbb{E}_1(J(s+t_0)\cap\mathbb{R}_-)}\right).\end{aligned}$$

Theorem 3.3(b) shows that $\|\psi\|_{\mathbb{E}_1([a,b])} \le c_* r$ for sufficiently small $r_{cu}^0 > 0$ since ψ solves (3.6) on $J(s+t_0)\cap[0,t_0] =: [a,b]$. Using $(J(s+t_0)\cap\mathbb{R}_-) \subset [-4,0]$ and (5.17), we estimate $\|\varphi\|_{\mathbb{E}_1(J(s+t_0)\cap\mathbb{R}_-)} \le c' r_{cu}^0$. Consequently, $\|R_- z\|_{\mathbb{E}_1(J(s))} \le c r_{cu}^0$, and so Remark 4.1 yields $\Gamma_-(t,z) = 1$ for $-t_0 - 2 \le t \le 0$, where we decrease r_{cu}^0 if necessary. The function z thus satisfies (4.6) for $-t_0 - 2 \le t \le 0$. Moreover, Remark 4.2 yields that $\Gamma_-(t,z) = \Gamma_-(t+t_0,\varphi)$ for $t \le -t_0-2$; and so z fulfills the equations (4.6) for $t \le -t_0 - 2$. Summing up, we have shown that z solves (4.6) on $\mathbb{R}_-$, and so $w_* + z(0) = w(t_0) \in \widetilde{\mathcal{M}}_{cu} \cap (\mathcal{M}^*(r) + w_*)$.

(e.ii) Let $w_0 \in \widetilde{\mathcal{M}}_{cu}$ and φ be given by (5.13). Assume that $\varphi(t) \in \mathcal{M}^*(r)$ for all $t \in [t_0,0]$ and some $t_0 < 0$ and $r \in (0, r_{cu}^0]$. We first consider the case that $t_0 \in [-2,0)$. Assertion (d) shows that $\Gamma_-(t,\varphi) = 1$ and φ solves (3.6) on $[t_0,0]$. We further set $\widetilde{\varphi}(t) = \varphi(t+t_0)$ for $t \le 0$. Remark 4.2 yields that $\Gamma_-(t,\widetilde{\varphi}) = \Gamma_-(t+t_0,\varphi)$ for $t \le -2$. Since $\|\widetilde{\varphi}\|_{\mathbb{E}_1([t-2,t+2]\cap\mathbb{R}_-)} \le \|\varphi\|_{\mathbb{E}_1([-6,0])} \le (c_W(1+c_R))^{-1}\eta$ for $-2 \le t \le 0$ by (5.17) and our choice of r_{cu}^0, we deduce $\Gamma_-(t,\widetilde{\varphi}) = 1$ for $-2 \le t \le 0$ from Remark 4.1. Finally, $\Gamma_-(t+t_0,\varphi) = 1$ for $-2 \le t \le 0$ due to part (d), and hence $\Gamma_-(t,\widetilde{\varphi}) = \Gamma_-(t+t_0,\varphi)$ for all $t \le 0$. As a result, $\widetilde{\varphi} \in \mathbb{E}_1(\mathbb{R}_-,\alpha)$ solves (4.6) on $J = \mathbb{R}_-$ with $\widetilde{\varphi}(0) = \varphi(t_0)$, i.e., $\varphi(t) + w_* \in \widetilde{\mathcal{M}}_{cu} \cap (\mathcal{M}^*(r) + w_*)$ for each $t \in [t_0,0]$. The general case $t_0 < -2$ is then established by repeating the above arguments finitely many times.

(f) Assume that $w_0 = w_* + \varphi_0 \in \widetilde{\mathcal{M}}_{cu} \cap \mathcal{M}_{\mathrm{s}} \cap (w_* + \mathcal{M}^*(r))$ with $r \in (0, r_{cu}^0]$. Let $\varphi \in \mathbb{E}_1(\mathbb{R}_-,\alpha)$ be the solution of (4.6) with $\varphi(0) = \varphi_0$ given by (5.13). For $\alpha + \epsilon \in (\alpha, \omega_s)$, there is a solution ψ of (3.6) on $\mathbb{R}_+$ with $\psi(0) = \varphi_0$ satisfying $\langle \psi(t) \rangle_\gamma \le \overline{c} e^{-(\alpha+\epsilon)t} \langle \varphi_0 \rangle_\gamma \le \overline{c} r$ for all $t \ge 0$, if we fix a sufficiently small $r_{cu} := r \in (0, r_{cu}^0]$ and use assertion (a). Set $\psi(t) = \varphi(t)$ for $t \le 0$. Part (e.i) of the proof now shows that $w_* + \psi(t) \in \widetilde{\mathcal{M}}_{cu}$ for $t \ge 0$ and that the function $z_t = \psi(\cdot + t)$ satisfies $\|z_t\|_{\mathbb{E}_1(\mathbb{R}_-,\alpha)} \le c\,\langle \psi(t) \rangle_\gamma$, where the constant does not depend on $t \ge 0$. Employing also (2.3), (2.7) and (2.13), we arrive at

$$\langle \varphi_0 \rangle_\gamma \le c e^{\alpha t} \left\langle e^{\alpha(-t)} z_t(-t) \right\rangle_\gamma \le c e^{\alpha t} \|z_t\|_{\mathbb{E}_1(\mathbb{R}_-,\alpha)} \le c e^{\alpha t} \langle \psi(t) \rangle_\gamma \le c e^{-\epsilon t} \langle \varphi_0 \rangle_\gamma$$

for constants independent on $t \ge 0$. As $t \to \infty$, it follows $w_0 - w_* = \varphi_0 = 0$.

(g), (h), (i) These parts are shown as Theorem 4.6(d), (e) and (f), making use of (5.17). □

Corollary 5.3. *Assume that Hypothesis 3.4 and (3.21) hold. Then there is a number* $\overline{r} > 0$ *such that* $\mathcal{M}_{\rm c} \cap (w_* + \mathcal{M}^*(\overline{r})) = \mathcal{M}_{\rm cs} \cap \mathcal{M}_{\rm cu} \cap (w_* + \mathcal{M}^*(\overline{r}))$, $\mathcal{M}_{\rm c} \cap \mathcal{M}_{\rm s} \cap (w_* + \mathcal{M}^*(\overline{r})) = \{w_*\}$, *and* $\mathcal{M}_{\rm c} \cap \mathcal{M}_{\rm u} \cap (w_* + \mathcal{M}^*(\overline{r})) = \{w_*\}$. *Here,* $\mathcal{M}_k$ *are the manifolds obtained in Theorems 4.6, 5.1 and 5.2.*

Proof. We set $\eta = \min\{\eta_c, \eta_{cs}, \eta_{cu}\} > 0$ and let r be less than or equal to the minimum of the numbers $r_k(\eta)$ obtained in Theorems 4.6, 5.1, and 5.2. For $w_0 \in \mathcal{M}_{\rm c} \cap (w_* + \mathcal{M}^*(\overline{r}))$, there exists the function φ from (4.10) with $\varphi(0) = w_0 - w_*$, where $\Gamma(t, \varphi) = 1$ for $|t| \le 2$. For $s \in [0, 9/4]$ and $s' \in [-9/4, 0]$, we have $\|R_+\varphi\|_{\mathbb{E}_1(J(s))} \le c\,\|\varphi\|_{\mathbb{E}_1([0,4])}$ and $\|R_-\varphi\|_{\mathbb{E}_1(J(s'))} \le c\,\|\varphi\|_{\mathbb{E}_1([-4,0])}$ for some constant c. In view of (4.15) and Remark 4.1, we can decrease $\overline{r} > 0$ in order to obtain $\Gamma_+(t, \varphi) = 1$ for $t \in [0, 2]$ and $\Gamma_-(t, \varphi) = 1$ for $t \in [-2, 0]$. Therefore $\Gamma(t, \varphi) = \Gamma_\pm(t, \varphi)$ for $t \in \mathbb{R}_\pm$ in view of Remark 4.2, and the restrictions of φ to $\mathbb{R}_+$ and $\mathbb{R}_-$ thus belong to $\widetilde{\mathcal{M}}_{cs}$ and $\widetilde{\mathcal{M}}_{cu}$ by (5.4) and (5.13), respectively. As a result, $w_0 \in \mathcal{M}_{\rm cs} \cap \mathcal{M}_{\rm cu}$. The converse inclusion can be shown similarly, thereby fixing a possibly smaller $\overline{r}$. The last two equalities then follow from Theorems 5.1 and 5.2. ☐

Example 5.4. As in Examples 2.2, 2.4 and 3.6 of [20] we consider the Stefan problem with surface tension. For times $t \ge 0$, we look at open subsets $D_i(t)$ of a fixed bounded domain $D \subseteq \mathbb{R}^n$ with $\partial D \in C^2$ and outer unit normal ν_D, where the liquid phase is contained in $D_1(t)$ and the solid one in $D_2(t)$, say. The domains have the compact interface $\Gamma(t) \subseteq D$ so that $D_1(t) \dot\cup \Gamma(t) \dot\cup D_2(t) = D$. We assume that $\Gamma(t) \cap \partial D = \emptyset$ for all $t \ge 0$. The phases have the temperatures $u_i(t)$. On the interface we have the mean curvature $H(\Gamma(t))$, which is chosen to be negative at $x \in \Gamma(t)$ if $D_1(t)$ is convex near x. The normal velocity of $\Gamma(t)$ is denoted by $V(t)$, where the normal ν of $\Gamma(t)$ is defined with respect to $D_1(t)$. Here the interface and the temperatures are unknown. We consider the system

$$\begin{aligned} \partial_t u_i - d_i \Delta u_i &= 0, \qquad t > 0,\ x \in D_i(t), \\ \partial_{\nu_D} u_2 &= 0, \qquad t \ge 0,\ x \in \partial D, \\ u_i &= \sigma H(\Gamma(t)), \qquad t \ge 0,\ x \in \Gamma(t), \\ d_2 \partial_\nu u_2 - d_1 \partial_\nu u_1 &= lV, \qquad t \ge 0,\ x \in \Gamma(t), \\ u_i(0) &= u_0^i, \quad x \in D_0^i, \qquad \Gamma(0) = \Gamma_0, \end{aligned} \tag{5.18}$$

for constants $d_1, d_2, \sigma, l > 0$, initial domains $D_0^i \subseteq D$ and a closed compact C^2 hypersurface $\Gamma_0 \subseteq D$ with with $\Gamma_0 = \partial D_0^1$ and $D_0^1 \dot\cup \Gamma_0 \dot\cup D_0^2 = D$, and initial temperature distributions u_0^i on D_0^i. Actually, this a simplified model and we refer to [17] for a thermodynamically consistent version allowing for different heat capacities in the phases, kinetic undercooling and coefficients depending on the temperature. This problem could also be treated by the methods in the present paper, but for simplicity we restrict ourselves to the system (5.18).

In Theorem 1.1 of [15] it was shown that for connected phases the equiliberia of (5.18) are spheres Σ of radius $R > 0$ in D with constant temperature σ/R. These form a $n+1$–dimensional manifold in the phase space. We fix such a sphere. As recalled in Examples 2.2, 2.4 and 3.6 of [20], one can transform

(5.18) into the form (1.1) where $\rho(t,y)$ is the normal distance of the evolving interface to a base point $y \in \Sigma$. Moreover, our Hypothesis 3.4 is satisfied. The linearization $-\Lambda_0$ at Σ (i.e., $u_* = \sigma/R$ and $\rho_* = 0$) has an $n+1$–dimensional kernel with $n+1$ linearly independent eigenvalues if the volume $|D|$ of D differs from $l\,|\Sigma| R^2/\sigma$, see Theorem 2.1 of [17]. If $\sigma|D| = l\,|\Sigma| r^2$ the dimension is $n+2$. Moreover the spectrum is real, only consists of eigenvalues of finite multiplicity, and it has a strictly positive simple eigenvalue if and only if $\sigma|D| > l\,|\Sigma| r^2$.

Assuming $\sigma|D| \neq l\,|\Sigma| r^2$ we thus obtain a $n+1$ dimensional center manifold. It contains the equilibria near the given one by Theorem 4.6(e), hence $\mathcal{M}_{\rm c}$ only consists of equilibria. If $\sigma|D| \neq l\,|\Sigma| r^2$ we still obtain a $n+2$ dimensional center manifold containing the equilibria near the given one. There is a one–dimensional unstable manifold if and only if $\sigma|D| > l\,|\Sigma| r^2$. In addition we have the stable, center–stable and center–unstable manifolds described by the above results. ♢

6. Convergence

Based on the analysis of the previous sections, we now study the attractivity properties of the center manifold, using the notation introduced in the above theorems. Related results were obtained for static nonlinear boundary conditions in the paper [8], which was inspired by Palmer's theory in the ode case [14]. Throughout we assume that Hypothesis 3.4 and the spectral gap conditions (3.21) or (5.1) hold, which is true if, e.g., the spatial domain Ω is bounded. In particular, we have the equilibrium $w_* = (u_*, \rho_*) \in W_1$ of (1.1). Recall that the solutions $w = (u, \rho) \in \mathbb{W}_1$ of (1.1) correspond to the solutions $\varphi = (v, \sigma) = w - w_* \in \mathbb{W}_1^*$ of (3.6), where the operators in (3.6) are given by (2.9) and (3.2) for the equilibrium w_* and we have $\mathcal{D}(w_*) = 0$ and $\dot{\rho}_*(0) = 0$. We further recall that the projections $P_{\rm c}$, $P_{\rm u}$ and $P_{\rm cu}$ map into $E_1^0 \subset E_1$ and that $P_{\rm s}$ and $P_{\rm cs}$ leave invariant our scale of 'E-spaces', cf. (3.15).

In our main results we also assume that the center–unstable or the center spectral subspace are finite dimensional, which again holds if Ω is bounded. In an intermediate step of the proof of the crucial Lemma 6.1 below, we loose control of the norm in $W_p^{1+\kappa_0}$ of the second component of an auxiliary function ψ. In the corresponding parts of the proof (but not in the statements) we thus have to replace the space $\mathbb{E}_\rho$ by the space

$$\mathbb{E}_\rho^\sharp(J) = L_p(J; Z_1) \cap W_p^1(J; Z) \cap \bigcap\nolimits_{j \in \widetilde{\mathcal{J}}} W_p^{\kappa_j}\big(J; W_p^{k_j}(\Sigma; V_\rho)\big), \tag{6.1}$$

cf. (2.6). We further set $\mathbb{E}_1^\sharp(J) = \mathbb{E}_u(J) \times \mathbb{E}_\rho^\sharp(J)$, use the same conventions as for $\mathbb{E}_1$, and define

$$\mathbb{W}_1^\sharp(J) = \{\varphi \in \mathbb{E}_1^\sharp(J) \,|\, \varphi(t) + w_* \in W_\gamma \quad (\forall\, t \in J)\},$$

cf. (2.8). The proof of the embedding (2.7), see §2 of [5] or §2 of [13], works also for $\mathbb{E}_\rho^\sharp(J)$ and thus

$$\mathbb{E}_1^\sharp(J) \hookrightarrow C_{ub}(J; E_\gamma), \tag{6.2}$$

where the same remarks as after (2.3) apply. Observe that for $(u,\rho) \in \mathbb{E}_1^\sharp(J)$, the function $\partial_t \rho$ does not possess extra time regularity. In order to work with this larger solution space and to obtain convergence with respect to $\langle\cdot\rangle_1$, we have to impose additional assumptions besides (RR). We require that $\mathcal{R}$ is affine in $\dot\rho$ and that G acts on $\mathbb{W}_1^\sharp(J)$. These assumptions hold for the Stefan problem, see Example 6.6.

(RR') We have $\mathcal{A} \in C^1(W_\gamma; \mathcal{L}(X_1, X))$, $\mathcal{R}(u, \rho, \dot\rho) = \mathcal{R}_0(u, \rho) + \mathcal{R}_1(u, \rho)\dot\rho$ with $\mathcal{R}_0 \in C^1(W_\gamma; X)$, $\mathcal{R}_1 \in C^1(W_\gamma; \mathcal{L}(Y_{0\gamma}, X))$, and $\mathcal{D} = (\mathcal{D}_0, \dots, \mathcal{D}_m) \in C^1(W_1; Y_1)$ induces a map $\mathcal{D} \in C^1(\mathbb{W}_1^\sharp(J); \mathbb{F}(J))$ for any compact J. The first derivatives of these maps are bounded and Lipschitz continuous on closed balls.

It is easy to see that then $F, \tilde F \in C^1(\mathbb{W}_1^\sharp(J); \mathbb{E}(J))$ and $G \in C^1(\mathbb{W}_1^\sharp(J); \mathbb{F}(J))$ with locally bounded derivatives, cf. (3.1), where $\tilde F = F - A_{*\dot\rho} G_0 = F + \mathcal{R}_1(w_*)G_0$. Further, let $\delta \subset [0, d]$ and $J \subset \mathbb{R}_+$ be a closed interval of length larger than 2. Given $r > 0$, we consider functions $w, \overline{w} \in \mathbb{W}_1^\sharp(J)$ whose norms in $\mathbb{E}_1^\sharp([t, t+2])$ are less or equal r for all intervals $[t, t+2] \subseteq J$. It then holds

$$\begin{aligned} \|F(w) - F(\overline{w})\|_{\mathbb{E}(J,\delta)} &\le \varepsilon(r)\, \|w - \overline{w}\|_{\mathbb{E}_1^\sharp(J,\delta)}, \\ \|G(w) - G(\overline{w})\|_{\mathbb{F}(J,\delta)} &\le \varepsilon(r)\, \|w - \overline{w}\|_{\mathbb{E}_1^\sharp(J,\delta)}, \end{aligned} \tag{6.3}$$

where ε does not depend on w, $\overline{w}$ or δ. To show (6.3), we continuously extend w and $\overline{w}$ to compactly supported functions in $\mathbb{E}_1^\sharp(\mathbb{R}_+, \delta)$, cf. (4.2). We can then argue as in Proposition 3.10, using Lemma 4.7 of [20] with $a = \frac{1}{2}$ and (6.2).

We start with a basic lemma that allows to shadow a small solution $\varphi(t)$, $t \in [0, T]$, of (3.6) by a solution on the center–unstable manifold, where one could replace the restriction $t \ge 1$ by $t \ge t_0$ for any $t_0 > 0$.

Lemma 6.1. *Assume that Hypothesis 3.4, condition (RR'), and (5.1) hold. Then there exist constants $r > 0$ and $\alpha \in (\underline{\omega_{cu}}, \omega_s)$ such that, for every solution $\varphi = (v, \sigma) \in \mathbb{W}_1^*([0, T])$ of (3.6) with $\langle\varphi(t)\rangle_\gamma < r$ for all $0 \le t \le T$ with some $T > 1$, there is a solution $\overline{\varphi} = (\overline{v}, \overline{\sigma}) \in \mathbb{W}_1^*([0, T])$ of (3.6) satisfying $w_* + \overline{\varphi}(t) \in \mathcal{M}_{\rm cu}$ for all $t \in [0, T]$, $P_{\rm cu}\overline{\varphi}(T) = P_{\rm cu}\varphi(T)$ and*

$$|\varphi(t) - \overline{\varphi}(t)|_{E_1} + |\partial_t\sigma(t) - \partial_t\overline{\sigma}(t)|_{Z_\gamma} \le c\mathrm{e}^{-\alpha t}\, \langle\varphi(0)\rangle_\gamma \le \overline{c} r \tag{6.4}$$

for all $1 \le t \le T$. Given $T_0 \in (1, T)$, the constants are uniform for $T \ge T_0$.

Proof. (1) We assume that $T \ge 3 =: T_0$. For a general $T_0 > 1$ the proof is similar. Let $\varphi = (v, \sigma) \in \mathbb{W}_1^*([0, T])$ be a solution of (3.6) such that $\langle\varphi(t)\rangle_\gamma < r$ for all $t \in [0, T]$, where a sufficiently small $r > 0$ is chosen below. The assumption (5.1) implies that $\|\mathrm{e}^{-t\Lambda_0 P_{\rm cu}} P_{\rm cu}\|_{\mathcal{B}(E_0)} \le N\mathrm{e}^{-\delta t}$ for all $t \le 0$ and some constants $\delta \in (\omega_{cu}, \omega_s)$ and $N \ge 1$. Theorem 5.2 gives a radius $r_0 > 0$ such that the restriction $\phi_{cu} : P_{\rm cu}E \cap B_E(0, r_0) \to E_1$ is Lipschitz with constant ℓ and such that $w_* + \xi + \phi_{cu}(\xi) \in \mathcal{M}_{\rm cu}$ for all $\xi \in P_{\rm cu}E \cap B_E(0, r_0)$. We set

$$\varepsilon_1(R) = \max_{z \in W_1 - w_*, |z|_{E_1} \le R} \{\|\widetilde{F}'(z)\|_{\mathcal{L}(E_1,X)}, \|G'(z)\|_{\mathcal{L}(E_1,Y_1)}\}. \tag{6.5}$$

Because of (3.1), we can fix a (small) number $R > 0$ such that

$$\begin{aligned} &d := N\varepsilon_1(R)\widehat{c}\,(\|P_{\rm cu}\|_{\mathcal{L}(E,E_0)} + \|P_{\rm cu}\Pi\|_{\mathcal{B}(\widehat{Y}_1,E_0)})(1+\ell) < \omega_s - \delta, \\ &R\, \|P_{\rm cu}\|_{\mathcal{B}(E_1,E)} < r_0\,, \end{aligned} \tag{6.6}$$

where $\widehat{c}$ is the norm of the embedding $E_0 \hookrightarrow E$. Theorem 3.3(c) (with $T = 1/2$) implies the inequality

$$\langle\varphi(t)\rangle_1 \le c\,\langle\varphi(t-1/2)\rangle_\gamma \le cr \qquad \text{for all} \qquad t \in [1/2, T].$$

Here and below the constants do not depend on v, T, t, R or r, and we choose a sufficiently small $r > 0$ to apply Theorem 3.3(c). Let $\overline{c}$ be the norm of the embedding $E_\gamma \to E$. We can now take small $r > 0$ such that

$$\begin{aligned} |\varphi(t)|_1 &\le R \qquad \text{for all} \qquad 1/2 \le t \le T, \\ r(1+\overline{c}\ell)\|P_{\rm cu}\|_{\mathcal{L}(E_\gamma,E_1)} &\le R/2 \qquad \text{and} \qquad r\,\|P_{\rm cu}\|_{\mathcal{L}(E_\gamma,E)} < r_0. \end{aligned} \tag{6.7}$$

(2) To control the distance between φ and $\mathcal{M}_{\rm cu} - w_*$, we define

$$\psi = P_{\rm s}\varphi - \phi_{cu}(P_{\rm cu}\varphi) \qquad \text{on} \quad [0, T].$$

The function ψ takes values in $P_{\rm s}E_\gamma$. Since $\varphi - \psi = P_{\rm cu}\varphi + \phi_{cu}(P_{\rm cu}\varphi)$ on $[0, T]$, we have $w_* + \varphi(t) - \psi(t) \in \mathcal{M}_{\rm cu}$, thanks to the last inequality in (6.7). Moreover, $\varphi-\psi$ and $\partial_t(\varphi-\psi) = P_{\rm cu}\dot\varphi + \phi'_{cu}(P_{\rm cu}\varphi)P_{\rm cu}\dot\varphi$ belong to $L_p([0, T]; E_1)$ since $\phi'_{cu}(P_{\rm cu}\varphi) : P_{\rm cu} \to E_1$ is uniformly bounded by Theorem 5.2. Let $J = [a, b] \subset [0, T]$ with $b - a \ge 1/4$. We then obtain

$$\|\psi\|_{\mathbb{E}_1^\sharp(J)} \le \|\varphi\|_{\mathbb{E}_1(J)} + \|\psi-\varphi\|_{\mathbb{E}_1^\sharp(J)} \le c\,\|\varphi\|_{\mathbb{E}_1(J)} \le c\,\langle\varphi(a)\rangle_\gamma \le c_1 r, \tag{6.8}$$

where we use Theorem 3.3 in the penultimate inequality. We can also suppose that the norms of φ and $\varphi - \psi$ in $\mathbb{E}_1^\sharp(J)$ are bounded by $c_1 r$. We put

$$f = F(\varphi) - F(\varphi-\psi), \qquad \widetilde{f} = \widetilde{F}(\varphi) - \widetilde{F}(\varphi-\psi) \qquad \text{and} \qquad g = G(\varphi) - G(\varphi-\psi).$$

Let $\alpha \in [0, \omega_s)$. Estimate (6.3) yields

$$\|f\|_{\mathbb{E}([a,b],\alpha)}, \|\widetilde{f}\|_{\mathbb{E}([a,b],\alpha)}, \|g\|_{\mathbb{F}([a,b],\alpha)} \le \varepsilon_2(r)\,\|\psi\|_{\mathbb{E}_1^\sharp([a,b],\alpha)}, \tag{6.9}$$

where $b - a \ge 2$, $\varepsilon_2(r)$ is proportional to $\varepsilon(c_1 r)$ and ε is given by (6.3). Note that $P_{\rm s} = I - P_{\rm cu}$, $(\widehat{B}, \widehat{C})P_{\rm cu} = 0$, and $P_{\rm cu}(\varphi - \psi) = P_{\rm cu}\varphi$. Equation (3.13) for φ, the identities (5.15) and (5.16) and Proposition 3.6(c) now imply

$$(\widehat{B}, \widehat{C})\psi = (\widehat{B}, \widehat{C})\varphi - (\widehat{B}, \widehat{C})\phi_{cu}(P_{\rm cu}\varphi) = \widehat{G}(\varphi) - \widehat{G}(\varphi-\psi) = \widehat{g}, \tag{6.10}$$

$$\begin{aligned} \dot\psi &= P_{\rm s}(-\Lambda_*\varphi + [\widetilde{F}(\varphi), G_0(\varphi)]) - \phi'_{cu}(P_{\rm cu}\varphi)P_{\rm cu}([\widetilde{F}(\varphi), G_0(\varphi)] - \Lambda_*\varphi) \\ &\qquad - \phi'_{cu}(P_{\rm cu}(\varphi-\psi))P_{\rm cu}(\Lambda_*(\varphi-\psi) - [\widetilde{F}(\varphi-\psi), G_0(\varphi-\psi)]) \\ &\qquad + P_{\rm s}\Lambda_*(\varphi-\psi) - P_{\rm s}[\widetilde{F}(\varphi-\psi), G_0(\varphi-\psi)] \\ &= -P_{\rm s}\Lambda_*\psi + P_{\rm s}[\widetilde{F}(\varphi), G_0(\varphi)] - P_{\rm s}[\widetilde{F}(\varphi-\psi), G_0(\varphi-\psi)] \\ &\qquad + \phi'_{cu}(P_{\rm cu}\varphi)P_{\rm cu}(\Lambda_*\psi - [\widetilde{F}(\varphi), G_0(\varphi)] + [\widetilde{F}(\varphi-\psi), G_0(\varphi-\psi)]) \\ &= -\Lambda_{-1}P_{\rm s}\psi + P_{\rm s}\Pi\widehat{g} + P_{\rm s}[\widetilde{f}, g_0] - \phi'_{cu}(P_{\rm cu}\varphi)P_{\rm cu}(\Pi\widehat{g} + [\widetilde{f}, g_0]) \\ &=: -\Lambda_{-1}P_{\rm s}\psi + P_{\rm s}\Pi\widehat{g} + P_{\rm s}[\widetilde{f}, g_0] + h \end{aligned} \tag{6.11}$$

on $[0, T]$. In the penultimate equality we used (6.10) and $P_{\rm cu}\Lambda_{-1}\psi = \Lambda_{-1}P_{\rm cu}\psi = 0$. The variation of constants formula for $T_{-1}(\cdot)$ now gives

$$\psi(t) = T(t-t_0)P_{\rm s}\psi(t_0) + \int_{t_0}^t T_{-1}(t-\tau)P_{\rm s}([\widetilde{f}(\tau), g_0(\tau)] + h(\tau) + \Pi\widehat{g}(\tau))\,{\rm d}\tau \tag{6.12}$$

for $0 \le t_0 < t \le T$. Since the operators $P_{\rm s}$ and $\phi'_{cu}(P_{\rm cu}\varphi)$ may mix the first and second component of E, it is a bit tricky to establish sharp estimates for ψ. We will treat the relevant parts separately, introducing the functions

$$\psi^{0fg}(t) = T(t-t_0)\psi(t_0) + \int_{t_0}^{t} T_{-1}(t-\tau)([\widetilde{f}(\tau), g_0(\tau)] + \Pi\widehat{g}(\tau))\,\mathrm{d}\tau,$$
$$\psi^{0h}(t) = \int_{t_0}^{t} T(t-\tau)h(\tau)\,\mathrm{d}\tau, \qquad \psi^0 = \psi^{0fg} + \psi^{0h}.$$

To apply maximal regularity results, we first recall that $P_{\rm s}\psi(t_0) = \psi(t_0) \in E_\gamma$, $(\widehat{B}, \widehat{C})\psi(t_0) = \widehat{g}(t_0)$, $\varphi(t_0) \in \mathcal{M}^*$, and $(B_0, C_0)P_{\rm cu}$ maps into $(B_0, C_0)E_1^0 \subset Z_\gamma^1$ by (2.14). Equation (5.15) thus shows that

$$\begin{aligned}(B_0, C_0)\psi(t_0) - g_0(t_0) &= (B_0, C_0)\varphi(t_0) - G_0(\varphi(t_0)) - (B_0, C_0)P_{\rm cu}\varphi(t_0) \\ &\quad - (B_0, C_0)\phi_{cu}(P_{\rm cu}(\varphi(t_0) - \psi(t_0))) + G_0(\varphi(t_0) - \psi(t_0)) \\ &=: \zeta(t_0) \end{aligned} \tag{6.13}$$

belongs to Z_γ^1. Further let $\chi = (\chi_1, \chi_2)$ be the solution of (3.6) with the initial value $\varphi(t_0) - \psi(t_0) = P_{\rm cu}\varphi(t_0) + \phi_{cu}(P_{\rm cu}\varphi(t_0)) \in \mathcal{M}_{\rm cu} - w_*$ at time t_0. Using (2.7) and Theorem 5.2, we then derive

$$\begin{aligned}|\zeta(t_0)|_{Z_\gamma^1} &\le |\dot{\sigma}(t_0)|_{Z_\gamma^1} + |\dot{\chi}_2(t_0)|_{Z_\gamma^1} \le \langle\varphi(t_0)\rangle_\gamma + c\,\|\chi\|_{\mathbb{E}_1([t_0,t_0+1])} \\ &\le \langle\varphi(t_0)\rangle_\gamma + c\,|P_{\rm cu}\varphi(t_0)|_E \le c\,\langle\varphi(t_0)\rangle_\gamma \le cr. \end{aligned} \tag{6.14}$$

So the problem (3.9) with data $\psi(t_0)$, f and g has a unique solution in $\mathbb{E}_1([t_0, T])$ which coincides with ψ^{0fg} due to Proposition 3.6. We put $J_0 = [t_0, t_0+2] \cap [t_0, T]$ and assume that $t_0 \le T - 1/4$. Proposition 3.5 and (6.9) yield

$$\begin{aligned}\|\psi^{0fg}\|_{\mathbb{E}_1(J_0)} &\le c\,(|\psi(t_0)|_{E_\gamma} + |\zeta(t_0)|_{Z_\gamma^1} + \|f\|_{\mathbb{E}(J_0)} + \|g\|_{\mathbb{F}(J_0)}) \\ &\le c\,\big(|\psi(t_0)|_{E_\gamma} + |\zeta(t_0)|_{Z_\gamma^1} + \varepsilon_2(r)\,\|\psi\|_{\mathbb{E}_1^\sharp(J_0)}\big).\end{aligned}$$

We have $h \in L_p([t_0, T]; E_\gamma) \hookrightarrow L_p([t_0, T]; E_0)$, see (2.13), but it is not clear whether one can control the norm of h_2 in $\mathbb{F}_0$ by the norm of ψ in $\mathbb{E}_1$. However, Corollary 2.3 of [5] (combined with a perturbation argument as in the proof of Corollary 2.6 of [20]) and (6.9) imply that

$$\|\psi^{0h}\|_{\mathbb{E}_1^\sharp(J_0)} \le c\,\|h\|_{L_p(J_0;E_0)} \le c\,(\|\widetilde{f}\|_{\mathbb{E}(J_0)} + \|g\|_{L_p(J_0;Y_1)}) \le c\varepsilon_2(r)\,\|\psi\|_{\mathbb{E}_1^\sharp(J_0)}.$$

Finally, the function $P_{\rm cu}\psi^0$ and its derivative can easily be estimated using that $P_{\rm cu}$ maps into $D(\Lambda_0^2)$ leading to

$$\begin{aligned}\|P_{\rm cu}\psi^0\|_{\mathbb{E}_1^\sharp(J_0)} &\le c\,\|P_{\rm cu}\psi^0\|_{W_p^1(J_0;E_1)} \le c\,(\|[\widetilde{f}, g_0] + h\|_{\mathbb{E}(J_0)} + \|\widehat{g}\|_{L_p(J_0;\widehat{Y})}) \\ &\le c\varepsilon_2(r)\,\|\psi\|_{\mathbb{E}_1^\sharp(J_0)},\end{aligned}$$

where we also use (6.9). To treat the remaining interval $[t_0+2, T]$ we argue as in (4.14) in [20]. Let $J_n = [t_0+n, t_0+n+1] \cap [t_0, T]$ and $J'_n = [t_0+n-1, t_0+n+1] \cap [t_0, T]$ for $n \in \mathbb{N}$ with $n \ge 2$, and take $\chi_n : J'_n \to \mathbb{R}$ such that χ_n, χ'_n and χ''_n

are uniformly bounded, $\chi_n(t_0+n-1) = 1$ and $\chi_n = 0$ on $[t_0+n-1/2, t_0+n+1]$ for $n \geq 2$. For $t \in J_n$, we then write

$$\begin{aligned}
\psi(t) = &\int_{t_0+n-1}^{t} T_{-1}(t-\tau)(1-\chi_n(\tau))([\widetilde{f}(\tau), g_0(\tau)] + \Pi\widehat{g}(\tau))\,\mathrm{d}\tau \\
&- \int_{t_0+n-1}^{t} T(t-\tau)P_{\mathrm{cu}}(1-\chi_n(\tau))([\widetilde{f}(\tau), g_0(\tau)] + \Pi\widehat{g}(\tau))\,\mathrm{d}\tau \\
&+ \int_{t_0+n-1}^{t} T(t-\tau)(1-\chi_n(\tau))P_{\mathrm{s}}h(\tau)\,\mathrm{d}\tau \\
&+ \int_{t_0+n-1}^{t_0+n-\frac{1}{2}} T(t-\tau)P_{\mathrm{s}}\chi_n(\tau)([\widetilde{f}(\tau), g_0(\tau)] + h(\tau) + \Pi\widehat{g}(\tau))\,\mathrm{d}\tau \\
&+ \int_{t_0}^{t_0+n-1} T(t-\tau)P_{\mathrm{s}}([\widetilde{f}(\tau), g_0(\tau)] + h(\tau) + \Pi\widehat{g}(\tau))\,\mathrm{d}\tau \\
&+ T(t-t_0)P_{\mathrm{s}}\psi(t_0).
\end{aligned}$$

We can now combine the arguments given above and in the proof in Proposition 3.7, using Propositions 3.5 and 3.6 for the first integral and standard semigroup theory and (3.15) for the other terms. Employing also (6.9) and Lemma 4.7 in [20], it follows

$$\|\psi\|_{\mathbb{E}_1^\sharp([t_0,T],\alpha)} \leq c\left(\mathrm{e}^{-\alpha t_0}\,|\psi(t_0)|_{E_\gamma} + |\zeta(t_0)|_{Z_\gamma^1} + \varepsilon_2(r)\,\|\psi\|_{\mathbb{E}_1^\sharp([t_0,T],\alpha)}\right).$$

Choosing a sufficiently small $r > 0$ and using (6.2), we arrive at

$$\mathrm{e}^{-\alpha t_0}\|\psi\|_{\mathbb{E}_1^\sharp([t_0,T],\alpha)} \leq c\,(|\psi(t_0)|_{E_\gamma} + |\zeta(t_0)|_{Z_\gamma^1}) \tag{6.15}$$

for all $t_0 \in [0, T-1/4]$ and $\alpha \in [0, \omega_s - \varepsilon]$, and any fixed $\varepsilon \in (0, \omega_s)$.

(3) We next introduce the candidate for the asserted shadowing solution on $\mathcal{M}_{\mathrm{cu}}$. Since $|P_{\mathrm{cu}}\varphi(T)|_E < r_0$ by (6.7), there exists the backward solution $\overline{\varphi} = (\overline{v}, \overline{\sigma}) = P_{\mathrm{cu}}\overline{\varphi} + \phi_{cu}(P_{\mathrm{cu}}\overline{\varphi})$ of (3.6) such that $w_* + \overline{\varphi}$ belongs to $\mathcal{M}_{\mathrm{cu}}$ and $P_{\mathrm{cu}}\overline{\varphi}(T) = P_{\mathrm{cu}}\varphi(T)$. Theorem 5.2 also yields that $\overline{\varphi}(t)$ exists at least for $t \in [T-3, T]$ and $\|\overline{\varphi}\|_{\mathbb{E}_1([T-3,T])} \leq c\,|P_{\mathrm{cu}}\overline{\varphi}(T)|_E \leq cr$. We then deduce $\langle\overline{\varphi}(T-3)\rangle_\gamma \leq cr$ using (2.3) and (2.7), as well as $|\overline{\varphi}(t)|_{E_1} \leq cr \leq R$ for all $t \in [T-2, T]$ using Theorem 3.3, after decreasing $r > 0$ if needed. Let $a \in [1/2, T-2]$ be the minimal time such that $\overline{\varphi}(t)$ exists, $w_* + \overline{\varphi}(t) \in \mathcal{M}_{\mathrm{cu}}$ and $|\overline{\varphi}(t)|_{E_1} \leq R$ holds for all $a \leq t \leq T$. We set $z = P_{\mathrm{cu}}(\varphi - \overline{\varphi})$, $\widetilde{f}^1 = \widetilde{F}(\varphi) - \widetilde{F}(\overline{\varphi})$ and $g^1 = G(\varphi) - G(\overline{\varphi})$. As in part (2), we then obtain

$$\dot{z} = -\Lambda_0 P_{\mathrm{cu}} z + P_{\mathrm{cu}}([\widetilde{f}^1, g_0^1] + \Pi\widehat{g}^1), \tag{6.16}$$

and hence

$$z(t) = -\int_t^T \mathrm{e}^{-(t-\tau)\Lambda_0 P_{\mathrm{cu}}} P_{\mathrm{cu}}([\widetilde{f}^1(\tau), g_0^1(\tau)] + \Pi\widehat{g}^1(\tau))\,\mathrm{d}\tau.$$

for all $t \in [a, T]$. Based on this formula and using (6.5) and (6.6), we can now proceed exactly as in part 3 of the proof of Lemma 3.1 in [8], where we take

$\alpha, \alpha' \in (d+\delta, \omega_s)$ with $\alpha < \alpha'$. We thus arrive at

$$\begin{aligned} |z(t)|_E &\le c\mathrm{e}^{-\alpha' t} \, \|\mathrm{e}_{\alpha'}\psi\|_{L_p([t,T];E_1)} \le c\mathrm{e}^{-\alpha' t} \, \|\psi\|_{\mathbb{E}_1^\sharp([t,T];\alpha')} \\ &\le c\,(|\psi(t)|_{E_\gamma} + |\zeta(t)|_{Z_\gamma^1}) \le cr \end{aligned} \tag{6.17}$$

for all $t \in [a, T]$, where the constants c are uniform for $t \le T$ in the first line, and for $t \le T - 1/4$ in the second one. (In the last two inequalities we have also employed the estimates (6.15) and (6.14).) If $t \in [T-1/4, T]$, we can estimate in (6.17) the $\mathbb{E}_1^\sharp$ norm on $[t, T]$ by that on $[t-1/4, T]$ and obtain

$$|z(t)|_E \le c\,(|\psi(t-1/4)|_{E_\gamma} + |\zeta(t-1/4)|_{Z_\gamma^1}) \le cr$$

with a uniform constant. Since $P_{\mathrm{cu}}\overline{\varphi} = P_{\mathrm{cu}}(\varphi - z)$, condition (6.6) yields $|P_{\mathrm{cu}}(\varphi - z)|_E = |P_{\mathrm{cu}}\overline{\varphi}|_E \le \|P_{\mathrm{cu}}\|_{\mathcal{B}(E_1,E)} R < r_0$ and it holds

$$\overline{\varphi} = P_{\mathrm{cu}}(\varphi - z) + \phi_{cu}(P_{\mathrm{cu}}(\varphi - z)).$$

Using also (6.7), we now conclude

$$\begin{aligned} |\overline{\varphi}(t_0)|_{E_1} &\le (1+\overline{c}\ell)|P_{\mathrm{cu}}(\varphi - z)|_{E_1} \le (1+\ell)\|P_{\mathrm{cu}}\|_{\mathcal{B}(E_\gamma,E_1)} \, |\varphi(t_0)|_{E_\gamma} + c\,|z(t_0)|_E \\ &\le R/2 + cr < R \end{aligned} \tag{6.18}$$

provided $r > 0$ is chosen small enough. It follows that $a = 1/2$. Since $w_* + \overline{\varphi}(1/2) \in \mathcal{M}_{\mathrm{cu}}$ we can extend $u_* + \overline{\varphi}$ on $\mathcal{M}_{\mathrm{cu}}$ to the time interval $[0, T]$ due to Theorem 5.2. This theorem also implies that $\|\overline{\varphi}\|_{\mathbb{E}_1([a,b])} \le c\,|P_{\mathrm{cu}}\overline{\varphi}(b)|_E \le c\,|\varphi(b) - z(b)|_E \le cr$, whenever $b - a \le 2$.

(4) To estimate $\varphi - \overline{\varphi}$, we deduce from (6.17) and (6.15) that

$$\|\mathrm{e}_\alpha z\|_{L_p([1/2,T];E)} \le c\,\|\psi\|_{\mathbb{E}_1^\sharp([0,T];\alpha')} \le c\,(|\psi(0)|_{E_\gamma} + |\zeta(0)|_{Z_\gamma^1}), \tag{6.19}$$

where the constants depend on $\alpha' - \alpha > 0$. Equation (6.16) and (RR') further yield that $|\dot{z}(t)|_E \le c\,|z(t)|_E + c\,|\varphi(t) - \overline{\varphi}(t)|_{E_1}$. Since

$$\varphi - \overline{\varphi} = P_{\mathrm{cu}}(\varphi - \overline{\varphi}) + P_{\mathrm{s}}(\varphi - \overline{\varphi}) = z + \psi + \phi_{cu}(P_{\mathrm{cu}}\varphi) - \phi_{cu}(P_{\mathrm{cu}}\overline{\varphi}),$$

we infer $|\dot{z}(t)|_E \le c\,(|z(t)|_E + |\psi(t)|_{E_1})$. Estimates (6.19) and (6.15) now lead to

$$\|\mathrm{e}_\alpha \dot{z}\|_{L_p(J;E)} \le c\,(\|\mathrm{e}_\alpha z\|_{L_p([J,T];E)} + \|\mathrm{e}_\alpha \psi\|_{L_p([J,T];E_1)}) \le c\,(|\psi(0)|_{E_\gamma} + |\zeta(0)|_{Z_\gamma^1}).$$

where $J = [\frac{1}{2}, T]$. Sobolev's embedding then gives

$$|P_{\mathrm{cu}}(\varphi(t) - \overline{\varphi}(t))|_E = |z(t)|_E \le c\,\mathrm{e}^{-\alpha t}\,(|\psi(0)|_{E_\gamma} + |\zeta(0)|_{Z_\gamma^1}) \tag{6.20}$$

for all $t \in [1/2, T]$. For the stable part $P_{\mathrm{s}}(\varphi - \overline{\varphi}) = \psi + \phi_{cu}(P_{\mathrm{cu}}\varphi) - \phi_{cu}(P_{\mathrm{cu}}\overline{\varphi})$, we combine the estimates (6.15) and (6.20) with the embedding (6.2), concluding

$$|P_{\mathrm{s}}(\varphi(t) - \overline{\varphi}(t))|_{E_\gamma} \le c\,(|\psi(t)|_{E_\gamma} + |z(t)|_E) \le c\,\mathrm{e}^{-\alpha t}\,(|\psi(0)|_{E_\gamma} + |\zeta(0)|_{Z_\gamma^1})$$

for $t \in [1/2, T]$. (Here we apply (6.15) with T replaced by t.) It follows that

$$|\varphi(t) - \overline{\varphi}(t)|_{E_\gamma} \le c\mathrm{e}^{-\alpha t}\,(|\psi(0)|_{E_\gamma} + |\zeta(0)|_{Z_\gamma^1}) \le c\mathrm{e}^{-\alpha t}\,\langle \varphi(0)\rangle_\gamma \tag{6.21}$$

for $t \in [1/2, T]$, employing also (6.14).

It remains to upgrade this estimate to (6.4). Proposition 3.6 implies

$$\partial_t(\varphi - \overline{\varphi}) = -\Lambda_{-1}(\varphi - \overline{\varphi}) + [\widetilde{f}^1, g_0^1] + \Pi\widehat{g}^1, \tag{6.22}$$

cf. part (3). Let $t \in [1,T]$. There is an $n \in \mathbb{N}$ with $t \in J_n := [n, n+1] \cap [0,T]$. We set $J_n' = [n-1, n+1] \cap [0,T]$. We will argue as in step (2), now using uniformly bounded cutoffs $\chi_n \in C^2([n-1, n+1])$ with $\chi_n(n-1/2) = 1$ and $\chi_n = 0$ on $[n-1/4, n+1]$. We observe that the norms of φ and $\overline{\varphi}$ in $\mathbb{E}_1(J_n')$ are bounded by cr due to Theorem 3.3 and the observation at the end of step (3). Equation (6.22) yields

$$\begin{aligned}
\varphi(t) - \overline{\varphi}(t) = &\int_{n-1}^{t} T_{-1}(t-\tau)(1-\chi_n(\tau))([\widetilde{f}^1(\tau), g_0^1(\tau)] + \Pi\widehat{g}^1(\tau))\,\mathrm{d}\tau \\
&+ \int_{n-1}^{n-1/4} T_{-1}(t-\tau)\chi_n(\tau)([\widetilde{f}^1(\tau), g_0^1(\tau)] + \Pi\widehat{g}^1(\tau))\,\mathrm{d}\tau \\
&+ T(t-n+1)(\varphi(n-1) - \overline{\varphi}(n-1)) =: D_1(t) + D_2(t) + D_3(t).
\end{aligned}$$

Due to Proposition 3.6, D_1 is the solution of (3.9) in $\mathbb{E}_1(J_n)$ with data $(0, (1-\chi_n)\widetilde{f}^1, (1-\chi_n)g^1)$, so that Proposition 3.5 and the embedding (2.7) yield

$$\begin{aligned}
|\,[\partial_t D_1(t)]_2|_{Z_\gamma^1} &\le c\,\|D_1\|_{\mathbb{E}_1(J_n)} \le c\,(\|\widetilde{f}^1\|_{\mathbb{E}(J_n)} + \|g^1\|_{\mathbb{F}(J_n)}) \le c\varepsilon(r)\,\|\varphi - \overline{\varphi}\|_{\mathbb{E}_1(J_n')} \\
&\le c\varepsilon(r)\,(|\varphi(n-1) - \overline{\varphi}(n-1)|_{E_\gamma} + |\partial_t\sigma(n-1) - \partial_t\overline{\sigma}(n-1)|_{Z_\gamma^1}).
\end{aligned}$$

Here we have also used (3.1) and Theorem 3.3(d). The other two terms can similarly be estimated using standard semigroup theory, leading to

$$\begin{aligned}
|\partial_t(D_2(t)+D_2(t))|_{E_1} &\le c\,(|\varphi(n-1) - \overline{\varphi}(n-1)|_{E_0} + \|\widetilde{f}^1\|_{\mathbb{E}(J_n')} + \|g^1\|_{L_p(J_n',Y_1)}) \\
&\le c\,|\varphi(n-1) - \overline{\varphi}(n-1)|_{E_0} + c\varepsilon(r)\,(|\varphi(n-1) - \overline{\varphi}(n-1)|_{E_\gamma} \\
&\qquad + |\partial_t\sigma(n-1) - \partial_t\overline{\sigma}(n-1)|_{Z_\gamma^1}).
\end{aligned}$$

We now combine the above two estimates with (6.21) and conclude that

$$\begin{aligned}
|\varphi(t) - \overline{\varphi}(t)|_{E_\gamma} + |\partial_t\sigma(t) - \partial_t\overline{\sigma}(t)|_{Z_\gamma^1} &\le c_1\mathrm{e}^{-\alpha t}\,\langle\varphi(0)\rangle_\gamma \\
&\quad + c_2\varepsilon(r)|\partial_t\sigma(n-1) - \partial_t\overline{\sigma}(n-1)|_{Z_\gamma^1})
\end{aligned}$$

for some constants. If necessary, we decrease $r > 0$ once more to obtain $c_2\varepsilon(r) \le \frac{1}{2c_1}\mathrm{e}^{-\alpha}$. Iteratively it then follows that

$$\begin{aligned}
&|\varphi(t) - \overline{\varphi}(t)|_{E_\gamma} + |\partial_t\sigma(t) - \partial_t\overline{\sigma}(t)|_{Z_\gamma^1} \\
&\le c\mathrm{e}^{-\alpha t}\,(\langle\varphi(0)\rangle_\gamma + |\varphi(0) - \overline{\varphi}(0)|_{E_\gamma} + |\partial_t\sigma(0) - \partial_t\overline{\sigma}(0)|_{Z_\gamma^1}) \le c\mathrm{e}^{-\alpha t}\,\langle\varphi(0)\rangle_\gamma
\end{aligned}$$

for all $t \in [1,T]$, using also (6.18), (6.20) and (6.14). We can now derive (6.4) from Theorem 3.3(d) and the above inequality. □

Remark 6.2. In view of the above proof, one could replace in (6.4) the factor $\langle\varphi(0)\rangle_\gamma$ by $|\psi(0)|_{E_\gamma} + |\zeta(0)|_{Z_\gamma^1}$, where $\zeta(0)$ is given by (6.13). Moreover, one can choose α arbitrarily close to ω_s for a possibly smaller radius $r > 0$. The same statements are true for the following results.

Our first convergence result says that the center–unstable manifold attracts solutions which stay in small ball around w_* for all $t \ge 0$ and that they approach a tracking solution $w_* + \overline{\varphi}$ on $\mathcal{M}_{\mathrm{cu}}$.

Theorem 6.3. *Assume that Hypothesis 3.4, conditions (RR') and (5.1), and $\dim P_{\rm cu}E < \infty$ hold. Then there exist constants $r > 0$ and $\alpha \in (\omega_{cu}, \omega_s)$ such that, for every solution $\varphi = (v, \sigma)$ of (3.6) on $\mathbb{R}_+$ with $\langle \varphi(t) \rangle_\gamma < r$ for all $t \geq 0$, there is a solution $\overline{\varphi} = (\overline{v}, \overline{\sigma})$ of (3.6) on $\mathbb{R}_+$ satisfying $w_* + \overline{\varphi}(t) \in \mathcal{M}_{\rm cu}$ for all $t \geq 0$ and*

$$|\varphi(t) - \overline{\varphi}(t)|_{E_1} + |\partial_t\sigma(t) - \partial_t\overline{\sigma}(t)|_{Z_\gamma} \leq c\mathrm{e}^{-\alpha t} \langle \varphi(0) \rangle_\gamma \tag{6.23}$$

for all $t \geq 1$. If even (3.21) holds, then $w_* + \overline{\varphi}(t) \in \mathcal{M}_{\rm c}$ for all $t \geq 0$

Proof. We choose $r > 0$ so small that Lemma 6.1 can be applied to φ. It gives solutions $\varphi_n = (v_n, \sigma_n)$ with $w_* + \varphi_n$ on $\mathcal{M}_{\rm cu}$ tracking φ on $[1, n]$ and satisfying

$$|P_{\rm cu}\varphi_n(1)|_E \leq |P_{\rm cu}\varphi(1)|_E + |P_{\rm cu}(\varphi_n(1) - \varphi(1))|_E \leq cr$$

for every $n \in \mathbb{N}$ with $n \geq 3$. There thus exists a subsequence $n_j \to \infty$ so that $P_{\rm cu}\varphi_{n_j}(1) \to \xi \in P_{\rm cu}E$ as $j \to \infty$. Theorem 5.2(d) and (e) combined with Theorem 3.3 provide a solution $\overline{\varphi} = (\overline{v}, \overline{\sigma})$ of (3.6) on $[-1, 3]$ such that $P_{\rm cu}\overline{\varphi}(1) = \xi$ and $w_* + \overline{\varphi}(t) \in \mathcal{M}_{\rm cu}$ for $t \in [-1, 3]$, decreasing $r > 0$ if needed to apply the mentioned theorems. We also obtain $\langle \overline{\varphi}(t) \rangle_\gamma \leq \langle \overline{\varphi}(1) \rangle_\gamma \leq c\,|\xi|_E \leq c_1 r$ for all $t \in [-1, 3]$ and some constant $c_1 > 0$. We further denote by c_2 the maximum of the embedding constants of $E_1 \hookrightarrow E_\gamma$ and $Z_\gamma \hookrightarrow Z_\gamma^1$, see (2.13).

Let T be the supremum of $t_1 > 1$ such that $\overline{\varphi}(t)$ exists and satisfies $\langle \overline{\varphi}(t) \rangle_\gamma \leq (2 + c_1 + c_2\overline{c})r$ for $t \in [0, t_1]$, where $\overline{c}$ is given by (6.4). We thus have $T \geq 2$. If we take a sufficiently small $r > 0$, Theorem 5.2(e) shows that $w_* + \overline{\varphi}(t) \in \mathcal{M}_{\rm cu}$ for $t \in [0, T)$. Moreover, the functions $P_{\rm cu}\varphi_n$ and $P_{\rm cu}\overline{\varphi}$ satisfy the ODE (5.14). Since $P_{\rm cu}\varphi_{n_j}(1) \to P_{\rm cu}\overline{\varphi}(1)$, the functions $P_{\rm cu}\varphi_{n_j}(t)$ tend to $P_{\rm cu}\overline{\varphi}(t)$ in E as $j \to \infty$ for $t \in [0, T)$. Theorem 5.2 then implies that φ_{n_j} converges to $\overline{\varphi}$ in $\mathbb{E}_1([t, t+1])$ for all intervals $[t, t+1] \subset [0, T)$, and hence $\varphi_{n_j}(t) \to \overline{\varphi}(t)$in E_γ and $\dot{\sigma}_{n_j}(t) \to \partial_t\overline{\sigma}$ in Z_γ^1, due (2.3) and (2.7). Lemma 6.1 now yields

$$\begin{aligned} \langle \overline{\varphi}(t) \rangle_\gamma &\leq \limsup_{j\to\infty} \Big(|\varphi_{n_j}(t) - \varphi(t)|_{E_\gamma} + |\varphi(t)|_{E_\gamma} + |\dot{\sigma}_{n_j}(t) - \dot{\sigma}(t)|_{Z_\gamma^1} + |\dot{\sigma}(t)|_{Z_\gamma^1} \Big) \\ &\leq \overline{c}c_2 r + r \end{aligned} \tag{6.24}$$

for all $t \in [0, T)$. As a result, $T = \infty$, and hence $\overline{\varphi}(t) \in \mathcal{M}_{\rm cu} - w_*$ exists and satisfies $\langle \overline{\varphi}(t) \rangle_\gamma \leq cr$ for all $t \geq 0$. In the same way we obtain the analogue of (6.23) with E_γ and Z_γ^1 on the left hand side. Estimate (6.23) now follows from Theorem 3.3(d). If (3.21) holds, Theorem 5.1(g) and (6.24) imply that also $w_* + \overline{\varphi}$ belongs to $\mathcal{M}_{\rm cs}$ (maybe after decreasing $r > 0$ once more). So the last assertion is a consequence of Corollary 5.3. □

In the next lemma we show that the tracking solution of Lemma 6.1 belongs to the center manifold if we have trichotomy and start on $\mathcal{M}_{\rm cs}$.

Lemma 6.4. *Assume that Hypothesis 3.4, condition (RR'), and (3.21) hold. Then there exist constants $r \in (0, r_{cs})$ and $\alpha \in (\underline{\omega}_c, \omega_s)$ such that, for every solution $\varphi = (v, \sigma) \in \mathbb{W}_1^*([0, T])$ of (3.6) with $w_* + \varphi(t) \in \mathcal{M}_{\rm cs}$ and $\langle \varphi(t) \rangle_\gamma < r$ for all $0 \leq t \leq T$ with some $T > 1$, there is a solution $\overline{\varphi} = (\overline{v}, \overline{\sigma}) \in \mathbb{W}_1^*([0, T])$ of (3.6) satisfying $w_* + \overline{\varphi}(t) \in \mathcal{M}_{\rm c}$ for all $t \in [0, T]$, $P_{\rm c}\overline{\varphi}(T) = P_{\rm c}\varphi(T)$ and*

$$|\varphi(t) - \overline{\varphi}(t)|_{E_1} + |\partial_t\sigma(t) - \partial_t\overline{\sigma}(t)|_{Z_\gamma} \leq c\mathrm{e}^{-\alpha t} \langle \varphi(0) \rangle_\gamma \leq \overline{c}r \tag{6.25}$$

for all $1 \le t \le T$*. Given* $T_0 > 1$*, the constants are uniform for* $T \ge T_0$*.*

Proof. We assume that $T \ge 3$. For a general $T_0 > 0$ the proof is similar. Formula (5.5) in Theorem 5.1 says that $\varphi = P_{\rm cs}\varphi + \phi_{cs}(P_{\rm cs}\mathcal{Q}(\varphi))$. As in the proof of Lemma 6.1 we define $\psi = P_{\rm s}\varphi - \phi_{cu}(P_{\rm cu}\varphi)$. The estimates (6.14) and (6.15) on ψ in this proof still hold because the present lemma has stronger assumptions. Since $|P_{\rm c}\varphi(T)|_E \le cr$, for sufficiently small $r > 0$ Theorem 4.6 provides a solution $\overline{\varphi}$ of (3.6) on $[T-3, T]$ such that $P_{\rm c}\overline{\varphi}(T) = P_{\rm c}\varphi(T)$ and $w_* + \overline{\varphi}$ belongs to $\mathcal{M}_{\rm c}$. Moreover, $|\overline{\varphi}(T)|_{E_1} \le c\,|P_{\rm c}\overline{\varphi}(T)|_E \le cr$. Here and below the constants do not depend on φ, T, t, r and the number $R > 0$ introduced later. Corollary 5.3 yields that $\mathcal{M}_{\rm c} = \mathcal{M}_{\rm cs} \cap \mathcal{M}_{\rm cu}$, and hence

$$\overline{\varphi} = P_{\rm c}\overline{\varphi} + \phi_c(P_{\rm c}\overline{\varphi}) = P_{\rm cs}\overline{\varphi} + \phi_{cs}(P_{\rm cs}\mathcal{Q}(\overline{\varphi})) = P_{\rm cu}\overline{\varphi} + \phi_{cu}(P_{\rm cu}\overline{\varphi}).$$

As a result, $P_{\rm s}\overline{\varphi} = \phi_{cu}(P_{\rm cu}\overline{\varphi})$ and $P_{\rm u}\overline{\varphi} = \phi_{cs}(P_{\rm cs}\mathcal{Q}(\overline{\varphi}))$. We thus infer

$$\begin{aligned} \varphi - \overline{\varphi} = \psi + \phi_{cu}(P_{\rm cu}\varphi) - \phi_{cu}(P_{\rm cu}\overline{\varphi}) + P_{\rm c}(\varphi - \overline{\varphi}) \\ + \phi_{cs}(P_{\rm cs}\mathcal{Q}(\varphi)) - \phi_{cs}(P_{\rm cs}\mathcal{Q}(\overline{\varphi})). \end{aligned} \tag{6.26}$$

We set $z = P_{\rm c}(\varphi - \overline{\varphi})$. Given a small $R > 0$ to be determined later, let $t_1 \in [1/2, T)$ be the minimal time such that the solution $\overline{\varphi}$ of (3.6) with $w_* + \overline{\varphi}$ on $\mathcal{M}_{\rm c}$ exists and the inequality $|\overline{\varphi}(t)|_{E_1} \le R$ holds for all $t_1 \le t \le T$. As in part (3) of the proof of Lemma 6.1, we obtain that $1/2 \le t_0 \le T - 2$ exists if $r > 0$ is chosen less than a number $\overline{r}(R)$. Theorem 3.3(c) further shows that $|\varphi(t)|_{E_1} \le c\,\langle\varphi(t-1/2)\rangle_\gamma \le cr \le R$ where we decrease $r > 0$ if needed.

From Theorems 5.1 and 5.2 we know that the maps $\phi_{cs} : P_{\rm cs}E_\gamma^0 \to P_{\rm u}X$ and $\phi_{cu} : P_{\rm cu}E \to P_{\rm s}E_k$ (with $k = 1, \gamma$) are Lipschitz with a constant $\varepsilon(R)$ on balls of radius R in the respective domain spaces. Moreover, $\mathcal{Q} : E_\gamma \to E_\gamma$ is locally Lipschitz by Lemma 3.2. Equation (6.26) thus yields

$$|\varphi(t) - \overline{\varphi}(t)|_{E_\gamma} \le |\psi(t)|_{E_\gamma} + \varepsilon(cR)\,|\varphi(t) - \overline{\varphi}(t)|_{E_\gamma} + |P_{\rm c}z(t)|_{E_\gamma}.$$

Decreasing $R > 0$ if needed, we infer

$$|\varphi(t) - \overline{\varphi}(t)|_{E_\gamma} \le c\,|\psi(t)|_{E_\gamma} + c\,|z(t)|_E \tag{6.27}$$

for $t_1 \le t \le T$. Arguing similarly and using Lemma 3.2, this estimate then leads to

$$\begin{aligned} |\varphi(t) - \overline{\varphi}(t)|_{E_1} &\le |\psi(t)|_{E_1} + c\left(|\varphi(t) - \overline{\varphi}(t)|_E + |\mathcal{Q}(\varphi(t)) - \mathcal{Q}(\overline{\varphi}(t))|_{E_\gamma} + |z(t)|_E\right) \\ &\le c\,(|\psi(t)|_{E_1} + |z(t)|_E) \end{aligned} \tag{6.28}$$

for all $t \in [t_1, T]$. We now proceed as in Lemma 3.4 of [8] starting from (3.27) there and modifying the reasoning as in step (3) of the proof of Lemma 6.1. Here we use the estimates (6.28), (6.15) and (6.14), and fix first a small $R > 0$ and then numbers $\alpha, \alpha' \in (\underline{\omega}_c, \omega_s)$ with $\alpha < \alpha'$. In this way we derive

$$|z(t)|_E \le c\mathrm{e}^{-\alpha' t}\,\|\psi\|_{\mathbb{E}_1^\sharp([t,T];\alpha')} \le c\,(|\psi(t)|_{E_\gamma} + |\zeta(t)|_{Z_\gamma^1}) \le cr \tag{6.29}$$

for all $t \in [t_1, T]$, where the first constant is uniform for $t \le T$ and the others for $t \le T - \frac{1}{4}$. If $t \in [T - \frac{1}{4}, T]$, as in part (3) of the proof of Lemma 6.1 we derive

$$|z(t)|_E \le c\,(|\psi(t - 1/4)|_{E_\gamma} + |\zeta(t - 1/4)|_{Z_\gamma^1}) \le cr$$

with a uniform constant. We observe that $P_c\overline{\varphi} = P_c(\varphi - z)$ and hence

$$\overline{\varphi} = P_c\overline{\varphi} + \phi_c(P_c\overline{\varphi}) = P_c(\varphi - z) + \phi_c(P_c(\varphi - z)).$$

The above inequalities thus lead to

$$|\overline{\varphi}(t_1)|_{E_1} \le c\,(|\varphi(t_1)|_E + |z(t_1)|_E) \le cr.$$

We finally conclude that $|\overline{\varphi}(t_1)|_{E_1} < R$, possibly after decreasing $r > 0$ again. As a result, $t_1 = 1/2$.

To show the asserted estimate on $\varphi - \overline{\varphi}$, we argue as in step (4) of the proof of Lemma 6.1 using (6.29) and (6.28). We first obtain

$$|z(t)|_E \le c\,\mathrm{e}^{-\alpha t}\,(|\psi(0)|_{E_\gamma} + |\zeta(0)|_{Z^1_\gamma})$$

for $t \in [1/2, T]$, see (6.20). The norm of $\psi(t)$ in E_γ can be bounded by $c\,\mathrm{e}^{-\alpha t}\,(|\psi(0)|_{E_\gamma} + |\zeta(0)|_{Z^1_\gamma})$ using the embedding (6.2) and the inequality (6.15). Estimate (6.27) thus implies

$$|\varphi(t) - \overline{\varphi}(t)|_{E_\gamma} \le c\,\mathrm{e}^{-\alpha t}\,(|\psi(0)|_{E_\gamma} + |\zeta(0)|_{Z^1_\gamma})$$

$t \in [1/2, T]$. The assertion then follows as in (4) of the proof of Lemma 6.1. □

By our second convergence theorem, the center manifold locally attracts the center–stable manifold with a tracking solution if the flow on $\mathcal{M}_c$ is *stable*, in the sense that for all $r > 0$ there is a radius $r_0 > 0$ such that for all $w_0 \in \mathcal{M}_c$ with $\langle w_0 - w_*\rangle_\gamma \le r_0$ there is a solution w of (1.1) on $\mathbb{R}_+$ staying on $\mathcal{M}_c$ such that $\langle w(t) - w_*\rangle_\gamma \le r$ holds for all $t \ge 0$. In particular, w_* is asymptotically stable for the full equation, if $\sigma_u = \emptyset$ and the flow on $\mathcal{M}_c$ is stable.

Theorem 6.5. *Assume that Hypothesis 3.4, conditions (RR') and* (3.21), *and* $\dim P_c E < \infty$ *hold. Suppose that* w_* *is stable for the flow on* $\mathcal{M}_c$. *Then there exist constants* $r > r_0 > 0$ *and* $\alpha \in (\underline{\omega}_c, \omega_s)$ *such that for* $w_* + \varphi(0) \in \mathcal{M}_{cs}$ *with* $\langle\varphi(0)\rangle_\gamma \le r_0$, *the solution* $w_* + \varphi(t) \in \mathcal{M}_{cs}$ *exists and satisfies* $\langle\varphi(t)\rangle_\gamma < r$ *for all* $t \ge 0$ *and there exists a solution* $\overline{\varphi} = (\overline{v}, \overline{\sigma})$ *of* (3.6) *on* $\mathbb{R}_+$ *such that* $w_* + \overline{\varphi}(t) \in \mathcal{M}_c$ *for all* $t \ge 0$ *and*

$$|\varphi(t) - \overline{\varphi}(t)|_{E_1} + |\partial_t\sigma(t) - \partial_t\overline{\sigma}(t)|_{Z_\gamma} \le c\mathrm{e}^{-\alpha t}\,\langle\varphi(0)\rangle_\gamma \tag{6.30}$$

for all $t \ge 1$. *If also* $\sigma_u = \emptyset$, *then* w_* *is stable for the full flow on* $\mathcal{M}$.

Proof. Let $r > 0$ and $\alpha \in (\underline{\omega}_c, \omega_s)$ be the numbers determined by Lemma 6.4. Take $r_0 \in (0, r)$ to be fixed later. Consider a function φ_0 satisfying $w_* + \varphi_0 \in \mathcal{M}_{cs}$ and $\langle\varphi_0\rangle_\gamma \le r_0$. We have the solution φ of (3.6) with $\varphi(0) = \varphi_0$ and denote by T the supremum of all $t > 0$ such that $\varphi(t)$ exists and satisfies $\langle\varphi(\tau)\rangle_\gamma < r$ for all $0 \le \tau \le t$. If r_0 is small enough, we have $T > 1$ due to Theorem 3.3. Suppose that T is finite. We then obtain $\langle\varphi(T)\rangle_\gamma = r$. Lemma 6.4 provides a solution φ_T of (3.6) on $[0, T]$ such that $P_c\varphi_T(T) = P_c\varphi(T)$, $w_* + \varphi_T(t) \in \mathcal{M}_c$ for $0 \le t \le T$, and

$$|\varphi(t) - \varphi_T(t)|_{E_1} + |\partial_t\sigma(t) - \partial_t\sigma_T(t)|_{Z_\gamma} \le c\mathrm{e}^{-\alpha t}\,\langle\varphi(0)\rangle_\gamma \le cr_0, \tag{6.31}$$

for $t \in [1, T]$, where $\varphi = (v, \sigma)$ and $\varphi_T = (v_T, \sigma_T)$. Here and below, c does not depend on T and r_0. The above estimate and Theorem 3.3 imply

$$\langle\varphi_T(1)\rangle_{E_\gamma} \le |\varphi_T(1) - \varphi(1)|_{E_\gamma} + |\dot\sigma_T(1) - \dot\sigma(1)|_{Z^1_\gamma} + \langle\varphi(1)\rangle_\gamma \le cr_0. \tag{6.32}$$

Since w_* is stable for the flow on $\mathcal{M}_c$, we can now choose $r_0 > 0$ such that $\langle \varphi_T(T) \rangle_{E_\gamma} \le r/2$. We then deduce

$$\langle \varphi(T) \rangle_\gamma \le |\varphi_T(T) - \varphi(T)|_{E_\gamma} + |\dot{\sigma}_T(T) - \dot{\sigma}(T)|_{Z^1_\gamma} + \langle \varphi_T(T) \rangle_{E_\gamma} \le cr_0 + r/2 < r,$$

possibly after decreasing $r_0 > 0$ once more. This strict inequality contradicts $\langle \varphi(T) \rangle_\gamma = r$, and hence $T = \infty$. Theorem 5.4 shows that $w_* + \varphi$ belongs to $\mathcal{M}_{cs}$. As a consequence, (6.32) holds for all $T > 1$. Since $P_c E$ is finite dimensional, there exists a sequence $T_n \to \infty$ such that $P_c \varphi_{T_n}(1)$ converge to some $\xi \in P_c E$ with $|\xi|_E \le cr_0$, as $n \to \infty$. If $r_0 > 0$ is small enough, Theorem 4.6 yields a solution $\overline{\varphi}$ of (3.6) on some time interval $[0, t_1)$ such that $P_c \overline{\varphi}(1) = \xi$, and hence $w_* + \overline{\varphi}$ is contained in $\mathcal{M}_c$ and $\langle \overline{\varphi}(1) \rangle_\gamma \le c\,|\xi|_0 \le cr_0$. The stability of w_* on $\mathcal{M}_c$ thus implies that $\overline{\varphi}(t)$ exists and $\langle \overline{\varphi}(t) \rangle_\gamma \le r$ for all $t \ge 0$, possibly after decreasing $r_0 > 0$. As in the proof of Theorem 6.3, we finally deduce the asserted convergence from (6.31).

If $\sigma_u = \emptyset$, then $\mathcal{M}_{cu} = \mathcal{M}_c$ and w_* is locally attractive. The stability of w_* now follows easily from $\varphi = \varphi - \overline{\varphi} + \overline{\varphi}$ and the stability on $\mathcal{M}_c$. □

Example 6.6. We continue Example 5.4 on the Stefan problem with surface tension. If $\sigma|D| \ne l\,|\Sigma| r^2$ we obtain a center manifold consisting of equilibria only, and the induced flow is of course stable. The condition (RR') follows from Example 2.4 in [20] and in particular from the formula for the term $\mathcal{R}^0(\rho, \partial_t \rho)$ given there. As a result, the center manifold attracts all solution starting near the given equilibrium if $\sigma|D| < l\,|\Sigma| r^2$ and it attracts the center–stable manifold if $\sigma|D| > l\,|\Sigma| r^2$. In the latter case the given equilibrium is unstable by Theorem 5.1(b). The solutions converge to an equlibrium. For a similar problem in Theorem 5.2 of [17] the stability for $\sigma|D| < l\,|\Sigma| r^2$ and instability for $\sigma|D| > l\,|\Sigma| r^2$ have been shown for different methods, but not the attraction the center–stable manifold if $\sigma|D| > l\,|\Sigma| r^2$. (The stable or center–stable manifolds w not considerd in [17].) In Theorem 5.3 of this paper also global results have been shown for this specific equation using Lyapunov functions. ◇

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