Lecture Notes
Evolution Equations

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These lecture notes are based on my course from winter semester 2023/24, though there are minor corrections and improvements. Typically, the proofs and calculations in the notes are a bit shorter than those given in class. The drawings and many additional oral remarks from the lectures are omitted here. On the other hand, the notes contain very few proofs (of peripheral statements) not presented during the course. Occasionally I use the notation and definitions of my lecture notes Analysis 1–4 and Functional Analysis without further notice. I want to thank Heiko Hoffmann for his support in the preparation of an earlier version of these notes.

Karlsruhe, October 3, 2023

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CHAPTER 1

Strongly continuous semigroups and their generators

Throughout, $X$ and $Y$ are non-zero Banach spaces over the field $F \in \{\mathbb{R}, \mathbb{C}\}$, where we mostly write $\| \cdot \|$ instead of $\| \cdot \|_X$ etc. for their norms. The space of all bounded linear maps $T : X \to Y$ is denoted by $B(X, Y)$ and endowed with the operator norm $\| T \|_{B(X,Y)} = \| T \| = \sup_{x \neq 0} \| Tx \|/\| x \|$. We abbreviate $B(X) = B(X,X)$. Further, $X^*$ is the dual space of $X$, where $x^* \in X^*$ act as $\langle x, x^* \rangle$ on $X$, and $I$ is the identity map on $X$. Let $A : D(A) \to Y$ be linear with domain $D(A)$ being a linear subspace of $X$. Then $R(A) = A D(A)$ denotes the range of $A$ and $N(A) = \{ x \in D(A) \mid Ax = 0 \}$ its kernel. For $\omega \in \mathbb{R}$, we set $\mathbb{R}_{\geq 0} = [0, \infty)$, $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}_{\leq 0} = (-\infty, 0]$, $\mathbb{R}_- = (-\infty, 0)$, $\mathbb{F}_\omega = \{ \lambda \in F \mid \Re \lambda > \omega \}$, $\mathbb{C}_+ = \mathbb{C}_0$, $\mathbb{C}_- = \{ \lambda \in \mathbb{C} \mid \Re \lambda < 0 \}$, $\omega_+ = \max\{\omega, 0\}$, $\omega_- = \max\{-\omega, 0\}$.

In this course we study linear evolution equations such as

$$u'(t) = Au(t), \quad t \geq 0, \quad u(0) = u_0,$$  

(EE)
on a state space $X$ for given linear operators $A$ and initial values $u_0 \in D(A)$. (For a moment we assume that $A$ is closed and densely defined.) We are looking for the state $u(t) \in X$ describing the system governed by $A$ at time $t \geq 0$. A reasonable description of the system requires a unique solution $u$ of (EE) that continuously depends on $u_0$. In this case (EE) is called wellposed, cf. Definitions 1.9 and 2.1. We will show in Section 2.1 that wellposedness is equivalent to the fact that $A$ generates a $C_0$-semigroup $T(\cdot)$ which yields the solutions via $u(t) = T(t)u_0$. In the next section we will define and investigate these concepts, before we characterize generators in Sections 1.2 and 1.3. In the final section the theory is then applied to operators like the Laplacian. Three intermezzi present basic notions and facts from the lecture notes [ST] on spectral theory.

1.1. Basic concepts and properties

We introduce the fundamental notions of these lectures.

**Definition 1.1.** A map $T(\cdot) : \mathbb{R}_{\geq 0} \to B(X)$ is called a strongly continuous operator semigroup or just $C_0$-semigroup if it satisfies

(a) $T(0) = I$ and $T(t+s) = T(t)T(s)$ for all $t, s \in \mathbb{R}_{\geq 0}$,

(b) for each $x \in X$ the orbit $T(\cdot)x : \mathbb{R}_{\geq 0} \to X; t \mapsto T(t)x$, is continuous. Here, (a) is the semigroup property and (b) the strong continuity of $T(\cdot)$. 

The generator $A$ of $T(\cdot)$ is given by
\[
\text{D}(A) = \{ x \in X \mid \text{the limit } \lim_{t \to 0, t \in \mathbb{R}_{>0}\setminus\{0\}} \frac{1}{t}(T(t)x - x) \text{ exists} \},
\]
\[
Ax = \lim_{t \to 0, t \in \mathbb{R}_{>0}\setminus\{0\}} \frac{1}{t}(T(t)x - x) \quad \text{for } x \in \text{D}(A).
\]
Replacing $\mathbb{R}_{>0}$ by $\mathbb{R}$, one obtains the concept of a $C_0$-group with generator $A$.

Observe that the domain $\text{D}(A)$ of the generator is defined in a ‘maximal’ way, in the sense that it contains all elements for which the orbit is differentiable at $t = 0$. In view of the introductory remarks, usually the generator is the given object and $T(\cdot)$ describes the unknown solution. We will first study basic properties of $C_0$-semigroups, starting with simple observations.

**Remark 1.2.** a) Let $A$ generate a $C_0$-semigroup or a $C_0$-group. Then its domain $\text{D}(A)$ is a linear subspace and $A$ is a linear map.

b) Let $(T(t))_{t \in \mathbb{R}}$ be a $C_0$-group with generator $A$. Then its restriction $(T(t))_{t \geq 0}$ is a $C_0$-semigroup whose generator extends $A$. (Actually these two operators coincide by Theorem 1.29.)

c) Let $T(\cdot) : \mathbb{R}_{\geq 0} \to \mathcal{B}(X)$ be a semigroup. We then have
\[
T(t)T(s) = T(t+s) = T(s+t) = T(s)T(t),
\]
\[
T(nt) = T(\sum_{j=1}^{n} t) = \prod_{j=1}^{n} T(t) = T(t)^n
\]
for all $t, s \geq 0$ and $n \in \mathbb{N}$.

Let $T(\cdot) : \mathbb{R} \to \mathcal{B}(X)$ be a group. Then the above properties are valid for all $s, t \in \mathbb{R}$, and hence $T(t)T(-t) = T(0) = I = T(-t)T(t)$. There thus exists the inverse $T(t)^{-1} = T(-t)$ for every $t \in \mathbb{R}$. ◊

We next construct a $C_0$-group with a bounded generator, which is actually differentiable in operator norm. Conversely, an exercise shows that a $C_0$-semigroup with $T(t) \to I$ in $\mathcal{B}(X)$ as $t \to 0^+$ must have a bounded generator.

**Example 1.3.** Let $A \in \mathcal{B}(X)$ and $b > 0$. For $t \in \mathbb{R}$ with $|t| \leq b$, the numbers $\| \frac{t^n}{n!} A^n \| \leq \frac{(b|A|)^n}{n!}$ are summable in $n \in \mathbb{N}_0$. As in Lemma 4.23 of [FA], the series $T(t) = e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$, $t \in \mathbb{R}$, thus converges in $\mathcal{B}(X)$ uniformly for $|t| \leq b$. In the same way one sees that
\[
\frac{d}{dt} \sum_{n=0}^{N} \frac{t^n}{n!} A^n = \sum_{n=1}^{N} \frac{t^{n-1}}{(n-1)!} A^n = A \sum_{k=0}^{N-1} \frac{t^k}{k!} A^k
\]
tends to $Ae^{tA}$ in $\mathcal{B}(X)$ as $N \to \infty$ locally uniformly in $t \in \mathbb{R}$. As in Analysis 1 or 4 one then shows that the map $\mathbb{R} \to \mathcal{B}(X); t \mapsto e^{tA}$, is continuously differentiable with derivative $Ae^{tA}$. Moreover, $(e^{tA})_{t \in \mathbb{R}}$ is a group (where one replaces $\mathbb{R}_{\geq 0}$ by $\mathbb{C}$ in Definition 1.1 (a) if $\mathbb{F} = \mathbb{C}$).

The case of a matrix $A$ on $X = \mathbb{C}^m$ was treated in Section 4.5 of [A4]. ◊

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1This concept is defined before Lemma 1.23.
In the next lemma the exponential boundedness of a semigroup follows from a mild extra assumption. This assumption is satisfied if \( \|T(t)\| \) is uniformly bounded on an interval \([0, b]\) with \( b > 0 \) or if \( T(\cdot) \) is strongly continuous. (We need both cases below.)

**Lemma 1.4.** Let \( T(\cdot): \mathbb{R}_{\geq 0} \rightarrow \mathcal{B}(X) \) satisfy condition (a) in Definition 1.1 as well as \( \limsup_{t \rightarrow 0} \|T(t)x\| < \infty \) for all \( x \in X \). Then there are constants \( M \geq 1 \) and \( \omega \in \mathbb{R} \) such that \( \|T(t)\| \leq Me^{\omega t} \) for all \( t \geq 0 \).

**Proof.** 1) We first claim that there are constants \( c \geq 1 \) and \( t_0 > 0 \) with \( \|T(t)\| \leq c \) for all \( t \in [0, t_0] \). To show this claim, we suppose that there is a null sequence \((t_n)\) in \( \mathbb{R}_{\geq 0} \) such that \( \lim_{n \rightarrow \infty} \|T(t_n)\| = \infty \). The principle of uniform boundedness (Theorem 4.4 in [FA]) then yields a vector \( x \in X \) with \( \sup_n \|T(t_n)x\| = \infty \). There thus exists a subsequence satisfying \( \|T(t_{n_j})x\| \rightarrow \infty \) as \( j \rightarrow \infty \). This fact contradicts the assumption, and so the claim is true.

2) Let \( t \geq 0 \). Then there are numbers \( n \in \mathbb{N}_0 \) and \( \tau \in [0, t_0) \) such that \( t = nt_0 + \tau \). Take \( \omega = t_0^{-1}\ln \|T(t_0)\| \) if \( T(t_0) \neq 0 \) and any \( \omega < 0 \) otherwise. Set \( M = e^{\omega t_0} \). Using Remark 1.2, we estimate

\[
\|T(t)\| = \|T(\tau)T(t_0)^n\| \leq c\|T(t_0)\|^n \leq ce^{\omega t_0} = ce^{t_0 e^{-\tau} \omega} \leq Me^{\omega t}.
\]

The above considerations lead to the following concept, which is discussed below and will be explored more thoroughly in Chapter 4.

**Definition 1.5.** Let \( T(\cdot) \) be a \( C_0 \)-semigroup with generator \( A \). The quantity

\[
\omega_0(T) = \omega_0(A) := \inf \{ \omega \in \mathbb{R} \mid \exists M \omega \geq 1 \ \forall t \geq 0 : \|T(t)\| \leq M\omega e^{\omega t} \} \in [-\infty, \infty)
\]

is called its (exponential) growth bound. If \( \sup_{t \geq 0} \|T(t)\| < \infty \), then \( T(\cdot) \) is bounded. (Similarly one defines \( \omega_0(f) \in [-\infty, +\infty] \) for any map \( f : \mathbb{R}_{\geq 0} \rightarrow Y \).

**Remark 1.6.** Let \( T(\cdot) \) be a \( C_0 \)-semigroup.

a) Lemma 1.4 implies that \( \omega_0(T) < \infty \).

b) There are \( C_0 \)-semigroups with \( \omega_0(T) = -\infty \), see Example 1.8.

c) In general the infimum in Definition 1.5 is not a minimum. For instance, let \( X = \mathbb{R}^2 \) be endowed with the 1-norm \( \| \cdot \|_1 \) and \( A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \). We then have \( T(t) = e^{tA} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \) and \( \|T(t)\| = 1 + t \) for \( t \geq 0 \). As a result,

\[
M_\varepsilon := \sup_{t \geq 0} e^{-\varepsilon t} \|T(t)\| = \sup_{t \geq 0}(1 + t)e^{-\varepsilon t} = \varepsilon^{-1} e^{-1}
\]

tends to infinity as \( \varepsilon \rightarrow 0^+ \), where \( \varepsilon \in (0, 1] \).

d) Let \( X = \mathbb{C}^m \) and \( A \in \mathbb{C}^{m \times m} \). Satz 4.22 and Theorem 6.3 of [A4] imply

\[
\omega_0(A) = s(A) := \max \{ \Re \lambda_j \mid \lambda_1, \ldots, \lambda_k \ \text{the are eigenvalues of} \ A \}.
\]

This result can be generalized to bounded \( A \) if \( \dim X = \infty \), cf. Example 5.4 of [ST]. Every generator satisfies \( \omega_0(A) \geq s(A) \) by Proposition 1.20. However, the converse inequality is much more important since \( A \) is the given object and \( T(\cdot)x \) the unknown solution. In Chapter 4 we will discuss this point in detail.

Similarly, a semigroup \( (e^{tA})_{t \geq 0} \) on \( X = \mathbb{C}^m \) is bounded if and only if \( s(A) \leq 0 \) and all eigenvalues of \( A \) on \( i\mathbb{R} \) are semi-simple. This indicates that boundedness of \( C_0 \)-semigroups is a more subtle property. \( \diamond \)
The next auxiliary result will often be used to check strong continuity.

**Lemma 1.7.** Let \(T(\cdot) : \mathbb{R}_{\geq 0} \to \mathcal{B}(X)\) be a map satisfying condition (a) in Definition 1.1. Then the following assertions are equivalent.

a) \(T(\cdot)\) is strongly continuous (and thus a \(C_0\)-semigroup).

b) \(T(t)x \to x\) in \(X\) as \(t \to 0^+\) for all \(x \in X\).

c) There are numbers \(c, t_0 > 0\) and a dense subspace \(D\) of \(X\) such that \(\|T(t)\| \leq c\) and \(T(t)x \to x\) in \(X\) as \(t \to 0^+\) for all \(t \in [0, t_0]\) and \(x \in D\).

For groups one has analogous equivalences.

**Proof.** Assertion c) follows from a) because of Lemma 1.4, and b) from c) by Lemma 4.10 in [FA].

Let statement b) be true. Take \(x \in X\) and \(t > 0\). For \(h > 0\), the semigroup property and b) imply the limit

\[
\|T(t + h)x - T(t)x\| = \|T(t)(T(h)x - x)\| \leq \|T(t)\| \|T(h)x - x\| \to 0
\]
as \(h \to 0^+\). Let \(h \in (-t, 0)\). Lemma 1.4 yields the bound

\[
\|T(t + h)\| \leq Me^{\omega(t+h)} \leq Me^{\omega t}
\]

for some constants \(M \geq 1\) and \(\omega \in \mathbb{R}\). We then derive

\[
\|T(t + h)x - T(t)x\| \leq \|T(t + h)\| \|x - T(-h)x\| \to 0
\]
as \(h \to 0^−\), so that a) is true. The addendum is shown similarly.

In the above lemma the implication ‘c) => a)’ can fail if one omits the boundedness assumption, cf. Exercise I.5.9(4) in [EN].

We now examine translation semigroups, which are easy to grasp and still illustrate many of the basic features of \(C_0\)-semigroups. Another important class of simple examples are multiplication semigroups as discussed in the exercises.

We recall that \(\text{supp}\ f\) is the support of a function \(f : M \to Y\) on a metric space \(M\); i.e., the closure in \(M\) of the set \(\{s \in M \mid f(s) \neq 0\}\).

**Example 1.8.** a) Let \(X = C_0(\mathbb{R}) := \{f \in C(\mathbb{R}) \mid f(s) \to 0\ as\ |s| \to \infty\}\) be endowed with \(\|\cdot\|_{\infty}\), which is a Banach space by Example 1.14 in [FA]. Take \(f \in X\) and \(t, r, s \in \mathbb{R}\). We define the translations

\[
(T(t)f)(s) = f(s + t).
\]

They shift the graph of \(f\) to the left if \(t > 0\), since \((T(t)f)(s)\) is equals the value of \(f\) at \(s + t > s\). Clearly, \(T(0) = I\) and \(T(t)\) is a linear isometry on \(X\) so that \(\|T(t)\| = 1\). We further obtain \(T(t)T(r) = T(t + r)\) noting

\[
(T(t)T(r)f)(s) = (T(r)f)(s + t) = f(r + s + t) = (T(t + r)f)(s).
\]

We claim that \(C_c(\mathbb{R}) = \{f \in C(\mathbb{R}) \mid \text{supp}\ f\ is\ compact\}\) is dense in \(C_0(\mathbb{R})\). Indeed, let \(f \in C_0(\mathbb{R})\) and choose cut-off functions \(\varphi_n \in C_c(\mathbb{R})\) satisfying \(\varphi_n = 1\) on \([-n, n]\) and \(0 \leq \varphi_n \leq 1\). Then the maps \(\varphi_n f\) belong to \(C_c(\mathbb{R})\) and

\[
\|f - \varphi_n f\|_{\infty} \leq \sup_{|s| \geq n} |(1 - \varphi_n(s))f(s)| \leq \sup_{|s| \geq n} |f(s)|
\]
tends to 0 as \(n \to \infty\).
Pick \( f \in C_c(\mathbb{R}) \) and a number \( a > 0 \) with \( \text{supp} \, f \subseteq [-a, a] \). Let \( t \in [-1, 1] \). If \( |s| > a + 1 \), we have \( |s + t| > a \) and thus \( f(s + t) = 0 \); i.e., \( \text{supp} \, T(t) f \) is contained in \([−a − 1, a + 1]\). It follows
\[
\|T(t)f - f\|_\infty \leq \sup_{|n| \leq a + 1} |f(s + t) - f(s)| \rightarrow 0
\]
as \( t \rightarrow 0 \), since \( f \) is uniformly continuous on \([−a − 1, a + 1]\). Lemma 1.7 then implies that \( T(\cdot) \) is a \( C_0 \)-group.

Similarly, one shows that \( T(\cdot) \) is an (isometric) \( C_0 \)-group on \( X = L^p(\mathbb{R}) \) with \( 1 \leq p < \infty \), see Example 4.12 in [FA].

In contrast to these results, \( T(\cdot) \) is not strongly continuous on \( X = L^\infty(\mathbb{R}) \). Indeed, consider \( f = \chi_{[0,1]} \) and observe that
\[
T(t)f = \chi_{[0,1]}(s + t) = \begin{cases} 
1, & s + t \in [0,1] \\
0, & \text{else}
\end{cases}
= \chi_{[-t,1-t]}(s)
\]
for \( s, t \in \mathbb{R} \). Thus, \( \|T(t)f - f\|_\infty = 1 \) for every \( t \neq 0 \).

In addition, \( T(\cdot) \) is not continuous as a \( B(X) \)-valued function for \( X = C_0(\mathbb{R}) \) (and neither for \( X = L^p(\mathbb{R}) \) by Example 4.12 in [FA]). In fact, pick functions \( f_n \in C_c(\mathbb{R}) \) with \( 0 \leq f_n \leq 1 \), \( f_n(n) = 1 \), and \( \text{supp} \, f_n \subseteq \left( n - \frac{3}{n}, n + \frac{1}{n} \right) \) for \( n \in \mathbb{N} \). Then the support of \( T(\frac{2}{n})f_n \) is contained in \( \left( n - \frac{3}{n}, n - \frac{1}{n} \right) \), implying
\[
\|T(\frac{2}{n}) - I\| \geq \|T(\frac{2}{n})f_n - f_n\|_\infty = 1 \quad \text{for all} \quad n \in \mathbb{N}.
\]

b) For an interval that is bounded from above, one has to prescribe the behavior of the left translation at the right boundary point. Here we simply prescribe the value 0. We work on the Banach space \( X = C_0([0,1]) := \{ f \in C([0,1]) | \lim_{s \uparrow 1} f(s) = 0 \} \) with \( \| \cdot \|_\infty \), see Example 1.14 in [FA]. Let \( t, r \geq 0 \), \( f \in X \), and \( s \in [0,1] \). We define
\[
(T(t)f)(s) := \begin{cases} 
f(s + t), & s + t < 1, \\
0, & s + t \geq 1.
\end{cases}
\]
Since \( f(s + t) \rightarrow 0 \) as \( s \rightarrow 1 - t \) if \( t < 1 \), the function \( T(t)f \) belongs to \( X \). Clearly, \( T(t) \) is linear on \( X \) and \( \|T(t)\| \leq 1 \). We stress that \( T(t) = 0 \) whenever \( t \geq 1 \). (One says that \( T(\cdot) \) is nilpotent.) As a consequence, \( \omega_0(T) = -\infty \) and \( T(\cdot) \) cannot be extended a group in view of Remark 1.2. We next compute
\[
(T(t)T(r)f)(s) = \begin{cases} 
(T(r)f)(s + t), & s < 1 - t, \\
0, & s \geq 1 - t,
\end{cases}
\]
\[
= \begin{cases} 
f(s + t + r), & s < 1 - t, \quad s + t < 1 - r, \\
0, & \text{else},
\end{cases}
\]
\[
= (T(t + r)f)(s).
\]
Hence, \( T(\cdot) \) is a semigroup.

As in part a) or in Example 1.19 of [FA], one sees that
\[
C_c([0,1]) := \{ f \in C([0,1]) | \exists b_f \in (0,1) : \text{supp} \, f \subseteq [0,b_f] \}.
\]
is a dense subspace of \( X \). For \( f \in C_c([0,1]) \) and \( t \in (0,1-b_f) \) we compute

\[
T(t)f(s) - f(s) = \begin{cases} 
  f(s + t) - f(s), & \text{if } s \in [0,1-t), \\
  0, & \text{if } s \in [1-t,1) \subseteq [b_f,1), 
\end{cases}
\]

and deduce \( \lim_{t \to 0} \|T(t)f - f\|_\infty = 0 \) using the uniform continuity of \( f \). According to Lemma 1.7, \( T(\cdot) \) is a \( C_0 \)-semigroup on \( X \).

We now introduce a solution concept for the problem (EE). Different ones will be discussed in Section 2.2.

**Definition 1.9.** Let \( A \) be a linear operator on \( X \) with domain \( D(A) \) and let \( x \in D(A) \). A function \( u : \mathbb{R}_{\geq 0} \to X \) solves the homogeneous evolution equation (or Cauchy problem)

\[
u'(t) = Au(t), \quad t \geq 0, \quad u(0) = x, \tag{1.1}
\]

if \( u \) belongs to \( C^1(\mathbb{R}_{\geq 0}, X) \) and satisfies \( u(t) \in D(A) \) and (1.1) for all \( t \geq 0 \).

We next show the fundamental regularity properties of \( C_0 \)-semigroups. Recall that the generator’s domain \( D(A) \) was ‘maximally’ defined as the set of all initial values for which the orbit is differentiable at \( t = 0 \). We now use the semigroup law to transfer this property to later times. The crucial invariance of the domain under the semigroup then directly follows from its definition.

**Proposition 1.10.** Let \( A \) generate the \( C_0 \)-semigroup \( T(\cdot) \) and \( x \in D(A) \). Then \( T(t)x \) belongs to \( D(A) \), the orbit \( T(\cdot)x \) to \( C^1(\mathbb{R}_{\geq 0}, X) \), and we have

\[
\frac{d}{dt}T(t)x = AT(t)(x) = T(t)Ax \quad \text{for all } t \geq 0.
\]

Moreover, the function \( u = T(\cdot)x \) is the only solution of (1.1).

**Proof.** 1) Let \( t > 0, h > 0 \) and \( x \in D(A) \). Remark 1.2 and the continuity of \( T(t) \) then imply the convergence

\[
\frac{1}{h}(T(h) - I)T(t)x = T(t)\frac{1}{h}(T(h)x - x) \to T(t)Ax
\]

as \( h \to 0 \). By Definition 1.1 of the generator, the vector \( T(t)x \) thus belongs to \( D(A) \) and satisfies \( AT(\cdot)x = T(\cdot)Ax \). Next, let \( 0 < h < t \). We then compute

\[
\frac{1}{h}(T(t-h)x - T(t)x) = T(t-h)\frac{1}{h}(T(h)x - x) \to T(t)Ax
\]

as \( h \to 0 \), by means of Lemma 1.12 below (with \( S(\tau,\sigma) = T(\tau - \sigma) \)). Together we have shown that the orbit \( u = T(\cdot)x \) has the derivative \( AT(\cdot)x \). Since \( AT(\cdot)x \) is continuous, \( u \) is contained in \( C^1(\mathbb{R}_{\geq 0}, X) \). Summing up, \( u \) solves (1.1).

2) Let also \( v \) solve (1.1). We show \( v = u \) by a standard trick. Take \( t > 0 \) and set \( w(s) = T(t-s)v(s) \) for \( s \in [0, t] \). Let \( h \in [-s, t-s] \setminus \{0\} \). We write

\[
\frac{1}{h}(w(s+h)-w(s)) = T(t-s-h)\frac{1}{h}(v(s+h)-v(s)) - \frac{1}{h}(T(t-s-h)-T(t-s))v(s).
\]

Using \( v \in C^1 \), Lemma 1.12, \( v(s) \in D(A) \) and the first step, we infer that \( w \) is differentiable with derivative

\[
w'(s) = T(t-s)v'(s) - T(t-s)Av(s) = 0,
\]
where the last equality follows from (1.1) for \( v \). Hence, for each \( x^* \in X^* \) the scalar function \( \langle w(\cdot), x^* \rangle \) is differentiable with vanishing derivative and thus constant, which leads to the equality
\[
\langle T(t)x, x^* \rangle = \langle w(0), x^* \rangle = \langle w(t), x^* \rangle = \langle v(t), x^* \rangle
\]
for all \( t > 0 \). The Hahn-Banach theorem now yields \( T(\cdot)x = v \) as asserted, see Corollary 5.10 of [FA].

\[\square\]

**Remark 1.11.** Let \( f \in C_0(\mathbb{R}) \setminus C^1(\mathbb{R}) \). Then the orbit \( T(\cdot)f = f(\cdot + t) \) of the translation semigroup on \( C_0(\mathbb{R}) \) is not differentiable (cf. Example 1.8). \[\diamondsuit\]

The following simple lemma is used in the above proof and also later on.

**Lemma 1.12.** Let \( D = \{(\tau, \sigma) \mid a \leq \sigma \leq \tau \leq b\} \) for some \( a < b \) in \( \mathbb{R} \), \( S : D \to \mathcal{B}(X) \) be strongly continuous, and \( f \) be contained in \( C([a, b], X) \). Then the function \( g : D \to X, g(\tau, \sigma) = S(\tau, \sigma)f(\sigma) \), is also continuous.

**Proof.** Observe that \( \sup_{(\tau, \sigma) \in D} \|S(\tau, \sigma)x\| < \infty \) for every \( x \in X \) by continuity. The uniform boundedness principle thus says that \( c := \sup_D \|S(\tau, \sigma)\| \) is finite. For \((t, s), (\tau, \sigma) \in D\) we then obtain
\[
\|S(t, s)f(s) - S(\tau, \sigma)f(\sigma)\| \leq \|(S(t, s) - S(\tau, \sigma))f(s)\| + c\|f(s) - f(\sigma)\|.
\]
The right-hand side of this inequality tends to 0 as \((\tau, \sigma) \to (t, s)\). \[\square\]

**Remark 1.13.** Let \( x_n \to x \) in \( X \) and \( T_n \to T \) strongly in \( \mathcal{B}(X, Y) \). As in the proof of Lemma 1.12 one then shows that \( T_nx_n \to Tx \) in \( Y \) as \( n \to \infty \). \[\diamondsuit\]

**Intermezzo 1:** **Closed operators, spectrum, and \( X \)-valued Riemann integrals.** As noted above, generators of \( C_0 \)-semigroups are unbounded unless the semigroup is continuous in \( \mathcal{B}(X) \). However, we will see in Proposition 1.19 that they still respect limits to some extent. We introduce the relevant concepts here. See Chapter 1 in [ST] for more details

Let \( D(A) \subseteq X \) be a linear subspace and \( A : D(A) \to X \) be linear. (One could also take \( Y \neq X \) as range space.) We often endow \( D(A) \) with the graph norm \( \|x\|_A := \|x\| + \|Ax\| \), writing \( [D(A)], X_1 \) or \( [D(A)], \|\cdot\|_A \) and \( \|x\|_A \) for \( \|x\|_A \). Observe that \([D(A)]\) is a normed vector space and that \( A \) is an element of \( \mathcal{B}([D(A)], X) \). Also, a map \( f \in C([a, b], X) \) belongs to \( C([a, b], [D(A)]) \) if and only if \( f \) takes values in \( D(A) \) and \( Af : [a, b] \to X \) is continuous.

The operator \( A \) is called *closed* if for every sequence \( (x_n) \) in \( D(A) \) possessing the limits
\[
\lim_{n \to \infty} x_n = x \quad \text{and} \quad \lim_{n \to \infty} Ax_n = y \quad \text{in} \ X,
\]
we obtain
\[
x \in D(A) \quad \text{and} \quad Ax = y.
\]
We start with prototypical examples.

**Example 1.14.** a) Every operator \( A \in \mathcal{B}(X) \) with \( D(A) = X \) is closed, since here \( Ax_n \to Ax \) if \( x_n \to x \) in \( X \) as \( n \to \infty \).

b) Let \( X \) be \( C([0, 1]) \) and \( Af = f' \) with \( D(A) = C^1([0, 1]) \). Let \( (f_n) \) be a sequence in \( D(A) \) such that \( (f_n) \) and \( (f'_n) \) converge in \( X \) to \( f \) and \( g \) respectively. By Analysis 1, the limit \( f \) then belongs to \( C^1([0, 1]) \) and satisfies \( f' = g \); i.e., \( A \) is
closed. Next, consider the map $A_0 f = f'$ with $D(A_0) = \{ f \in C^1([0,1]) \mid f'(0) = 0 \}$. Let $(f_n)$ be a sequence in $D(A)$ such that $f_n \to f$ and $f'_n \to g$ in $X$ as $n \to \infty$. We again obtain $f \in C^1([0,1])$ and $f' = g$. It further follows $f'(0) = g(0) = \lim_{n \to \infty} f'_n(0) = 0$, so that also $A_0$ is closed.

Before we discuss basic properties of closed operators, we define the Riemann integral for $X$-valued functions. Let $a < b$ be real numbers. A (tagged) partition $Z$ of $[a,b]$ is a finite set of numbers $a = t_0 < t_1 < \ldots < t_m = b$ together with ‘tags’ $\tau_k \in [t_{k-1}, t_k]$ for all $k \in \{1, \ldots, m\}$. Set $\delta(Z) = \max_{k \in \{1, \ldots, m\}} (t_k - t_{k-1})$.

For a map $f \in C([a,b], X)$ and a partition $Z$, the Riemann sum is given by

$$S(f, Z) = \sum_{k=1}^m f(\tau_k)(t_k - t_{k-1}) \in X.$$ 

As for real-valued $f$ it can be shown that for any sequence $(Z_n)$ of (tagged) partitions with $\lim_{n \to \infty} \delta(Z_n) = 0$ the sequence $(S(f, Z_n))_n$ converges in $X$ and that the limit $J$ does not depend on the choice of such $(Z_n)$. We then say that $S(f, Z)$ tends in $X$ to $J$ as $\delta(Z) \to 0$. The Riemann integral is now defined as

$$\int_a^b f(t) \, dt = \lim_{\delta(Z) \to 0} S(f, Z).$$

We also set $\int_a^a f(t) \, dt = -\int_a^b f(t) \, dt$. As in the real-valued case, one shows the basic properties the integral (except for monotony), e.g., linearity, additivity and validity of the standard estimate. Moreover, the same definition and results work for piecewise continuous functions. The fundamental theorem of calculus and a result on dominated convergence are shown in the next remark.

**Remark 1.15.** For a linear operator $A$ in $X$ the following assertions hold.

a) The operator $A$ is closed if and only if its graph $\text{Gr}(A) = \{ (x, Ax) \mid x \in D(A) \}$ is closed in $X \times X$ (endowed with the product metric) if and only if $D(A)$ is a Banach space with respect to the graph norm $\| \cdot \|_A$.

b) If $A$ is closed with $D(A) = X$, then $A$ is continuous (closed graph theorem).

c) Let $A$ be injective. Set $D(A^{-1}) := \text{R}(A) = \{ Ax \mid x \in D(A) \}$. Then $A$ is closed if and only if $A^{-1}$ is closed.

d) Let $A$ be closed and $f \in C([a,b], [D(A)])$. We then have

$$\int_a^b f(t) \, dt \in D(A) \quad \text{and} \quad A \int_a^b f(t) \, dt = \int_a^b Af(t) \, dt.$$ 

An analogous result is valid for piecewise continuous $f$ and $Af$. So we can commute the Riemann integral and bounded linear operators, since $[D(A)]$ is just $X$ (with an equivalent norm) if $A \in B(X)$.

e) Let $f_n, f \in C([a,b], X)$ for $n \in \mathbb{N}$ such that $f_n(s) \to f(s)$ in $X$ as $n \to \infty$ for each $s \in [a,b]$ and $\|f_n(\cdot)\| \leq \varphi$ for a map $\varphi \in L^1(a,b)$ and all $n \in \mathbb{N}$. Then there exists the limit

$$\lim_{n \to \infty} \int_a^b f_n(s) \, ds = \int_a^b f(s) \, ds.$$ 

The assumptions are satisfied if $f_n \to f$ uniformly as $n \to \infty$, of course.
f) For \( f \in C([a, b], X) \), the function

\[
[a, b] \to X; \ t \mapsto \int_a^t f(s) \, ds,
\]

is continuously differentiable with derivative

\[
\frac{d}{dt} \int_a^t f(s) \, ds = f(t)
\]

for each \( t \in [a, b] \). For \( g \in C^1([a, b], X) \), we have

\[
\int_a^b g'(s) \, ds = g(b) - g(a).
\]

(1.3)

g) Let \( J \subseteq \mathbb{R} \) be an interval. Take a sequence \( (f_n) \) in \( C^1(J, X) \) and maps \( f, g \in C^1(J, X) \) such that \( f_n \to f \) and \( f_n' \to g \) uniformly on \( J \) as \( n \to \infty \). We then obtain \( f \in C^1(J, X) \) and \( f' = g \).

**Proof.** Parts a) and c) are shown in Lemma 1.4 of [ST], and b) is established in Theorem 1.5 of [ST].

For d), let \( f \) be as in the statement. Note that for each partition \( Z \) of \([a, b]\)
the Riemann sum \( \mathcal{S}(f, Z) \) belongs to \( \mathcal{D}(A) \). Since \( Af \) is continuous, we obtain

\[
AS(f, Z) = \sum_{k=1}^m (Af)(\tau_k)(t_k - t_{k-1}) = S(Af, Z) \to \int_a^b Af(t) \, dt
\]
as \( \delta(Z) \to 0 \). Claim d) now follows from the closedness of \( A \).

Dominated convergence with majorant \( \|f\|_\infty 1 + \varphi \) yields assertion e) because

\[
\left\| \int_a^b f(s) \, ds - \int_a^b f_n(s) \, ds \right\| \leq \int_a^b \|f(s) - f_n(s)\| \, ds.
\]

For f), take \( t \in [a, b] \) and \( h \neq 0 \) such that \( t + h \in [a, b] \). We can then estimate

\[
\left\| \frac{1}{h} \left( \int_a^{t+h} f(s) \, ds - \int_a^t f(s) \, ds \right) - f(t) \right\| = \left\| \frac{1}{h} \int_t^{t+h} (f(s) - f(t)) \, ds \right\| \to 0
\]

as \( h \to 0 \). So we have shown (1.2). In the proof of Proposition 1.10 we have seen that a function in \( C^1([a, b]) \) is constant if its derivative vanishes. Equation (1.3) can thus be deduced from (1.2) as in Analysis 2.

Let \( f_n, f, \) and \( g \) be as in part g). Take \( a \in J \). Formula (1.3) says that

\[
f_n(t) = f_n(a) + \int_a^t f_n'(s) \, ds
\]

for all \( t \in J \). Letting \( n \to 0 \), from e) we deduce

\[
f(t) = f(a) + \int_a^t g(s) \, ds
\]

for all \( t \in J \). Due to (1.2), the map \( f \) belongs \( C^1(J, X) \) and satisfies \( f' = g \). □
For a closed operator $A$ we define its resolvent set
$$\rho(A) = \{ \lambda \in \mathbb{F} \mid \lambda I - A : D(A) \to X \text{ is bijective} \}.$$ 
If $\lambda \in \rho(A)$, we write $R(\lambda, A)$ for $(\lambda I - A)^{-1}$ and call it resolvent.\(^2\) The spectrum of $A$ is the set
$$\sigma(A) = \mathbb{F} \setminus \rho(A).$$

The point spectrum
$$\sigma_p(A) = \{ \lambda \in \mathbb{F} \mid \exists v \in D(A) \setminus \{0\} \text{ with } Av = \lambda v \}$$
is a subset of $\sigma(A)$ which can be empty if $\dim X = \infty$, see Example 1.25 in [ST]. We discuss basic properties of spectrum and resolvent which will be used throughout these lectures.

**Remark 1.16.** a) Let $A$ be closed and $\lambda \in \rho(A)$. It is easy to check that also the operator $\lambda I - A$ is closed (see Corollary 1.8 in [ST]), and hence $R(\lambda, A)$ is closed by Remark 1.15(c). Assertion d) of this remark then shows the boundedness of $R(\lambda, A)$.

b) Let $A$ be a linear operator and $\lambda \in \mathbb{F}$ such that $\lambda I - A : D(A) \to X$ is bijective with bounded inverse. Then $(\lambda I - A)^{-1}$ is closed, so that Remark 1.15(c) implies the closedness of $A$. In particular, $\lambda$ belongs to $\rho(A)$.

c) We list several important statements of Theorem 1.13 in [ST]. The set $\rho(A)$ is open and so $\sigma(A)$ is closed. More precisely, for $\lambda \in \rho(A)$ all $\mu$ with $|\mu - \lambda| < 1/\|R(\lambda, A)\|$ are also contained in $\rho(A)$ and we have the power series
$$R(\mu, A) = \sum_{n=0}^{\infty} (\lambda - \mu)^n R(\lambda, A)^{n+1}. \quad (1.5)$$
This series converges absolutely in $\mathcal{B}(X, D(A))$ and uniformly for $\mu$ with $|\mu - \lambda| \leq \delta/\|R(\lambda, A)\|$ and $\delta \in (0, 1)$, where one also obtains the inequality $\|R(\mu, A)\| \leq \|R(\lambda, A)\|/(1 - \delta)$. The resolvent has the derivatives
$$(\frac{d}{d\lambda})^n R(\lambda, A) = (-1)^n n! R(\lambda, A)^{n+1} \quad (1.6)$$
for all $\lambda \in \rho(A)$ and $n \in \mathbb{N}_0$. It further fulfills the resolvent equation
$$R(\mu, A) - R(\lambda, A) = (\lambda - \mu) R(\lambda, A) R(\mu, A) = (\lambda - \mu) R(\mu, A) R(\lambda, A), \quad (1.7)$$
for $\lambda, \mu \in \rho(A)$, and we have
$$\|R(\lambda, A)\| \geq d(\lambda, \sigma(A))^{-1}. \quad (1.8)$$
d) Let $T \in \mathcal{B}(X)$ and $\mathbb{F} = \mathbb{C}$. By Theorem 1.16 of [ST], the spectrum $\sigma(T)$ is even compact and always non-empty, and the spectral radius of $T$ is given by
$$r(T) := \max \{ |\lambda| \mid \lambda \in \sigma(A) \} = \inf_{n \in \mathbb{N}} \|T^n\|^{\frac{1}{n}} = \lim_{n \to \infty} \|T^n\|^{\frac{1}{n}}.$$
e) Example 1.21 provides closed operators $A$ with $\sigma(A) = \emptyset$ or $\sigma(A) = \mathbb{F}$. \(\diamondsuit\)

This ends the intermezzo, and we come back to the investigation of $C_0$-semigroups. We first note a simple rescaling lemma which is often used to simplify the reasoning.

\(^2\)Usually one takes $\mathbb{F} = \mathbb{C}$ in spectral theory, but many facts also hold for $\mathbb{F} = \mathbb{R}$. Sometimes real scalars are more convenient, and so we treat both fields if it is feasible. In examples we often restrict to $\mathbb{F} = \mathbb{C}$, as already real matrices may only have non-real eigenvalues.
1.1. Basic concepts and properties

Lemma 1.17. Let \( T(\cdot) \) be a \( C_0 \) semigroup with generator \( A \), \( \lambda \in \mathbb{F} \), and \( a > 0 \). Set \( S(t) = e^{\lambda t}T(at) \) for \( t \geq 0 \). Then \( S(\cdot) \) is a \( C_0 \) semigroup and has the generator \( B = \lambda I + aA \) with \( D(B) = D(A) \).

Proof. Let \( t, s > 0 \). We compute \( S(t + s) = e^{\lambda t}e^{\lambda s}T(at)T(as) = S(t)S(s) \).

The strong continuity of \( S(\cdot) \) and the identity \( S(0) = I \) are clear. Let \( B \) be the generator of \( S(\cdot) \). Because of

\[
\frac{1}{t}(S(t)x - x) = ae^{\lambda \frac{1}{t}}(T(at)x - x) + \lambda \frac{1}{t}(e^{\lambda t} - 1)x,
\]

\( x \) belongs to \( D(B) \) if and only if \( x \in D(A) \), and we then have \( Bx = aAx + \lambda x \). \( \square \)

Below we will derive key features of generators, which are consequences of the next fundamental lemma.

Lemma 1.18. Let \( T(\cdot) \) be a \( C_0 \) semigroup with generator \( A \), \( \lambda \in \mathbb{F}, t > 0 \), and \( x \in X \). Then the integral \( \int_0^t e^{-\lambda s}T(s)x \, ds \) belongs to \( D(A) \) and satisfies

\[
e^{-\lambda t}T(t)x - x = (A - \lambda I)\int_0^t e^{-\lambda s}T(s)x \, ds. \tag{1.9}
\]

Furthermore, for \( x \in D(A) \) we have

\[
e^{-\lambda t}T(t)x - x = \int_0^t e^{-\lambda s}T(s)(A - \lambda I)x \, ds. \tag{1.10}
\]

Proof. We only consider \( \lambda = 0 \) since the general case then follows by means of Lemma 1.17. For \( h > 0 \) and \( t > 0 \) we compute

\[
\frac{1}{t}(T(h) - I)\int_0^t T(s)x \, ds = \frac{1}{h}\left(\int_0^t T(s + h)x \, ds - \int_0^t T(s)x \, ds\right)
= \frac{1}{h}\left(\int_h^{t+h} T(r)x \, dr - \int_0^t T(s)x \, ds\right)
= \frac{1}{h}\int_h^{t+h} T(s)x \, ds - \frac{1}{h}\int_0^h T(s)x \, ds, \tag{1.11}
\]

where we substituted \( r = s + h \). The last line tends to \( T(t)x - x \) as \( h \to 0 \) due to the continuity of the orbits and (1.4). By the definition of the generator, this means that \( \int_0^t T(s)x \, ds \) is an element of \( D(A) \) and (1.9) holds. Let \( x \in D(A) \). Proposition 1.10 then shows that \( T(\cdot)x \) belongs to \( C^1(\mathbb{R}_{\geq 0}, X) \) with derivative \( \frac{d}{dt}T(\cdot)x = T(\cdot)Ax \). Hence, formula (1.10) follows from (1.3). \( \square \)

We can now show basic properties of generators. Recall that they commute with their semigroup by Proposition 1.10.

Proposition 1.19. Let \( A \) generate a \( C_0 \) semigroup \( T(\cdot) \). Then \( A \) is closed and densely defined. Moreover, \( T(\cdot) \) is the only \( C_0 \) semigroup generated by \( A \). If \( \lambda \in \rho(A) \), then we have \( R(\lambda, A)T(t) = T(t)R(\lambda, A) \) for all \( t \geq 0 \).

Proof. 1) To show closedness, we take a sequence \( (x_n) \) in \( D(A) \) with limit \( x \) in \( X \) such that \( (Ax_n) \) converges to some \( y \) in \( X \). Equation (1.10) yields

\[
\frac{1}{t}(T(t)x_n - x_n) = \frac{1}{t}\int_0^t T(s)Ax_n \, ds
\]
for all \( n \in \mathbb{N} \) and \( t > 0 \). Letting \( n \to \infty \), we infer
\[
\frac{1}{t}(T(t)x - x) = \frac{1}{t} \int_0^t T(s)y \, ds
\]
by means of Remark 1.15 e). Because of (1.4), the right-hand side tends to \( y \) as \( t \to 0 \). This exactly means that \( x \) belongs \( D(A) \) and \( Ax = y \); i.e., \( A \) is closed.

2) Let \( x \in \mathcal{X} \). For \( n \in \mathbb{N} \), we define the vector
\[
x_n = n \int_0^\frac{1}{n} T(s)x \, ds
\]
which belongs to \( D(A) \) by Lemma 1.18. Formula (1.4) shows that \( (x_n) \) tends to \( x \), and hence the domain \( D(A) \) is dense in \( \mathcal{X} \).

3) Let \( A \) generate another \( C_0 \)-semigroup \( S(\cdot) \). The function \( S(\cdot)x \) then solves (1.1) for each \( x \in D(A) \) by Proposition 1.10. The uniqueness statement in this result thus implies that \( T(t)x = S(t)x \) for all \( t \geq 0 \) and \( x \in D(A) \). Since these operators are bounded, step 2) leads to \( T(\cdot) = S(\cdot) \) as desired.

4) Let \( \lambda \in \rho(A) \), \( t \geq 0 \), and \( x \in \mathcal{X} \). Set \( y = R(\lambda, A)x \in D(A) \). Proposition 1.10 implies the identity \( T(t)(\lambda y - Ay) = (\lambda I - A)T(t)y \). Applying \( R(\lambda, A) \), we conclude that \( R(\lambda, A)T(t)x = T(t)R(\lambda, A)x \). \( \square \)

We next derive important information about spectrum and resolvent of generators. Actually we show a bit more than needed later on.

**Proposition 1.20.** Let \( A \) generate the \( C_0 \)-semigroup \( T(\cdot) \) and \( \lambda \in \mathbb{F} \). Then the following assertions hold.

a) If the improper integral
\[
R(\lambda)x := \int_0^\infty e^{-\lambda s}T(s)x \, ds := \lim_{t \to \infty} \int_0^t e^{-\lambda s}T(s)x \, ds
\]
exists in \( \mathcal{X} \) for all \( x \in \mathcal{X} \), then \( \lambda \in \rho(A) \) and \( R(\lambda) = R(\lambda, A) \).

b) The integral in a) exists even absolutely for all \( x \in \mathcal{X} \) if \( \Re \lambda > \omega_0(T) \). Hence, the spectral bound (of \( A \))
\[
s(A) := \sup \{ \Re \lambda \mid \lambda \in \sigma(A) \}
\]
(1.12)
is less or equal than \( \omega_0(T) \).

c) Let \( M \geq 1 \) and \( \omega \in \mathbb{R} \) with \( \|T(t)\| \leq Me^{\omega t} \) for all \( t \geq 0 \). Take \( n \in \mathbb{N} \) and \( \lambda \in \mathbb{F}_\omega \) (i.e., \( \Re \lambda > \omega \)). We then have
\[
\|R(\lambda, A)^n\| \leq \frac{M}{(\Re \lambda - \omega)^n}.
\]

We recall from Definition 1.5 and Lemma 1.4 that the exponent \( \omega \) in part c) has to satisfy \( \omega \geq \omega_0(T) \) and that any number \( \omega \in (\omega_0(T), \infty) \) fulfills the conditions in c).

The integral in part a) is called the Laplace transform of \( T(\cdot)x \). It can be used for alternative approaches to the theory of \( C_0 \)-semigroups (and their generalizations), cf. [ABHN]. In Chapter 4 we will study whether the equality \( s(A) = \omega_0(T) \) can be shown in b). This property would allow to control the growth (or decay) of the semigroup in terms of the given object \( A \).
Proof of Proposition 1.20. a) Let $h > 0$ and $x \in X$. By Lemma 1.17, we have the $C_0$-semigroup $T_\lambda(t) = (e^{-\lambda t}T(s))_{s \geq 0}$ with generator $A - \lambda I$ on the domain $D(A)$. Equation (1.11) yields
\[
\frac{1}{h}(T_\lambda(h) - I)R(\lambda)x = \lim_{t \to \infty} \frac{1}{h}(T_\lambda(h) - I) \int_0^t T_\lambda(s)x \, ds
\]
\[
= \lim_{t \to \infty} \frac{1}{h} \int_t^{t+h} T_\lambda(s)x \, ds - \frac{1}{h} \int_0^h T_\lambda(s)x \, ds
\]
\[
= - \frac{1}{h} \int_0^h T_\lambda(s)x \, ds,
\]
due to the convergence of $\int_0^\infty T_\lambda(s)x \, ds$. The right-hand side tends to $-x$ as $h \to 0$ by (1.4), so that $R(\lambda)x$ belongs to $D(A - \lambda I) = D(A)$ and satisfies $(\lambda I - A)R(\lambda)x = x$.

Let $x \in D(A)$. Proposition 1.10 says that $T(s)Ax = AT(s)x$ for $s \geq 0$, and $A$ is closed due to Proposition 1.19. Using also Remark 1.15 d), we deduce
\[
R(\lambda)(\lambda I - A)x = \lim_{t \to \infty} \int_0^t e^{-\lambda s}T(s)(\lambda I - A)x \, ds = \lim_{t \to \infty} (\lambda I - A) \int_0^t e^{-\lambda s}T(s)x \, ds
\]
\[
= (\lambda I - A) \lim_{t \to \infty} \int_0^t e^{-\lambda s}T(s)x \, ds = (\lambda I - A)R(\lambda)x = x.
\]
Hence, part a) is shown.

e) Let $x \in X$. Fix a number $\omega \in (\omega_0(T), \Re \lambda)$. It follows $\|e^{-\lambda s}T(s)x\| \leq Me^{(\omega - \Re \lambda)s}$ for some $M \geq 1$ and all $s \geq 0$. For $0 < a < b$ we can thus estimate
\[
\left\| \int_0^b T_\lambda(s)x \, ds - \int_0^a T_\lambda(s)x \, ds \right\| \leq \int_a^b \|T_\lambda(s)x\| \, ds \leq M\|x\| \int_a^b e^{(\omega - \Re \lambda)s} \, ds \to 0
\]
as $a, b \to \infty$. Consequently, $\int_0^t T_\lambda(s)x \, ds$ converges (absolutely) in $X$ as $t \to \infty$ for all $x \in X$, and thus assertion b) follows from a).

c) Let $n \in \mathbb{N}$, $x \in X$, and $t \geq 0$. Arguing as in Analysis 2, one can differentiate
\[
\left( \frac{d}{d\lambda} \right)^{n-1} \int_0^t e^{-\lambda s}T(s)x \, ds = \int_0^t (-1)^{n-1}s^{n-1}e^{-\lambda s}T(s)x \, ds.
\]
As in part b), the integrals converge as $t \to \infty$ uniformly for $\Re \lambda \geq \omega + \varepsilon$ and any $\varepsilon > 0$. Hence, (1.6) and a variant of Remark 1.15 g) imply
\[
R(\lambda, A)^n x = \frac{(-1)^{n-1}}{(n-1)!} \lim_{t \to \infty} \left( \frac{d}{d\lambda} \right)^{n-1} \int_0^t T_\lambda(s)x \, ds
\]
\[
= \lim_{t \to \infty} \frac{1}{(n-1)!} \int_0^t s^{n-1}T_\lambda(s)x \, ds = \frac{1}{(n-1)!} \int_0^\infty s^{n-1}e^{-\lambda s}T(s)x \, ds.
\]
Computing an elementary integral, one can now estimate
\[
\|R(\lambda, A)^n x\| \leq \frac{M\|x\|}{(n-1)!} \int_0^\infty s^{n-1}e^{(\omega - \Re \lambda)s} \, ds = \frac{M}{(\Re \lambda - \omega)^n} \|x\|
\]
for all $\Re \lambda > \omega$ since $\varepsilon$ is arbitrary. \qed
We calculate the generators of the translation semigroups from Example 1.8 and discuss their spectra. They turn out to be the first derivative endowed with appropriate domains. We also use the above necessary conditions to show that on certain domains the first derivative fails to be a generator.

**Example 1.21.** a) Let \( T(t)f = f(\cdot + t) \) be the translation group on \( X = C_0(\mathbb{R}) \). We compute the generator \( A \) and its spectrum.

1) Below we use that a function \( g \in X \) is uniformly continuous since \( C_c(\mathbb{R}) \) is dense in \( X \) and uniform continuity is preserved by uniform limits.

For \( f \in D(A) \), \( t \neq 0 \) and \( s \in \mathbb{R} \), there exist the pointwise limits

\[
A f(s) = \lim_{t \to 0} \frac{1}{t} (T(t)f(s) - f(s)) = \lim_{t \to 0} \frac{1}{t} (f(s + t) - f(s)) = f'(s)
\]

so that \( f \) is differentiable with \( f' = Af \in C_0(\mathbb{R}) \). We have shown the inclusion

\[
D(A) \subseteq C_0^1(\mathbb{R}) := \{ f \in C^1(\mathbb{R}) \mid f, f' \in X \}.
\]

Conversely, let \( f \in C_0^1(\mathbb{R}) \). For \( s \in \mathbb{R} \), we compute

\[
| \frac{1}{t} (T(t)f(s) - f(s)) - f'(s) | = \left| \frac{1}{t} (f(s + t) - f(s)) - f'(s) \right|
\]

\[
= \left| \int_0^t (f'(s + \tau) - f'(s)) \, d\tau \right|
\]

\[
\leq \sup_{0 \leq |\tau| \leq |t|} | f'(s + \tau) - f'(s) |.
\]

The right-hand side tends to 0 as \( t \to 0 \) uniformly in \( s \in \mathbb{R} \) since \( f' \in C_0(\mathbb{R}) \) is uniformly continuous. This means that \( f \) belongs to \( D(A) \), and so we obtain

\[
A = \frac{d}{ds} \text{ with } D(A) = C_0^1(\mathbb{R}).
\]

2) For the spectrum, we let \( \mathcal{F} = \mathbb{C} \). In Theorem 1.29 we will see that \( A \) generates the \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \) and \( -A \) is the generator of \( (S(t))_{t \geq 0} = (T(-t))_{t \geq 0} \). Proposition 1.20 yields the inequalities \( s(A) \leq \omega_0(A) = 0 \) and \( s(-A) \leq 0 \). Observing \( -(\lambda I - (-A)) = -\lambda I - A \), we conclude \( \sigma(-A) = -\sigma(A) \) as well as \( -R(\lambda, -A) = R(-\lambda, A) \). So we have proven the inclusion \( \sigma(A) \subseteq i\mathbb{R} \).

To show the converse, let \( \lambda \in \mathbb{C}_+ \), \( f \in X \), and \( s \in \mathbb{R} \). Since all of the following limits exist with respect to the supremum norm in \( s \), Proposition 1.20 yields

\[
(R(\lambda, A)f)(s) = \left( \lim_{b \to \infty} \int_0^b e^{-\lambda t} T(t)f(t) \, dt \right)(s) = \lim_{b \to \infty} \int_0^b e^{-\lambda t} T(t)f(t)(s) \, dt
\]

\[
= \lim_{b \to \infty} \int_0^b e^{-\lambda t} f(t + s) \, dt = \lim_{b \to \infty} \int_s^{b+s} e^{\lambda(s-\tau)} f(\tau) \, d\tau
\]

\[
= \int_s^{\infty} e^{\lambda(s-\tau)} f(\tau) \, d\tau.
\]

We pick functions \( \varphi_n \in C_c(\mathbb{R}) \) with \( 0 \leq \varphi_n \leq 1 \) and \( \varphi_n = 1 \) on \( [0, n] \) for \( n \in \mathbb{N} \), and set \( \alpha = \text{Re} \lambda > 0 \), \( \beta = \text{Im} \lambda \), as well as \( f_n(\tau) = e^{ib\tau} \varphi_n(\tau) \). Since \( \|f_n\|_\infty = 1 \), the above formula leads to the lower bound

\[
\|R(\lambda, A)\| \geq \|R(\lambda, A)f_n\|_\infty \geq |R(\lambda, A)f_n(0)| = \left| \int_0^{\infty} e^{-\alpha \tau} e^{-ib\tau} f_n(\tau) \, d\tau \right|
\]

\[
= \int_0^{\infty} e^{-\alpha \tau} \varphi_n(\tau) \, d\tau \geq \int_0^{n} e^{-\alpha \tau} \, d\tau = \frac{1 - e^{-\alpha n}}{\alpha}.
\]
Letting $n \to \infty$, we arrive at $\| R(\lambda, A) \| \geq \frac{1}{\Re \lambda}$. Proposition 1.20 then yields the equality $\| R(\lambda, A) \| = \frac{1}{\Re \lambda}$ (take $M = 1$, $\omega = 0$, and $n = 1$ there). If $i\beta$ belonged to $\rho(A)$ for some $\beta \in \mathbb{R}$, then we would infer

$$\frac{1}{\alpha} = \| R(\alpha + i\beta, A) \| \to \| R(i\beta, A) \|$$

as $\alpha \to 0$, which is impossible. We thus obtain $\sigma(A) = i\mathbb{R}$.

b) We treat the nilpotent left translation semigroup on $X = C_0([0, 1])$; i.e.,

$$(T(t)f)(s) = \begin{cases} f(s + t), & s + t < 1, \\ 0, & s + t \geq 1, \end{cases}$$

for $f \in X$, $t \geq 0$ and $s \in [0, 1)$. Let $A$ be its generator. Take $f \in D(A)$. As in part a), one shows that the right derivative $\frac{d}{dt}f$ exists and $\frac{d}{dt}f = Af$.

(Here we can only consider $t \to 0^+$. ) However, since $f$ and $Af$ are continuous, Corollary 2.1.2 of [Pa] says that $f \in C^1([0, 1])$, and so we have the inclusion

$$D(A) \subseteq C_0^1([0, 1]) := \{ f \in C^1([0, 1]) \mid f, f' \in X \}$$

as well as $Af = f'$. Let $f \in C_0^1([0, 1])$ and note that its 0-extension $\tilde{f}$ to $\mathbb{R}_{\geq 0}$ belongs to $C_0^1(\mathbb{R}_{\geq 0})$ and has compact support. As in part a), it follows

$$\frac{1}{t}(T(t)f(s) - f(s)) = \begin{cases} \frac{1}{t}(f(s + t) - f(s)), & 0 \leq s < 1 - t, \\ -\frac{1}{t}f(s), & 1 - t \leq s < 1, \end{cases}$$

$$= \frac{1}{t}(\tilde{f}(s + t) - \tilde{f}(s)) \to \tilde{f}'(s) = f'(s)$$

as $t \to 0^+$ uniformly in $s \in [0, 1)$, since $\tilde{f}'$ is uniformly continuous. Hence, $D(A) = C_0^1([0, 1])$ and $Af = f'$. Because of $\omega_0(A) = -\infty$, Proposition 1.20 yields $\sigma(A) = \emptyset$ and $\rho(A) = \mathbb{F}$.

c) The operator $Af = f'$ with $D(A) = C^1([0, 1])$ on $X = C([0, 1])$ has the spectrum $\sigma(A) = \mathbb{F}$. In fact, for each $\lambda \in \mathbb{F}$ the function $t \mapsto e_{\lambda}(t) := e^{\lambda t}$ belongs to $D(A)$ with $Ae_{\lambda} = \lambda e_{\lambda}$ so that even $\lambda \in \sigma_p(A)$. Hence, $A$ is not a generator in view of Proposition 1.20.

d) Let $X = C_0(\mathbb{R}_{\leq 0}) := \{ f \in C(\mathbb{R}_{\leq 0}) \mid f(s) \to 0 \text{ as } s \to -\infty \}$ and $A = \frac{d}{ds}$ with $D(A) = C_0^1(\mathbb{R}_{\leq 0}) := \{ f \in C^1(\mathbb{R}_{\leq 0}) \mid f, f' \in X \}$. Then $A$ is not a generator. Indeed, for all $\lambda \in \mathbb{F}_+$ we have $e_{\lambda} \in D(A)$ and $Ae_{\lambda} = \lambda e_{\lambda}$ so that $\lambda \in \sigma(A)$, violating $\sigma(A) < \infty$ in Proposition 1.20.

e) On $X = C([0, 1])$ the map $A = \frac{d}{dt}$ with $D(A) = \{ f \in C^1([0, 1]) \mid f(1) = 0 \}$ is not a generator as $D(A) = \{ f \in X \mid f(1) = 0 \} \neq X$, cf. Proposition 1.19. ♦

We stress that in parts c) and d) one does not impose conditions at the upper boundary of the spatial interval, as needed for a left translation, in contrast to a) and b). This lack of boundary conditions leads to spectral properties of $A$ ruling out that it is a generator. We will come back to this point in Example 1.36.
1.2. Characterization of generators

Proposition 1.19 and 1.20 contain necessary conditions to be a generator. In this section we want to show their sufficiency. This is the content of Hille–Yosida Theorem 1.26 which is the core of the theory of $C_0$-semigroups. Our approach is based on the so-called Yosida approximations which are defined by

$$A_λ := λAR(λ, A) = λ^2R(λ, A) − λI ∈ B(X).$$

(1.13)

for $λ ∈ ρ(A)$. Here we note the basic identities

$$AR(λ, A) = λR(λ, A) − I \text{ and } AR(λ, A)x = R(λ, A)Ax$$

(1.14)

for $x ∈ D(A)$. The next lemma is stated in somewhat greater generality than needed later on. In view of Proposition 1.19 and 1.20, for a generator $A$ it says that the bounded operators $A_λ$ approximate $A$ strongly on $D(A)$ as $λ → ∞$.

LEMMA 1.22. Let $A$ be a closed operator satisfying $(ω, ∞) ⊆ ρ(A)$ and $∥R(λ, A)∥ ≤ \frac{M}{λ}$ for some $M ≥ 1$ and $ω ∈ ℝ$ and all $λ > ω$. As $λ → ∞$, we then have

$$∀ x ∈ \overline{D(A)} : \ λR(λ, A)x → x,$$

$$∀ y ∈ D(A) \text{ with } Ay ∈ \overline{D(A)} : \ λAR(λ, A)y → Ay.$$

PROOF. Let $x ∈ D(A)$ and $λ ≥ ω + 1$. The assumption and (1.14) yield

$$∥λR(λ, A)∥ ≤ M \frac{λ}{λ − ω} ≤ M \max\{|ω + 1|, 1\},$$

$$∥λR(λ, A)x − x∥ = ∥R(λ, A)Ax∥ ≤ \frac{M}{λ − ω}∥Ax∥ → 0, \ λ → ∞.$$  

By density, the first assertion follows. Taking $x = Ay$ and using (1.14), one then deduces the second assertion from the first one. □

For linear operators $A, B$ on $X$ we write $A ⊆ B$ if $\text{Gr}(A) ⊆ \text{Gr}(B)$; i.e., if $D(A) ⊆ D(B)$ and $Ax = Bx$ for all $x ∈ D(A)$. In this case we call $B$ an extension of $A$. Equality of $A$ and $B$ is then often shown by means of the next observation, requiring that $D(A)$ is not ‘too small’ and $D(B)$ is not ‘too large’.

LEMMA 1.23. Let $A$ and $B$ be linear operators with $A ⊆ B$ such that $A$ is surjective and $B$ is injective. We then have $A = B$. In particular, $A$ and $B$ are equal if they satisfy $A ⊆ B$ and $ρ(A) ∩ ρ(B) ≠ ∅$.

PROOF. We have to prove the inclusion $D(B) ⊆ D(A)$. Let $x ∈ D(B)$. By the assumptions, there is a vector $y ∈ D(A)$ with $Bx = Ay = By$. Since $B$ is injective, we obtain $x = y$ so that $x$ belongs to $D(A)$.

Let $λ ∈ ρ(A) ∩ ρ(B)$. The first part then shows the equality $λI − A = λI − B$, and hence $A = B$. □

We introduce a class of $C_0$-semigroups which is easier to handle in many respects, cf. Theorem 1.39.

DEFINITION 1.24. Let $ω ∈ ℝ$. An $ω$-contraction semigroup is a $C_0$-semigroup $T(·)$ satisfying $∥T(t)∥ ≤ e^{ωt}$ for all $t ≥ 0$. Such a semigroup is also said to be quasi-contractive. If $ω = 0$, we call $T(·)$ a contraction semigroup.
We first discuss this concept and its relation to the exponential bound from Lemma 1.4, also noting the dependence on the choice of the norm on $X$.

**Remark 1.25.** a) Let $T(\cdot)$ be a contraction semigroup. Then the norm of the orbit $t \mapsto T(t)x$ is non-increasing since
\[ \|T(t)x\| = \|T(t-s)T(s)x\| \leq \|T(s)x\| \]
for $x \in X$ and $t \geq s \geq 0$. This fact is important since often $\|x\|$ is related significant quantities in applications, e.g., the energy of the state $x$.

b) Let $A \in \mathcal{B}(X)$. Estimating the power series in Example 1.3, we derive $\|e^{tA}\| \leq e^{\|A\|}$; i.e., $A$ generates a $\|A\|$-contractive semigroup. However, its growth bound $\omega_0(A)$ is possibly much smaller than $\|A\|$ by Remark 1.6 d).

c) There are unbounded generators $A$ of a $C_0$-semigroup having norms $\|T(t)\| \geq M$ for all $t > 0$ and some $M > 1$. Hence, they cannot be $\omega$-contractive for any $\omega \in \mathbb{R}$. As an example, let $X = C_0(\mathbb{R})$ be endowed with the norm
\[ \|f\| = \max\{ \sup_{s \geq 0} |f(s)|, M \sup_{s < 0} |f(s)| \} \]
for some $M > 1$, which is equivalent to the supremum norm. The translations $T(t)f = f(\cdot + t)$ thus yield a $C_0$-semigroup on $(X, \|\cdot\|)$. Take any $t > 0$. Choose a function $f \in C_0(\mathbb{R})$ with $\|f\|_\infty = 1$ and supp $f \subseteq (0, t)$. We then obtain $\|f\| = 1$, supp $T(t)f \subseteq (-t, 0)$, and so
\[ \|T(t)\| \geq \|T(t)f\| = M \sup_{-t \leq s \leq 0} |f(s + t)| = M. \]
Since $\|T(t)\| \leq M$, we actually have $\|T(t)\| = M$ for all $t > 0$.

d) However, for each $C_0$-semigroup $T(\cdot)$ on a Banach space $X$ one can find an equivalent norm on $X$ for which $T(\cdot)$ becomes $\omega$-contractive. Indeed, take numbers $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq M e^{\omega t}$ for all $t \geq 0$. We set
\[ \|\|x\|| = \sup_{s \geq 0} e^{-\omega s} \|T(s)x\| \]
for $x \in X$, which defines an equivalent norm since $\|x\| \leq \|\|x\|| \leq M \|x\|$. We further obtain
\[ \|e^{-\omega t}T(t)x\| = \sup_{s \geq 0} e^{-\omega(s+t)} \|T(s+t)x\| \leq \|\|x\|| \]
so that $T(\cdot)$ is $\omega$-contractive for this norm. However, this renorming can destroy additional properties as the Hilbert space structure, and in general one cannot do it for two $C_0$-semigroups at the same time. See Remark I.2.19 in [Go]. ⊢

The following major theorem characterizes the generators of $C_0$-semigroups. It was shown in the contraction case independently by Hille and Yosida in 1948. Yosida’s proof extends very easily to the general case and is presented below. As we see in Theorem 2.2, the generator property of $A$ is equivalent to ‘wellposedness’ of (1.1). In other words, the Hille–Yosida Theorem describes the class of operators for which (1.1) is solvable in a reasonable sense. It is thus the fundament of the theory of linear evolution equations, which is actually concerned with many topics beyond wellposedness – below we treat regularity, perturbation, approximation, and long-time behavior, for instance.
Theorem 1.26. Let $M \geq 1$ and $\omega \in \mathbb{R}$. A linear operator $A$ generates a $C_0$-semigroup on $X$ satisfying $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ if and only if

$A$ is closed, $\overline{D(A)} = X$, $(\omega, \infty) \subseteq \rho(A), \quad \forall n \in \mathbb{N}, \lambda > \omega : \|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n}$. (1.15)

In this case, if $\mathbb{F} = \mathbb{C}$ one even has $\mathbb{C}_\omega = \{\lambda \in \mathbb{C} \mid \text{Re} \lambda > \omega\} \subseteq \rho(A)$ and

$\forall n \in \mathbb{N}, \lambda \in \mathbb{C}_\omega : \|R(\lambda, A)^n\| \leq \frac{M}{(\text{Re} \lambda - \omega)^n}$. (1.16)

The operator $A$ generates an $\omega$-contraction semigroup if and only if

$A$ closed, $\overline{D(A)} = X$, $(\omega, \infty) \subseteq \rho(A), \quad \forall \lambda > \omega : \|R(\lambda, A)\| \leq \frac{1}{\lambda - \omega}$. (1.17)

In this case (1.16) is true with $M = 1$, if $\mathbb{F} = \mathbb{C}$.

In applications it is of course much easier check the assumptions in the quasi-contractive case. Based on the above result, Theorem 1.39 will provide another, even more convenient characterization of generators in this case.

Proof of Theorem 1.26. It is clear (1.17) implies (1.15) for $M = 1$. Propositions 1.19 and 1.20 imply (1.16) and the necessity of (1.15), respectively (1.17). If (1.15) is true, then the shifted operator $A - \omega I$ satisfies (1.15) with ‘$\omega = 0$.’ Below we show that $A - \omega I$ generates a bounded semigroup. Lemma 1.17 then yields the assertion.

We establish the sufficiency of (1.15) in two steps. We first use the semigroups $e^{tA_n}$ generated by the (bounded) Yosida approximations $A_n = n^2 R(n, A) - nI$ for $n \in \mathbb{N}$ and prove that they converge to a $C_0$-semigroup $T(\cdot)$ as $n \to \infty$. In a second step we show that it is generated by $A$.

1) Let (1.15) be true with $\omega = 0$. Take $n, m \in \mathbb{N}$ and $t \geq 0$. Employing Lemma 1.17, the powers series representation of $e^{tA_n}$ in Example 1.3 and (1.15), we estimate

$$
\|e^{tA_n}\| = \|e^{-tn}e^{n^2R(n,A)t}\| \leq e^{-tn}\sum_{j=0}^{\infty} \frac{(nt)^j}{j!}\|R(n,A)\|^j \leq Me^{-tn}\sum_{j=0}^{\infty} \frac{(nt)^j}{j!} = M.
$$

(1.18)

We further have $A_nA_m = A_mA_n$ and hence

$$
A_ne^{tA_m} = A_n \sum_{j=0}^{\infty} \frac{t^j}{j!}A_m^j = \sum_{j=0}^{\infty} \frac{t^j}{j!}A_m^jA_n = e^{tA_m}A_n.
$$

Take $t_0 > 0$, $y \in D(A)$, and $t \in [0, t_0]$. Using (1.3), we compute

$$
e^{tA_n}y - e^{tA_m}y = \int_0^t \frac{d}{ds}e^{(t-s)A_m}e^{sA_n}y \, ds = \int_0^t e^{(t-s)A_m}e^{sA_n}(A_n - A_m)y \, ds.
$$

Estimate (1.18) and Lemma 1.22 then lead to the limit

$$
\|e^{tA_n}y - e^{tA_m}y\| \leq t_0M^2\|A_ny - A_my\| \to 0
$$

(1.19)
as \( n, m \to \infty \). Because of the density of \( D(A) \) and the bound (1.18), we can apply Lemma 4.10 of [FA]. Since \( t_0 > 0 \) is arbitrary, it yields operators \( T(t) \) in \( \mathcal{B}(X) \) such that \( e^{tA_n}x \to T(t)x \) as \( n \to \infty \) and \( \|T(t)\| \leq M \) for all \( t \geq 0 \) and \( x \in X \). Clearly, \( T(0) = I \) and

\[
T(t+s)x = \lim_{n \to \infty} e^{(t+s)A_n}x = \lim_{n \to \infty} e^{tA_n}e^{sA_n}x = T(t)T(s)x
\]

for all \( t, s \geq 0 \) (use Remark 1.13). Letting \( m \to \infty \) in (1.19), we further deduce

\[
\|e^{tA_n}y - T(t)y\| \leq t_0 M^2 \|A_n y - Ay\|
\]

for all \( t \in [0, t_0] \). This means that \( e^{tA_n}y \) converges to \( T(t)y \) uniformly for \( t \in [0, t_0], \) and hence \( T(\cdot)y \) is continuous for all \( y \in D(A) \). Lemma 1.7 and the density of \( D(A) \) now imply that \( T(\cdot) \) is a (bounded) \( C_0 \)-semigroup.

2) Let \( B \) be the generator of \( T(\cdot) \). We have \( \mathbb{R}_+ \subseteq \rho(A) \cap \rho(B) \) by Proposition 1.20 and the assumptions. In view of Lemma 1.23 it thus remains to show \( A \subseteq B \). For \( t > 0 \) and \( y \in D(A) \), Lemma 1.18 and Remarks 1.13 and 1.15 e) yield

\[
\frac{1}{t}(T(t)y - y) = \lim_{n \to \infty} \frac{1}{t}(e^{tA_n}y - y) = \lim_{n \to \infty} \frac{1}{t} \int_0^t e^{sA_n}A_n y \, ds = \frac{1}{t} \int_0^t T(s)Ay \, ds.
\]

As \( t \to 0 \), from (1.4) we conclude that \( y \in D(B) \) and \( By = Ay \); i.e., \( A \subseteq B \). \( \square \)

We illustrate the above theorem by some examples. Applications to more complicated partial differential operators will be discussed in Section 1.4.

**Example 1.27.** a) Let \( X = C_0(\mathbb{R}_{\leq 0}) \) with \( \mathbb{F} = \mathbb{C} \) and \( A = -\frac{d^2}{dx^2} \) with \( D(A) = C^1(\mathbb{R}_{\leq 0}) \), cf. Example 1.21. Then \( A \) generates the \( C_0 \)-semigroup given by \( B(t)f = f(\cdot - t) \) for \( t \geq 0 \) and \( f \in X \). It has the spectrum \( \sigma(A) = \mathbb{C}^- \).

**Proof.** We first check in several steps the conditions (1.17).

1) Let \( f \in X \) and \( \varepsilon > 0 \). We extend \( f \) to a function \( \tilde{f} \in C_0(\mathbb{R}) \). As in Example 1.8 one finds a map \( \tilde{g} \in C_c(\mathbb{R}) \) with \( \|\tilde{f} - \tilde{g}\|_\infty \leq \varepsilon \). By the proof of Proposition 4.13 in [FA] there is function \( \tilde{h} \in C_c(\mathbb{R}) \) with \( \|\tilde{g} - \tilde{h}\|_\infty \leq \varepsilon \). The restriction \( h \) of \( \tilde{h} \) to \( \mathbb{R}_{\leq 0} \) thus belongs to \( D(A) \) and satisfies \( \|f - h\|_\infty \leq 2\varepsilon \), so that \( A \) is densely defined.

2) Let the sequence \( (u_n) \) in \( D(A) \) tend in \( X \) to a function \( u \), and \( (Au_n) \) to some \( f \) in \( X \). The map \( u \) is thus differentiable with \( -u' = f \in X \). As a result \( u \in D(A) \) and \( Au = f \); i.e., \( A \) is closed.

3) Let \( f \in X \) and \( \lambda > 0 \). To show the bijectivity of \( \lambda I - A \), we note that a function \( u \) belongs to \( D(A) \) and satisfies \( \lambda u - Au = f \) if and only if

\[
u' = -\lambda u + f \quad \text{and} \quad u \in C^1(\mathbb{R}_{\leq 0}) \cap X
\]

(using that then \( u' = -\lambda u + f \in X \)). This condition is equivalent to

\[
u \in C^1(\mathbb{R}_{\leq 0}) \cap X, \quad \forall t_0 \leq s \leq 0 : \quad u(s) = e^{-\lambda(s-t_0)}u(t_0) + \int_{t_0}^s e^{-\lambda(s-\tau)}f(\tau) \, d\tau.
\]

Since \( u \) and \( f \) are bounded and \( \lambda > 0 \), here one can let \( t_0 \to -\infty \) and derive

\[
u(s) = \int_{-\infty}^s e^{-\lambda(s-\tau)}f(\tau) \, d\tau =: R(\lambda)f(s) \quad \text{for all} \ s \leq 0, \quad \lim_{s \to -\infty} \nu(s) = 0.
\]
Conversely, if the function \( v := R(\lambda)f \) belongs to \( X \), a direct calculation shows that it is an element of \( D(A) \) and satisfies \( \lambda v - Av = f \).

We now show \( R(\lambda)f \in X \), where the continuity is clear. Let \( \varepsilon > 0 \). There is a number \( s_\varepsilon \leq 0 \) with \( |f(\tau)| \leq \varepsilon \) for all \( \tau \leq s_\varepsilon \). For \( s \leq s_\varepsilon \) we then estimate

\[
|R(\lambda)f(s)| \leq \int_{-\infty}^{s} e^{-\lambda(s-\tau)}|f(\tau)|\, d\tau \leq \varepsilon \int_{0}^{\infty} e^{-\lambda r} \, dr = \frac{\varepsilon}{\lambda},
\]

substituting \( r = s - \tau \). As a result, \( R(\lambda)f(s) \) tends to 0 as \( s \to -\infty \), and so \( \lambda \) is contained in \( \rho(A) \) and \( R(\lambda) = R(\lambda, A) \).

4) Employing the above formula for the resolvent, we calculate

\[
\|R(\lambda, A)f\|_\infty \leq \sup_{s \leq 0} \int_{s}^{0} e^{-\lambda(s-\tau)}\|f\|_\infty \, d\tau = \|f\|_\infty \int_{0}^{\infty} e^{-\lambda r} \, dr = \frac{\|f\|_\infty}{\lambda}
\]

for all \( f \in X \) and \( \lambda > 0 \). Theorem 1.26 now implies that \( A \) generates a contraction semigroup \( T(\cdot) \). In particular, \( \sigma(A) \) is contained in \( \mathbb{C}_- \). For \( \lambda \in \mathbb{C}_- \), the function \( e^{-\lambda} \) belongs to \( D(A) \) and satisfies \( Ae_{-\lambda} = -e^{-\lambda} = \lambda e_{-\lambda} \) so that \( \mathbb{C}_- \subseteq \sigma(A) \). The closedness of \( \sigma(A) \) then implies the second assertion.

5) To determine \( T(\cdot) \), we take \( \varphi \in D(A) \). We set \( u(t, s) = (u(t))(s) = (T(t)\varphi)(s) \) and for \( t \geq 0 \) and \( s \leq 0 \). By Proposition 1.10, the function \( u \) belongs to \( C^1(\mathbb{R}_\geq 0, X) \cap C(\mathbb{R}_\geq 0, [D(A)]) \) and solves the problem

\[
\frac{\partial}{\partial t} u(t, s) = -\partial_s u(t, s), \quad t \geq 0, \quad s \leq 0,
\]

\[
u(0, s) = \varphi(s), \quad s \leq 0.
\]

(Note that \( D(A) \) includes the ‘boundary condition’ \( u(t, s) \to 0 \) as \( s \to -\infty \).) It is straightforward to see that via \( v(t, s) = \varphi(s-t) \) one defines another solution in the same function spaces. The uniqueness statement in Proposition 1.10 then yields \( u = v \) and hence \( T(t)\varphi = \varphi(\cdot - t) \) for all \( t \geq 0 \). The last claim now follows from the density of \( D(A) \). □

b) We provide an operator \( A \) which fulfills (1.15) for \( n = 1 \) and some \( M > 1 \), but which is not generator. So one cannot omit the powers \( n \) in (1.15).

Let \( X = C_0(\mathbb{R})^2 \) with \( F = \mathbb{C} \), \( \|(f, g)\| = \max\{\|f\|_\infty, \|g\|_\infty\} \), \( m(s) = is \), and

\[
A \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} m u + mv \\ mv \end{pmatrix} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}
\]

for \( (u, v) \in D(A) = \{(u, v) \in X \mid (mu, mv) \in X\} \).

Since \( C_c(\mathbb{R}) \times C_c(\mathbb{R}) \subseteq D(A) \), the domain \( D(A) \) is dense in \( X \). Take \( (u_n, v_n) \) in \( D(A) \) such that \( (u_n, v_n) \to (u, v) \) and \( A(u_n, v_n) \to (f, g) \) in \( X \) as \( n \to \infty \). By pointwise limits, we infer that \( mu_n + mv_n = f \) and \( mv_n = g \in C_c(\mathbb{R}) \), so that also \( mu_n \in C_0(\mathbb{R}) \). As a result, the vector \( (u, v) \) belongs to \( D(A) \) and \( A \) is closed.

Let \( \lambda \in \mathbb{C}_+ \). Since \( 1/(\lambda - m) \) and \( m/(\lambda - m) \) are bounded, the operator

\[
R(\lambda) = \begin{pmatrix} 1/\lambda - m & m/(\lambda - m)^2 \\ 0 & 1/\lambda - m \end{pmatrix}
\]

maps \( X \) into \( D(A) \). We further compute

\[
(\lambda I - A)R(\lambda) = \begin{pmatrix} \lambda - m & -m \\ 0 & \lambda - m \end{pmatrix} \begin{pmatrix} 1/\lambda - m & m/(\lambda - m)^2 \\ 0 & 1/\lambda - m \end{pmatrix} = I,
\]
and similarly \( R(\lambda)(\lambda w - Aw) = w \) for \( w \in D(A) \). So we have shown that \( C_+ \subseteq \rho(A) \) and \( R(\lambda) = R(\lambda, A) \).

For \( \lambda > 0 \) and \( \|(f, g)\| \leq 1 \) we next estimate
\[
\|R(\lambda, A) \left( \begin{array}{c} f \\ g \end{array} \right) \| \leq \max \left\{ \| \frac{f}{\lambda - m} \|_{\infty} + \| \frac{mg}{(\lambda - m)^2} \|_{\infty}, \| \frac{g}{\lambda - m} \|_{\infty} \right\}
\leq \sup_{s \in \mathbb{R}} \left( \frac{1}{|\lambda - is|} + \frac{|s|}{|\lambda - is|^2} \right) \leq \frac{1}{\lambda} + \sup_{s \in \mathbb{R}} \frac{|s|}{\lambda^2 + s^2} = \frac{3/2}{\lambda}.
\]

On the other hand, for \( a > 0 \) and \( n \in \mathbb{N} \) we choose \( g_n \in C_0(\mathbb{R}) \) such that \( g_n(n) = 1 \) and \( \|g_n\|_{\infty} = 1 \). It then follows
\[
\|R(a + in, A)\| \geq \|R(a + in, A) \left( \begin{array}{c} 0 \\ g_n \end{array} \right) \| \geq \| \frac{m}{(a + in - m)^2} g_n \|_{\infty}
\geq \left| \frac{in}{(a + in - in)^2} g_n(n) \right| = \frac{n}{a^2}.
\]
The resolvent \( R(\lambda, A) \) is thus unbounded on every imaginary line \( \text{Re}\lambda = a \), violating Proposition 1.20c); i.e., \( A \) does not generate a \( C_0 \)-semigroup.

There are operators satisfying even \( \|R(\lambda, A)\| \leq \frac{c}{\text{Re}(\lambda)} \) for some \( c > 1 \) and all \( \lambda \in C_+ \) which fail to be a generator (see Example 2 in §12.4 of [HP]).

We now turn our attention to the generation of groups. We will reduce this question to the semigroup case, using the following simple fact.

**Lemma 1.28.** Let \( T(\cdot) \) be a \( C_0 \)-semigroup and \( t_0 > 0 \) such that \( T(t_0) \) is invertible. Then \( T(\cdot) \) can be extended to a \( C_0 \)-group \( (T(t))_{t \in \mathbb{R}} \).

**Proof.** Take constants \( M \geq 1 \) and \( \omega \in \mathbb{R} \) with \( \|T(t)\| \leq Me^{\omega t} \) for all \( t \geq 0 \). Set \( c = \|T(t_0)^{-1}\| \). Let \( 0 \leq t \leq t_0 \). We then compute
\[
T(t_0) = T(t_0 - t)T(t) = T(t)T(t_0 - t),
\]
\[
I = T(t_0)^{-1}T(t_0 - t)T(t) = T(t)T(t_0 - t)T(t_0)^{-1}.
\]
The operator \( T(t) \) thus has the inverse \( T(t_0)^{-1}T(t_0 - t) \) with norm less than or equal to \( M_1 := cMe^{\omega t_0} \). Next, let \( n = nt_0 + \tau \) for some \( n \in \mathbb{N} \) and \( \tau \in [0, t_0) \). In this case \( T(t) = T(\tau)T(t_0)^n \) has the inverse \( T(t_0)^{-n}T(\tau)^{-1} \).

We now define \( T(t) := T(-t)^{-1} \) for \( t \leq 0 \). This definition gives a group, since for \( t, s \geq 0 \) we can calculate
\[
T(-t)T(-s) = T(t)^{-1}T(s)^{-1} = (T(s)T(t))^{-1} = T(s + t)^{-1} = T(-s - t),
\]
\[
T(-t)T(s) = (T(s)T(t - s))^{-1}T(s) = T(t - s)^{-1}T(s)^{-1}T(s) = T(t - s)^{-1} = T(s - t)
\quad \text{for } t \geq s,
\]
\[
T(-t)T(s) = T(t)^{-1}T(t)T(s - t) = T(s - t)
\quad \text{for } s \geq t,
\]
and similarly for \( T(s)T(-t) \). Let \( t \in [0, t_0] \) and \( x \in X \). We then obtain
\[
\|T(-t)x - x\| = \|T(-t)(x - T(t)x)\| \leq M_1\|x - T(t)x\| \to 0
\]
as \( t \to 0 \). So \( (T(t))_{t \in \mathbb{R}} \) is a \( C_0 \)-group by Lemma 1.7. \( \square \)
The next theorem characterizes generators of $C_0$-groups in the same way as in the Hille–Yosida Theorem 1.26, but now requiring resolvent bounds also for negative $\lambda$.

Moreover, $A$ generates the $C_0$-group $(T(t))_{t \in \mathbb{R}}$ if and only if $A$ and $-A$ generate the $C_0$-semigroups $(T(t))_{t \geq 0}$ and $(T(-t))_{t \geq 0}$, respectively.

**Theorem 1.29.** Let $A$ be a linear operator, $M \geq 1$, and $\omega \geq 0$. The following assertions are equivalent.

a) $A$ generates a $C_0$-group $(T(t))_{t \in \mathbb{R}}$ with $\|T(t)\| \leq Me^{\omega |t|}$ for all $t \in \mathbb{R}$.

b) $A$ generates a $C_0$-semigroup $(T_+(t))_{t \geq 0}$, and $-A$ with $D(-A) := D(A)$ generates a $C_0$-semigroup $(T_-^\pm(t))_{t \geq 0}$ with $\|T_\pm(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.

c) $A$ is closed, $\overline{D(A)} = X$, and for all $\lambda \in \mathbb{R}$ with $|\lambda| > \omega$ we have $\lambda \in \rho(A)$ and $\|(|\lambda| - \omega)^n R(\lambda, A)^n\| \leq M$ for all $n \in \mathbb{N}$.

If one (and thus all) of these conditions is (are) fulfilled, one has $T_+(t) = T(t)$ and $T_-(t) = T(-t)$ for every $t \geq 0$. Moreover, in part c) one can then replace ‘$\lambda \in \mathbb{R}$’ by ‘$\lambda \in \mathbb{C}$’ and ‘$|\lambda|$’ by ‘$\|\text{Re } \lambda\|$’ (provided that $\mathbb{F} = \mathbb{C}$).

**Proof.** 1) We first deduce statement b) from a). Assuming a), we set $T_+(t) = T(t)$ and $T_-(t) = T(-t)$ for each $t \geq 0$. Recall from Remark 1.2 that $T(-t) = T(t)^{-1}$. It is easy to check that one thus obtains two $C_0$-semigroups. We denote their generators by $A_+$.

For $x \in D(A)$, there exists $\frac{d}{dt} T(0)x = Ax$ implying $A \subseteq A_+$ and $A \subseteq -A_-$. To show the inverse inclusion, let $x \in D(A_-)$ and $t > 0$. We then compute

\[
\frac{1}{-t}(T(-t)x - x) = \frac{1}{-t}(T_-(t)x - x) \rightarrow -A_-x,
\]

\[
\frac{1}{t}(T(t)x - x) = -T(t)\frac{1}{t}T_-(t)x - x) \rightarrow -A_-x
\]
as $t \rightarrow 0$, so that $x \in D(A)$ and hence $A = -A_-$. One proves $A = A_+$ similarly. Therefore, property b) and the first addendum are true.

2) Let b) be valid. For $\lambda > \omega$, assertion c) follows from Theorem 1.26. For $\lambda < -\omega$, we use that $\sigma(-A) = -\sigma(A)$ with $R(-\lambda, -A) = -R(\lambda, A)$, cf. Example 1.21a). Theorem 1.26 thus also yields the estimate in part c) for $\lambda < -\omega$ since here $-\lambda = |\lambda|$. The second addendum is shown in the same way.

3) We assume claim c) and derive statement a). Theorem 1.26 implies that $A$ generates a $C_0$-semigroup $(T_+(t))_{t \geq 0}$ and $-A$ generates a $C_0$-semigroup $(T_-(t))_{t \geq 0}$ (arguing for $-A$ as in the previous step). Let $x \in D(A) = D(-A)$ and $t \geq s \geq 0$. Proposition 1.10 and its proof imply

\[
\frac{d}{ds}T_+(s)T_-(s)x = T_+(s)AT_-(s)x + T_+(s)(-A)T_-(s)x = 0
\]
and then $T_+(t)T_-(t)x = x$. Analogously, one obtains $T_-(t)T_+(t)x = x$. It follows that $I = T_+(t)T_-(t) = T_-(t)T_+(t)$ since $D(A)$ is dense. By Lemma 1.28, $T_+(\cdot)$ can thus be extended to a $C_0$-group. Let $B$ be its generator. We have $B \subseteq A$ by definition and $s(B) < \infty$ by step 1) and Proposition 1.20. Condition c) and Lemma 1.23 then yield $A = B$ and hence assertion a). \qed
1.3. Dissipative operators

Even in the contraction case, the Hille-Yosida Theorem 1.26 poses the difficult task to show a resolvent estimate for all $\lambda > 0$. In this section we prove the Lumer-Phillips Theorem 1.39 which reduces this task to checking the dissipativity and a certain range condition of $A$. The former property can often be verified by direct computations, and for the latter there are powerful (also functional analytic) tools to solve the occurring equations. Below these matters will be illustrated by the first derivative again, more involved applications will be treated in the following section.

We start with an auxiliary notion. The duality set $J(x)$ of a vector $x \in X$ is defined by

$$J(x) = \{ x^* \in X^* \mid \langle x, x^* \rangle = \| x \|^2, \| x \| = \| x^* \| \},$$

where $\langle x, x^* \rangle = x^*(x)$ for all $x \in X$ and $x^* \in X^*$. The Hahn-Banach theorem ensures that $J(x) \neq \emptyset$, cf. Corollary 5.10 in [FA]. In standard function spaces one can compute elements in the duality set explicitly.

**Example 1.30.** a) Let $X$ be a Hilbert space with scalar product $(\cdot, \cdot)$. By Riesz’ Theorem 3.10 in [FA], for each functional $y^* \in X^*$ there is a unique vector $y \in X$ satisfying $(x, y^*) = (x, y)$ for all $x \in X$, and one has $\| y \| = \| y^* \|$. As a result, $y^* \in J(x)$ is equivalent to $\| x \| = \| y \|$ and $(x, y) = (x, y)^* = \| x \|^2$, or to $\| x \| = \| y \|$ and $(x, y) = \| x \| \| y \|$. These conditions are valid if and only if $y = \alpha x$ for some $\alpha \in \mathbb{F}$ with $|\alpha| = 1$ (due to the characterization of equality in the Cauchy-Schwarz inequality). Inserting this expression in $(x, y) = \| x \|^2$, we see that $x = y$. The converse implication is clear. Consequently, $J(x) = \{ \varphi_x \}$ for the functional given by $\varphi_x(z) = (z|z)$.

b) Let $X = L^p(\mu)$ for an exponent $p \in [1, \infty)$ and a measure space $(S, \mathcal{A}, \mu)$ which has to be $\sigma$-finite if $p = 1$. We identify $X^*$ with $L^{p'}(\mu)$ via the usual duality pairing $\langle f, g \rangle_{L^p \times L^{p'}} = \int_S f g d\mu$, where $p' = \frac{p}{p-1}$ for $p > 1$ and $1' = \infty$, see Theorem 5.4 in [FA]. Take $f \in X \setminus \{0\}$. We set

$$g = \| f \|_2^{-p} |f|^{p-2}$$

writing $\frac{0}{0} := 0$. For $p = 1$, we have $\| g \|_{\infty} = \| f \|_1$. For $p > 1$, we compute

$$\| g \|_{p'} = \| f \|_2^{-p} \left( \int_S |f|^{(p-1) \cdot \frac{p}{p-1}} d\mu \right)^{\frac{p-1}{p}} = \| f \|_p^{2-p} \| f \|_p^{-1} = \| f \|_p.$$

Since also

$$\langle f, g \rangle = \| f \|_p^{2-p} \int_S f |f|^{p-2} d\mu = \| f \|_p^{2-p} \| f \|_p^{p-1} = \| f \|_p^2,$$

we obtain $g \in J(f)$. It follows from an exercise that $J(f) = \{ g \}$ if $p \in (1, \infty)$. Note that $g = \overline{f}$ for $p = 2$ which corresponds to part a).

c) Let $\emptyset \neq U \subseteq \mathbb{R}^m$ be open and $E = C_0(U)$ with

$$C_0(U) := \{ f \in C(U) \mid f(x) \to 0 \text{ as } x \to \partial U \text{ and as } |x| \to \infty \text{ for unbounded } U \},$$

which is a Banach space for the supremum norm. For $f \in E$ there is a point $x_0 \in U$ with $|f(x_0)| = \| f \|_{\infty}$. Set $\varphi(g) = \overline{f}(x_0) g(x_0)$ for $g \in E$; i.e., $\varphi = f(x_0) \delta_{x_0}$.
As in Example 2.8 of [FA] one checks that \( \varphi \in \mathcal{E}^* \) with \( \| \varphi \| = |f(x_0)| = \| f \|_\infty \). We clearly have \( \varphi(f) = |f(x_0)|^2 = \| f \|_\infty^2 \). Hence, \( \varphi \) belongs to \( J(f) \). The same construction works on \( E = C(K) \) for a compact metric space \( K \).

We now state the core concept of this section.

**Definition 1.31.** A linear operator \( A \) is called dissipative if for each vector \( x \in D(A) \) there is a functional \( x^* \in J(x) \) such that \( \operatorname{Re} \langle Ax, x^* \rangle \leq 0 \). The operator \( A \) is called accretive if \(-A\) is dissipative.

The next fundamental characterization provides the link between dissipativity and the resolvent condition (1.17) in the Hille–Yosida theorem. We also show that a generator of a contraction semigroup is dissipative in a somewhat stronger sense, which will be used in Chapter 3.

**Proposition 1.32.** A linear operator \( A \) is dissipative if and only if it satisfies 
\[
\| \lambda x - Ax \| \geq \lambda \| x \|
\]
for all \( \lambda > 0 \) and \( x \in D(A) \). If \( A \) generates a contraction semigroup, then we have \( \operatorname{Re} \langle Ax, x^* \rangle \leq 0 \) for every \( x \in D(A) \) and all \( x^* \in J(x) \).

**Proof.** 1) Let \( A \) generate the contraction semigroup \( T(\cdot) \). Take \( x \in D(A) \) and \( x^* \in J(x) \). Using \( x^* \in J(x) \) and the contractivity, we estimate
\[
\operatorname{Re} \langle Ax, x^* \rangle = \lim_{t \to 0^+} \frac{1}{t} \langle T(t)x - x, x^* \rangle = \lim_{t \to 0^+} \frac{1}{t} \left( \operatorname{Re} \langle T(t)x, x^* \rangle - \| x^* \|^2 \right) 
\leq \limsup_{t \to 0^+} \frac{1}{t} (\| x \| \| x^* \| - \| x \|^2) = 0.
\]

2) Let \( A \) be dissipative. Take \( x \in D(A) \) and \( \lambda > 0 \). There thus exists a functional \( x^* \in J(x) \) with \( \operatorname{Re} \langle Ax, x^* \rangle \leq 0 \). These facts imply the inequalities
\[
\lambda \| x \|^2 \leq \operatorname{Re} \langle \lambda x, x^* \rangle - \operatorname{Re} \langle Ax, x^* \rangle \leq \| \lambda x - Ax, x^* \| \leq \| \lambda x - Ax \| \| x^* \|. 
\]
Since \( \| x \| = \| x^* \| \), it follows \( \lambda \| x \| \leq \| \lambda x - Ax \| \).

3) Conversely, let \( \| \lambda x - Ax \| \geq \lambda \| x \| \) be true for all \( \lambda > 0 \) and \( x \in D(A) \). If \( x = 0 \) we can take \( x^* = 0 \) in the definition of dissipativity. Otherwise, we replace \( x \) by \( \| x \|^{-1} x \), and will thus assume that \( \| x \| = 1 \).

Take \( y_\lambda^* \in J(\lambda x - Ax) \). This functional is not zero since \( \| y_\lambda^* \| = \| \lambda x - Ax \| \geq \lambda \| x \| = \lambda > 0 \) by the assumptions. We now set \( x_\lambda^* = \| y_\lambda^* \|^{-1} y_\lambda^* \) and note that \( \| x_\lambda^* \| = 1 \). Using the assumptions again, we deduce
\[
\lambda \leq \| \lambda x - Ax \| = \frac{1}{\| y_\lambda^* \|} \langle \lambda x - Ax, y_\lambda^* \rangle = \operatorname{Re} \langle \lambda x - Ax, x_\lambda^* \rangle 
\leq \lambda \operatorname{Re} \langle x, x_\lambda^* \rangle - \operatorname{Re} \langle Ax, x_\lambda^* \rangle \leq \min \{ \lambda - \operatorname{Re} \langle Ax, x_\lambda^* \rangle, \lambda \operatorname{Re} \langle x, x_\lambda^* \rangle + \| Ax \| \}.
\]
This inequality implies the core bounds
\[
\operatorname{Re} \langle Ax, x_\lambda^* \rangle \leq 0 \quad \text{and} \quad 1 - \frac{1}{\lambda} \| Ax \| \leq \operatorname{Re} \langle x, x_\lambda^* \rangle.
\]
Let \( \tilde{x}_\lambda^* \) be the restriction of \( x_\lambda^* \) to the space \( E = \operatorname{lin} \{ x, Ax \} \) equipped with the norm of \( X \). Because of \( \| \tilde{x}_\lambda^* \| \leq \| x_\lambda^* \| = 1 \), the Bolzano–Weierstraß theorem yields a sequence \( (\lambda_j) \) in \( \mathbb{R}_+ \) and a vector \( y^* \in \mathcal{E}^* \) such that \( \lambda_j \to \infty \) and \( \tilde{x}_{\lambda_j}^* \to y^* \) as \( j \to \infty \). Applying these limits to the above estimates, we derive
\[
\| y^* \| \leq 1, \quad \operatorname{Re} \langle Ax, y^* \rangle \leq 0 \quad \text{and} \quad 1 \leq \operatorname{Re} \langle x, y^* \rangle.
\]
The Hahn-Banach theorem allows us to extend $y^*$ to a functional $x^* \in X^*$ with $\|x^*\| = \|y^*\| \leq 1$. It then satisfies $\text{Re}(Ax, x^*) \leq 0$ and

$$1 \leq \text{Re}(x, x^*) \leq |\langle x, x^* \rangle| \leq \|x^*\| \leq 1$$

since $\|x\| = 1$. So we have equalities in the above formula, which means that $\|x^*\| = 1 = \|x\|$ and $\langle x, x^* \rangle = 1 = \|x\|^2$; i.e., $x^* \in J(x)$ and $A$ is dissipative. □

The dissipativity of differential operators $A$ heavily depends on the boundary conditions, as we now discuss for first-order operators on an interval.

Example 1.33. a) Let $X = C_0(\mathbb{R})$, $b,c \in C_0(\mathbb{R})$ be real-valued, and $Au = bu' + cu$ with $D(A) = C_0^1(\mathbb{R})$. Take $u \in D(A)$ and some $s_0 \in \mathbb{R}$ with $|u(s_0)| = \|u\|_\infty$. Then $\varphi = \overline{u}(s_0)\delta_{s_0}$ belongs to $J(u)$ by Example 1.30. We then compute

$$r := \text{Re}(Au - \|c_+\|_\infty u, \varphi) = b(s_0) \text{Re}(u'(s_0)\overline{u}(s_0)) + (c(s_0) - \|c_+\|_\infty) |u(s_0)|^2$$

$$\leq b(s_0) \text{Re}(u'(s_0)\overline{u}(s_0)).$$

We set $h(s) = \text{Re}(\overline{u}(s_0)u(s))$ for $s \in \mathbb{R}$. Clearly, $h \in C_0^1(\mathbb{R})$ is real-valued and $|u(s_0)|^2 = h(s_0) \leq h\|\infty \leq |u(s_0)| \|u\|_\infty = |u(s_0)|^2$

so that $h$ attains its maximum at $s_0$. Therefore, $h'(s_0) = 0$ and $r \leq 0$. This means that $A - \|c_+\|_\infty I$ is dissipative.

b) Let $X = C([0,1])$, $b,c \in X$ be real-valued, $b(0) \geq 0$ for simplicity, and $A_j = bu' + cu$ with $D(A_j) = \{u \in C((0,1]) | u'(j) = 0 \}$ for $j \in \{0,1\}$. Then $A_1 - \|c_+\|_\infty I$ is dissipative. If $b(1) \leq 0$, also $A_0 - \|c_+\|_\infty I$ is dissipative. On the other hand, if $b(1) > 0$ the operator $A_0 - \omega I$ does not generate a contraction semigroup for any $\omega \in \mathbb{R}$. (Using a more sophisticated construction, one can show that it is not dissipative.)

Proof. For $u \in D(A_j)$, we use the functional $\varphi(v) = \overline{u}(s_0)v(s_0)$ on $X$, where $|u(s_0)| = \|u\|_\infty$ for some $s_0 \in [0,1]$. We also set $h(s) = \text{Re}(\overline{u}(s_0)u(s))$ for $s \in [0,1]$. As in a), one sees that $\varphi$ belongs to $J(u)$, $h \in C((0,1])$ attains its maximum at $s_0$, and

$$r := \text{Re}(A_ju - \|c_+\|_\infty u, \varphi) \leq b(s_0) \text{Re}(u'(s_0)\overline{u}(s_0)) = b(s_0)h'(s_0).$$

If $s_0 \in (0,1)$, this inequality again yields $r \leq 0$. Similarly, for $s_0 = 0$ we obtain

$$h'(0) = \lim_{s \to 0^+} \frac{1}{s}(h(s) - h(0)) \leq 0$$

since $h(0)$ is a maximum of $h$. Using $b(0) \geq 0$, we infer $r \leq 0$.

Finally, let $s_0 = 1$. In this case the above argument yields $h'(1) \geq 0$. We first look at $j = 0$. For $b(1) \leq 0$, we derive $r \leq b(1)h'(1) \leq 0$ so that $A_0 - \|c_+\|_\infty I$ is dissipative in this case. Next, let $b(1) > 0$. Fix $\omega \in \mathbb{R}$. Choose a real-valued function $u \in D(A_0)$ with maximum $u(1) = 1$ and $u'(1) > (\omega - c(1))/b(1)$. Since then $\varphi = \delta_1$, we obtain the inequality

$$\text{Re}(A_0u - \omega u, \varphi) = b(1)u'(1) + c(1) - \omega > 0.$$ 

Hence, $A_0 - \omega I$ cannot generate a contraction semigroup by Proposition 1.32. (Note that we did not show that $\text{Re}(A_0u - \omega u, \psi) > 0$ for all $\psi \in J(u)$.)

For $j = 1$ we have the boundary condition $u'(1) = 0$ and thus $h'(1) = 0$. It follows that $r \leq b(1)h'(1) = 0$ and so $A_1 - \|c_+\|_\infty I$ is dissipative. □
c) Let $X = L^2(\mathbb{R})$ and $A = \frac{d^2}{dx^2}$ with $D(A) = C^1_c(\mathbb{R})$. For $u \in D(A)$ we have $\bar{u} \in J(u)$ by Example 1.30. Integration by parts yields

$$2 \text{Re}\langle Au, \bar{u} \rangle = \langle Au, \bar{u} \rangle + \langle Au, \bar{u} \rangle = \int_{\mathbb{R}} u' \bar{u} \, ds + \int_{\mathbb{R}} \bar{u}' u \, ds = 0;$$

i.e., $A$ is dissipative (but not closed by Example 1.42). In the same way one checks the dissipativity of $-A$.

d) Let $X = L^2(0,1)$, $A_j = \frac{d}{dx}$, and $D(A_j) = \{ u \in C^1([0,1]) : |u(j)| = 0 \}$ for $j \in \{0,1\}$. For $u \in D(A_j)$ we take again $\bar{u} \in J(u)$ and obtain

$$2 \text{Re}\langle Au, \bar{u} \rangle = \int_{0}^{1} u' \bar{u} \, ds + \int_{0}^{1} \bar{u}' u \, ds = u\bar{u}|_0^1 = |u(1)|^2 - |u(0)|^2.$$

It follows that $A_1$ is dissipative. However, $A_0 - \omega I$ is not dissipative for any $\omega \in \mathbb{R}$, since we can find a map $u$ in $D(A_0)$ satisfying $|u(1)|^2 > 2\omega ||u||_2^2$ and so

$$\text{Re}\langle A_0u - \omega u, \bar{u} \rangle = \frac{1}{2} |u(1)|^2 - \omega ||u||_2^2 > 0.$$

Examples c) and d) can be extended to $L^p$ with $p \in [1,\infty)$, cf. Example 1.48.

Above we have encountered rather natural dissipative, but non-closed operators. To treat such operators, we introduce a concept extending closedness.

**Intermezzo 2: Closable operators.**

**Definition 1.34.** A linear operator $A$ is called closable if it possesses a closed extension $B$.

Note that a closed operator is closable since $A \subseteq A$. We first characterize closability and construct the closure $\overline{A}$ of a closable operator $A$, which is the smallest closed extension of $A$.

**Lemma 1.35.** For a linear operator $A$, the following statements are equivalent.

a) The operator $A$ is closable.

b) Let $(x_n)$ be a sequence in $D(A)$ with $x_n \to 0$ and $Ax_n \to y$ in $X$ as $n \to \infty$. Then $y = 0$.

c) In the set $D(\overline{A}) = \{ x \in X | \exists (x_n) \text{ in } D(A), y \in X : x_n \to x, Ax_n \to y, n \to \infty \}$ the vector $y$ is uniquely determined by $x$. Letting $\overline{A} : D(\overline{A}) \to X : \overline{A}x = y$, one thus defines a map. The operator $\overline{A}$ is linear, closed, and extends $A$.

Let one and hence all of the properties a)–c) are valid. Then $\text{Gr}(\overline{A}) = \text{Gr}(A)$, $D(A)$ is dense in $[D(\overline{A})]$, and we have $\overline{A} \subseteq B$ for every closed operator $B \supseteq A$.

**Proof.** Part c) clearly implies a). Let a) be true and $B$ be a closed extension of $A$. Take $(x_n)$ as in statement b). Then the vectors $Ax_n = Bx_n$ tend to $y = B0 = 0$ since $B$ is closed.

We assume that property b) holds. Let $(x_n)$ and $(z_n)$ be sequences in $D(A)$ with limit $x$ in $X$ such that $(Ax_n)_n$ converges to $y$ and $(Az_n)_n$ to $w$ in $X$. Then $(x_n - z_n)$ is a null sequence in $X$ with $A(x_n - z_n) = Ax_n - Az_n \to y - w$ as $n \to \infty$. Part b) thus implies $y = w$, so that $\overline{A}$ is a mapping. One easily verifies that $\overline{A}$ is linear and that $\text{Gr}(\overline{A}) = \text{Gr}(A)$, which shows the first part of the addendum. Hence, $\overline{A}$ is closed due to Remark 1.15 and $\overline{A}$ extends $A$. Therefore assertion c) is shown.
Let \( B \) be another closed extension of \( A \). We then have \( \text{Gr}(A) \subseteq \text{Gr}(B) \) and so \( \text{Gr}(\overline{A}) = \text{Gr}(\overline{A}) \subseteq \text{Gr}(B) \) because of the closedness of \( B \). In particular, \( B \) extends \( \overline{A} \). The density assertion is an immediate consequence of \( \text{Gr}(A) = \text{Gr}(\overline{A}) \) and the definition of the graph norm.

As consequence of this lemma, a linear operator is closed if and only if it is its own closure. We illustrate the concepts of extension and closure by the first derivative, again stressing the role of the boundary conditions.

**Example 1.36.** a) Let \( X = L^1(0, 1) \) and \( Af = f(0) \mathbb{1} \) with \( D(A) = C([0, 1]) \). This operator is not closable. In fact, the functions \( f_n \in D(A) \) given by \( f_n(s) = \max\{1 - ns, 0\} \) satisfy \( \|f_n\|_1 = \frac{1}{2n} \to 0 \) as \( n \to \infty \), but \( Af_n = \mathbb{1} \) for all \( n \in \mathbb{N} \), contradicting Lemma 1.35 b).

b) Let \( X = C([0, 1]) \) and \( A_0u = u' \) with \( D(A_0) = C^1_c((0, 1)) \), as well as \( Au = u' \) with \( D(A) = C^1_0(0, 1) := C^1_c((0, 1)) \). As in Example 1.14 we see that \( A \) is closed. Hence, \( A_0 \) is closable and \( A_0 \subseteq A \) since \( A_0 \subseteq A \). To check equality, let \( u \in C^1_0(0, 1) \). Take \( \varphi_n \in C^1_c(0, 1) \) such that \( \varphi = 1 \) on \( (\frac{1}{n}, 1 - \frac{1}{n}) \), \( 0 \leq \varphi_n \leq 1 \) and \( \|\varphi_n'\|_\infty \leq c n \) for some \( c > 0 \) and all \( n \in \mathbb{N} \) with \( n \geq 2 \). (For instance, one can take

\[
\varphi_n(s) = \begin{cases} 
0, & 0 < s < \frac{1}{n}, \\
8n^2(s - \frac{1}{4n})^2, & \frac{1}{4n} \leq s \leq \frac{1}{2n}, \\
1 - 8n(\frac{3}{4n} - s)^2, & \frac{1}{2n} \leq s \leq \frac{3}{4n}, \\
1, & \frac{3}{4n} < s \leq \frac{1}{2}, \\
\varphi_n(1 - s), & \frac{1}{2} < s < 1,
\end{cases}
\]

where \( c = 4 \).) Then the function \( u_n = \varphi_n u \) belongs to \( D(A_0) \), and we have

\[
\|u_n - u\|_\infty = \sup_{0 \leq s \leq 1} |(\varphi_n(s) - 1)u(s)| \leq \sup_{0 \leq s \leq 1} |u(s)| \to 0,
\]

\[
\|\varphi_n' - u'|_\infty \leq \sup_{0 \leq s \leq 1} |(\varphi_n(s) - 1)u'(s)| \to 0
\]

as \( n \to \infty \) since \( u, u' \in C_0(0, 1) \). We further obtain

\[
\|\varphi_n u\| \leq \sup_{s \in [0, \frac{1}{n}]} |\varphi_n'(s)u(s)| + \sup_{s \in [1 - \frac{1}{n}, 1]} |\varphi_n'(s)u(s)|
\]

\[
\leq \sup_{s \in [0, \frac{1}{n}]} cn \left| \int_0^s u'(\tau) \, d\tau \right| + \sup_{s \in [1 - \frac{1}{n}, 1]} cn \left| \int_s^1 u'(\tau) \, d\tau \right|
\]

\[
\leq cn \int_0^\frac{1}{n} |u'(\tau)| \, d\tau + cn \int_1^{1 - \frac{1}{n}} |u'(\tau)| \, d\tau \to 0
\]

as \( n \to \infty \), because of (1.4) and \( u' \in C_0(0, 1) \). Hence, \( A_0(\varphi_n u) = \varphi_n u + \varphi_n u' \) converges to \( Au = u' \). This means that \( A \subseteq \overline{A} \) and thus \( \overline{A} = A \). In particular \( A_0 \) is not closed and thus fails to be a generator.

We discuss further closed extensions of \( A_0 \) given by \( A_j u = u' \) for \( j \in \{1, 2, 3\} \).

1) Let \( D(A_1) = \{u \in C^1([0, 1]) \mid u'(1) = 0\} \). By an exercise, \( A_1 \) generates a \( C_0 \)-semigroup on \( X \) and \( \sigma(A_1) = \{0\} \). Observe that \( A_1 \) is a strict extension of
A. Lemma 1.23 thus implies that $\rho(A) \cap \rho(A_1) = \emptyset$ and hence $F \setminus \{0\} \subseteq \sigma(A)$. (Actually, we have $\sigma(A) = F$ since $1 \not\in AD(A)$.) As a result, $A$ is not generator – it has too many boundary conditions, namely four instead of one as in $D(A_1)$.

2) Let $D(A_3) = C^1([0,1])$. Example 1.21 says that $\sigma(A_3) = F$. So $A_3$ is not a generator because it has not enough boundary conditions, namely none. We have $A \lneqq A_1 \lneqq A_3$.

3) Let $D(A_2) = \{u \in C^1([0,1]) \mid u(1) = 0\}$. Also $A_2$ is ‘sandwiched’ between $A$ and $A_3$; i.e., $A \lneqq A_2 \lneqq A_3$, but $A_1$ and $A_2$ are not comparable. The operator $A_2$ is not a generator as its domain is not dense, see Example 1.21.

Summing up, the ‘minimal’ operator $A$ and the ‘maximal’ operator $A_3$ do not generate $C_0$-semigroups. Between them there are various, partly noncomparable operators (so-called ‘realizations’ of $\frac{d}{dt}$) which may or may not be generators. Their domains are often determined by boundary conditions.

We come back to the investigation of $C_0$-semigroups. Below we use closures in a generation result, but at first we establish sufficient conditions for a subspace $D$ to be dense in $D(A)$ in the graph norm. Such a subspace is called core of a closed operator $A$. Observe that $D$ is a core if and only if $A$ is the closure of $A|_D$. In Example 1.36 b) the set $C^1_c(0,1)$ is a core for $A$. One can often extend properties from cores to the full domain, see e.g. Proposition 1.38 c).

It is often difficult to decide whether a subspace $D$ is a core of an operator $A$. The next result gives a convenient sufficient condition involving the semigroup.

**Proposition 1.37.** Let $A$ generate the $C_0$-semigroup $T(\cdot)$ on $X$. Let $D$ be a linear subspace of $D(A)$ which is dense in $X$ and invariant under the semigroup; i.e., $T(t)D \subseteq D$ for all $t \geq 0$. Then $D$ is dense in $[D(A)]$.

**Proof.** Set $C = \sup_{0 \leq t \leq 1} \|T(t)\| < \infty$. Let $x \in D(A)$. The map $T(\cdot)x : \mathbb{R}_\geq 0 \to [D(A)]$ is continuous by Proposition 1.10. Take $\varepsilon > 0$. There is a time $\tau = \tau(\varepsilon, x) \in (0,1]$ with $\|T(t)x - x\|_A \leq \varepsilon$ for all $t \in [0,\tau]$. It follows

$$\left\| \frac{1}{\tau} \int_0^\tau T(t)x\, dt - x \right\|_A \leq \frac{1}{\tau} \int_0^\tau \|T(t)x - x\|_A\, dt \leq \varepsilon.$$  

Using the density of $D$ in $X$, we find a vector $y \in D$ with

$$\|x - y\| \leq \left( C + \frac{C + 1}{\tau} \right)^{-1} \varepsilon.$$  

Let $\tilde{D}$ be the closure of $D$ in $[D(A)]$. We want to replace $y$ by a vector $z$ in $\tilde{D}$ that is close to $x$ for $\| \cdot \|_A$. To this aim, we set

$$z = \frac{1}{\tau} \int_0^\tau T(t)y\, dt.$$  

The integrand $T(t)y$ takes values in $D$ by assumption, and as above it is continuous in $[D(A)]$. In view of the definition of the integral, $z$ thus belongs to $\tilde{D}$. The previous inequalities and Lemma 1.18 imply the bound

$$\|x - z\|_A \leq \left\| x - \frac{1}{\tau} \int_0^\tau T(t)x\, dt \right\|_A + \frac{1}{\tau} \left\| \int_0^\tau T(t)(x - y)\, dt \right\|_A + \frac{1}{\tau} \left\| A \int_0^\tau T(t)(x - y)\, dt \right\|_A.$$
Remark 1.16 c) shows that $y \in \mathcal{X}$ in $\lambda \mathcal{X}$ by Remark 1.16 b). Let $\lambda > 0$ and $\lambda > 1$ with norm less than or equal to $\frac{1}{\lambda_0}$. In particular, $A$ is closed by Remark 1.16 b). Let $\lambda \in (0, 2\lambda_0)$. Since $|\lambda - \lambda_0| < \lambda_0 \leq \|R(\lambda_0, A)^{-1}\|^{-1}$, Remark 1.16 c) shows that $\lambda$ belongs to $\rho(A)$. Step a) also yields the estimate $\|R(\lambda, A)\| \leq \frac{1}{\lambda}$. In view of (1.8) also $2\lambda_0$ is contained in $\rho(A)$ and thus $\|R(2\lambda_0, A)\| \leq \frac{1}{2\lambda_0}$. We can now iterate this argument, deriving assertion b).

c) Assume that $D(A)$ is dense in $X$. To check the closability of $A$, we choose a sequence $(x_n)$ in $D(A)$ with limit 0 in $X$ such that $(Ax_n)$ converges in $X$ to some $x \in X$. By density, there are vectors $y_k$ in $D(A)$ tending to $y$ in $X$ as $k \to \infty$. Take $\lambda > 0$ and $n, k \in \mathbb{N}$. Proposition 1.32 implies the lower bound

$$\|\lambda^2 x_n - \lambda Ax_n + \lambda y_k - Ay_k\| = \|(\lambda I - A)(\lambda x_n + y_k)\| \geq \lambda \|\lambda x_n + y_k\|.$$  

Letting $n \to \infty$, we deduce

$$\|\lambda y + \lambda y_k - Ay_k\| \geq \lambda \|y_k\|,$$

$$\|y + y_k - \lambda^{-1} Ay_k\| \geq \|y_k\|.$$  

As $\lambda \to \infty$, it follows that $\|y + y_k\| \geq \|y_k\|$. Taking the limit $k \to \infty$, we conclude $y = 0$. Due to Lemma 1.35, the operator $A$ is closable.

Let $x \in D(\overline{A})$. Then there are vectors $z_n \in D(A)$ with $z_n \to x$ and $Az_n \to \overline{A}x$ in $X$ as $n \to \infty$. Using Proposition 1.32, we now infer the estimate

$$\|\lambda x - \overline{A}x\| = \lim_{n \to \infty} \|\lambda z_n - A z_n\| \geq \lim_{n \to \infty} \lambda \|z_n\| = \lambda \|x\|,$$

and thus the dissipativity of $\overline{A}$.

The following theorem by Lumer and Phillips from 1961 is the most important tool to verify the generator property in concrete cases (besides a result in
Section 2.3). To show that an operator \( A \) (or its closure) generates a contraction semigroup, one only has to establish the density of \( D(A) \), the dissipativity of \( A \), and that \( \lambda_0 I - A \) is surjective (or has dense range) for some \( \lambda_0 > 0 \). The first two properties can often be checked by direct computations using the given information on \( A \). The final range conditions are usually harder to show. One has to solve the 'stationary problem'

\[
\exists u \in D(A) : \quad \lambda_0 u - Au = f
\]

at least for \( f \) from a dense set of 'good' vectors. Fortunately, there are various tools to solve this problem which we partly discuss in the next section.

Based on our preparations, the Lumer–Phillips theorem can easily be deduced from the contraction case of the Hille–Yosida Theorem 1.26. In Example 1.49 we will see that one cannot omit the range conditions in parts a) or b).

**Theorem 1.39.** Let \( A \) be a linear and densely defined operator. The following assertions hold.

- a) Let \( A \) be dissipative and \( \lambda_0 > 0 \) such that \( \lambda_0 I - A \) has dense range. Then \( \lambda_0 I - A \) is surjective. Then \( A \) generates a contraction semigroup.
- b) Let \( A \) be dissipative and \( \lambda_0 > 0 \) such that \( \lambda_0 I - A \) is surjective. Then \( A \) generates a contraction semigroup.
- c) Let \( A \) generate a contraction semigroup. Then \( A \) is dissipative, \( F_+ \subseteq \rho(A) \), and \( \| R(\lambda, A) \| \leq 1/ \text{Re}(\lambda) \) for \( \lambda \in F_+ \).

One can replace ‘contraction’ by ‘\( \omega \)-contraction’ and \( A \) by \( A - \omega I \) for \( \omega \in \mathbb{R} \).

Operators satisfying the assumptions in assertion b) are called *maximally dissipative* or *m-dissipative*. (Such maps cannot have non-trivial dissipative extensions because of Lemma 1.23 and Proposition 1.38 a.) If a closed operator \( A \) satisfies the hypotheses of part a), then \( A \) generates a contraction semigroup since \( A = \overline{A} \). This variant of the result is often very useful in applications. Concerning the addendum, one can easily check that the closure of \( A - \omega I \) is equal to \( \overline{A} - \omega I \).

**Proof of Theorem 1.39.** Let the conditions in a) be true. Proposition 1.38 then tells us that \( A \) possesses a dissipative closure \( \overline{A} \). Let \( y \in X \). By assumption, there are vectors \( x_n \in D(A) \) such that the images \( y_n = \lambda_0 x_n - Ax_n \) tend to \( y \) in \( X \) as \( n \to \infty \). The dissipativity of \( A \) yields the inequality

\[
\| x_n - x_m \| \leq \frac{1}{\lambda_0} \| (\lambda_0 - A)(x_n - x_m) \| = \frac{1}{\lambda_0} \| y_n - y_m \|
\]

for all \( n, m \in \mathbb{N} \) thanks to Proposition 1.32. This means that \( (x_n) \) has a limit \( x \) in \( X \), and hence the vectors \( \overline{A} x_n = Ax_n = \lambda_0 x_n - y_n \) tend to \( \lambda_0 x - y \) as \( n \to \infty \). Since \( \overline{A} \) is closed, \( x \) belongs to \( D(\overline{A}) \) and satisfies \( \overline{A} x = \lambda_0 x - y \) so that \( \lambda_0 I - \overline{A} \) is surjective. Proposition 1.38 and Theorem 1.26 now imply the assertion.

By Proposition 1.38, \( A \) is closed if \( \lambda_0 I - A \) is surjective, and then part a) shows that \( A = \overline{A} \) generates a contraction semigroup. Assertion c) is a consequence of Propositions 1.32 and 1.20. The addendum follows by a rescaling argument based on Lemma 1.17. \( \square \)
We will reformulate the range condition in the Lumer–Phillips theorem using duality. To this aim, we recall the following concept from the lecture Spectral Theory. For a densely defined linear operator $A$, we define its adjoint $A^\ast$ by

$$A^\ast x^\ast = y^\ast$$

for all $x^\ast \in D(A^\ast)$, where

$$D(A^\ast) = \{ x^\ast \in X^* \mid \exists y^\ast \in X^* \ \forall x \in D(A) : \langle Ax, x^\ast \rangle = \langle x, y^\ast \rangle \}.$$  

This means that $\langle Ax, x^\ast \rangle = \langle x, A^\ast x^\ast \rangle$ for all $x \in D(A)$ and $x^\ast \in D(A^\ast)$. Recall from Remark 1.23 in [ST] that $A^\ast$ is a closed linear operator. The domain $D(A^\ast)$ in (1.20) is defined in a ‘maximal way’ which is convenient for the theory, but for concrete operators it is often very difficult to calculate $D(A^\ast)$ explicitly. The next result replaces the range condition by the injectivity of $\lambda_0 I - A^\ast$ (or the dissipativity of $A^\ast$), cf. Theorem 1.24 in [ST]. In Example 1.49 we present a closed and densely defined dissipative operator having a non-dissipative adjoint.

**Corollary 1.40.** Let $A$ be dissipative and densely defined, and let $\lambda_0 I - A^\ast$ be injective for some $\lambda_0 > 0$. Then $\overline{A}$ generates a contraction semigroup. If $A^\ast$ is dissipative, then $\lambda I - A^\ast$ is injective for all $\lambda > 0$.

**Proof.** The addendum follows from Proposition 1.38. Let $\lambda_0 I - A^\ast$ be injective. Take a functional $x^\ast \in X^*$ such that $\langle \lambda_0 x - Ax, x^\ast \rangle = 0$ for all $x \in D(A)$. From (1.20) we then deduce that $x^\ast$ belongs to $D(A^\ast)$ and $A^\ast x^\ast = \lambda_0 x^\ast$. Hence, $x^\ast = 0$. The Hahn-Banach theorem now implies the density of $R(\lambda_0 I - A)$, see Corollary 5.13 in [FA]. Theorem 1.39 thus yields the assertion. \qed

Examples 1.33 c) and d) indicate that integration by parts is a very convenient tool to check dissipativity for differential operators in an $L^2$-context. To tackle such problems, we briefly discuss concepts and basic facts from Section 4.2 of [FA] and also from Chapter 3 of [ST], where the topic is treated in much greater detail. The material below is needed in many of our examples.

**Intermezzo 3: Weak derivatives and Sobolev spaces.** Let $\emptyset \neq G \subseteq \mathbb{R}^m$ be open, $k \in \mathbb{N}$, $j \in \{1, \ldots, m\}$, and $p \in [1, \infty]$. A function $u \in L^p(G)$ has a weak derivative in $L^p(G)$ with respect to the $j$th coordinate if there is a map $v \in L^p(G)$ satisfying

$$\int_G u \partial_j \varphi \, dx = - \int_G v \varphi \, dx$$

for all $\varphi \in C^\infty_c(G)$. The function $v$ is uniquely determined a.e. by Lemma 4.15 in [FA]. We set $\partial_j u := v$ in the above situation. The *Sobolev space* $W^{1,p}(G) := \{ u \in L^p(G) \mid \forall j \in \{1, \ldots, m\} \exists \partial_j u \in L^p(G) \}$ is a Banach space when endowed with the norm

$$\|u\|_{1,p} = \begin{cases} \left( \|u\|^p_p + \sum_{j=1}^m \|\partial_j u\|^p_p \right)^{\frac{1}{p}}, & p < \infty, \\ \max_{j \in \{1, \ldots, m\}} \{ \|u\|_\infty, \|\partial_j u\|_\infty \}, & p = \infty, \end{cases}$$

see Proposition 4.19 of [FA]. (As usual we identify functions which are equal almost everywhere.) Hence, by definition weak derivatives can be integrated by parts against ‘test functions’ $\varphi \in C^\infty_c(G)$. This norm is equivalent to the one
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given by \( \|u\|_p + \sum_{j=1}^m \|\partial_j u\|_p \) due to Remark 4.16 in [FA]. Analogously one defines the Sobolev spaces \( W^{k,p}(G) \) and higher-order weak derivatives \( \partial^{\alpha} = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \) for \( \alpha \in \mathbb{N}_0^n \) and \( |\alpha| = \alpha_1 + \cdots + \alpha_n \leq k. \) We put \( u = \partial^\beta u. \)

One often writes \( H^k \) instead of \( W^{k,2} \) which is a Hilbert space. We summarize properties of Sobolev spaces and weak derivatives that are needed later on.

Remark 1.41. a) Let \( u \in C^k(G) \) such that \( u \) and all its derivatives up to order \( k \) are contained in \( L^p(G) \). Then \( u \) belongs to \( W^{k,p}(G) \) and its classical and weak derivatives coincide by Remark 4.16 of [FA].

b) Let \( u, u_n, v \in L^p(G) \) and \( \alpha \in \mathbb{N}_0^n \) such that \( u_n \to u \) and \( \partial^{\alpha} u_n \to v \) in \( L^p(G) \) as \( n \to \infty. \) Then \( u \) possesses the weak derivative \( \partial^{\alpha} u = v \) as shown in Lemma 4.17 in [FA] or Lemma 3.16 in [ST]. In other words, the operator \( \partial^\alpha \) with (maximal) domain \( \{ u \in L^p(G) \mid \exists \partial^\alpha u \in L^p(G) \} \) is closed in \( L^p(G). \)

c) Let \( p < \infty. \) Theorem 3.27 of [ST] says that \( C_\infty^\infty(\mathbb{R}^m) \) is dense in \( W^{k,p}(\mathbb{R}^m) \) and that \( C^\infty(\mathbb{R}^m) \cap W^{k,p}(G) \) is dense in \( W^{k,p}(G). \) (See also Theorem 4.21 of [FA] for the first result.)

d) Let \( -\infty \leq a < b \leq \infty, \ J = (a,b) \) and \( u \in L^p(J). \) Then the function \( u \) belongs to \( W^{1,p}(J) = W^{1,p}(a,b) \) if and only if (a representative of) \( u \) is continuous and there is a map \( v \in L^p(J) \) satisfying

\[
u(t) = u(s) + \int_s^t v(\tau) \, d\tau \quad \text{for all } t, s \in J.
\]

We then have \( u' = \partial u := \partial_1 u = v \) and \( u \) has a continuous extension to \( a \) (or \( b \)) if \( a > -\infty \) (or \( b < \infty \)). Moreover, \( W^{1,p}(J) \) is continuously embedded into \( C_0(J). \) See Theorem 3.22 and Remark 3.33 in [ST].

As an example, take a function \( u \in C_c(\mathbb{R}) \) whose restrictions \( u^+ \) and \( u^- \) to \( \mathbb{R}_{\geq 0} \) and \( \mathbb{R}_{\leq 0} \), respectively, are continuously differentiable. The map \( u \) then belongs to \( W^{1,p}(\mathbb{R}) \) for all \( p \in [1, \infty) \) and its derivative is given by \( (u^\pm)' \) on \( \mathbb{R}_{\pm} \) by Example 4.18 of [FA], where one also finds a multidimensional example.

e) Let \( u \in W^{1,p}(G) \) and \( v \in W^{1,p'}(G) \) with \( \frac{1}{p} + \frac{1}{p'} = 1. \) Proposition 4.20 of [FA] yields that \( uv \) is an element of \( W^{1,1}(G) \) and satisfies the product rule \( \partial_j (uv) = u\partial_j v + v\partial_j u. \) Analogous results hold for higher-order derivatives.

f) Let \( G \) have a \( C^1 \)-boundary \( \partial G \) or \( G \) be bounded with a Lipschitz boundary. See the beginning of Section 3.3 in [ST] for these concepts. By the Trace Theorem 3.38 in [ST], the map \( W^{1,p}(G) \cap C(\bar{G}) \to L^p(\partial G, d\sigma); u \mapsto u|_{\partial G} \), has a continuous extension \( \text{tr} : W^{1,p}(G) \to L^p(\partial G, d\sigma) \) called the trace operator. Its kernel is the closure \( W^{1,p}_0(G) \) of the test functions \( C_\infty^\infty(G) \) in \( W^{1,p}(G). \) If \( \text{tr} u = 0, \) one says that \( u \) vanishes on \( \partial G \) ‘in the sense of trace.’

Let \( G \) have a bounded Lipschitz boundary, \( f \in W^{1,p}(G)^m, \) and \( u \in W^{1,p'}(G). \) The Divergence Theorem 3.41 in [ST] then yields

\[
\int_G u \, \text{div} \, f \, dx = -\int_G f \cdot \nabla u \, dx + \int_{\partial G} \text{tr}(u) \nu \cdot \text{tr}(f) \, d\sigma.
\]

Here \( \nu \) is the unit outer normal and the dot denotes the scalar product in \( \mathbb{R}^m. \) We usually omit the trace operator in the boundary integral. If \( G = \mathbb{R}^m \) the formula is true without the boundary integral. \( \diamond \)
Coming back to semigroups, we illustrate the above concepts by a simple example concerning generation properties of $\frac{d}{dt}$ in $L^2(\mathbb{R})$.

**Example 1.42.** Let $X = L^2(\mathbb{R})$ and $A = \frac{d}{ds}$ with $D(A) = C^1_c(\mathbb{R})$.

1) The operators $\pm A$ are densely defined and dissipative by Example 1.33. Proposition 1.38 then yields their closability and the dissipativity of their closures, where $-\lambda$ has the closure $-\lambda$. We next show that $\overline{A} = (\partial, W^{1,2}(\mathbb{R}))$.

For each $u \in D(A)$ there are functions $u_n \in C^1_c(\mathbb{R})$ such that $u_n \to u$ and $u'_n = Au_n \to \overline{A}u$ in $L^2(\mathbb{R})$ as $n \to \infty$. In view of Remark 1.41 b), the map $u$ thus belongs to $W^{1,2}(\mathbb{R})$ and $\overline{A}u = \partial u$; i.e., $\overline{A} \subseteq (\partial, W^{1,2}(\mathbb{R}))$. For the converse, take $u \in W^{1,2}(\mathbb{R})$. Remark 1.41 c) then provides a sequence $(u_n)$ in $C^1_c(\mathbb{R})$ with limit $u$ in $W^{1,2}(\mathbb{R})$. Hence, $u_n \to u$ and $u'_n \to \partial u$ in $L^2(\mathbb{R})$ so that $u$ is an element of $D(\overline{A})$.

2) We compute $\overline{A}^\ast$. Let $u, v \in W^{1,2}(\mathbb{R})$. Formula (1.22) then yields

$$\langle \overline{A}u, v \rangle = \int_\mathbb{R} \partial u v \, ds = -\int_\mathbb{R} u \partial v \, ds = \langle u, -\partial v \rangle,$$

so that $(-\partial, W^{1,2}(\mathbb{R}))$ is a restriction of $\overline{A}^\ast$, see (1.20). Conversely, let $v \in D(\overline{A}^\ast)$. The functions $v$ and $\overline{A}^\ast v$ thus belong to $L^2(\mathbb{R})$ and satisfy

$$\int_\mathbb{R} u \overline{A}^\ast v \, ds = \langle u, \overline{A}^\ast v \rangle = \langle Au, v \rangle = \int_\mathbb{R} u' v \, ds$$

for all $u \in C_c^\infty(\mathbb{R}) \subseteq D(A) \subseteq D(\overline{A})$, which means that $v \in W^{1,2}(\mathbb{R})$ and $\overline{A}^\ast v = -\partial v = -\overline{A}v$. As a result, $\overline{A}^\ast = -\overline{A}$. Corollary 1.40 then shows that $\pm \overline{A}$ generate contraction semigroups.

3) To determine these semigroups, we recall from Example 1.8 that the translation group $T(t)f = f(\cdot + t)$ on $X$ has a generator $B$. For $f \in D(A)$ the functions $w(t) = \frac{1}{t}(T(t)f - f)$ converge uniformly to $f'$ as $t \to 0^+$. Moreover, the supports $\text{supp} w(t)$ are contained in the bounded set $\text{supp} f + [-1, 0]$ for all $0 \leq t \leq 1$, so that $w(t)$ tends to $f'$ in $X$. This means $A \subseteq B$ and so $\overline{A} \subseteq B$. Lemma 1.23 now yields $\overline{A} = B$ and hence $\overline{A}$ generates $T(\cdot)$. \hfill \Box

We conclude this section with a discussion of isometric groups.

**Corollary 1.43.** Let $A$ be linear. The following statements are equivalent.

a) The operator $A$ generates an isometric $C_0$-group $T(\cdot)$; i.e., $\|T(t)x\| = \|x\|$ for all $x \in X$ and $t \in \mathbb{R}$.

b) The operator $A$ is closed, densely defined, $\pm A$ are dissipative, and $\lambda_0 I \pm A$ are surjective for some $\lambda_0 > 0$.

c) The operator $A$ is closed, densely defined, $\Re \lambda \in \rho(A)$, and $\|R(\lambda, A)\| \leq \frac{1}{\lambda}$ for all $\lambda \in \mathbb{R} \setminus \{0\}$.

If $\mathbb{F} = \mathbb{C}$, one can also replace in c) the set $\mathbb{R} \setminus \{0\}$ by $\mathbb{C} \setminus i\mathbb{R}$ and $|\lambda|$ by $|\Re \lambda|$.

**Proof.** The Lumer-Phillips Theorem 1.39 says that b) holds if and only if $A$ and $-A$ generate contraction semigroups. Theorem 1.29 thus implies the equivalence of assertions b) and c), the addendum, and that b) is true if and
only if \( A \) generates a contractive \( C_0 \)-group \( T(\cdot) \). It remains to show that a contractive \( C_0 \)-group \( T(\cdot) \) is already isometric. Indeed, in this case we have
\[
\|T(t)x\| \leq \|x\| = \|T(-t)T(t)x\| \leq \|T(-t)\|\|T(t)x\| \leq \|T(t)x\|
\]
for all \( x \in X \) and \( t \in \mathbb{R} \), so that \( T(t) \) is isometric. \( \square \)

We want to show an important variant of the above corollary on Hilbert spaces which requires a few more concepts from [ST]. Let \( X \) be a Hilbert space. For a linear operator on \( X \) with dense domain we define the Hilbert space adjoint \( A' \) of \( A \) as in (1.20) replacing the duality pairing \( \langle x,x^* \rangle \) by the inner product \( (x|y) \). A linear operator \( A \) on \( X \) is called symmetric if
\[
\forall x,y \in D(A) : \quad (Ax|y) = (x| Ay),
\]
which means that \( A \subseteq A' \) if \( D(A) \) is dense. If \( A \) is densely defined, we say that it is self-adjoint if \( A = A' \); i.e., if \( A \) is symmetric and
\[
D(A) = \{ y \in X \mid \exists z \in X \forall x \in D(A) : (Ax|y) = (x|z) \}
\]
\[= \{ y \in X \mid (D(A), \| \cdot \|) \to \mathbb{F}; x \mapsto (Ax|y), \text{ is continuous} \}.
\]
(The last equality is a consequence Riesz’ representation Theorem 3.10 in [FA].)

A densely defined, linear operator \( A \) is called skew-adjoint if \( A = -A' \) which is equivalent to the self-adjointness of \( iA \). Finally, \( T \in \mathcal{B}(X) \) is unitary if it has the inverse \( T^{-1} = T' \).

We recall a very useful criterion from Theorem 4.7 of [ST]. A symmetric, densely defined, closed operator \( A \) is self-adjoint if and only if its spectrum \( \sigma(A) \) belongs to \( \mathbb{R} \), which in turn follows from the condition \( \rho(A) \cap \mathbb{R} \neq \emptyset \).

As in Remark 1.23 in [ST] one can check that \( A' \) is a closed linear map. Hence, every densely defined, symmetric operator is closable with \( \overline{A} \subseteq A' \) (cf. Lemma 1.35) and each self-adjoint operator is closed. Let \( A \) be symmetric and densely defined. Take \( u,v \in D(\overline{A}) \). There are sequences \( (u_n) \) and \( (v_n) \) in \( D(A) \) with limits \( u \) and \( v \) in \( X \), respectively, such that \( Au_n \to \overline{A}u \) and \( Av_n \to \overline{A}v \) in \( X \) as \( n \to \infty \). We then compute
\[
(\overline{A}u|v) = \lim_{n \to \infty} (Au_n|v_n) = \lim_{n \to \infty} (u_n|Av_n) = (u|\overline{A}v),
\]
so that also the closure \( \overline{A} \) is symmetric.

There are densely defined, symmetric, closed operators that are not self-adjoint. (By Example 4.8 of [ST] this is the case for \( A = i\partial \) with \( D(A) = \{ u \in W^{1,2}(\mathbb{R}_+) \mid u(0) = 0 \} \) on \( X = L^2(\mathbb{R}_+) \). Here one has \( D(A') = W^{1,2}(\mathbb{R}_+) \).)

The next result due to Stone from 1930 belongs to the mathematical foundations of quantum mechanics.

**Theorem 1.44.** Let \( X \) be a Hilbert space and \( A \) be a linear operator on \( X \) with a dense domain. Then \( A \) generates a \( C_0 \)-group of unitary operators if and only if \( A \) is skew-adjoint.

**Proof.** 1) Let \( A' = -A \). Hence, \( A \) is closed. For \( x \in D(A) \), we have \( J(x) = \{ \varphi_x \} \) with \( \varphi_x = (\cdot|x) \) by Example 1.30. We thus obtain
\[
(Ax, \varphi_x) = (Ax|x) = -(x|Ax) = -(Ax|x) = -\langle Ax, \varphi_x \rangle
\]
and so \( \text{Re}(Ax, \varphi_x) = 0 \). Therefore, \( A', A' = -A, \text{ and } (A')' = A \) are dissipative.
From Corollary 1.40 we then deduce that $A$ and $-A$ generate contraction semigroups. Corollary 1.43 now shows that $A$ generates a $C_0$-group $T(\cdot)$ of invertible isometries. Hence, each $T(t)$ is unitary by Proposition 5.52 in [FA].

2) Let $A$ generate a unitary $C_0$-group $T(\cdot)$. We infer $T(t)' = T(t)^{-1} = T(-t)$ for all $t \in \mathbb{R}$ by Remark 1.2, and hence $T(\cdot)'$ is a unitary $C_0$-group with the generator $-A$. For $x, y \in D(A)$ we thus obtain

$$ (Ax|y) = \lim_{t \to 0} \left( \frac{1}{2} (T(t)x - x)|y) = \lim_{t \to 0} \left( x| \frac{1}{2} (T(t)'y - y)) = (x|-Ay). $$

This means that $-A \subseteq A'$. We further know from Theorem 1.29 that $\sigma(A)$ and $\sigma(-A)$ are contained in $i \mathbb{R}$. Equation (4.3) in [ST] then yields $\sigma(A') = \sigma(A) \subseteq i \mathbb{R}$. The assertion $-A = A'$ now follows from Lemma 1.23.

### 1.4. The Laplacian and related operators

In this section we discuss generation and related properties of the Laplacian

$$ \Delta = \partial_1^2 + \cdots + \partial_m^2 = \text{div} \nabla $$

in various settings, where we partly treat more general operators. To apply the Lumer–Phillips Theorem 1.39, we have to check three conditions. The density of the domain often follows from standard results on function spaces. With the right tools one can usually verify dissipativity in a straightforward way (imposing appropriate boundary conditions). For the range condition one has to solve the ‘elliptic problem’ $u - \Delta u = f$ plus boundary conditions for given $f$. Using differing methods, this will be done first on $\mathbb{R}^m$, then on intervals, and finally with Dirichlet boundary conditions on bounded domains. As we will see in the next chapter, these results will allow us to solve diffusion equations, actually with improved regularity. We will further use the Dirichlet–Laplacian in the wave equation, cf. Example 1.55. We strive for a self-contained presentation (employing the lectures Functional Analysis and Spectral Theory), but for certain additional facts we have to cite deeper results from the theory of partial differential equations.

**A) The Laplacian on $\mathbb{R}^m$.** Since the Laplacian has constant coefficients, on the full space $\mathbb{R}^m$ the Fourier transform is a very powerful tool to deal with it, for instance, to check the range condition. We first recall relevant results from Spectral Theory, taken from Sections 3.1 and 3.2 of [ST]. Let $\mathbb{F} = \mathbb{C}$. For a function $f \in L^1(\mathbb{R}^m)$ we define its *Fourier transform*

$$ (\mathcal{F}f)(\xi) = \hat{f}(\xi) := \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{-i \xi \cdot x} f(x) \, dx, \quad \xi \in \mathbb{R}^m, $$

where we put $\xi \cdot x = \sum_{j=1}^m \xi_j x_j$. This formula clearly defines a function $\mathcal{F}f : \mathbb{R}^m \to \mathbb{R}^m$ which is bounded by $(2\pi)^{-m/2}\|f\|_1$. Actually, $\mathcal{F}f$ belongs to $C_0(\mathbb{R}^m)$ by Corollary 3.8 in [ST]. For further investigations the *Schwartz space*

$$ S_m = \{ f \in C^\infty(\mathbb{R}^m) \mid \forall k \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^m : k_{\alpha} := \sup_{x \in \mathbb{R}^m} |x|_2^k \|\partial^\alpha f(x)\| < \infty \}. $$

turns out to be very useful.

By Remark 3.6 of [ST] the family of seminorms $\{ \|p_{k,\alpha}\| \mid k \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^m \}$ yields a complete metric on $S_m$. The space $C_c^\infty(\mathbb{R}^m)$ and also the Gaussian
\[ \gamma(x) = e^{-\frac{1}{2}|x|^2} \] are contained in \( S_m \). Proposition 3.10 of [ST] shows that the restriction \( \mathcal{F} : S_m \to S_m \) is bijective and continuous with the continuous inverse given by

\[ \mathcal{F}^{-1}g(y) = (\mathcal{F}g)(-y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{iy\cdot\xi}g(\xi)\,d\xi, \quad y \in \mathbb{R}^n, \]

for \( g \in S_m \). In our context the core fact is Plancherel’s theorem, which says that one can extend \( \mathcal{F} : S_m \to S_m \) to a unitary map \( \mathcal{F}_2 : L^2(\mathbb{R}^m) \to L^2(\mathbb{R}^m) \) satisfying \( \mathcal{F}_2 f = \mathcal{F} f \) for \( f \in L^2(\mathbb{R}^m) \), see Theorem 3.11 in [ST]. We stress that \( \mathcal{F}_2 f \) is not given by the above integral formula if \( f \in L^2(\mathbb{R}^m) \) is not integrable; but we still write \( \mathcal{F} \) instead of \( \mathcal{F}_2 \) and \( \hat{f} \) instead of \( \mathcal{F}_2 f \). We again have the inversion formula \( \mathcal{F}^{-1}g(y) = \mathcal{F}g(-y) \) for \( y \in \mathbb{R}^n \) and \( g \in L^2(\mathbb{R}^m) \).

To apply the Fourier transform to differential operators, one needs the following properties. Lemma 3.7 of [ST] yields the differentiation formulas

\[ \mathcal{F}(\partial^\alpha u) = i^{\alpha}\xi^\alpha \mathcal{F}u \quad \text{and} \quad \partial^\alpha \mathcal{F}u = (-i)^{\alpha} \mathcal{F}(x^\alpha u) \] (1.23)

for \( u \in S_m \) and \( \alpha \in \mathbb{N}_+^m \), where we write \( \xi^\alpha \) for the map \( \xi \mapsto \xi_1 \cdots \xi_m \) and so on. Due to Theorem 3.25 in [ST], we have the crucial description

\[ W^{k,2}(\mathbb{R}^m) = \{ u \in L^2(\mathbb{R}^m) : |\xi|^k \hat{u} \in L^2(\mathbb{R}^m) \} \] (1.24)

with equivalent norms \( \|u\|_{k,2} \simeq \|u\|_2 + \||\xi|^k \hat{u}\|_2 \) for \( k \in \mathbb{N}_0 \), and also that the first part of (1.23) is true for \( u \in W^{\alpha,2}(\mathbb{R}^m) \).

To check the range condition for the Laplacian on \( \mathbb{R}^m \), we take \( f \in L^2(\mathbb{R}^m) \) and \( \lambda \in \mathbb{C} \setminus \mathbb{R}_{\leq 0} \). We look for a function \( u \in W^{2,2}(\mathbb{R}^m) \) satisfying \( \lambda u - \Delta u = f \). Because of formula (1.23), such a solution fulfills the problem

\[ \hat{\dot{f}} = \lambda \hat{u} - \sum_{k=1}^m i^2\xi_k^2 \hat{u} = (\lambda + |\xi|^2_{\mathbb{H}})\hat{u}. \]

The unique function solving this equation is given by \( \hat{u} = (\lambda + |\xi|^2_{\mathbb{H}})^{-1}\hat{\dot{f}} \), which is an element of \( L^2(\mathbb{R}^m) \) by (1.26) below and since \( \hat{\dot{f}} \in L^2(\mathbb{R}^m) \). We now define

\[ u := R(\lambda)f = \mathcal{F}^{-1}\left( \frac{\hat{\dot{f}}}{\lambda + |\xi|^2_{\mathbb{H}}} \right). \] (1.25)

Since \( \mathcal{F} \) is bijective on \( L^2(\mathbb{R}^m) \), this function belongs to \( L^2(\mathbb{R}^m) \). Based on these observations we can now establish our first generation result for the Laplacian.

**Example 1.45.** Let \( E = L^2(\mathbb{R}^m) \) with \( F = \mathbb{C} \), \( A = \Delta \), and \( D(A) = W^{2,2}(\mathbb{R}^m) \). The operator \( A \) is self-adjoint and generates a contraction semigroup on \( E \). Moreover, its graph norm is equivalent to that of \( W^{2,2}(\mathbb{R}^m) \).

**Proof.** The asserted norm equivalence follows from (1.24) and Plancherel’s theorem since \( \mathcal{F}(\Delta u) = -|\xi|^2_{\mathbb{H}} \hat{u} \) by (1.23) for \( u \in W^{2,2}(\mathbb{R}^m) \). The domain \( D(A) \) is dense in \( E \) since it contains \( C^\infty_c(\mathbb{R}^m) \), see Proposition 4.13 of [FA].

Let \( f \in E \) and \( \lambda \in \mathbb{C} \setminus \mathbb{R}_{\leq 0} \). To check the range condition, we estimate

\[ \left| \frac{\hat{\dot{f}}}{\lambda + |\xi|^2_{\mathbb{H}}} \right| \leq c_\lambda |\hat{\dot{f}}| \quad \text{with} \quad c_\lambda := \begin{cases} \frac{1}{|\lambda|}, & \Re \lambda \geq 0, \\ \frac{1}{|\Im \lambda|}, & \Re \lambda < 0. \end{cases} \] (1.26)
Since \( \hat{f} \in E \) by Plancherel’s theorem, the term in parentheses in (1.25) thus belongs to \( E \). Using Plancherel once more, we now define \( u = R(\lambda)\hat{f} \in E \) as in (1.25) and estimate
\[
\|u\|_2 = \|\hat{u}\|_2 \leq c_\lambda \|f\|_2; \quad \text{i.e.,} \quad \|R(\lambda)\|_{B(E)} \leq c_\lambda. \tag{1.27}
\]
We further compute
\[
|\xi|^2 |\hat{u}| = |\xi|^2 \frac{\lambda + |\xi|^2}{\lambda^2 + |\xi|^2} |\hat{f}| \leq (1 + |\lambda| c_\lambda) |\hat{f}|. \]
Equation (1.24) now implies that \( u \) belongs to \( W^{2,2}(\mathbb{R}^m) \) with norm \( \|u\|_{2,2} \leq c'_\lambda \|f\|_2 \). Therefore \( R(\lambda) \) maps \( E \) continuously into \( W^{2,2}(\mathbb{R}^m) \). From the first part of (1.23) and (1.25) we then deduce
\[
\mathcal{F}(\lambda u - \Delta u) = (\lambda + |\xi|^2)\hat{u} = \hat{f},
\]
obtaining \( \lambda u - \Delta u = f \) in \( E \) by the bijectivity of \( \mathcal{F} \).

This means that \( \lambda I - A \) is bijective with the bounded inverse \( R(\lambda) \). Hence, \( A \) is closed by Remark 1.16. Moreover, the spectrum \( \sigma(A) \) is contained in \( \mathbb{R}_{\leq 0} \), and inequality (1.27) implies the Hille–Yosida estimate for \( \lambda > 0 \). As a result, \( E \) generates a contraction semigroup on \( A \) by Theorem 1.26.

Let \( u, v \in W^{2,2}(\mathbb{R}^m) \). Gauß’ formula (1.22) and \( \Delta = \text{div} \nabla \) yield
\[
(Au|v) = \int_{\mathbb{R}^m} \text{div}(\nabla u) v \, dx = -\int_{\mathbb{R}^m} \nabla u \cdot \nabla v \, dx = \int_{\mathbb{R}^m} u \text{div}(\nabla v) \, dx = (u|Av),
\]
so that \( A \) is symmetric. Since \( \sigma(A) \subseteq \mathbb{R}_{\leq 0} \), the self-adjointness of \( A \) finally follows from Theorem 4.7 of [ST]. \( \square \)

We stress that the above norm equivalence says that one can bound in \( L^2(\mathbb{R}^m) \) each derivative of \( u \in D(A) \) up to order 2 just by \( u \) and the sum \( \Delta u \) of unmixed second derivatives. In particular, if \( m \geq 2 \) the possible cancellations in \( \Delta u \) do not play a role! On \( C_0(\mathbb{R}^m) \) the situation is quite different. Here we use of the version of the Lumer–Phillips theorem involving the closure. With the available tools we can compute its domain only for \( m = 1 \), see the comments below.

**Example 1.46.** Let \( E = C_0(\mathbb{R}) \), \( D(A_0) = \{ u \in C^2(\mathbb{R}) \mid u, \Delta u \in E \} \), and \( A_0 = \Delta \). The operator \( A_0 \) has a closure \( A \) that generates a contraction semigroup on \( E \). If \( m = 1 \), we have \( Au = u'' \) and \( D(A) = D(A_0) = C^2(\mathbb{R}) := \{ u \in C^2(\mathbb{R}) \mid u, u', u'' \in E \} \).

**Proof.** 1) The domain of \( A_0 \) is dense in \( E \) because of \( C^\infty_c(\mathbb{R}^m) \subseteq D(A_0) \), cf. the proof of Proposition 4.13 in [FA]. Let \( u \in D(A_0) \). Example 1.30 says that the functional \( \varphi = u(x_0)\delta_{x_0} \) belongs to \( J(u) \), where \( x_0 \in \mathbb{R}^m \) satisfies \( |u(x_0)| = \|u\|_\infty \). Setting \( h = \text{Re}(u(x_0)u) \in D(A_0) \), we obtain
\[
\text{Re}(A_0 u, \varphi) = \text{Re}(u(x_0)\Delta u(x_0)) = \Delta h(x_0).
\]
As in Example 1.33 we see that \( h(x_0) \) is a maximum of \( h \). By Analysis 2, the matrix \( D^2 h(x_0) \) is thus negative semidefinite and hence \( \Delta h(x_0) = \text{tr}(D^2 h(x_0)) \leq 0 \); i.e., \( A_0 \) is dissipative.

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3Actually we have the equality \( \sigma(A) = \mathbb{R}_{\leq 0} \) by Example 3.47 in [ST].
Let \( f \in S_m \) and define \( u = R(1)f \) by (1.25). Lemma 3.7 in [ST] implies that \( u \) is an element of \( S_m \subseteq D(A_0) \). As seen in the previous proof we have \( u - \Delta u = f \), so that \( R(J - A_0) \) contains the dense subspace \( S_m \). The first assertion now follows from the Lumer–Phillips Theorem 1.39.

2) Let \( m = 1 \) and \( u \in D(A) \). Because of \( A = \mathcal{A}_0 \) there are functions \( u_n \in D(A_0) \) with \( u_n \rightarrow u \) and \( u''_n \rightarrow Au \) in \( E \) as \( n \rightarrow \infty \). We further need to control the first derivative. For later use, the argument is presented in somewhat greater generality. We look at an interval \( J \) of length \( |J| > 0 \), a function \( v \in C^2(J) \) with bounded \( v \) and \( v'' \), \( \delta \in (0, |J|) \), and points \( r, s \in J \) with \( \delta < s - r < 2\delta \). Taylor’s theorem provides a number \( \sigma \in (r, s) \) such that

\[
\begin{align*}
v(s) &= v(r) + v'(r)(s - r) + \frac{1}{2}v''(\sigma)(s - r)^2, \\
v'(r) &= \frac{v(s) - v(r)}{s - r} - \frac{1}{2}v''(\sigma)(s - r).
\end{align*}
\]

The last equation yields

\[
|v'(r)| \leq \frac{2}{\delta} \max_{\tau \in [r, r + \delta]} |v(\tau)| + \delta \max_{\tau \in [r, r + \delta]} |v''(\tau)|, \tag{1.28}
\]

\[
\|v'\|_{\infty} \leq \frac{2}{\delta} \||v|\|_{\infty} + \delta \||v''|\|_{\infty}.
\]

Inserting \( v = u_n \) into (1.28), we infer that \( u_n' \in E \). With \( v = u_n - u_m \), it also follows that \( u_n' \) tends in \( E \) to a map \( f \). As a result, \( u \) belongs to \( C^1(\mathbb{R}) \) with \( u' = f \in E \). The limit \( u'' \rightarrow Au \) in \( E \) then leads to \( u \in C^2_0(\mathbb{R}) \) and \( Au = u'' \). \( \square \)

For \( m \geq 2 \) the domain \( D(A) \) is not \( C^2_0(\mathbb{R}^m) \) in Example 1.46. To make this fact plausible, we look at the function

\[
\tilde{u}(x, y) = \begin{cases} (x^2 - y^2) \ln(x^2 + y^2), & (x, y) \neq (0, 0), \\
0, & (x, y) = (0, 0). \end{cases}
\]

By a straightforward computation, the second derivative

\[
\partial_{xx}\tilde{u}(x, y) = 2\ln(x^2 + y^2) + \frac{4x^2}{x^2 + y^2} + \frac{(6x^2 - 2y^2)(x^2 + y^2) - 4x^2(x^2 - y^2)}{(x^2 + y^2)^2}
\]

is unbounded on \( B(0, 1) \), but the functions \( \tilde{u}, \nabla \tilde{u}, \) and \( \Delta \tilde{u}(x, y) = 8x^2 - \frac{y^2}{x^2 + y^2} \) are bounded on \( B(0, 2) \). To deal with larger \( (x, y) \), we simply take a smooth map \( \varphi \) with \( \text{supp } \varphi \subseteq B(0, 2) \) which is equal to 1 on \( B(0, 1) \). Then the functions \( u = \varphi \tilde{u} \) and \( \Delta u = \varphi \Delta \tilde{u} + 2\nabla \varphi \cdot \nabla \tilde{u} + \tilde{u} \Delta \varphi \) are bounded and have compact support on \( \mathbb{R}^m \), but \( u \) does not belong to \( W^{2, \infty}(\mathbb{R}^m) \). (One can construct an analogous example in \( C_0(\mathbb{R}^m) \) instead of \( L^\infty(\mathbb{R}^m) \) using \( \ln \).

With much more effort and deeper tools, Corollary 3.1.9 in [Lu] shows that the operator \( A_1 = \Delta \), with domain

\[
D(A_1) = \{ u \in C_0(\mathbb{R}^m) \mid \forall p \in (1, \infty), r > 0 : u \in W^{2,p}(B(0, r)), \Delta u \in C_0(\mathbb{R}^m) \}
\]

is closed in \( E \) and that \( \rho(A_1) \) contains a halfline \( (\omega, \infty) \). Since \( D(A_0) \subseteq D(A_1) \), we first obtain \( A = \mathcal{A}_0 \subseteq A_1 \), and then \( A = A_1 \) by Lemma 1.23.
B) The second derivative on an interval. In the one-dimensional case the equation \( \lambda u - \Delta u = f \) with boundary conditions becomes an ordinary boundary value problem. In [ST] we have solved this problem explicitly and thus obtained a concrete formula for the resolvent. We only look at Dirichlet conditions, others can be treated similarly. We start with the sup-norm case.

Example 1.47. Let \( E = C_0(0,1) \), \( D(A) = \{ u \in C^2(0,1) \mid u, u'' \in E \} \), and \( Au = u'' \). Then the operator \( A \) generates a contraction semigroup on \( E \), and its graph norm is equivalent to that of \( C^2([0,1]) \).

Proof. The equivalence of the norms can be deduced from (1.28), which is also true with intervals \( [r - \delta, r] \subseteq (0,1) \). Let \( f \in E \). Take \( \varepsilon > 0 \). As in Example 1.8 we find a map \( f \in C_c(0,1) \) with \( \| f - \hat{f} \|_\infty \leq \varepsilon \). Moreover, proceeding as in the proof of Proposition 4.13 in [FA] one constructs a function \( g \in C_c^\infty(0,1) \subseteq D(A) \) satisfying \( \| \hat{f} - g \|_\infty \leq \varepsilon \). Hence, \( A \) is densely defined. The dissipativity of \( A \) is shown as in Example 1.46, where the argument \( x_0 \) of the maximum of \( |u| \) belongs to \( (0,1) \) since the cases \( x_0 \in \{0,1\} \) are excluded by the boundary conditions.

Let \( f \in E \). If \( F = \mathbb{C} \), take \( \lambda \in \mathbb{C} \setminus \mathbb{R}_{\leq 0} \) and \( \sqrt{\lambda} =: \mu \in \mathbb{C}_+ \). If \( F = \mathbb{R} \), let \( \lambda > 0 \) and \( \mu = \sqrt{\lambda} > 0 \). Set

\[
(a(f, \mu)) = \frac{1}{2\mu(e^-\mu - e^\mu)} \left( e^{\mu\tau} \int_0^1 (e^{\mu\tau} - e^{-\mu\tau}) f(\tau) \, d\tau - \int_0^1 (e^{\mu\tau} - e^{-\mu\tau}) f(\tau) \, d\tau \right).
\]

In Example 2.16 of [ST] we have shown that the map \( u : [0,1] \to F \);

\[
u(s) = a(f, \mu)e^{\mu s} + b(f, \mu)e^{-\mu s} + \frac{1}{2\mu} \int_0^1 e^{-\mu|s-\tau|} f(\tau) \, d\tau,
\]

(1.29)

belongs to \( C^2([0,1]) \) and satisfies \( \lambda u - u'' = f \) as well as \( u(0) = u(1) = 0 \). Hence, \( u \) an element of \( D(A) \) and \( \lambda I - A \) is surjective. The Lumer–Phillips Theorem 1.39 now implies that \( A \) is closed and generates a contraction semigroup on \( E \). Moreover, (1.29) gives the resolvent via \( R(\lambda, A)f = u \).

We next show the analogous result for \( L^p(0,1) \). Here we check dissipativity on \( L^p \) now also for \( p \neq 2 \).

Example 1.48. Let \( E = L^p(0,1) \), \( 1 \leq p < \infty \), \( Au = \partial^2 u \), and

\[ D(A) = \{ u \in W^{2,p}(0,1) \mid u(0) = u(1) = 0 \} = W^{2,p}(0,1) \cap W_0^{1,p}(0,1). \]

(Remark 1.41 yields \( W^{1,p}(0,1) \) \( \to C([0,1]) \).) The operator \( A \) generates a contraction semigroup on \( E \) and its graph norm is equivalent to \( \| \cdot \|_{2,p} \).

Proof. The last assertion follows from Proposition 3.37 of [ST], cf. (1.28). The domain \( D(A) \) is dense due to Proposition 4.13 in [FA] since it contains \( C_c^\infty(0,1) \). One can extend the operator \( R(\lambda, A) \) from (1.29) to a map \( R(\lambda) \) on \( E = L^p(0,1) \) for \( \lambda \in F \setminus \mathbb{R}_{\leq 0} \) where \( \mu = \sqrt{\lambda} \in \mathbb{F}_+ \). Omitting a factor, we rewrite the last summand of (1.29) as

\[
v(s) := \int_s^1 e^{-\mu|s-\tau|} f(\tau) \, d\tau = e^{-\mu s} \int_0^s e^{\mu \tau} f(\tau) \, d\tau + e^{\mu s} \int_s^1 e^{-\mu \tau} f(\tau) \, d\tau
\]
for \( f \in E \) and \( s \in [0,1] \). Using (1.21), we can now differentiate
\[
v'(s) = -\mu e^{-\mu s} \int_0^s e^{\mu \tau} f(\tau) \, d\tau + f(s) + \mu e^{\mu s} \int_s^1 e^{-\mu \tau} f(\tau) \, d\tau - f(s).
\]
Since the summands \( \pm f(s) \) cancel, \( v \) belongs to \( C^1([0,1]) \). Analogously one checks that the weak derivative \( \partial^2 v \in L^p(0,1) \) exists and satisfies \( \lambda v - \partial^2 v = 2\mu f \). The other two summands \( u_j \) in (1.29) are smooth and fulfill \( \lambda u_j = u''_j \). Moreover, we have \( u(0) = 0 = u(1) \) as in Example 2.16 of [ST] for \( f \in C([0,1]) \).

Summing up, \( u = R(\lambda)f \) is an element of \( D(A) \) and solves \( \lambda u - Au = f \).

To apply the Lumer–Phillips theorem, it remains to check the dissipativity. To avoid certain technical problems we restrict ourselves to \( p \in [2,\infty) \), see an example in Section 2.3 for the case \( p \in [1,2) \). Let \( u \in D(A) \). We set \( w = |u|^{p-2} \pi \) which belongs to \( J(u) \) by Example 1.30. Note that \( w(0) = 0 = w(1) \) by the boundary conditions. Remark 1.41 yields the embedding \( W^{2,p}(0,1) \hookrightarrow C^1([0,1]) \). Since \( p \geq 2 \), we can now compute
\[
\begin{align*}
w' &= \frac{d}{ds} ((u\pi)^{\frac{p-2}{2}} \pi) = |u|^{p-4} |\pi|^2 \pi' + \frac{p-2}{2} |u|^2 |\pi'|^{\frac{p-2}{2}} (u' \pi + u \pi') \\
&= |u|^{p-4} (|\pi|^2 \pi' + (p-2) \text{Re}(\pi u') \pi)
\end{align*}
\]
Formula (1.22) and \( w(0) = 0 = w(1) \) now imply
\[
\text{Re}(Au, w) = \text{Re} \left( \int_0^1 \partial^2 uw \, ds \right) = -\int_0^1 \text{Re}(u'w') \, ds + \text{Re}(u'w) \bigg|_0^1 \\
&= -\int_0^1 |u|^{p-4} (|\pi u'|^2 + (p-2)(\text{Re}(\pi u'))^2) \, ds \\
&= -\int_0^1 |u|^{p-4} (|\text{Im}(\pi u')|^2 + (p-1)(\text{Re}(\pi u'))^2) \, ds \leq 0.
\]

Theorem 1.39 now implies the assertion, and \( R(\lambda) \) is the resolvent of \( A \). \( \square \)

We add an example where \( A \) is dissipative, but not a generator, and \( A^* \) is not dissipative, cf. Corollary 1.40. This can happen since we impose too many (four) boundary conditions instead of two (for two derivatives) as above.

**Example 1.49.** Let \( E = L^2(0,1) \), \( Au = \partial^2 u \), and
\[
D(A) = \{ u \in W^{2,2}(0,1) \mid u(0) = u'(0) = u(1) = u'(1) = 0 \} = W_0^{2,2}(0,1).
\]
(The last space is the closure of \( C_c^\infty(0,1) \) in \( W^{2,2}(0,1) \); the final equality follows as in Remark 1.41.) Then \( A \) is closed, densely defined, dissipative, and symmetric, but not a generator and not self-adjoint, and \( A^* \) is not dissipative.

**Proof.** The density of \( D(A) \) follows again from Proposition 4.13 in [FA]. To check closedness, take maps \( u_n \in D(A) \) such that \( u_n \to u \) and \( u''_n \to v \) in \( E \) as \( n \to \infty \). Proposition 3.37 in [ST] then shows that also \( (u_n') \) converges in \( E \), cf. (1.28). From Remark 1.41 we now deduce that \( u \) belongs to \( W^{2,2}(0,1) \) and \( u_n \to u \) in \( W^{2,2}(0,1) \). The boundary conditions for \( u_n \) transfer to \( u \) via the limits of \( (u_n) \) and \( (u'_n) \) since \( W^{1,2}(0,1) \hookrightarrow C([0,1]) \) by Remark 1.41. Hence, \( u \) belongs to \( D(A) \) and \( A \) is closed.
Let $u \in D(A)$ and $v \in W^{2,2}(0,1)$. Using integration by parts (1.22) and the boundary conditions of $u$, we compute

$$(Au|v) = \int_0^1 \partial^2 u \tau \, ds = - \int_0^1 u' \tau' \, ds + u|\tau|_0^1 = \int_0^1 u \partial^2 \tau \, ds + u|\tau|_0^1 = (u|\partial^2 v).$$

Hence, $A$ is symmetric (take $v \in D(A)$) and dissipative (take $v = u$). Moreover, the operator $\partial^2$ with domain $W^{2,2}(0,1)$ is a restriction of $A'$ and also of $A^*$. Let $v \in D(A^*)$. As in Example 1.42 one can see that $A^* v \in E$ is the second weak derivative of $v \in E$. Proposition 3.37 in [ST] thus implies that $v$ belongs to $W^{2,2}(0,1)$. It follows $A^* = \partial^2$ with $D(A^*) = W^{2,2}(0,1) \neq D(A)$. Hence, $A$ is not self-adjoint.

Since $\partial^2 e^{is} = \lambda e^{is}$ for $\mu = \sqrt{\lambda}$ and $\lambda \in \mathbb{F} \setminus \mathbb{R}_{\leq 0}$, the operator $\lambda I - A^*$ is not injective. As a result, $A^*$ is not dissipative in view of Proposition 1.38 and the spectrum of $A$ contains $\mathbb{F} \setminus \mathbb{R}_{\leq 0}$ by Theorem 1.24 of [ST]. In particular, $A$ is not a generator. \hfill \Box

C) Operators defined by sesquilinear forms. In many applications one looks at the Laplacian or related ‘elliptic operators in divergence form’ on a domain in $\mathbb{R}^m$. In an $L^2$-context we can show generation properties of these operators, though it is not possible to describe their domains precisely by our means. (This point is discussed below.) We restrict ourselves again to Dirichlet boundary conditions for simplicity; others are treated in the exercises. Most of the results are presented for a larger class of operators (defined by sesquilinear forms) since the analysis is almost the same as for the Laplacian itself. The main tool is the Lax–Milgram lemma which is a core consequence of Riesz’ representation of Hilbert space duals.

**Theorem 1.50.** Let $Y$ be a Hilbert space and $g : Y \times Y \to \mathbb{F}$ be a sesquilinear map which is bounded and strictly accretive; i.e.,

$$|g(x,y)| \leq C\|x\|\|y\| \quad \text{and} \quad \Re g(y,y) \geq \eta\|y\|^2$$

for all $x,y \in Y$ and some constants $C, \eta > 0$. Then for each functional $\psi \in Y^*$ there is a unique vector $z \in Y$ satisfying $g(y,z) = \psi(y)$ for all $y \in Y$. The map $\psi \mapsto z$ is antilinear and bounded.

**Proof.** Let $y \in Y$. The map $\varphi_y := g(\cdot,y)$ belongs to $Y^*$ with $\|\varphi_y\| \leq C\|y\|$. Riesz’ Theorem 3.10 in [FA] yields a unique element $Sy$ of $Y$ satisfying $\|Sy\| = \|\varphi_y\| \leq C\|y\|$ and $(\cdot|Sy) = \varphi_y$. Moreover, $S : Y \to Y$ is linear since both maps $y \mapsto \varphi_y$ and $\varphi_y \mapsto Sy$ are antilinear in $y$. We next estimate

$$\eta\|y\|^2 \leq \Re g(y,y) = \Re(y|Sy) \leq |(y|Sy)| \leq \|y\|\|Sy\|,$$

and hence $\|Sy\| \geq \eta\|y\|$ for every $y \in Y$. As a consequence, $S$ is bounded, injective and has a closed range $R(S)$ by Corollary 4.31 in [FA]. For a vector $y \perp R(S)$ we also obtain

$$0 = (y|Sy) = \Re(y|Sy) = \Re g(y,y) \geq \eta\|y\|^2 \quad (1.30)$$

so that $y = 0$. It follows that $R(S) = Y$ by Theorem 3.8 in [FA] (or a corollary to Hahn–Banach), and so $S$ is invertible with $\|S^{-1}\| \leq \frac{1}{\eta}$. 


Let \( \psi \in Y^* \). There is a unique vector \( v \in Y \) such that \( \psi = (\cdot | v) \) thanks to Riesz’ theorem. The above construction implies the identity
\[
\langle y, S^{-1}v \rangle = \langle y | SS^{-1}v \rangle = \langle y | v \rangle = \psi(y)
\]
for all \( y \in Y \). We set \( z = S^{-1}v = S^{-1}\Phi_Y^{-1}\psi \), where \( \Phi_Y : Y \to Y^* \) denotes the antilinear isomorphism from Riesz’ theorem.

Let also \( w \in Y \) satisfy \( \langle y, w \rangle = \psi(y) \) for all \( y \in Y \). Setting \( y = z - w \), we infer
\[
0 = \langle z - w, z - w \rangle \geq \eta \|z - w\| \quad \text{and thus } w = z.
\]

In the typical applications of Theorem 1.50, \( Y \) is taken as a subspace of \( W^{1,2}(G) \) for an open set \( G \subseteq \mathbb{R}^m \) (say, with a regular boundary), where we focus on \( Y = W^{1,2}_0(G) \). One is then mainly interested in properties related to the \( L^2 \)-norm, so that one also looks at \( L^2(G) \). We note that \( W^{1,2}_0(G) \) is densely embedded in \( L^2(G) \). We extend the above setting to the cover this framework.

Let \( \varrho : Y \times Y \to F \) be as given in Theorem 1.50. We also assume that \( Y \) is densely embedded in a Hilbert space \( X \) by \( J_Y : Y \to X \). (Often we omit \( J_Y \) in our notation, in our examples it is just the inclusion.) By means of the isometric, antilinear Riesz’ isomorphism \( \Phi = \Phi_X : X \to X^* \) from Theorem 3.10 in [FA], we identify \( X \) and \( X^* \) most of the time. Proposition 5.46 in [FA] yields the dense embeddings
\[
J_Y^* : X^* \hookrightarrow Y^*, \quad Y \hookrightarrow X \equiv X^* \hookrightarrow Y^*,
\]
where we have \( \langle x, J_Y^* \Phi x \rangle_Y = \langle J_Y y | x \rangle_X = \langle y | x \rangle_X \) for \( x \in X \) and \( y \in Y \).

We stress that we not identify \( Y \) with \( Y^* \) since this would require the Riesz isomorphism \( \Phi_Y \), which is quite different from \( \Phi_X \) in our examples.

To associate an operator in \( X \) with \( \varrho \), we define
\[
\mathcal{D}(A) = \{ x \in Y \mid \exists c > 0 \ \forall y \in Y : |\varrho(y, x)| \leq c \|y\| \}.
\]

Since \( Y \) is dense in \( X \), we can extend \( -\varrho(\cdot, x) \) to an element \( \varphi_x \) of \( X^* \). Thanks to Riesz’ theorem it can be represented by a unique element \( Ax \in X \) in the sense that
\[
\forall y \in Y : \quad -\varrho(y, x) = \langle y | Ax \rangle_X = \langle y, J_Y^* \Phi Ax \rangle_Y = \langle y, Ax \rangle_Y.
\]

In the last equality we consider \( Ax \in X \) as element in \( Y^* \), as one usually does. Formula (1.34) determines \( Ax \) uniquely by the density of \( Y \). Moreover, \( A \) is linear as in the proof of Theorem 1.50. We further need the adjoint form
\[
\varrho' : X \times Y \to F; \quad \varrho'(y, z) = \varrho(z, y).
\]

Note that \( \varrho' \) is also sesquilinear, bounded, and strictly accretive. We call \( \varrho \) symmetric if \( \varrho = \varrho' \). We can now show that \( A \) has very convenient properties.

**Theorem 1.51.** Let \( X \) and \( Y \) be Hilbert spaces with an embedding \( J_Y : Y \hookrightarrow X \) having dense range and norm \( \kappa \). Assume that \( \varrho : X \times Y \to F \) is sesquilinear, bounded, and strictly accretive. Define \( A \) by (1.33) and (1.34). Then \( A \) generates an \( \omega \)-contraction semigroup on \( X \) with \( \omega := \kappa^{-2} \), and hence \( s(A) \leq \omega < 0 \). The adjoint \( A' \) of \( A \) is given by the form \( \varrho' \) as in (1.33) and (1.34), so that \( A \) is self-adjoint if \( \varrho \) is symmetric.
Proof. 1) Let \( x \in D(A) \). The definition \((1.34)\) and the assumptions yield
\[
\Re(Ax|x)_X = \Re(x|Ax)_X = -\Re a(x, x) \leq -\eta \|x\|_Y^2 \leq -\eta \kappa^{-2} \|x\|_X^2 \leq 0,
\]
so that \( A \) and \( A_\omega := A + \eta \kappa^{-2}I \) are dissipative. As \( A = A_\omega - \eta \kappa^{-2}I \), Proposition 1.38 a) yields the injectivity of \( A \).

Take \( z \in X \) and set \( \psi = (\cdot | z)_X \in X^* \rightarrow Y^* \). Theorem 1.50 thus yields an element \( x \in Y \) with \( a(y, x) = -(y|z)_X \) for all \( y \in Y \). Since then \( |a(y, x)| \leq \|z\|_X \|y\|_X \) by Cauchy–Schwarz, \( x \) belongs to \( D(A) \). Moreover, we have \( (y|z)_X = (y|Ax)_X \) due to \((1.34)\) and so \( Ax = z \) due to the density of \( Y \). Summing up, \( A \) is bijective and its inverse is bounded thanks to the dissipativity of \( A_\omega \) and Proposition 1.38 a).

To check the density of \( D(A) \) in \( X \), let \( z \in X \) be orthogonal to \( D(A) \) and set \( y = A^{-1}z \in D(A) \). Strict accretivity then implies
\[
0 = (y|Ay)_X = -a(y, y) = -\Re a(y, y) \leq -\eta \|y\|_Y^2
\]
so that \( y = 0 \) and \( z = 0 \). Hence, \( D(A) \) is dense in \( X \) by Theorem 3.8 in [FA]. Theorem 1.39 now shows that \( A \) generates an \( \omega \)-contraction semigroup on \( X \) and \( s(A) \leq \omega < 0 \).

2) We next compute \( A' \). Let \( A^\dagger \) be the operator associated with \( a' \). Take \( x \in D(A) \) and \( y \in D(A^\dagger) \). The definitions yield
\[
(Ax|y)_X = (y|Ax)_X = -a(y, x) = -a'(x, y) = (x|A^\dagger y)_X;
\]
i.e., \( A^\dagger \subseteq A' \). Both operators are invertible by part 1) and also (4.3) of [ST]. Lemma 1.23 now yields \( A^\dagger = A' \). \( \Box \)

Chapter VI of [Ka] or Chapter 1 of [Ou] provide more general versions of this result and many related facts, see also Section 2.3 below.

The next straightforward application is a version of Example 5.11 of [ST]. Here we need Poincaré’s inequality. For every bounded open set \( G \subseteq \mathbb{R}^m \) and \( p \in [1, \infty) \), there is a constant \( c = c(G, p) > 0 \) such that
\[
\|u\|_p \leq c \|\nabla u\|_2
\]
(1.35)
for all \( u \in W^{1,p}_0(G) \), see Theorem 3.36 in [ST]. The latter space is the closure of \( C^\infty_0(G) \) in \( W^{1,p}(G) \).

Example 1.52. Let \( G \subseteq \mathbb{R}^m \) be open and bounded, and the coefficients \( a_{jk} \in L^\infty(G, \mathbb{F}) \) for \( j, k \in \{1, \ldots, m\} \) be strictly elliptic; i.e.,
\[
\Re \sum_{j,k=1}^m a_{jk}(x) z_j z_k \geq \eta |z|^2_2
\]
(1.36)
for some \( \eta > 0 \), all \( z \in \mathbb{F}^m \), and a.e. \( x \in G \). We write \( a = (a_{jk})_{j,k} \). Let \( E = L^2(G) \) and \( V = W^{1,2}_0(G) \), where \( \|\cdot\| = \|\cdot\|_2 \) and \( V \) is equipped the norm \( \|v\|_V = \|\nabla u\|_2 \) which is equivalent to the usual one by (1.35). We define
\[
a : V \times V \rightarrow \mathbb{F}; \quad a(v, w) = \sum_{j,k=1}^m \int_G \partial_j v \bar{\partial}_j w \bar{w} \ dx.
\]
Then the conditions of Theorem 1.51 are satisfied with \( C = \|a\|_\infty \), \( \eta \) and \( \kappa = c(G, 2) \), where \( |a(x)|_2 \) is the operator norm for \( |z|^2_2 \) in \( \mathbb{F}^m \).
So $a$ induces an invertible generator $A$ of a contraction semigroup on $E$, and $A$ is self-adjoint if $a$ is Hermitian. After complex conjugation, for $f \in E$ the function $u \in D(A)$ with $Au = f$ is given by

$$
\int_G \nabla f \, dx = - \sum_{j,k=1}^m \int_G \partial_j a_{jk} \partial_k u \, dx, \tag{1.37}
$$

for all $v \in V$, where $D(A)$ is the set of $u \in V$ such that the right-hand side is bounded by $c\|v\|_2$ for some $c = c(a,u)$ and all $v \in V$. ⊢

One calls $u \in V$ satisfying (1.37) a ‘weak solution’ of $Au = f$. To obtain a better understanding of this equation, we impose stronger conditions on $G$ and $a$. In particular, let $a_{jk} \in W^{1,\infty}(G)$ for $j, k \in \{1, \ldots, m\}$. We then define an ‘elliptic’ operator in ‘divergence form’ with Dirichlet boundary conditions via

$$
A_0 u = \text{div}(a \nabla u) = \sum_{j,k=1}^m \partial_j (a_{jk} \partial_k u), \quad u \in D(A_0) = W^{2,2}(G) \cap W^{1,2}_0(G).
$$

One sets $W^{-1,2}(G) = W^{1,2}_0(G)^*$ if $W^{1,2}_0(G)$ is equipped with the full norm $\| \cdot \|_1$. We write $V^*$ instead since we use the equivalent norm $\| \cdot \|_V$ on $V = W^{1,2}_0(G)$. The second part of the next result is also true in the framework of Theorem 1.51, cf. Section 1.4.2 in [Ou].

**Example 1.53.** In addition to the hypotheses of Example 1.52, we assume that $a_{jk} \in W^{1,\infty}(G)$ for all $j, k \in \{1, \ldots, m\}$ and that $\partial G$ is Lipschitz. Integrating by parts via (1.22), we then obtain $(v|A_0 u) = -g(v,u) = (v|Au)$ for all $v \in V$ and $u \in D(A_0)$. By density of $V$, we have $A_0 u = Au$ and hence $A_0 \subseteq A$.

Observe that $D(A)$ is dense in $V$ because of $C_c^\infty(G) \subseteq D(A_0) \subseteq D(A)$. For $u \in D(A)$, the definition (1.34) yields

$$
\|Au\|_{V^*} = \sup_{\|v\|_V \leq 1} |(v, Au)_V| = \sup_{\|v\|_V \leq 1} |a(v, u)| \leq C \|u\|_V.
$$

Therefore we can extend $A$ to a bounded operator $\tilde{A} : V \to V^*$, which is the ‘weak extension’ of $A$. Its range contains $L^2(G)$ and is thus dense in $V^*$. We further obtain

$$
\eta \|u\|_V^2 \leq |a(u, u)| = |(u, Au)_V| \leq \|u\|_V \|Au\|_{V^*}, \quad \eta \|u\|_V \leq \|Au\|_{V^*}.
$$

By density, the last inequality can be extended to $\|\tilde{A}u\|_{V^*} \geq \eta \|u\|_V$ for all $u \in V$. Corollary 4.31 in [FA] then implies the invertibility of $\tilde{A}$. ⊢

The equality $A_0 = A$ is not true in the above example, in general. If $\partial G \in C^2$ and $a_{jk} = a_{kj} \in C^2(\overline{G}, \mathbb{R})$, Theorem 6.3.4 of [Ev] shows that $A_0 = A$ and that the graph norm of $A$ and $\| \cdot \|_{2,2}$ are equivalent. The proof uses PDE methods.

**D) The Dirichlet–Laplacian and the wave equation.** Examples 1.52 and 1.54 can be applied to the case of $a = I$ yielding the Dirichlet–Laplacian $\Delta_D$. For later reference, we restate the results.

**Example 1.54.** Let $G \subseteq \mathbb{R}^m$ be open and bounded with Lipschitz boundary $\partial G$, $E = L^2(G)$, and $A_0 = \Delta$ with $D(A_0) = W^{2,2}(G) \cap W^{1,2}_0(G)$. Then $A_0$ is densely defined, symmetric, and dissipative. The operator $A_0$ has an extension $\Delta_D$ which is self-adjoint, invertible and generates an $\omega$-contraction semigroup,
where $\omega = -c(G, 2)^{-2} < 0$ is given by (1.35). Moreover, $D(\Delta_D)$ is densely embedded in $W^{1,2}_0(G)$.

The domain $D(\Delta_D)$ contains all maps $u \in W^{1,2}_0(G)$ for which there is a function $f =: \Delta_D u$ in $L^2(G)$ such that

$$\forall v \in W^{1,2}_0(G) : \quad (v|\Delta_D u)_{L^2} = -\int_G \nabla v \cdot \nabla u \, dx.$$ 

The operator $\Delta_D$ has an invertible bounded extension $\tilde{\Delta}_D : W^{1,2}_0(G) \to W^{-1,2}_0(G)$ (the weak Dirichlet–Laplacian) which acts as

$$\forall u, v \in W^{1,2}_0(G) : \quad \langle v, \tilde{\Delta}_D u \rangle_{W^{1,2}_0(G)} = -\int_G \nabla v \cdot \nabla u \, dx. \quad \diamond$$

The next operator will be used to solve the wave equation as explained in Section 2.1. We again write $V$ for $W^{1,2}_0(G)$ equipped with the norm $\| |\nabla v| \|^2_{L^2}$.

**Example 1.55.** Let $G \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary $\partial G$ and $\Delta_D$ be given on $L^2(G)$ by Example 1.54. Set $E = V \times L^2(G)$, $D(A) = D(\Delta_D) \times V$, and

$$A = \begin{pmatrix} 0 & I \\ \Delta_D & 0 \end{pmatrix}.$$ 

Then $A$ is skew-adjoint, and thus generates a unitary $C_0$-group on $E$ due to Stone’s Theorem 1.44. Moreover, $D(A)$ and $D(\Delta_D) \times V$ have equivalent norms.

**Proof.** Let $(u_1, v_1)$ and $(u_2, v_2)$ belong to $D(A)$. We compute

$$\langle A\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} | \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \rangle_E = \langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} | \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \rangle_E = \int_G \left( \nabla v_1 \cdot \nabla u_2 + \Delta_D u_1 v_2 \right) \, dx$$

$$= -\int_G \left( v_1 \Delta_D \nabla u + \nabla u_1 \cdot \nabla v_2 \right) \, dx = -\langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} | \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \rangle_E$$

using the scalar product of $V$ and the definition of $\Delta_D$. We thus arrive at

$$\langle A\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} | \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \rangle_E = \langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} | -A \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \rangle_E.$$ 

Hence, $-A$ is a restriction of $A'$ and so $iA \subseteq (iA)^T$. We define

$$R = \begin{pmatrix} 0 & \Delta_D^{-1} \\ I & 0 \end{pmatrix} : E \to D(\Delta_D) \times V = D(A),$$

where $\Delta_D^{-1}$ exists thanks to Example 1.54. It is easy to see that $AR = I$ and $RAw = w$ for every $w \in D(A)$. Hence, $A$ is invertible so that $0 \notin \rho(iA)$ and $iA$ is self-adjoint by Theorem 4.7 in [ST]; i.e., $A$ is skew-adjoint.

Finally, both $D(A)$ and $D(\Delta_D) \times V$ are complete and the norm of the former space is stronger than the latter one. So, the norms are equivalent by Corollary 4.29 in [FA] (to the open mapping theorem). \[\square\]
Bibliography


