ON GROWTH AND INSTABILITY FOR SEMILINEAR EVOLUTION EQUATIONS: AN ABSTRACT APPROACH

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Abstract. We propose a new approach to the study of (nonlinear) growth and instability for semilinear evolution equations with compact nonlinearities. We show, in particular, that compact nonlinear perturbations of a linear evolution equations can be treated as linear ones as far as the growth of their solutions is concerned. We obtain exponential lower bounds of solutions for initial values from a dense set if, e.g., the resolvent of the generator is unbounded on a vertical line in the right halfplane.

1. Introduction

The paper is devoted to the study of instability of solutions to semilinear evolution equations

$$x'(t) = Ax(t) + K(t, x(t)), \quad t \geq 0, \qquad x(0) = x_0 \in X, \tag{1.1}$$

on a Banach space X, where A generates a C_0-semigroup on X and K is a nonlinear map on X subject to appropriate conditions ensuring the existence of global mild solutions to (1.1). While the problem of finding instability conditions for (1.1) in terms of A and K is of fundamental importance, very few results in this direction were obtained so far even when K is stationary, i.e., $K(t, x) = K(x)$ for $t \geq 0$.

One of the basic and commonly used instability criteria is due to Shatah and Strauss, [64]. It says that the system governed by (1.1) is unstable near the origin if the spectral bound of A is greater than zero and K is small enough in a metric sense (that is, $K(x) = \mathrm{O}(\|x\|^{1+\delta})$ with $\delta > 0$ as $\|x\| \to 0$) and continuous. The notion of smallness can be refined by replacing a polynomial bound for K with a bound of more general type, but the scheme of proof remains the same all the time. For bounded A such criteria go back to [20]. This linearized instability principle appeared to be

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very useful in a great number of applications. Other versions of such results are investigated in [31], [32] and [33].

Recently, the problem of keeping control over asymptotic behavior of trajectories of linear infinite-dimensional systems under "small" nonlinear perturbations was revived in [27], [28], and [57]. There the opposite problem of stabilization by the nonlinearity was emphasized. Note that these papers treated the situation of discrete time, where a number of difficulties (e.g. failure of the spectral mapping theorem) is missing.

In this paper, we address the situation when K is a compact nonlinear map, i.e., K is small in a topological sense, but otherwise it can be large metrically. This type of perturbations allows for global instability results, instead of just local ones. Recall that the map $K : X \to X$ is called compact if it is continuous (this assumption varies) and maps bounded subsets of X to precompact sets. Assuming that K is a compact C^1-map, the paper [71] considered the case that either the (Browder) essential spectrum of A intersects the open right half-plane $\mathbb{C}_+$ or that there are infinitely many eigenvalues of A in $\varepsilon + \mathbb{C}_+$ for some $\varepsilon > 0$. Then for a residual set of initial values x_0 the mild solutions $\{x(t, x_0) : 0 \le t < T\}$ of (1.1) are unbounded on their maximal existence interval $[0, T)$, where $0 < T \le \infty$. The proof relied on a discrete version of this result: If A is a bounded linear operator on X with the essential spectral radius $r_e(A) > 1$ and $T = A + K$, then the trajectory $\bigcup_{n\in\mathbb{N}} T^n(B)$ of the unit ball $B \subset X$ is unbounded in X.

We substantially improve these results and, for the continuous time version (1.1), replace the spectral terms used in [71] by much weaker assumptions on resolvent bounds. While the results in [71] yield merely unboundedness (and then instability), we derive an optimal exponential lower bound. To the best of our knowledge, such results were absent in the literature. In fact, our approach much goes further and allows one to treat K depending on time. Moreover, the obtained results cannot be improved even for stationary linear compact perturbations, see Remark 7.13. To this aim we use techniques for the study of orbits of linear operators from [48] that, with certain modifications, work in the context of nonlinear maps as well. This approach is used for the treatment of nonlinear equations for the first time, and we hope it will be useful in a number of instances.

The following statement is a partial case of our main results. See Theorem 7.6 combined with Theorem 2.2, Proposition 6.3 and Corollary 6.6. (A linear version of this result is given in Corollary 7.11.)

Theorem 1.1. *Let A generate a C_0-semigroup on a Banach space X, and let $K : [0,\infty) \times X \to X$ be separately continuous, map bounded sets in compact ones and satisfy $\|K(t,x)\| \le c(t)\|x\|$ for some $c \in L^1_{loc}([0,\infty))$ and all $t \ge 0$ and $x \in X$. Let $a : [0,\infty) \to [0,\infty)$ be non-increasing with $\lim_{t\to\infty} a(t) = 0$, and let $r > 0$ and $t_0 = t_0(r) \ge 0$ be such that $a(t_0) < \frac{r}{2M}$.*

(i) *Assume that the resolvent $(\omega + ib - A)^{-1}$ is well-defined and unbounded for $b \in \mathbb{R}$ and some $\omega \in \mathbb{R}$. Then for every $r > 0$ there* *exists* $x_0 \in B(y,r)$ *and a mild solution* $x(t,x_0)$ *of* (1.1) *such that*

$$\|x(t,x_0)\| \geq a(t)e^{\omega t}, \qquad t \geq t_0.$$

(ii) *In any case, for every* $r > 0$ *there exists* $x_0 \in B(y,r)$ *and a mild solution* $x(t,x_0)$ *of* (1.1) *such that*

$$\|x(t,x_0)\| \geq a(t)e^{s_e(A)t}, \qquad t \geq t_0.$$

Here $s_e(A)$ *is the supremum of the real parts of* λ *from the essential spectrum of* A.

To the best of our knowledge, the result is genuinely new in three respects:

(a) it is the first lower bound for global growth of solutions to a general class of nonlinear evolution equations in the literature (and in fact, it is new even in the setting of linear equations);
(b) the result is formulated in explicit (a priori) spectral terms;
(c) the results is sharp, even in a linear context (see Remark 7.13).

To provide a necessary insight and because of independent interest, we first develop a similar theory for the discrete counterpart of (1.1)

$$x_{n+1} = Ax_n + K_n(x_n), \qquad x_0 = x \in X, \qquad n \geq 0, \tag{1.2}$$

where now A is a bounded linear operator on a Banach space X, and K_n are compact maps on X. In particular, we obtain an analogue of Theorem 1.1 for (1.2) only assuming compactness of each map K_n, where the lower bounds depend on the essential spectral radius of the operator coefficient.

Theorem 1.2. *Let* A *be a bounded linear operator on a Banach space* X, $(K_n)_{n=1}^{\infty}$ *be a sequence of compact maps on* X, *and* $(x_n(x_0))_{n=1}^{\infty}$ *be given by* (1.2). *Take a non-increasing sequence* $(a_n)_{n=1}^{\infty} \subset \mathbb{R}_+$ *satisfying* $\lim_{n\to\infty} a_n = 0$. *Fix* $y \in X$, $r > 0$, *and* $n_0 \in \mathbb{N}$ *with* $a_{n_0} < \frac{r}{2}$. *Then there exists* $x_0 \in B(y,r)$ *such that*

$$\|x_n(x_0)\| \geq a_n r_e(A)^n, \qquad n \geq n_0,$$

where $r_e(A)$ *is the essential spectral radius of* A.

This result is proved in Theorem 4.6. If $r_e(A) > 1$, then the orbits $x_n(x_0)$ grow exponentially for a dense set of initial values x_0. Thus, if the nonlinear part K is small in a topological sense rather than in the sense of norm, Theorem 1.2 provides a global generalization of the classical (discrete) principle of linearized instability, as discussed e.g. in [36, Theorem 5.1.5] or [27, Sections 1.2 and 4]. Note that the proofs of a number of statements on instability for continuous time are reduced to similar considerations in the discrete setting (as e.g. in [36]). We obtain a number of similar statements, using for instance measures of non-compactness, and also producing residual sets of solutions to (1.1) and (1.2) with exponentially growing orbits. Also our discrete results are essentially optimal, cf. [47, Sections V.37 and V.39] for the case $K = 0$.

To explain the general effects, we note that as remarked already in [71], roughly speaking, the linear semigroup $(T(t))_{t\geq 0}$ is expanding along infinitely many independent directions and the non-linear perturbation K (being relatively compact) is not able to compensate this expansion, except perhaps in a finite number of directions. Our results say that up to a small multiplicative correction the expansion takes place as if the nonlinear part in (1.1) is absent.

The next toy example illustrates the specifics of the infinite-dimensional setting very well. Let X be a separable Hilbert space. In the Banach space $B(X)$ of bounded linear operators on X, consider the difference equation

$$Y_{n+1} = N(Y_n) = AY_n + KY_n^2, \qquad Y_0 \in B(X), \qquad n \in \mathbb{N}.$$

If X is finite-dimensional, then even if $\dim X = 1$ the asymptotic behavior of $\{Y_n\}$ could be extremely complicated (see e.g. [11]), and any bounds for $\|Y_n\|$ can hardly put under control for *individual* Y_0. However, if $\dim X = \infty$, and $K \in B(X)$ is compact, then the quadratic part $Y \mapsto KY^2$, being compact in $B(X)$, becomes "small" with respect to the linear part $Y \mapsto AY$. Hence, by our Theorem 4.6 below, the (exponential) growth of trajectories $\{Y_n\}$ for a residual set of initial values is determined by the essential spectral radius of the linear part $L_A : Y \mapsto AY$ (i.e., by the essential spectral radius of A if one notes that $\sigma_e(L_A) = \sigma(A)$ by e.g. [25, Theorem 3.1]).

Aiming at generalizations of local instability results as in e.g. [20], [36], [64] and [27], it is natural to consider also nonlinear perturbations $K + F$ with a compact operator K and a metrically small map $F : B(0, r) \to X$ satisfying $F(x) = O(\|x\|^{1+\delta})$ as $\|x\| \to 0$. However, at least in the time discrete case given by (1.2) we show in Section 5 that a local instability result analogous to Theorem 1.2 cannot hold for such perturbations $K + F$.

As an illustration of our abstract results, we apply them to certain excited and backward damped wave equations in Examples 8.3, 8.4, 8.5, and 8.6. Here we allow for nonlinear forcing terms $f(t, x, u)$ in which the scalar function f may grow superlinearly (if it has the right sign) and is only continuous in the last argument. In general, the study of damped wave equations is an extremely vast and challenging area of research with many open problems stemming from mathematical physics and control theory. One may consult, e.g., the book [44, Chapter 6], the survey [17], and the papers [1], [4], [13], [18], [23], [39], [40], [41], [53], [56], [60], [61], [65] for some of its developments in the linear setting, relevant for the nonlinear studies in the present paper. However, we are not aware of any results similar to Theorem 1.1 in the context of nonlinear damped wave equations.

We believe there are many other frameworks, where our instability criteria could be useful, e.g. in the context of reaction-diffusion systems as in [71]. However, they would require a separate treatment.

In Sections 2, 3 and 9 we provide tools for our analysis and discuss the background. The main results are proved in Section 4 and 7 for discrete and continuous time, respectively. Section 5 contains counterexamples to local results in discrete time. The necessary information on nonlinear evolution equations is collected in Section 6 and then used in Section 8 for our examples.

Finally, we fix some notation used throughout the paper. All of the Banach (and Hilbert) spaces considered in this paper will be complex. To avoid trivialities, we will always assume that these spaces are infinite-dimensional. For a densely defined closed operator A on a Banach space we denote by $\rho(A)$ its resolvent set, by $\sigma(A)$ its spectrum, and by $\sigma_p(A)$ its point spectrum. We let $D(A)$ stand for the domain of A, $\mathrm{Ker}\,(A)$ for the kernel of A, $\mathrm{Im}(A)$ for its range, and $R(\lambda, A) = (\lambda - A)^{-1}$ for the resolvent of A. The space of bounded linear operators on a Banach space X will be denoted by $B(X)$. For a subspace M of a Banach space, $\dim M$ is the dimension of M, and $\mathrm{codim}\, M$ its codimension. For a subset S of a topological space, ∂S designates its boundary, and $\mathrm{card}\, S$ the cardinality of an arbitrary set S.

2. Preliminaries: a toolkit for getting (nonlinear) instability

In this section, we review several tools and techniques used for deriving instability. Some of them appear in the context of nonlinear evolution equations for the first time.

2.1. **Fine spectral theory.**

We start with a short reminder of fine spectral theory. Recall that for a closed, densely defined linear operator A on a Banach space X its essential spectrum $\sigma_e(A)$ is defined as

$$\sigma_e(A) = \{\lambda \in \mathbb{C} : \lambda - A \text{ is not Fredholm}\}$$

Clearly $\sigma_e(A) \subset \sigma(A)$. Moreover, by [59, Theorem 7.25], $\sigma_e(A)$ is closed (but can be empty). There are many other "essential spectra" in the literature, e.g. Browder's essential spectrum used in [71]. A crucial property of $\sigma_e(A)$ is that it is invariant is invariant under relatively compact perturbations if $\rho(A) \neq \emptyset$, see e.g. [21, Theorem 11.2.6]. Moreover, by [21, Theorem 11.2.2], if there exists $\mu \in \rho(A)$, then

$$\sigma_e(R(\mu, A)) \setminus \{0\} = \{(\mu - \lambda)^{-1} : \ \lambda \in \sigma_e(A)\}. \tag{2.1}$$

The property is a consequence of a more general spectral mapping theorem for essential spectrum ([29]), and it allows one to reduce many statements on essential spectrum for unbounded operators to their counterparts for bounded ones.

If A is bounded, then $\sigma_e(A)$ is a non-empty compact subset of $\mathbb{C}$. In this case, if $r_e(A)$ denotes the essential spectral radius of A, then one has $r_e(A^n) = r_e(A)^n$ for all $n \in \mathbb{N}$, as a consequence of the spectral mapping theorem for the essential spectrum. Note that $\sigma(A) \setminus \{\lambda : |\lambda| \leq r_e(A)\}$ is at most countable and consists of eigenvalues of finite multiplicity, see e.g. [47, Theorem III.19.4].

Some generalizations of Fredholm operators will also play a role. Recall that a closed, densely defined operator A on X is called upper-Fredholm if $\dim \mathrm{Ker}\,(A) < \infty$ and $\mathrm{Im}\,(A)$ is closed. If $\sigma_e(A)$ is large and $\rho(A) \neq \emptyset$, then the set of $\lambda \in \mathbb{C}$ such that $\lambda - A$ is not upper Fredholm is large as well, since it contains the topological boundary $\partial\sigma_e(A)$. More precisely, if $\lambda \in \partial\sigma_e(A)$, then for every $\varepsilon > 0$ and every subspace $M \subset X$ of finite codimension there exists a unit vector $u \in M \cap D(A)$ such that $\|(A-\lambda)u\| < \varepsilon$. See Lemma 9.2.

The following useful proposition can be found in e.g. [59, Theorem 9.43]. As several arguments below hold "up to compact perturbations", the proposition allows one to deal with point spectrum rather than generic essential spectrum, and that is technically more convenient.

Proposition 2.1. *Let $A \in B(X)$. The operator A has closed range and finite-dimensional kernel (i.e., A is upper Fredholm) if and only if* $\dim \operatorname{Ker}(A+K) < \infty$ *for all compact operators K on X.*

While there is a version of Proposition 2.1 for unbounded A, the version above will be sufficient for our purposes.

2.2. **Measures of non-compactness.** As far the essential spectrum is involved, measures of non-compactness naturally come into play, although their role in our studies will rather be supplementary in contrast to [71]. For $A \in B(X)$ let

$$\|A\|_\mu = \inf\big\{\|A \restriction_M \| : M \subset X,\, \operatorname{codim} M < \infty\big\}.$$

The mapping $A \mapsto \|A\|_\mu$ is called μ-measure of non-compactness on $B(X)$. It defines a semi-norm on $B(X)$, where $\|A\|_\mu = 0$ if and only if A is compact. Moreover, we have $\|AB\|_\mu \le \|A\|_\mu \cdot \|B\|_\mu$ for all $A, B \in B(X)$.

For $A \in B(X)$, we introduce the essential norm of A by

$$\|A\|_e := \inf\{\|A - K\| : K \in K(X)\};$$

i.e., $\|A\|_e$ is the norm of the image of A in the Calkin algebra $B(X)/K(X)$ under the corresponding quotient map. If X is a Hilbert space, then according to [78] (see also [75]) we have

$$\|A\|_\mu = \|A\|_e.$$

In general, $\|A\|_\mu$ and $\|A\|_e$ are equivalent norms on $B(X)/K(X)$ if and only if X has a so-called compact approximation property, see [5]. While substantial classes of Banach spaces possess this property, there are reflexive Banach spaces failing to satisfy it. However, by the well-known Nussbaum formula

$$r_e(A) = \lim_{n\to\infty} \|A^n\|_\mu^{1/n} = \lim_{n\to\infty} \|A^n\|_e^{1/n},$$

valid for all definitions of the essential spectrum, the quantities $\|\cdot\|_\mu$ and $\|\cdot\|_e$ are asymptotically equivalent in a sense. Moreover, the limits above can be replaced with the infimums. Hence, $\|A\|_\mu^n \ge r_e(A)^n$ and $\|A\|_e^n \ge r_e(A)^n$ for all $n \in \mathbb{N}$.

2.3. **Spectral theory for operator semigroups and resolvent bounds.** The spectral theory for C_0-semigroups is rather involved due to the unboundedness of their generators, and one has to invoke the size of resolvents to partially remedy the situation.

First recall that if $(T(t))_{t\geq 0}$ a C_0-semigroup on a Banach space X, with generator A, then

$$e^{t\sigma_e(A)} \subset \sigma_e(T(t)) \qquad \text{for all} \quad t \geq 0, \tag{2.2}$$

see e.g. [49] for even finer versions of the above inclusion. This inclusion is strict in general. One can replace in (2.2) the essential spectrum by the spectrum, where again the inclusion can be strict.

The failure of the spectral mapping theorems for semigroups leads to a number of major difficulties in the semigroup theory. To discuss some of them define the exponential growth bound of $(T(t))_{t\geq 0}$ by

$$\omega_0(T) := \lim_{t\to\infty} \frac{\ln \|T(t)\|}{t},$$

the spectral bound of its generator A by

$$s(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\},$$

and the pseudo-spectral bound of A (or abscissa of uniform boundedness of the resolvent of A) by

$$s_0(A) := \inf\{\omega > s(A) : R(\lambda, A) \text{ is uniformly bounded for } \operatorname{Re}\lambda \geq \omega\}.$$

The first two bounds possess "essential analogues" given by

$$\omega_e(T) := \lim_{t\to\infty} \frac{\ln \|T(t)\|_e}{t} \qquad \text{and} \qquad s_e(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma_e(A)\}.$$

Note that the first limit exists, and $r_e(T(t)) = e^{\omega_e(T)t}$ for $t \geq 0$, see e.g. [74]. It is also crucial to observe that from [24, Corollary IV.2.11] it follows that $\omega_0(T) = \max\{s(A), \omega_e(T)\}$, and moreover the set $\{\lambda \in \sigma(A) : \operatorname{Re}\lambda > s_e(A)\}$ is at most countable, and consists of eigenvalues of A (with finite multiplicity). Clearly,

$$s_e(A) \leq s(A) \leq s_0(A) \leq \omega_0(T) \qquad \text{and} \qquad s_e(A) \leq \omega_e(T) \leq \omega(T).$$

There are various examples of C_0-semigroups making all or some of the above inequalities strict, see e.g. [3, Chapter 5.1]. So that, in general, neither the spectrum nor the resolvent of A determine the exponential norm bounds for $(T(t))_{t\geq 0}$. Semigroups with $s(A) < \omega_0(T)$ also arise from concrete partial differential equations, such as damped wave equations, cf. Section 8.

On the other hand, if X is a Hilbert space, then the well-known Gearhart-Herbst-Prüss theorem guarantees that

$$\omega_0(T) = s_0(A), \tag{2.3}$$

and the exponential decay of $(T(t))_{t\geq 0}$ is equivalent to $s_0(A) < 0$. For an exhaustive discussion of relations between these two and other related bounds see e.g. [3, Chapters 5.1-5.3] or [52, Chapters 1-4].

To be able to obtain sharp lower bounds for the trajectories of (1.1) we need to introduce the new resolvent bound $s_R(A)$ as the infimum of the set S_R of $a > s_e(A)$ satisfying

$$\text{card}\,(\sigma_p(A) \cap (a + i\mathbb{R})) < \infty \quad \text{and} \quad \limsup_{|b|\to\infty} \|R(a + ib, A)\| < \infty. \tag{2.4}$$

This bound will play a crucial role in the sequel. Note that every vertical line $a + i\mathbb{R}$ with $a > s_e(A)$ contains at most countablly many eigenvalues. Using the discreteness of the set $\{\lambda \in \sigma(A) : \operatorname{Re}\lambda > s_e(A)\}$ and the Neumann series expansion for the resolvent, one shows that the set S_R is open. Thus the infimum is not attained, and we have

$$\begin{aligned}
&s_R(A) = s_e(A) \qquad \text{or} \qquad \text{card}\,(\sigma_p(A) \cap (s_R(A) + i\mathbb{R})) = \infty \\
&\text{or} \quad \text{card}\,(\sigma_p(A) \cap (s_R(A) + i\mathbb{R})) < \infty \text{ and } \limsup_{|b|\to\infty} \|R(s_R(A) + ib, A)\| = \infty.
\end{aligned}$$

Moreover, as the set $\{\lambda \in \sigma(A) : \operatorname{Re}\lambda > s_e(A)\}$ is discrete, one may also define $s_R(A)$ as the infimum of the set of $a > s_e(A)$ such that

$$\exists\, \beta = \beta(a) > 0 : \ a + i(\mathbb{R} \setminus (-\beta, \beta)) \subset \rho(A) \text{ and } \sup_{|b|\geq\beta} \|R(a + i\beta, A)\| < \infty.$$

If the adjoint A^* generates a C_0-semigroup on X^*, the above statement implies that

$$s_R(A) = s_R(A^*). \tag{2.5}$$

To explain the relevance of $s_R(A)$ and to relate it to the spectral properties of $(T(t))_{t\geq0}$, we introduce the notion of admissibility. We say that $\omega \in \mathbb{R}$ is *admissible* if for every $t_0 > 0$, every $\varepsilon > 0$ and every subspace $M \subset X$ of the form $M = \bigcap_{1\leq j\leq n} \operatorname{Ker} x_j^*$ with $x_j \in D(A^*)$ for $j \in \{1, \ldots n\}$, there exist $x \in M$ with $\|x\| = 1$ and $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu = \omega$ such that

$$\|T(t)x - e^{\mu t}x\| < \varepsilon, \qquad 0 \leq t \leq t_0. \tag{2.6}$$

Note that $\operatorname{codim} M < \infty$. Observe also that x and μ depend on t_0, ϵ and M, and so x is *not* an approximate eigenvalue of $T(t)$, $0 \leq t \leq t_0$, in general. However, the notion of admissibility will help us to "emulate" the approximate eigenvalues of $T(t)$ to an extent that sufficices for the construction of growing solutions to nonlinear evolution equations.

Using Lemma 9.2 it is easy to show that $s_e(A)$ is admissible. In this case, there exists μ with $\operatorname{Re}\mu = s_e(A)$ such that for any $\epsilon, t_0 > 0$ and any closed subspace of finite codimension M, one can find a unit vector $x \in M$ satisfying (2.6). However, Theorem 2.2 provides a more general statement showing that in fact $s_R(A)$ is admissible. Thus the next result is one of the basic tools in this paper.

Theorem 2.2. *Let $(T(t))_{t\geq0}$ be a C_0-semigroup on a Banach space X with generator A, and $s_R(A)$ be defined by* (2.4)*. Then $s_R(A)$ is admissible.*

The proof of the theorem f is given in the appendix. It is similar to the proof [48, Proposition 4.4], though it is technically more demanding.

Finally, the next elementary observation will be instructive and helpful. Let $(T(t))_{t\in\mathbb{R}}$ be a C_0-group on a Hilbert space X. Note then that for fixed $\mu_0 \in \rho(A)$ and any $x \in X$ and $\alpha > 0$, one has

$$\begin{aligned} \|R(\mu_0, A)^\alpha x\|^2 =& |\langle T(t)R(\mu_0, A)^\alpha x, T^*(-t)R(\bar{\mu}_0, A^*)^\alpha x\rangle| \\ \le& \|T(t)R(\mu_0, A)^\alpha x\| \|T^*(-t)R(\bar{\mu}_0, A^*)^\alpha x\|, \qquad t \in \mathbb{R}. \end{aligned} \tag{2.7}$$

Thus, if there exist $M_\alpha > 0$ and $\omega_\alpha > 0$ such that

$$\|T^*(-t)R(\bar{\mu}_0, A^*)^\alpha\| \le M_\alpha e^{-\omega_\alpha t}, \qquad t \ge 0,$$

then for y from the dense subspace $D((\mu_0 - A)^\alpha)$ of X we have

$$\|T(t)y\| \ge M(y)e^{\omega_\alpha t}, \qquad t \ge 0.$$

The examples of such groups $(T(t))_{t\in\mathbb{R}}$ occur in applications, e.g. when $s_0(A) \ge 0$, but the resolvent of A growths at most polynomially in the half-plane $\{\lambda : \operatorname{Re}\lambda \ge -a\}$ for some $a > 0$. Here [76] provides an appropriate abstract framework. This observation links the study of the growth of semigroup orbits to a well-studied area centered around orbits decay. A similar situation will be discussed in a concrete case of damped wave equations in Section 8 below.

3. Warm-up: initial observations and comments

3.1. **Spectrum does not suffice.** It is well-known that the spectral radius is not continuous on $B(X)$. This leads to the fact that, in general, the instability of (1.1) is not preserved under small Lipschitz perturbations.

For a fixed $a > 0$ let $X = L^2(0, 2a)$ and consider the selfadjoint operator $(Af)(s) = sf(s)$ on X. Clearly, $\sigma(A - a) = [-a, a]$ and the system

$$x'(t) = (A - a)x(t), \qquad t \ge 0, \quad x(0) = x_0, \tag{3.1}$$

is exponentially unstable. By a classical result due to Herrero [37] (see [35] for a simple proof), if the spectrum of a bounded normal operator on X is connected and contains zero, then the operator is a limit of a sequence (or net) of bounded nilpotent operators on X in the uniform operator topology. Hence there exists a family of nilpotent operators $(A_\varepsilon)_{\varepsilon>0}$ on X such that $A_\varepsilon \to A$ in $L(X)$ as $\varepsilon \to 0$. Consider the perturbed sysem

$$x'(t) = (A - a)x(t) + (A_\varepsilon - A)x(t) = (A_\varepsilon - a)x(t), \qquad t \ge 0.$$

Note that for any $\delta > 0$ there exists $\varepsilon_0 > 0$ such that

$$\sigma(A_\varepsilon - a) = \{-a\} \qquad \text{and} \qquad \|A_\varepsilon - A\| < \delta, \qquad \varepsilon \in (0, \varepsilon_0).$$

So, the exponential instability of (3.1) is not preserved by arbitrarily small Lipschitz perturbations $A_\varepsilon - A$ (even if the linear part A is selfadjoint). This general phenomena was observed in [67] and rediscovered recently in [57]. However, [57] went much further, see Subsection 3.3 below. We add that in [68] a Lipschitz perturbation K is constructed which destroys instability of a linear system $x'(t) = Ax(t)$ and satisfies the strictly sublinear growth assumption $\|K(x)\|/\|x\| \to 0$ as $\|x\| \to 0$. Our examples in Section 5 differ from these treatments.

3.2. **Unstable orbits may co-exist with very stable ones.** On the Hilbert space $X = L^2((0,\infty), e^{-2t}dt)$ let the operator $Af = f'$ be defined on its maximal domain in X. Then A generates the C_0-semigroup $(T(t))_{t\geq 0}$ of left shifts on X, and $\sigma(A) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq 2\}$. While $(T(t))_{t\geq 0}$ is hypercyclic, i.e., its trajectories are dense in X for a dense set of initial values (see e.g. [22]), the trajectories vanish eventually for every initial value with compact support. Thus, we have a dense set of initial values x_0 for the abstract Cauchy problem

$$x'(t) = Ax(t), \qquad x_0 = x_0,$$

yielding "superstable" mild solutions and, at the same time, a dense set of initial values x_0 whose trajectories have a totally unstable behavior. For more information on hypercyclic semigroups, we refer to e.g. [22].

3.3. **Sublinear perturbations.** Rodriguez and Solá-Morales proved in [57] that in an infinite-dimensional separable real Hilbert space X there exists a C^1 map $T : X \mapsto X$ of the form $T = A + K$, where A is a bounded linear operator on X with the spectral radius larger than one and the nonlinearity K satisfies $\|K(x)\|/\|x\| \to 0$ as $x \to 0$ and $K(0) = 0$. Nevertheless the fixed point $x = 0$ of T is exponentially asymptotically stable. Moreover, one has

$$\left(\frac{1}{-\log\|x\|}\right)^{c_2} < \frac{\|Kx\|}{\|x\|} < 4\left(\frac{1}{-\log(\|x\|)}\right)^{c_1}$$

as $x \to 0$ for some $0 < c_1 < c_2$.

3.4. **Rank-one perturbations.** The instability properties for (1.2) become rather arbitrary if the essential spectrum of A is merely contained in the closed unit disc. A good illustration for that phenomena is provided by the fact that there exist a unitary operator A on a Hilbert space X and rank-one operator K on X such that the operator $A + K$ is hypercyclic, see e.g. [30], [8]. Clearly, $r_e(A + K) = r_e(A) = 1$. On the other hand, it is well-known that there exist a unitary operator A_1 and a rank-one operator K_1 such that $A_1 + K_1$ is strongly stable, that is, $(A_1 + K_1)^n x \to 0$ as $n \to \infty$ for every $x \in X$. Indeed, if S is a unilateral shift on the Hardy space $H^2(\mathbb{D})$, then [15] shows that certain unitary operators on $H^2(\mathbb{D})$ arise as rank-one perturbations of compressions of S to the invariant subspaces of S^*. Moreover, clearly $S^{*n} \to 0$ as $n \to \infty$, strongly. A more general set-up for such a construction can be found in [50].

Apparently, a similar example can be constructed in a continuous framework. Recall that for any selfadjoint operator A on a Hilbert space X the semigroup $(e^{iAt})_{t\geq 0}$ is unitary. It is plausible that there exist bounded selfadjoint operators A_1 and A_2 on X and rank-one perturbation K_1 and K_2 such that semigroup $(e^{i(A_1+K_1)t})_{t\geq 0}$ is hypercyclic, while the semigroup

$(e^{i(A_2+K_2)t})_{t\geq 0}$ is strongly stable. Although, no results have been published in this context.

4. Instability for discrete time

In this section, we assume that X is a Banach space, $A \in B(X)$, and we let $(K_n)_{n=1}^{\infty}$ be a sequence of compact maps $K_n : X \to X$ (in general, non-linear). For $x \in X$ and $n \in \mathbb{N}$ let

$$f_n(x) = (A + K_n) \cdots (A + K_1)x, \tag{4.1}$$

and set $f_0(x) = x$. In other words, $(f_n(x))_{n=0}^{\infty}$ is a solution of the difference equation

$$x_{n+1} = (A + K_n)x_n, \qquad x_0 = x. \tag{4.2}$$

The next simple lemma replaces the difference equation (4.2) by a "difference inclusion". In this way, we are able to get rid of nonlinear perturbations and can argue up to compact sets.

Lemma 4.1. *Let $(f_n)_{n=1}^{\infty}$ be given by* (4.1)*, $n \in \mathbb{N}$ be fixed, and $X_0 \subset X$ be a bounded set. Then there exists a compact set $C \subset X$ such that*

$$f_j(x) \in A^j x + C$$

for all $x \in X_0$ and $j \in \{1, \dots, n\}$.

Proof. It is easy to see by induction on j that

$$f_j(x) = A^j x + \sum_{s=0}^{j-1} A^{j-s-1} K_{s+1} f_s(x)$$

for each $x \in X$ and each $j \in \{1, \dots, n\}$. Using induction again, we infer that $f_s(X_0)$ is a bounded set for each s. Hence, $K_{s+1} f_s(X_0)$ is precompact, which implies the precompactness of the set

$$\bigcup_{j=1}^{n} \sum_{s=0}^{j-1} A^{j-s-1} K_{s+1} f_s(X_0).$$

Let C be its closure. Then $f_j(x) \in A^j x + C$ for all $x \in X_0$ and $j \in \{1, \dots, n\}$. □

Now we are able to give a lower bound for a finite piece of the trajectory in terms of the deviation of its linear part from a finite-dimensional subspace. The next simple lemma provides an intuition behind the crucial Lemma 4.3.

Lemma 4.2. *Let $(f_n)_{n=1}^{\infty}$ be given by* (4.1)*. Let $n \in \mathbb{N}$ be fixed and let $X_0 \subset X$ be a bounded set. Then for every $\varepsilon > 0$ there exists a finite-dimensional subspace $F \subset X$ such that*

$$\|f_n(x)\| \geq \operatorname{dist}\{A^n x, F\} - \varepsilon$$

for all $x \in X_0$.

Proof. Let C be the compact set constructed in Lemma 4.1. Since C is compact, there exists a finite-dimensional subspace $F \subset X$ such that $\operatorname{dist}\{c, F\} < \varepsilon$ for every $c \in C$. For each $x \in X_0$ we have $f_n(x) \in A^n x + C$ implying

$$\|f_n(x)\| \geq \operatorname{dist}\{A^n x, C\} \geq \operatorname{dist}\{A^n x, F\} - \varepsilon. \qquad \square$$

Having obtained the estimate for a finite piece of trajectory, we can spread it out to a finite-dimensional subspace of X, as the next lemma shows.

Lemma 4.3. *Let $A \in B(X)$, $n \in \mathbb{N}$, $\varepsilon > 0$, and let $G \subset X$ be a finite-dimensional subspace. Then the following assertions hold.*

(i) *There exists a unit vector $u \in X$ such that* $\operatorname{dist}\{A^n u, G\} \geq \frac{1-\varepsilon}{2}\|A^n\|_\mu$.

(ii) *There exists a unit vector $u \in X$ such that* $\operatorname{dist}\{A^j u, G\} \geq \frac{1-\varepsilon}{2} r_e(A)^j$ *for all $j = 1, \dots, n$.*

Proof. To prove (i), note that by Lemma 9.1 in the appendix there exists a closed subspace $M \subset X$ of a finite codimension such that

$$\|g + m\| \geq (1 - \varepsilon/2) \max\{\|g\|, \|m\|/2\}$$

for all $g \in G$ and $m \in M$. Since $L := A^{-n}(M)$ is a subspace of a finite codimension, there exists a unit vector $u \in L$ with $\|A^n u\| \geq (1-\varepsilon/2)\|A^n\|_\mu$. Then

$$\operatorname{dist}\{A^n u, G\} = \inf\{\|A^n u + g\| : g \in G\} \geq \frac{1 - \varepsilon/2}{2}\|A^n u\| \geq \frac{1-\varepsilon}{2}\|A^n\|_\mu,$$

which gives (i).

Let $\mu \in \sigma_e(A)$ with $|\mu| = r_e(A)$. Then $\mu \in \partial\sigma_e(A)$, and by Lemma 9.2 it belongs to the essential approximate point spectrum of A, i.e., for every closed subspace $L \subset X$ of finite codimension and each $\delta > 0$ there exists a unit vector $u \in L$ with $\|(A - \mu)u\| < \delta$. For each $j = 1, \dots, n$ we have

$$\|(A^j - \mu^j)u\| = \left\|\sum_{k=0}^{j-1} A^{j-k-1}\mu^k (A - \mu)u\right\| \leq j\|A\|^j \delta.$$

We now choose M as in the proof of (i) and set $L = \bigcap_{j=1}^n A^{-j}M$. Then there is a unit vector $u \in L$ such that $\|(A - \mu)u\|$ is so small that $\|(A^j - \mu^j)u\| \leq r_e(A^j)\varepsilon/2$ for all $j = 1, \dots, n$. Thus $\|A^j u\| \geq (1 - \varepsilon/2) r_e(A^j)$, and as above we deduce that $\operatorname{dist}\{A^j u, G\} \geq \frac{1-\varepsilon}{2} r_e(A^j)$ for all $j = 1, \dots, n$. $\square$

The above lemmas allow us to obtain exponential lower bounds for the norms of trajectories $\|f_n(x)\|$ for residual set of initial values x along some subsequences $\{n_k\}$. Apart from a usual Baire's category theorem, the geometrical Lemma 4.3 is indispensable here.

Theorem 4.4. *Let $A \in B(X)$ and $K_n : X \to X$ be compact for $n \in \mathbb{N}$. Take a sequence $(a_n)_{n=0}^\infty$ in $\mathbb{R}_+$ such that $\lim_{n\to\infty} a_n = 0$, and let $L \subset \mathbb{N}$ be* *infinite. For $x \in X$ define $f_n(x)$ by* (4.1)*. Then the set*

$$\big\{x \in X : \textit{ there are infinitely many } n \in L \textit{ with } \|f_n(x)\| \geq a_n \|A^n\|_\mu\big\}$$

is residual in X.

Proof. The statement is clear if $\|A^n\|_\mu = 0$ for an infinite number of $n \in L$. Thus, without loss of generality, we may assume that $\|A^n\|_\mu > 0$ for all $n \in L$. For $k \in \mathbb{N}$ let

$$M_k = \{x \in X : \text{ there exists } \; n \in L \; \text{ with } \; n \geq k \; \text{ and } \; \|f_n(x)\| > a_n \|A^n\|_\mu\}.$$

Clearly M_k is an open set. We show that it is dense in X. Let $y \in X$ and fix $\varepsilon > 0$. Choose $n \in L$ with $n \geq k$ and $a_n < \varepsilon/4$. By Lemma 4.1, there exists a compact set $C \subset X$ such that

$$f_n(y+v) \in A^n(y+v) + C$$

for all $v \in X$ with $\|v\| \leq \varepsilon$. Since C is compact, there exists a finite-dimensional subspace $F \subset X$ such that $\operatorname{dist}\{c, F\} < \frac{\varepsilon\|A^n\|_\mu}{12}$ for all $c \in C$. Lemma 4.3 then yields a unit vector $u \in X$ with $\operatorname{dist}\{A^n u, F\} > \frac{\|A^n\|_\mu}{3}$. Note that

$$\operatorname{dist}\{A^n(y+\varepsilon u), F\} + \operatorname{dist}\{A^n(y-\varepsilon u), F\} \geq 2\varepsilon \operatorname{dist}\{A^n u, F\} \geq \frac{2\varepsilon}{3}\|A^n\|_\mu.$$

Indeed, passing to the quotient space X/F, the former inequality follows from

$$\|\pi(A^n y) + \varepsilon\pi(A^n u)\|_{X/F} + \|\pi(A^n y) - \varepsilon\pi(A^n u)\|_{X/F} \geq \frac{2\varepsilon}{3}\|\pi(A^n u)\|_{X/F}$$

via the triangle inequality, where π denotes the corresponding quotient mapping. So $x := y + \varepsilon u$ or $x := y - \varepsilon u$ satisfy

$$\|x - y\| \leq \varepsilon \quad \text{and} \quad \operatorname{dist}\{A^n x, F\} \geq \frac{\varepsilon}{3}\|A^n\|_\mu.$$

We conclude

$$\|f_n(x)\| \geq \operatorname{dist}\{A^n x, F\} - \frac{\varepsilon\|A^n\|_\mu}{12} \geq \frac{\varepsilon\|A^n\|_\mu}{4} \geq a_n\|A^n\|_\mu.$$

Hence $x \in M_k$, and M_k is dense since the choice of y and ε was arbitrary. By the Baire category theorem, $\bigcap_{k=1}^\infty M_k$ is a dense G_δ set and thus residual. □

Corollary 4.5. *Let $A \in B(X)$ satisfy* $\sup\{\|A^n\|_\mu : n \in \mathbb{N}\} = \infty$*. Then the set*

$$\big\{x \in X : \sup_{n\in\mathbb{N}} \|f_n(x)\| = \infty\big\}$$

is residual. In particular, this is true if $r_e(A) > 1$.

If we concentrate on merely dense sets of initial values rather than residual ones, then we can construct orbits $(f_n(x))_{n=1}^\infty$ with exponential growing norm lower bounds for *all* large n.

Theorem 4.6. *Let $A \in B(X)$, $(K_n)_{n=1}^\infty$ be a sequence of compact maps on X, and $(f_n)_{n=1}^\infty$ be given by (4.1). Take a non-increasing sequence $(a_n)_{n=1}^\infty \subset \mathbb{R}_+$ satisfying $\lim_{n\to\infty} a_n = 0$. Fix $y \in X$, $r > 0$, and $n_0 \in \mathbb{N}$ with $a_{n_0} < \frac{r}{2}$. Then there exists $x \in \overline{B}(y, r)$ such that*

$$\|f_n(x)\| \geq a_n r_e(A)^n, \qquad n \geq n_0.$$

Proof. Denote $r_e := r_e(A)$ for shorthand. The statement is clear if $r_e = 0$.

Let $r_e > 0$. We start with several convenient reductions. Without loss of generality we may assume that $r_e(A) = 1$. If not, then consider the operator $r_e^{-1}A$ and the compact mappings $r_e^{-n}K_n(r_e^{n-1}\cdot)$ for $n \geq 1$. Replacing A with λA for some $|\lambda| = 1$ and K_n with $\lambda^n K_n(\lambda^{-n+1}\cdot)$ if necessary, we may assume that $1 \in \sigma_e(A)$. The operator $A - I$ is not upper Fredholm since $1 \in \partial\sigma_e(A)$, see Lemma 9.2.

Proposition 2.1 thus gives a compact linear operator $\widetilde{K}$ on X with $\dim \mathrm{Ker}\,(A - I - \widetilde{K}) = \infty$. Thus, passing to $A - \widetilde{K}$ and $K_n + \widetilde{K}$ if necessary, we may assume that $\dim \mathrm{Ker}\,(A - I) = \infty$.

Take some $c_1 \in (2a_{n_0}, r)$. Fix integers $n_0 < n_1 < n_2 < \cdots$ such that $a_{n_k} < 2^{-(k+2)}(r - c_1)$. Set

$$c_k = 2^{-(k-1)}(r - c_1), \qquad k \geq 2.$$

Choose positive numbers ε_k such that $\varepsilon_{k+1} < \varepsilon_k$ and

$$\frac{(1-\varepsilon_k)^2}{2} c_k - \varepsilon_k \geq a_{n_{k-1}}$$

for all $k \in \mathbb{N}$.

Set $x_0 = y$, $F_0 = \{0\}$ and $M_0 = X$. Inductively, we construct finite-dimensional subspaces $F_1 \subset F_2 \subset \cdots$, finite-codimensional closed subspaces $M_1 \supset M_2 \supset \cdots$, and unit vectors $x_k \in M_k \cap \mathrm{Ker}\,(A - I)$ for $k \geq 1$ such that (4.3) below is true.

Let $k \geq 1$ and suppose that the vectors $x_0, \ldots, x_{k-1}$ and spaces $F_0, \ldots, F_{k-1}$ and $M_0, \ldots, M_{k-1}$ have already been constructed. Lemma 4.1 provides a compact set $C_k \subset X$ such that

$$f_n(u) \in A^n u + C_k$$

for all $n \leq n_k$ and $u \in X$ with $\|u\| \leq \|y\| + r$. Using the compactness of C_k, we find a finite-dimensional subspace $F_k \supset F_{k-1}$ such that

$$\{y, Ay, \ldots, A^{n_k}y, x_1, \ldots, x_{k-1}\} \subset F_k \quad \text{and} \quad \mathrm{dist}\,\{d, F_k\} \leq \varepsilon_k \tag{4.3}$$

for all $d \in C_k$. Lemma 9.1 yields a closed subspace $M_k \subset M_{k-1}$ of finite codimension such that

$$\|f + m\| \geq (1 - \varepsilon_k) \max\Big\{\|f\|, \frac{\|m\|}{2}\Big\}$$

for all $f \in F_k$ and $m \in M_k$. Since $\mathrm{Ker}\,(A - I)$ has infinite dimension, there exists a unit vector $x_k \in M_k \cap \mathrm{Ker}\,(A - I)$.

For the elements x_k, $k \in \mathbb{N}$, constructed above, set

$$x = y + \sum_{j=1}^{\infty} c_j x_j.$$

We show that x satisfies the assertions. First note that

$$\|x - y\| \leq \sum_{j=1}^{\infty} c_j = c_1 + \sum_{j=2}^{\infty} \frac{r - c_1}{2^{j-1}} = r,$$

so that $x \in \overline{B}(y, r)$. Let $k \geq 1$ and $n_{k-1} \leq n \leq n_k$. We estimate

$$\begin{aligned}\|f_n(x)\| &\geq \operatorname{dist}\{A^n x, C_k\} \geq \operatorname{dist}\{A^n x, F_k\} - \varepsilon_k \\ &= \operatorname{dist}\Big\{\sum_{j=k}^{\infty} c_j x_j, F_k\Big\} - \varepsilon_k,\end{aligned}$$

employing that

$$A^n x = A^n y + \sum_{j=1}^{\infty} c_j x_j \quad \text{and} \quad A^n y, x_1, \ldots, x_{k-1} \in F_k.$$

For $j \geq k$, the vector x_j belongs to $M_j \subset M_k$, and so

$$\|f_n(x)\| \geq \frac{1 - \varepsilon_k}{2} \cdot \Big\|\sum_{j=k}^{\infty} c_j x_j\Big\| - \varepsilon_k.$$

Since $x_k \in F_{k+1}$ and $x_j \in M_j \subset M_{k+1}$ for $j \geq k+1$, we obtain

$$\begin{aligned}\|f_n(x)\| &\geq \frac{(1 - \varepsilon_k)(1 - \varepsilon_{k+1})}{2} \|c_k x_k\| - \varepsilon_k \\ &\geq \frac{(1 - \varepsilon_k)^2}{2} \cdot c_k - \varepsilon_k \geq a_{n_{k-1}} \geq a_n.\end{aligned}$$

Hence

$$\|f_n(x)\| \geq a_n$$

for all $n \geq n_0$, and the statement follows. □

Remark 4.7. If X is a Hilbert space, then in the proof of Theorem 4.6 one can take $M_k = F_k^{\perp}$ in each step. It is then possible to obtain a slightly better estimate: Let $y \in X$, $n_0 \geq 0$, and $a_{n_0} < r$. Then there exists $x \in X$ with $\|x - y\| \leq r$ and $\|f_n(x)\| \geq a_n r_e(A)^n$ for all $n \geq n_0$.

Corollary 4.8. *Under the assumptions of Theorem 4.6, let $(a_n)_{n=1}^{\infty} \subset \mathbb{R}_+$ be a non-increasing sequence satisfying $\lim_{n\to\infty} a_n = 0$. Then*

(i) *there exists $x \in X$ such that $\|f_n(x)\| \geq a_n r_e(A)^n$ for all $n \in \mathbb{N}$;*
(ii) *there exists a dense subset of vectors $x \in X$ such that $\|f_n(x)\| \geq a_n r_e(A)^n$ for all n sufficiently large.*

Proof. To prove (i), it suffices to set $y = 0$ and to choose a big enough radius r in Theorem 4.6.

Let now $y \in X$ and $\varepsilon > 0$ be fixed. Let n_0 be such that $a_{n_0} < \frac{\varepsilon}{2}$. By Theorem 4.6, there exists $x \in X$ such that $\|x - y\| \le \varepsilon$ and $\|f_n(x)\| \ge a_n r_e(A)^n$ for all $n \ge n_0$, and assertion (ii) follows. □

Remark 4.9. If one is interested in merely local instability properties of (4.2) assuming $r_e(A) > 1$, then results close in spirit can be found in [69].

5. Local instability for discrete time: counterexamples

As we mentioned in the introduction, it is natural to try to combine "metric" instability results, such as e.g. in [20], [36], [64], [27], and "topological" instability conditions as above, and to obtain local instability for

$$x_n = (A + K + F)^n x_0, \qquad n \in \mathbb{N}, \tag{5.1}$$

where A is a linear operator on X with $r_e(A) > 1$, K is a compact operator on X with $K(0) = 0$, and F is a continuous map defined on a ball $B(0, r) \subset X$ such that $\|F(x)\| = O(\|x\|^{1+\delta})$ as $\|x\| \to 0$ for some $\delta > 0$. Unfortunately, this is not possible, in general, even if X is a Hilbert space, as the following examples show. We first note an auxiliary simple lemma.

Lemma 5.1. *Let $\varepsilon \in (0, 2^{-7})$. Then there exist continuous functions $f : [0,\infty) \to [0, \varepsilon/2]$ and $g : [0,\infty) \times [0,\infty) \to [0, 2]$ such that*

(i) $f(x) = 0, \qquad 0 \le x \le \varepsilon^3 \quad \text{or} \quad x \ge \varepsilon;$
(ii) $f(x) = \varepsilon/2, \qquad 4\varepsilon^3 \le x \le \frac{\varepsilon}{2};$

and

(iii) $g(x, t) = 0, \qquad 0 \le x \le \varepsilon^3 \quad \text{or} \quad x \ge \varepsilon;$
(iv) $g(x, \frac{\varepsilon}{2}) = 2, \qquad 4\varepsilon^3 \le x \le 16\varepsilon^3;$
(v) $g(x, t) \le x + \frac{t^2}{x}, \qquad x > 0.$

Proof. The existence of f is obvious.

Let $\widetilde{g} : [0,\infty) \times [0,\infty) \to [0, 2]$ be any continuous function satisfying (iii) and (iv). Set $g = \min\{\widetilde{g}, x + \frac{t^2}{x}\}$ for $x > 0$ and $g(0, t) = 0$ for all $t \ge 0$. Clearly, g fulfills (iii) and (v). If $4\varepsilon^3 \le x \le 16\varepsilon^3$ and $t = \frac{\varepsilon}{2}$, then

$$x + \frac{t^2}{x} \ge \frac{t^2}{x} \ge \frac{\varepsilon^2}{4 \cdot 16\varepsilon^3} = \frac{1}{64\varepsilon} > 2 = \tilde{g}(t, x).$$

So g satisfies (iv). □

Next we provide a counterexample for local *instability* of (5.1), where the nonlinear part is given by the sum of two nonlinear parts, each guaranteeing local instability when considered separately.

Example 5.2. Let H_0 be an infinite-dimensional Hilbert space, and set $H = H_0 \oplus \mathbb{C}$ with the Hilbert space norm. Let ε, f and g be given as above. Consider the mappings $A \in B(H)$, $K : H \to H$, and $F : H \to H$ defined by

$$A(y, a) = (2y, 0), \qquad (y, a) \in H;$$

$$K(y,a) = (0, f(\|y\|)), \qquad (y,a) \in H;$$
$$F(y,a) = \big(-g(\|y\|, |a|)y, 0\big), \qquad (y,a) \in H.$$

It is clear that $A \in B(H)$, $r_e(A) = 2$, K is compact, and F satisfies

$$\|F(y,a)\| = g\big(\|y\|, |a|\big) \cdot \|y\| \le \Big(\|y\| + \frac{|a|^2}{\|y\|}\Big)\|y\|$$
$$= \|y\|^2 + |a|^2 = \|(y,a)\|^2.$$

Moreover,

$$(A+K+F)(y,a) = \Big(\big(2 - g\big(\|y\|, |a|\big)\big)y, f(\|y\|)\Big).$$

Let $u_0 = (y_0, a_0) \in H$ be such that $\|y_0\| \le \varepsilon^3$ and $|a_0| \le \varepsilon^3$. Set

$$u_n := (y_n, a_n) = (A+K+F)u_{n-1}$$

for all $n \ge 1$. We show that $\|u_n\| \le \varepsilon$ for all $n \ge 1$.

First note that $|a_n| \le \frac{\varepsilon}{2}$ for $n \in \mathbb{N}$, by the definition of f. We prove that $\|y_n\| \le 16\varepsilon^3$ for all $n \in \mathbb{N}$. Observe that $\|y_n\| \le 2\|y_{n-1}\|$. We distinguish two cases.

A) There exists n such that $\|y_n\| > 4\varepsilon^3$ and $\|y_{n+1}\| > 4\varepsilon^3$. Let n_0 be the smallest integer with this property. We then obain

$$\|y_{n_0}\| \le 2\|y_{n_0-1}\| \le 8\varepsilon^3,$$

and so

$$a_{n_0+1} = f(\|y_{n_0}\|) = \frac{\varepsilon}{2} \quad \text{and} \quad 4\varepsilon^3 \le \|y_{n_0+1}\| \le 2\|y_{n_0}\| \le 16\varepsilon^3.$$

Hence, $g\big(\|y_{n_0+1}\|, |a_{n_0+1}|\big) = 2$ and $y_{n_0+2} = 0$. Thus $y_k = 0$ for all $k \ge n_0+2$. Moreover, $\min\{\|y_m\|, \|y_{m+1}\|\} \le 4\varepsilon^3$ for all $m \le n_0 - 1$. It follows

$$\text{either} \qquad \|y_{m+1}\| \le 4\varepsilon^3 \qquad \text{or} \qquad \|y_{m+1}\| \le 2\|y_m\| \le 8\varepsilon^3.$$

We have shown $\sup_k \|y_k\| \le 16\varepsilon^3$ in this case.

B) Let $\min\{\|y_n\|, \|y_{n+1}\} \le 4\varepsilon^3$ for all n. We again have

$$\text{either} \qquad \|y_{n+1}\| \le 4\varepsilon^3 \qquad \text{or} \qquad \|y_{n+1}\| \le 2\|y_n\| \le 8\varepsilon^3,$$

which yields $\sup_n \|y_n\| \le 8\varepsilon^3$.

In both cases we have shown $\sup_n \|y_n\| \le 16\varepsilon^3 \le \frac{\varepsilon}{2}$, and so $\sup_n \|u_n\| \le \varepsilon$.

The previous example can be substantially strengthened to provide a finer control over the (in-)stability properties of (5.1).

Example 5.3. Let $(\varepsilon_k)_{k=1}^{\infty}$ be a fast decreasing sequence of positive numbers such that $\varepsilon_1 < 2^{-7}$ and $\varepsilon_{k+1} < \frac{\varepsilon_k^3}{4}$ for all $k \ge 1$. As above, let $H = H_0 \oplus \ell^2$ where $\dim H_0 = \infty$. Take functions f_k and g_k as defined in Lemma 5.1 for $\varepsilon = \varepsilon_k$. Introduce mappings A, K, and $F : H \to H$ by

$$A\big(y, (a_k)\big) = \big(2y, (0)\big), \qquad \big(y, (a_k)\big) \in H;$$
$$K\big(y, (a_k)\big) = \big(0, (f_k(\|y\|))\big), \qquad \big(y, (a_k)\big) \in H;$$

$$F\big(u,(a_k)\big)=\Big(-y\sum_{k=1}^\infty g_k\big(\|y\|,|a_k|\big),0\Big),\qquad \big(y,(a_k)\big)\in H.$$

Note that for all $(y,(a_k))\in H$ there is at most one k such that $g_k\big(y,|a_k|\big)\neq 0$. So F is well-defined. Similarly, there is at most one k with $f_k(\|y\|)\neq 0$.

Clearly, $A\in B(H)$, $r_e(A)=2$ and K is compact since

$$KH\subset\Big\{(0,(a_k)):|a_k|\le\frac{\varepsilon_k}{2}\Big\}.$$

Furthermore,

$$\begin{aligned}\|F(y,(a_k))\|&\le\|y\|\cdot\sum_{k=1}^\infty g_k\big(\|y\|,|a_k|\big)=\|y\|\cdot\max_k g_k\big(\|y\|,|a_k|\big)\\&\le\max_k\big(\|y\|^2+|a_k|^2\big)\le\|(y,(a_k))\|^2.\end{aligned}$$

We have

$$(A+K+F)\big(y,(a_k)\big)=\Big((2-\sum_{k=1}^\infty g_k\big(\|y\|,|a_k|\big))y,\big(f_k(\|y\|)\big)\Big).$$

Let $u_0=\big(y_0,(a_{0,j})\big)$ be such that $\|u_0\|\le\frac{\varepsilon_k^3}{4}$ for some $k\ge1$, and let

$$u_n=\big(y_n,(a_{n,j})\big)=(A+K+F)u_{n-1}$$

for all $n\ge1$.

We show first that $\|y_n\|\le\frac{\varepsilon_k}{2}$ for all n. Suppose the contrary. Let n_0 be the smallest integer with $\|y_{n_0}\|>\frac{\varepsilon_k}{2}$. Let n_1 be the largest integer with $n_1<n_0$ and $\|y_{n_1}\|<\frac{\varepsilon_k^3}{2}$. Note that

$$0\le\sum_{j=1}^\infty g_j\big(\|y\|,|a_j|\big)\le2$$

for all $\big(y,(a_j)\big)\in H$, and hence

$$\|y_{n+1}\|\le2\|y_n\|$$

for all n. We infer

$$\|y_{n_1}\|\ge\frac{\varepsilon_k^3}{4},\quad\text{and so}\quad\frac{\varepsilon_k^3}{4}\le\|y_m\|\le\frac{\varepsilon_k}{2}$$

for all $m=n_1,\dots,n_0-1$, implying

$$f_j(\|y_m\|)=0=g_j(\|y_m\|,|a_j|)$$

for all $j\neq k$ and $m=n_1,\dots,n_0-1$. We further obtain

$$\|y_{n_1+1}\|\le2\|y_{n_1}\|<\varepsilon_k^3\quad\text{and}\quad a_{n_1+1,k}=f_k(\|y_{n_1}\|)=0.$$

Consider the orthogonal projection $P:H\to H_0\oplus\mathbb{C}$ defined by

$$P(y,(a_j))=(y,a_k).$$

By the above observations, $Pu_{n_1+1} = (y_{n_1+1}, 0)$ satisfies

$$\|Pu_{n_1+1}\| \leq \varepsilon_k^3$$

and $Pu_{n_1+1}, Pu_{n_1+2}, \ldots, Pu_{n_0}$ are the iterations described in the previous example for $\varepsilon = \varepsilon_k$. The proof of this example then yields $\|y_{n_0}\| \leq \frac{\varepsilon_k}{2}$, a contradiction. We have shown that

$$\sup_n \|y_n\| \leq \frac{\varepsilon_k}{2} \quad \text{and} \quad \sup_{j,n} f_j(\|y_n\|) \leq \frac{\varepsilon_k}{2}.$$

Hence $\sup_n \|u_n\| \leq \varepsilon_k$, and the mapping $A + K + F$ is stable.

We are short of a similar construction for continuous time. However, we suspect that an example similar to the above for (1.1) may not exist.

6. Well-posedness for continuous time and global existence

We are primarily interested in nonlinear evolution equations of the form

$$x'(t) = Ax(t) + K(t, x(t)), \qquad x(0) = x_0, \quad t \geq 0, \tag{6.1}$$

where A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X and K is a function with values in compact (nonlinear) operators on X subject to additional regularity assumptions. One can study classical solutions of (6.1), that is, maps $x \in C^1([0,T), X) \cap C([0,T), D(A))$ solving (6.1). However, it is often convenient to deal with the integrated version of (6.1), namely

$$x(t) = T(t)x_0 + \int_0^t T(t-s)K(s, x(s))\, ds, \qquad t \geq 0. \tag{6.2}$$

Recall that a continuous function $x : [0,T) \to X$ is called a *mild* solution of (6.1) if it satisfies (6.2). Each classical solution of (6.1) is a mild solution, but not conversely. A mild solution is a classical one if $x_0 \in D(A)$ and, for instance, $K : [0, \infty) \times X \to X$ is C^1. In the sequel we only treat mild solutions, which we will call 'solutions' from now on, for simplicity.

If X has nice geometric properties, e.g. it is reflexive, then mild solutions can be identified with so-called weak solutions of (6.1), that is functions $x : [0, \infty) \mapsto X$ satisfying (6.1) in a weak sense. This notion will not be used in this paper, however.

Let the solution $x(t) = x(t, x_0)$ of (6.2) exist for all $t \geq 0$. It is called (nonlinearly) stable if for every $\varepsilon > 0$ there is radius $\delta > 0$ such for all $y_0 \in B(x_0, \delta)$ all solutions y with $y(0) = y_0$ are defined on $[0, \infty)$ and satisfy $\|y(t) - x(t)\| < \varepsilon$ for all $t \geq 0$. Without loss of generality, one may assume here that $x_0 = 0$. If the solution x is not stable, then it is said to be unstable.

Instability is a local and rather weak property. It is often of interest to show *global* properties of x which are stronger than mere instability. Such properties are the main topic of this paper. To not overshadow the study of asymptotics of the solutions to (6.2) with assumptions on its well-posedness, we will just postulate the existence and minimal regularity properties of solutions that we need in the sequel. They can be satisfied in many situations of interest as we will make clear below. Aiming at the long-term behavior, it is natural to consider a set-up when the solutions of (6.2) exist globally, i.e., on the whole of $[0,\infty)$. However, we do not need uniqueness of solutions in our main Theorem 7.6. Below we will mention several statements which could be used as additional assumptions to our asymptotic results, so that the results could be formulated in a priori terms.

The next local existence result going back to [62] is well-known, cf. e.g. [14, Section 4], [54, Section 6].

Theorem 6.1. *Let A generate the C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X and let $K : [0,\infty)\times X \to X$ be continuous and Lipschitz in x on bounded sets: for every $T>0$ and every $r>0$ there is $C_{T,r}>0$ such that*

$$\|K(t,x)-K(t,y)\| \leq C_{T,r}\|x-y\|$$

for $t\in[0,T]$ and $x,y\in B(0,r)$. Then for every $x_0\in X$ there exists a maximal existence time $T=T(x_0)\in(0,\infty]$ such that the following holds.

(i) *There is a unique solution $x=x(\cdot,x_0)\in C([0,T),X)$ of (6.2).*
(ii) *If $T<\infty$, then $\lim_{t\to T}\|x(t,x_0)\|=\infty$.*
(iii) *For any $T^*\in(0,T)$ there exists a radius $\delta>0$ such that $T=T(y_0)>T^*$ for all $y_0\in B(x_0,\delta)$. Moreover, the map $B(x_0,\delta)\to C([0,T^*],X), y_0\to x(\cdot,y_0)$, is Lipschitz continuous.*

The Gronwall inequality implies the following global existence result.

Corollary 6.2. *Besides the conditions of Theorem 6.1, assume that there exists $c\in L^1_{loc}([0,\infty))$ such that*

$$\|K(t,x)\|\leq c(t)(1+\|x\|) \qquad \text{all } x\in X,\ t\geq 0. \tag{6.3}$$

We then have $T=\infty$ for all $x_0\in X$.

See e.g. [62, Theorem 1 and Remark 1] or [54, Sections 6.1, and Cor. 2.3, Section 6.2] (where in Corollary 2.3 of the last reference the compactness of $(T(t))_{t\geq0}$ is irrelevant under our assumptions).

In a similar way, the linear growth condition of the above corollary leads to bounds on the solution without assuming Lipschitz continuity of $K(t,\cdot)$. For the refinement in part c), we refer to [75].

Proposition 6.3. *Let A generate the C_0-semigroup $(T(t))_{t\geq0}$ on a Banach space X with $\|T(t)\|\leq Me^{\omega t}$ for $t\geq 0$ and some $\omega\geq 0$, and let $K : [0,\infty)\times X\to X$ be continuous separately in both variables.*

a) Assume that K satisfies (6.3). Let $x:[0,T)\to X$ be a solution of (6.2) and $t_0\in(0,T)$. Then $\|x(t)\|\leq C(t_0)(1+\|x_0\|)$ for $t\in[0,t_0]$ and a constant only depending on t_0, M, ω and $c(\cdot)$.

Let now (6.2) have a global solution x for all $x_0\in X_0$ and some subset $X_0\subseteq X$.

b) If (6.3) holds, then for every bounded set $B\subset X_0$ the set $\{x(t,x_0) : 0\leq t\leq t_0,\ x_0\in B\}$ is bounded for each $t_0>0$.

c) Assume that K is bounded on bounded sets and that there exist $r_0 > 1$ and a non-increasing function $c : [r_0, \infty) \to [0, \infty)$ such that $\|K(t,x)\| \le c(\|x\|)\|x\|$ for all $t \ge 0$ and $x \in X$ with $\|x\| \ge r_0$. If $\gamma > \omega$ and $\lim_{r\to\infty} c(r) = 0$, then there exists a constant M_{γ,x_0} such that $\|x(t,x_0)\| \le M_{\gamma,x_0} e^{\gamma t}$ for $t \ge 0$ and $x_0 \in X_0$.

So, if K is sublinear as $\|x\| \to \infty$, under the above assumptions the solutions of (6.2) have the same growth bound as those for its linear part.

Recall that a continuous mapping $S : X \to X$ is called compact if it maps bounded subsets of X into precompact subsets of X. If the mapping $x \mapsto K(t,x)$ is compact on X, then one may drop Lipschitz type assumptions on K at the price of loosing the uniqueness of mild solutions to (6.2). (Roughly, one replaces Banach's fixed point theorem, guaranteeing the uniquneess of a fixed point, with Schauder's fixed point theorem, where the uniquenesss is hardly available.) However, as we have already remarked above, for our purposes mere existence suffices. The next result from [58] is an example of existence theorems based on compactness properties of the nonlinearity. We note that in this and related papers (such as [12] cited below) typically the concept of an integral solution is used. However, in our setting integral and mild solutions coincide, see Proposition 2 of [58].

Theorem 6.4. *Let A be the generator of a C_0-semigroup on a Banach space X, and $K : [0,\infty) \times X \to X$ be separately continuous and map bounded sets in $[0,\infty) \times X$ to relatively compact sets in X. Then for every $x_0 \in X$ there exists a maximal existence time $T = T(x_0)$ such that (6.2) admits a mild solution x on $[0,T)$. If $T < \infty$, then x is unbounded and $\{K(t, x(t,x_0)) : 0 < t < T\}$ is not relatively compact. Thus, in particular, if the range of K is relatively compact, then T must be infinite.*

Remark 6.5. The statement is formulated in [58, Theorem 2] combined with the remark following it for a semigroup of contractions. However, one can consider an arbitrary semigroup by passing to an equivalent norm and by an appropriate rescaling.

Combined with Proposition 6.3, the above theorem yields the following result which we already used in Theorem 1.1. For real Banach spaces, it was proved in [12] in a more general framework of differential inclusions and with weaker compactness and regularity assumptions on K.

Corollary 6.6. *Assume the conditions of Theorem 6.4 and* (6.3)*. Then all solutions of* (6.2) *are global.*

Finally, we will need a result yielding the existence of a unique propagator to (6.1) in the linear setting, see e.g. [24, Corollary VI.9.20].

Proposition 6.7. *Let A be the generator of a C_0-semigroup on Banach space X, and let $K : [0,\infty) \to B(X)$ be strongly continuous. Then there* *exists a unique evolution family* $(U(t,\tau))_{t\geq\tau\geq 0} \subset B(X)$ *such that*

$$U(t,\tau)x = T(t-\tau)x + \int_\tau^t T(t-s)B(s)U(t,s)x\,ds, \qquad t \geq \tau. \tag{6.4}$$

7. Instability for continuous time

Following a similar strategy as in Section 4, we now obtain lower bounds for solutions of semilinear abstract differential equations with a linear part being a generator of a C_0-semigroup. In view of the failure of the spectral mapping theorem for C_0-semigroups, this task is more demanding than the one treated in Section 4. However, the ideas remain the same as for the systems with discrete time.

We start with proving auxiliary (and probably known) results on compactness. Below we will assume that the mapping $K : [0,\infty) \times X \to X$ is separately continuous and *collectively compact*, i.e., the set $K\big([0,t_0] \times B\big)$ is precompact in X for each bounded set $B \subset X$ and each $t_0 \geq 0$. For this notion one may consult e.g. [2].

A simple condition for collective compactness is provided by assuming continuity in t uniformly for x in bounded subsets.

Lemma 7.1. *Let* $K : [0,\infty) \times X \to X$ *be separately continuous such that* $K(s,\cdot) : X \to X$ *is compact for all* $s \geq 0$. *Suppose that for each* $t_0, \varepsilon > 0$ *and each bounded set* $B \subset X$ *there exists a* $\delta_\varepsilon > 0$ *such that*

$$\|K(s,x) - K(t,x)\| \leq \varepsilon$$

for all $s,t \in [0,t_0]$ *with* $|s-t| \leq \delta_\varepsilon$ *and* $x \in B$. *Then* K *is collectively compact.*

Proof. Let $t_0 > 0$ and $B \subset X$ be a bounded set. Take $\varepsilon > 0$. Let $\{t_1,\dots,t_n\}$ be a finite $\delta_{\varepsilon/2}$-net in the interval $[0,t_0]$. For each $j = 1,\dots,n$ the set $K(t_j)B$ is precompact and so is the set $C = \{K(t_j,b) : j \in \{1,\dots,n\}, b \in B\}$. Let $x_1,\dots,x_m$ be a finite $\frac{\varepsilon}{2}$-net for C.

Let $0 \leq s \leq t_0$ and $u \in B$. There exist $j \in \{1,\dots,n\}$ with $|s-t_j| \leq \delta_{\varepsilon/2}$ and $i \in \{1,\dots,m\}$ with $\|K(t_j,u) - x_i\| < \varepsilon/2$. Using also the assumption, we obtain

$$\|K(s,u) - x_i\| \leq \|K(s,u) - K(t_j,u)\| + \|K(t_j,u) - x_i\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $\{x_1,\dots,x_m\}$ is a finite ε-net for the set $K\big([0,t_0]\times B\big)$. Since $\varepsilon > 0$ was arbitrary, the set $K\big([0,t_0]\times B\big)$ is precompact. □

We collect our standing assumptions for semilinear evolution equations.

- (A0) Assume that $K : [0,\infty] \times X \to X$ is collectively compact.
- (A1) Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X with generator A and $K : [0,\infty) \times X \to X$ be continuous separately in both

variables. Let $X_0 \subset X$ and assume that for every $x_0 \in X_0$ there exists a continuous solution $x : [0, \infty) \to X$ of

$$x(t, x_0) = T(t)x_0 + \int_0^t T(t-s)K(s, x(s, x_0))\, ds, \qquad t \geq 0. \tag{7.1}$$

(A1') Let (A1) hold and assume that for every $x_0 \in X_0$ the solution $x(\cdot, x_0)$ of (7.1) is unique. So the map $x : [0, \infty) \times X_0 \to X$ is well-defined, and assume that it is continuous separately in both variables.

(A2) Let (A1) hold. Assume that for a bounded set $B \subset X_0$ and $x_0 \in B$ there exist continuous solutions $x = x(t, x_0)$ of (7.1) such that the set $\{x(t, x_0) : 0 \leq t \leq t_0,\ x_0 \in B\}$ is bounded for each $t_0 > 0$.

If the solutions of (7.1) are unique, then (A2) says that the map $x : [0, \infty) \times X_0 \to X$ maps bounded into bounded sets. Otherwise, the map can be defined for each bounded subset of X_0, though it depends on this subset and the choice of corresponding solutions. Recall that Corollary 6.6 provides an example of K satisfying (A1) and (A2). One can treat the non-unique case in a more systematic manner, for instance using "generalized semi-flows". See e.g. [7], where also nonlinear damped wave equations are treated. In this paper, we prefered however to avoid such concepts.

Proposition 7.2. *Let assumptions (A0), (A1) and (A2) hold, and let $t_0 > 0$ be fixed. If C be the set whose elements are of the form*

$$\int_0^t T(t-s)K(s, x(s, y))\, ds$$

for all $x_0 \in B$ and $t \in [0, t_0]$, then C is precompact in X.

Proof. Set $k = \sup\{\|T(t)\| : t \in [0, t_0]\} < \infty$ and $k' = \sup\{\|x(t)\| : t \in [0, t_0],\ x_0 \in B\} < \infty$. Fix $\varepsilon > 0$.

By assumption, the set $C_0 = \bigcup_{0 \leq s \leq t_0} \{K(s, y) : y \in X,\ \|y\| \leq k'\}$ is precompact. Let E be a finite $\frac{\varepsilon}{3kt_0}$-net in C_0. Since the set $\bigcup_{s \leq t_0} T(s)(E)$ is precompact, Mazur's theorem yields the compactness of

$$M := [0, t_0] \cdot \overline{\text{conv}} \Big(\bigcup_{s \leq t_0} T(s)(E) \Big).$$

Choose a finite $\frac{\varepsilon}{3}$-net E' in M. Let $t \leq t_0$, $x_0 \in B$, and $\varepsilon > 0$. We have

$$\Big\| \int_0^t T(t-s)K(s, x(s))\, ds - S \Big\| < \varepsilon/3.$$

where

$$S = \sum_{j=1}^{N} T(t - s_j)K(s_j, x(s_j))\lambda(\Delta_j)$$

for some $N \in \mathbb{N}$, $s_j \in \Delta_j$ and a partition $\{\Delta_j : j \in \{1 \ldots, N\}\}$ of $[0, t]$. For $j = 1, \ldots, N$ there are $e_j \in E$ such that

$$\|K(s_j, x(s_j)) - e_j\Big\| < \frac{\varepsilon}{3kt_0}$$

Hence,

$$\Big\| S - \sum_{j=1}^{N} T(t-s_j)e_j\lambda(\Delta_j)\Big\| < \frac{\varepsilon}{3},$$

Since $\sum_{j=1}^{N} T(t-s_j)e_j\lambda(\Delta_j)$ belongs to M, we find $e' \in E'$ with

$$\Big\|\sum_{j=1}^{N} T(t-s_j)e_j\lambda(\Delta_j) - e'\Big\| < \frac{\varepsilon}{3}.$$

It follows

$$\Big\|\int_0^t T(t-s)K(s,x(s))\,ds - e'\Big\| < \varepsilon,$$

i.e., E' is a finite ε-net for the set C. Since $\varepsilon > 0$ was arbitrary, C is precompact. □

We will use Proposition 7.2 in form of the next immediate corollary, which below plays the role of Lemma 4.1.

Corollary 7.3. *Under the assumptions of Proposition 7.2, there exists a compact set $C_0 = C_0(t_0, B) \subset X$ such that*

$$x(t, x_0) \in T(t)x_0 + C_0$$

for all $x_0 \in B$ and $t \in [0, t_0]$.

The next statement is a counterpart of Theorem 4.4 providing an exponential growth bound on a resudual set at the expense of stronger assumptions on K and a much smaller set where the bound holds.

Theorem 7.4. *Assume that (A0) holds, (A1') is true with $X_0 = X$, and (A2) is satisfied for any bounded subset of X. Let $a : [0,\infty) \to \mathbb{R}_+$ be a non-increasing function with $\lim_{t\to\infty} a(t) = 0$ and $L \subset \mathbb{R}_+$ be unbounded. Then the set*

$$\big\{y \in X : \text{there are infinitely many } t_n \in L \text{ with } \|x(t_n, y)\| \geq a(t_n)\|T(t_n)\|_\mu\big\}$$

is residual.

Proof. The proof follows that of Theorem 4.4, though we do not reduce it to this result. Without loss of generality we assume that $\|T(t_0)\|_\mu \neq 0$ for all $t_0 \geq 0$ (since otherwise $\|T(t)\|_\mu \leq \|T(t_0)\|_\mu \cdot \|T(t-t_0)\|_\mu = 0$ for all $t \geq t_0$.) For $k \in \mathbb{N}$ set

$$M_k = \{y \in X : \text{ there exists } t \in L \text{ with } t \geq k,\ \|x(t,y)\| > a(t)\|T(t)\|_\mu\}.$$

Note that M_k is open since x is continuous in y. We show that M_k is dense.

Let $y \in X$ and fix $\varepsilon > 0$. Choose $t \in L$ with $t \geq k$ and $a(t) < \varepsilon/4$. By Corollary 7.3 there is a compact set $C \subset X$ such that

$$x(t, y') \in T(t)y' + C$$

for all $y' \in X$ with $\|y' - y\| \leq \varepsilon$. Since C is compact, there exists a finite-dimensional subspace $F \subset X$ such that $\operatorname{dist}\{c, F\} < \frac{\varepsilon\|T(t)\|_\mu}{12}$ for all $c \in C$.

Lemma 4.3 provides a unit vector $u \in X$ with $\operatorname{dist}\{T(t)u, F\} > \frac{\|T(t)\|_\mu}{3}$. We compute

$$\operatorname{dist}\{T(t)(y+\varepsilon u), F\}+\operatorname{dist}\{T(t)(y-\varepsilon u), F\} \geq 2\varepsilon \operatorname{dist}\{T(t)u, F\} \geq \tfrac{2\varepsilon}{3}\|T(t)\|_\mu.$$

So $x_0 := y + \varepsilon u$ or $x_0 := y - \varepsilon u$ satisfy

$$\|x_0 - y\| \leq \varepsilon \quad \text{and} \quad \operatorname{dist}\{T(t)x_0, F\} \geq \frac{\varepsilon}{3}\|T(t)\|_\mu.$$

It follows

$$\|x(t, x_0)\| \geq \operatorname{dist}\{T(t)x_0, F\} - \frac{\varepsilon\|T(t)\|_\mu}{12} \geq \frac{\varepsilon\|T(t)\|_\mu}{4} \geq a(t)\|T(t)\|_\mu.$$

Hence $x_0 \in M_k$, and M_k is dense since $y \in X$ and $\varepsilon > 0$ were arbitrary. The Baire category theorem shows that $\bigcap_{k=1}^{\infty} M_k$ is a dense G_δ set, and thus residual. □

Corollary 7.5. *Assume that all of the conditions of Theorem 7.4 hold. Let* $a : [0, \infty) \to \mathbb{R}_+$ *be a non-increasing function satisfying* $\lim_{t\to\infty} a(t) = 0$. *Then there is a residual set* $M \subset X$ *such that for every* $x_0 \in M$ *there exist an unbounded sequence* $(t_n)_{n\geq 1}$ *with*

$$\|x(t_n, x_0)\| \geq a(t_n)e^{\omega_e(T)t_n}, \qquad n \in \mathbb{N}.$$

We next prove a continuous analogue of Theorem 4.6 which is one of the main results of this paper. It provides global and sharp exponential lower bounds for solutions of semilinear differential equations with unbounded linear part. Recall here the notion of admissibility introduced in Section 2.

Theorem 7.6. *Let (A0) and (A1) hold, assume that* $B(y, r) \subset X_0$ *and let (A2) hold for* $B = B(y, r)$. *Setting* $\alpha = \limsup_{t\to 0} \|T(t)\|$, *let* $a : [0, \infty) \to \mathbb{R}_+$ *be a non-increasing function satisfying* $\lim_{t\to\infty} a(t) = 0$, $t_0 \geq 0$ *with* $a(t_0) < \frac{r}{2\alpha}$. *Assume that* $\omega \in \mathbb{R}$ *is admissible. Then there exists* $z \in \overline{B}(y, r)$ *such that*

$$\|x(t, z)\| \geq a(t)e^{\omega t}, \qquad t \geq t_0.$$

Proof. Without loss of generality we may assume that $\omega = 0$. If not, then consider the semigroup $(e^{-\omega t}T(t))_{t\geq 0}$ and compact perturbations $e^{-\omega s}K(e^{\omega s}\cdot)$. Consider a new norm $\|\cdot\|'$ on X defined by

$$\|x\|' = \sup\{|\langle x, f\rangle| : f \in D(A^*), \|f\| = 1\}.$$

Then, by (9.2), $\|\cdot\|'$ is equivalent to $\|\cdot\|$, and

$$\|x\|' \leq \|x\| \leq \alpha\|x\|'$$

for all $x \in X$.

Fix c_1 such that $2a(t_0) < c_1 < r$ and $r - c_1 < 1$. Pick an increasing sequence $(t_k)_{k\geq 0}$ with $a(t_k) < \frac{r-c_1}{\alpha 2^{k+2}}$ for all $k \in \mathbb{N} \cup \{0\}$. Set

$$c_k = \frac{r - c_1}{2^{k-1}}, \qquad k \geq 1.$$

For each $k \in \mathbb{N}$ choose $\varepsilon_k > 0$ such that $\varepsilon_{k+1} < \varepsilon_k$ and

$$\frac{(1-\varepsilon_k)^2}{2\alpha} c_k - 2\varepsilon_k \geq a(t_{k-1})$$

for all $k \geq 1$.

The required element z will be constructed as the sum of a series of appropriate approximate eigenvectors x_k for $(T(t))_{t\geq 0}$ and $e^{\mu_k t}$. We set $F_0 = \{0\}$ and $M_0 = X$. Inductively we construct unit vectors $x_1, x_2, \dots$, finite-dimensional subspaces $F_1 \subset F_2 \subset \cdots$, and finite-codimensional subspaces $M_1 \supset M_2 \supset \cdots$ with $x_k \in M_k \cap F_{k+1}$ which satisfy (7.2)–(7.5) below.

Let $k \geq 1$ and suppose that the vectors $x_0, \dots, x_{k-1}$ and subspaces $F_1, \dots, F_{k-1}$ and $M_1, \dots, M_{k-1}$ have already been constructed in case $k \neq 1$. Corollary 7.3 yields a compact set $C_k \subset X$ such that $x(t,u) \in T(t)u + C_k$ for all $t \leq t_k$ and $u \in B(y,r)$. There thus exists a finite-dimensional subspace $F_k \supset F_{k-1}$ such that $x_{k-1} \in F_k$ if $k > 1$ and

$$\operatorname{dist}'\Big\{T(t)\Big(y + \sum_{j=1}^{k-1} \alpha^{-1} c_j x_j\Big), F_k\Big\} \leq \varepsilon_k/2 \tag{7.2}$$

for all $t \leq t_k$ and

$$\operatorname{dist}'\{d, F_k\} \leq \varepsilon_k/2 \tag{7.3}$$

for all $d \in C_k$, where dist' is the distance in the new norm $\|\cdot\|'$. Lemma 9.1 yields a subspace $M_k \subset M_{k-1}$ of finite codimension such that

$$\|f+m\|' \geq (1-\varepsilon_k) \max\Big\{\|f\|', \frac{\|m\|'}{2}\Big\} \tag{7.4}$$

for all $f \in F_k$ and $m \in M_k$, where

$$M_k = \bigcap_{1\leq j\leq n} \operatorname{Ker} x_j^*$$

for some $x_j^* \in D(A^*)$ for $1 \leq j \leq n$. Since 0 is admissible, we can choose $x_k \in M_k$ with $\|x_k\|' = 1$ and $\mu_k \in \mathbb{C}$ with $\operatorname{Re}\mu_k = 0$ such that

$$\|T(t)x_k - e^{\mu_k t} x_k\|' \leq \varepsilon_k \tag{7.5}$$

for all $t \leq t_k$.

Suppose that the vectors x_k, $k \in \mathbb{N}$, have been constructed as above. Set

$$z = y + \sum_{j=1}^{\infty} \alpha^{-1} c_j x_j.$$

We show that z meets the requirements. First, z belongs to $\overline{B}(y,r)$ since

$$\|z-y\| \leq \alpha\|z-y\|' \leq \sum_{j=1}^{\infty} c_j = c_1 + \sum_{j=2}^{\infty} \frac{r-c_1}{2^{j-1}} = r.$$

Fix now $k \geq 1$ and consider $t \in [t_{k-1}, t_k]$. Properties (7.3) and (7.2) yield

$$\|x(t,z)\|' \geq \operatorname{dist}'\{T(t)z, C_k\} \geq \operatorname{dist}'\{T(t)z, F_k\} - \varepsilon_k/2$$

$$\geq \operatorname{dist}'\Big\{\sum_{j=k}^{\infty}\alpha^{-1}c_jT(t)x_j, F_k\Big\} - \varepsilon_k.$$

By means of (7.5) we estimate

$$\|x(t,z)\|' \geq \operatorname{dist}'\Big\{\sum_{j=k}^{\infty}\alpha^{-1}c_je^{\mu_jt}x_j, F_k\Big\} - \sum_{j=k}^{\infty}\alpha^{-1}c_j\varepsilon_j - \varepsilon_k$$
$$\geq \operatorname{dist}'\Big\{\sum_{j=k}^{\infty}\alpha^{-1}c_je^{\mu_jt}x_j, F_k\Big\} - 2\varepsilon_k.$$

Taking into account (7.4) and that $x_j \in M_j \subset M_k$ for all $j \geq k$, we infer

$$\|x(t,z)\|' \geq \frac{1-\varepsilon_k}{2}\Big\|\sum_{j=k}^{\infty}\alpha^{-1}c_je^{\mu_jt}x_j\Big\| - 2\varepsilon_k.$$

Since $x_k \in F_{k+1}$ and $x_j \in M_j \subset M_{k+1}$ for $j \geq k+1$, (7.4) also implies

$$\|x(t,z)\|' \geq \frac{(1-\varepsilon_k)(1-\varepsilon_{k+1})}{2}\|\alpha^{-1}c_ke^{\mu_kt}x_k\| - 2\varepsilon_k$$
$$\geq \frac{(1-\varepsilon_k)^2}{2}\cdot\alpha^{-1}c_k - 2\varepsilon_k \geq a(t_{k-1}) \geq a(t).$$

Hence

$$\|x(t,x_0)\| \geq \|x(t,x_0)\|' \geq a(t)$$

for all $t \geq t_0$, as required. □

Remark 7.7. If X is a reflexive Banach space, then the proof of the previous theorem is simpler. In this case it is not necessary to do the renormalization since $\|\cdot\|' = \|\cdot\|$ by the density of $D(A^*)$, and the subspace M in the statement above can be any subspace of finite codimension.

Corollary 7.8. *(i) Let the assumptions of Theorem 7.6 hold, and let $r > 2a(0)\alpha$. Then there exists $y \in X_0$ with $\|x(t,y)\| \geq a(t)e^{\omega t}$ for all $t \geq 0$.*

(ii) Let assumptions (A0) and (A1) hold, and let (A2) be satisfied for all bounded subsets B of X_0. Then there exists a dense subset of vectors $y \in \operatorname{Int} X_0$ such that $\|x(t,y)\| \geq a(t)e^{\omega t}$ for all t sufficiently large.

Proof. To deduce (i), it suffices to set $t_0 = 0$, and to let y be the center of the ball in X_0 of radius greater than $2a(0)\alpha$.

To prove (ii), we let $y \in \operatorname{Int} X_0$ and $\varepsilon = \operatorname{dist}\{y, \partial X_0\}$. Find $t_0 \geq 0$ such that $a(t_0) < \frac{\varepsilon}{2}$. By Theorem 7.6, there exists $x_0 \in X$ such that $\|x_0 - y\| \leq \varepsilon$ and $\|x(t,x_0)\| \geq a(t)e^{\omega t}$ for all $t \geq t_0$. □

Remark 7.9. If X is a Hilbert space, as in Section 4 one can obtain a better estimate. Let $y \in X$, $\{u : \|u-y\| \leq r\} \subset X_0$, $t_0 \geq 0$ and $a(t_0) < r$. Then there exists $x_0 \in X$ with $\|x_0 - y\| \leq r$ and $\|x(t,x_0)\| \geq a(t)e^{\omega t}$ for $t \geq t_0$.

The next corollary is one of the main results of the paper. It is a direct consequence of Theorem 2.2 and Corollary 7.8.

Corollary 7.10. *Under the conditions of Theorem 7.6, there exists* $x_0 \in B(y,r)$ *and* $t_0 \geq 0$ *such that*

$$\|x(t,x_0)\| \geq a(t)e^{s_R(A)t}, \qquad t \geq t_0. \tag{7.6}$$

If the assumptions (A0) and (A1) hold, and (A2) is true for all bounded subsets B *of* X_0*, then the set of* x_0 *satisfying* (7.6) *is dense in* X.

If in (7.6) the operator A is bounded and $s_e(A) > 0$, then under appropriate local assumptions on K (involving compactness) the local instability of zero solution to (7.6) was shown in [70].

Our results on lower bounds are also new in the framework of linear equations (6.1). Here Theorems 2.2 and Corollary 7.8 yield the next estimate.

Corollary 7.11. *Let* A *be the generator of a* C_0*-semigroup on Banach space* X *and* $K : [0,\infty) \to B(X)$ *be a strongly continuous function such that* $K(t)$ *is a compact operator for each* $t \geq 0$*. Let* $U(t,\tau)$ *be the evolution family given by Proposition 6.7. Assume that* $a : [0,\infty) \to [0,\infty)$ *is a decreasing function satisfying* $\lim_{t\infty} a(t) = 0$*. Then there exists a dense set of vectors* x *such that*

$$\|U(t,0)x\| \geq a(t)e^{s_R(A)t}, \qquad t \geq t_0 = t_0(x).$$

We discuss our results a bit. First, note that one may treat $s_R(A)$ "up to compact perturbations": In the above result one may consider $s_R(A + S)$ for a compact perturbation $S \in B(X)$ of A and substract S from the nonlinearity K in (7.1), without changing assumptions on K.

Remark 7.12. Theorem 7.6 and Corollaries 7.8 and 7.10 in general do not hold for K being merely relatively compact with respect to A, i.e., such that $K : D(A) \to X$ is compact, where $D(A)$ is equipped with the grapgh norm. For instance, even in the setting of linear damped wave equations, if $A = A_-$ is the operator given by (8.5) below, then

$$A_- = D + K, \qquad D := \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \qquad K := \begin{pmatrix} 0 & 0 \\ 0 & -b \end{pmatrix},$$

where D generates a unitary C_0-group, $s_R(D) = 0$, K is relatively compact, but in view of [55] or [9], the operator A_- generates an exponentially stable semigroup if $b \in C^\infty(\mathcal{M})$ and b satisfies the so-called geometric control condition. See also e.g. [1], [40], [46] for a relevant discussion and some examples. If K is compact, then the situation changes as we show in the next section.

Finally, we address the optimality of results stated in Theorem 7.6 and Corollary 7.10.

Remark 7.13. We recall a known result, see e.g. [48, Theorem 5.3]. Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a Hilbert space X with generator A such that $s_e(A) = 0$ (or even $s_0(A) = 0$) and $T(t) \to 0$ as $t \to \infty$ strongly. Then for any decreasing map $a : [0,\infty) \to [0,\infty)$ with $\|a\|_\infty < 1$

there exists a unit vector $x \in X$ such that $\|T(t)x\| \geq a(t)$ for $t \geq 0$. Here $s_R(A) = 0$, and thus Theorem 7.6 and Corollary 7.10 cannot be improved by removing the function a from their formulations, even for linear equations (6.1) with $K = 0$. For a concrete examples of such semigroups one may consider the semigroup of left shifts on $L^2([0,\infty))$ or multiplication semigroups $(T(t)f)(z) = e^{-tz}f(z)$, $t \geq 0$, on appropriate spaces $L^2(\Omega)$ with $\Omega \subset \mathbb{C}$.

8. Backward damped wave equations and other applications

In this section, we illustrate our abstract results by applying them to the study of backward damped wave equations or, equivalently, excited wave equations, subject to nonlinear forcing terms of the form $-f(t,\cdot,u)$. In this basic PDE setting a positive abscissa $s_R(A) > 0$ and compact nonlinear perturbations $K(t,\cdot)$ occur naturally, including situations where positivity of s_R stems from the resolvent growth. In contrast to earlier work, as in e.g. in [71], we allow for time-dependent and merely continuous f so that we cannot expect uniqueness in general. As we focus on asymptotic properties of solutions, our assumptions on nonlinearities are, of course, not best possible, and serve first of all to create the right framework to the study of lower bounds for solutions in spectral terms.

We first look at the excited wave equation

$$\begin{aligned} &\partial_{tt}u(t,x) - \Delta u(t,x) - b\partial_t u(t,x) = -f(t,x,u(t,x)), \qquad x \in \mathcal{M}, \quad t \geq 0, \\ &u(0,x) = u_0(x), \quad \partial_t u(t,0) = u_1(x), \qquad x \in \mathcal{M}, \end{aligned} \tag{8.1}$$

on a d-dimensional, compact, smooth and connected Riemannian manifold $\mathcal{M}$ without or with boundary $\partial\mathcal{M}$. Here Δ is the Laplace-Beltrami operator on $\mathcal{M}$, depending in general on a metric on $\mathcal{M}$. We do not indicate this dependence since it will not be relevant. If $\partial\mathcal{M} \neq \emptyset$, we additionally impose Dirichlet boundary conditions in (8.1).

In order to apply the results from the previous section, we have to consider complex-valued u in (8.1) and related equations. Since we do not want to restrict ourselves to holomorphic maps $\zeta \mapsto f(t,x,\zeta)$, we identify $\mathbb{C}$ with $\mathbb{R}^2$ equipped with the Euclidean scalar product $\zeta_1 \cdot \zeta_2 = \xi_1\xi_2 + \eta_1\eta_2 = \mathrm{Re}\,(\zeta_1\overline{\zeta_2})$ where $\zeta_j = \xi_j + i\eta_j = (\xi_j, \eta_j)$. Differentiability is then understood in the real sense and derivatives are only $\mathbb{R}$-linear. Fortunately, this does not affect the basic rules from calculus that we use here. Throughout we assume that

$$\begin{aligned} &b \in L^\infty(\mathcal{M}), \qquad f : \mathbb{R} \times \mathcal{M} \times \mathbb{C} \to \mathbb{C} \text{ is continuous in } \zeta \text{ and measurable,} \\ &|f(t,x,\zeta)| \leq \kappa(t)(1 + |\xi|^\alpha), \qquad |f(t,x,\zeta) - f(s,x,\zeta)| \leq \omega(|t-s|)(1 + |\zeta|^\alpha) \end{aligned} \tag{8.2}$$

for some $0 \leq \alpha < d/(d-2)_+$, a locally integrable function $\kappa : \mathbb{R} \to [0,\infty)$, a map $\omega : [0,\infty) \to [0,\infty)$ with $\omega(\tau) \to 0$ as $\tau \to 0$ and all $t,s \in \mathbb{R}$, $\zeta \in \mathbb{C}$, $x \in \mathcal{M}$. (Here $y_+ = \max\{y,0\}$ for $y \in \mathbb{R}$ and $\frac{d}{0} := \infty$.) In our examples below we (mostly) have $b \geq 0$, fitting to the interpretation of (8.1) as excited wave equation. We also consider functions $f : [0,\infty)\times\mathcal{M}\times\mathbb{C} \to \mathbb{C}$, extending them by $f(t,x,\zeta) := f(0,x,\zeta)$ for $t \le 0$.

Observe that (8.2) implies the continuity of f in (t,ζ) and that one can choose a locally bounded map κ in (8.2).

To formulate (8.1) as an evolution equation of the form (1.1), we set $V = H^1(\mathcal{M})$ if $\mathcal{M}$ has no boundary and $V = H^1_0(\mathcal{M})$ otherwise. Recall that V is compactly embedded into $L^{2\alpha}(\mathcal{M})$, see Theorem 2.34 of [6]. On the state space $X = V \times L^2(\mathcal{M})$ we introduce the operator matrix

$$A_+ = \begin{pmatrix} 0 & I \\ \Delta & b \end{pmatrix}, \qquad D(A_+) = (H^2(\mathcal{M}) \cap V) \times V. \tag{8.3}$$

It is well known that A_+ generates a C_0-group $(T_+(t))_{t\in\mathbb{R}}$ on X. We write elements of X as $w = (u, v)$.

The forcing term is expressed by $K(t,w) = (0, -f(t,\cdot,u))$. We see below that $K : \mathbb{R}\times X \to X$ is continuous. Using standard properties of the wave equation with the (continuous) right-hand side $t \mapsto h(t) := -f(t,\cdot,u(t))$, it can be checked that a mild solution w to

$$w'(t) = A_+w(t) + K(t,w(t)), \quad t \ge 0, \quad w(0) = w_0 := (u_0,u_1) \in X, \tag{8.4}$$

on a time interval $J = [0,T)$ is of the form $w = (u, \partial_t u)$ for a function u in $C^2(J,H^{-1}(\mathcal{M})) \cap C^1(J,L^2(\mathcal{M})) \cap C(J,V)$ which solves (8.1).

For the damped case, we replace $+b$ by $-b$ obtaining the operator

$$A_- = \begin{pmatrix} 0 & I \\ \Delta & -b \end{pmatrix}, \qquad D(A_-) = D(A_+). \tag{8.5}$$

It generates a C_0-group $(T_-(t))_{t\in\mathbb{R}}$, which is contractive in forward time if $b \ge 0$, and it corresponds to the damped wave equation

$$\begin{aligned} &\partial_{tt}u(t,x) - \Delta u(t,x) + b\partial_t u(t,x) = -f(t,x,u(t,x)), \qquad x \in \mathcal{M}, \ \ t \ge 0, \\ &u(0,x) = u_0(x), \quad \partial_t u(t,0) = u_1(x), \qquad x \in \mathcal{M}. \end{aligned} \tag{8.6}$$

The above remarks on the solution also apply here. We study (8.6) backwards in time, i.e., for $t \le 0$. To bring it in standard forward form, one looks at $\tilde{u}(t) = u(-t)$ for $t \ge 0$ satisfying

$$\begin{aligned} &\partial_{tt}\tilde{u}(t,x) - \Delta\tilde{u}(t,x) - b\partial_t\tilde{u}(t,x) = -f(-t,x,\tilde{u}(t,x)), \qquad x \in \mathcal{M}, \ \ t \ge 0, \\ &\tilde{u}(0,x) = u_0(x), \quad \partial_t\tilde{u}(t,0) = -u_1(x), \qquad x \in \mathcal{M}. \end{aligned} \tag{8.7}$$

This system coincides with (8.1) except for the additional minus in f and before u_1. We drop the tilde. To rewrite this problem as (1.1), we use the operators $B = -A_-$ and $w \mapsto -K(-t,w) = (0, f(-t,\cdot,u))$. Observe that (8.7) can be reformulated as the first-order problem

$$w'(t) = Bw(t) - K(-t,w(t)), \quad t \ge 0, \qquad w(0) = (u_0, -u_1). \tag{8.8}$$

We also recall that $(A_+)^* = -A_-$. Indeed, arguing as in [77, Lemma 1, p.75], it is easy to check that $D(-A_-) \subset D((A_+)^*)$ and $-A_- \subset (A_+)^*$. As far as both operators $-A_-$ and $(A_+)^*$ generate C_0-semigroups, we have

$\rho(-A_-) \cap \rho(A_+) \neq \emptyset$ and then $D(-A_-) = D((A_+)^*)$. Therefore, $(T_+(t))^* = T_-(-t)$ for $t \in \mathbb{R}$, and, in view of the observation (2.5),

$$s_R(-A_-) = s_R(A_+). \tag{8.9}$$

It is well-known that $\sigma(A_-) \subset \{\lambda : -\|b\|_\infty \leq \operatorname{Re}\lambda \leq 0\}$ if $b \geq 0$, that $\sigma(A_-)$ is invariant under conjugation, and that it consists of a discrete set of eigenvalues since A_- has compact resolvent. Moreover if $b \geq 0$ is non-zero, we have $\sigma(A_-) \cap i\mathbb{R} = \emptyset$ if $\mathcal{M} \neq \emptyset$ and $\sigma(A_-) \cap i\mathbb{R} = \{0\}$ (with constants as eigenfunctions) if $\mathcal{M}$ has no boundary. (See e.g. [46].)

In the latter case, let P_0 be the Riesz projection corresponding to 0 and equip $X_0 = (I - P_0)X$ with the inner product norm

$$\|(u_0, u_1)\|_{X_0} = \|(-\Delta)^{1/2} u_0\|_{L^2} + \|u_1\|_{L^2}, \qquad (u_0, u_1) \in X_0.$$

Then $\dot{T}_-(t) := T_-(t) \restriction_{X_0}, t \in \mathbb{R}$, is a C_0-group generated by $\dot{A}_- := A_- \restriction_{X_0}$, which is contractive for $t \geq 0$. Moreover, by e.g. [1, Section 4], $\sigma(\dot{A}_-) = \sigma(A) \setminus \{0\} \subset \{\lambda : \operatorname{Re}\lambda \leq 0\}$, and in particular there are $c_1, c_2 > 0$ such that

$$c_1 \|R(\lambda, A_-)\| \leq \|R(\lambda, \dot{A}_-)\| \leq c_2 \|R(\lambda, A_-)\|$$

for $\lambda \in \rho(A_-)$ with $|\lambda| \geq \epsilon_0$ for an appropriate $\epsilon_0 > 0$. This construction allows one, in particular, to study the energy decay for (8.6) in a unified manner, see e.g. [1], and also [17] and [10]. We will also use $\dot{A}_-$ in the sequel to study the resolvent of A_-, see Example 8.6.

Before we consider spectrum and resolvent of $A_\pm$ in specific cases, we first establish the required properties of K,

Proposition 8.1. *Let f and b satisfy* (8.2). *Then the map $K : \mathbb{R} \times X \to X$ defined above is (jointly) continuous and collectively compact. If also*

$$|f(t, x, \zeta)| \leq \kappa(t)(1 + |\zeta|) \tag{8.10}$$

for κ from (8.2) *and all $(t, x, \zeta) \in \mathbb{R} \times \mathcal{M} \times \mathbb{C}$, we obtain $\|K(t, w)\| \leq c\kappa(t)(1 + \|w\|)$.*

Proof. The last claim is clear. For the first, let $t_n \to t$ in $\mathbb{R}$ and $w_n = (u_n, v_n) \to w = (u, v)$ in X. Since V is compactly embedded in $L^{2\alpha}(\mathcal{M})$, there is a subsequence and a map $g \in L^{2\alpha}(\mathcal{M})$ such that $u_{n_k} \to u$ in $L^{2\alpha}(\mathcal{M})$ and pointwise a.e. as $k \to \infty$ and $|u_{n_k}| \leq g$ a.e. for all k. As noted above, f is jointly continuous in (t, ζ) and κ can be chosen to be locally bounded so that $m := \sup_n \kappa(t_n) < \infty$. Hence, $f(t_{n_k}, \cdot, u_{n_k})$ tends pointwise a.e. to $f(t, \cdot, u)$ and $|f(t_{n_k}, \cdot, u_{n_k})| \leq m(1 + |g|^\alpha) \in L^2(\mathcal{M})$ by (8.2). it follows $f(t_{n_k}, \cdot, u_{n_k}) \to f(t, \cdot, u)$ in $L^2(\mathcal{M})$, and so K is continuous.

To prove compactness, take $t \in \mathbb{R}$ and a bounded sequence (w_n) in X. Again, there is a subsequence and maps $g \in L^{2\alpha}(\mathcal{M})$ and $u \in V$ such that $u_{n_k} \to u$ in $L^{2\alpha}(\mathcal{M})$ and pointwise a.e. as $k \to \infty$ and $|u_{n_k}| \leq g$ a.e. for all k. As above, we infer that $f(t, \cdot, u_{n_k})$ tends to $f(t, \cdot, u)$ in $L^2(\mathcal{M})$, and thus $K(t, \cdot) : X \to X$ is compact. To use Lemma 7.1, let $t_0, r > 0$, $t, s \in [-t_0, t_0]$

and $\|w\| \le r$. We then obtain $\|u\|_{L^{2\alpha}} \le Cr$ due to Sobolev's embedding, and (8.2) yields

$$\begin{aligned}\|f(t,\cdot,u)-f(s,\cdot,u)\|_{L^2(\mathcal{M})} \le& \omega(|t-s|)\|1+|u|^\alpha\|_{L^2(\mathcal{M})}\\ \le& \omega(|t-s|)(\mathrm{vol}(\mathcal{M})^{\frac12}+C^\alpha r^\alpha).\end{aligned}$$

As $\omega(\tau)\to 0$ as $\tau\to 0$, the map K is collectively compact by Lemma 7.1. □

Due to this result, Corollary 6.6 and Proposition 6.3, if $\alpha \le 1$ in (8.2) the operators $\pm K(\pm t,\cdot)$ fit to Theorem 7.6 and its corollaries. If $1<\alpha< d/(d-2)_+$, we need a sign condition and extra time regularity of f to show global existence by means of a standard energy estimate, cf. e.g. Chapter 6 of [14]. We define the potential of $\zeta\mapsto f(t,x,\zeta)$ by the (real) line integral

$$\varphi(t,x,\zeta)=\int_0^1 f(t,x,\tau\zeta)\cdot\zeta\,d\tau \tag{8.11}$$

for $(t,x,\zeta)\in\mathbb{R}\times\mathcal{M}\times\mathbb{C}$. Notice that

$$|\varphi(t,x,\zeta)|\le c\kappa(t)(1+|\zeta|^{\alpha+1}),\qquad (t,x,\zeta)\in\mathbb{R}\times\mathcal{M}\times\mathbb{C}, \tag{8.12}$$

by (8.2). We assume that φ is differentiable in ζ with $\nabla_\zeta\varphi=f$. If f is C^1 in $\zeta=(\xi,\eta)$, this property is fulfilled if and only if $\partial_\eta f_1=\partial_\xi f_2$. Moreover, it holds for the standard example $f(t,x,\zeta)=\psi(t,x,|\zeta|^2)\zeta$ for any map $\psi:\mathbb{R}\times\mathcal{M}\times\mathbb{R}\to\mathbb{R}$ which is continuous in the third variable. Here one has

$$\varphi(t,x,\zeta)=\frac12\int_0^{|\zeta|^2}\psi(t,x,r)dr.$$

We note that in the next proposition we do no use the results discussed in Section 6, besides a variant of the basic Theorem 6.1. Instead we construct the solutions as weak limits of (subsequences of) solutions to regularized problems. This standard method is based on uniform bounds for the energy and the compactness of K. (See e.g. Section 4.4 in [72] for a similar approach using a Galerkin approximation.)

Proposition 8.2. *Let b and f satisfy* (8.2) *with $1\le\alpha<d/(d-2)_+$ and φ be given by* (8.11). *Assume that*

$$\varphi\ge 0,\qquad \nabla_\zeta\varphi=f,\qquad |f(t,x,\zeta)-f(s,x,\zeta)|\le \ell(t_0)|t-s|(1+|\zeta|) \tag{8.13}$$

for $t_0>0$, $t,s\in[-t_0,t_0]$, $\zeta\in\mathbb{C}$, $x\in\mathcal{M}$, and a locally bounded map $\ell:[0,\infty)\to[0,\infty)$. Then (8.4) *and* (8.8) *have global solutions w such that $\|w(t)\|\le c(t_0,r)$ for $0\le t\le t_0$ and $\|(u_0,u_1)\|\le r$ and every $r,t_0>0$.*

Proof. We only treat (8.4) since (8.8) is completely analogous. For the energy estimate we need classical solutions of (8.1) so that we approximate f and $w_0=(u_0,u_1)\in X$ by more regular functions. Let $t\in\mathbb{R}$, $x\in\mathcal{M}$, $\zeta\in\mathbb{C}$, $t_0>0$, and $n\in\mathbb{N}$. Take standard mollifiers $\rho_n:\mathbb{R}^2\to[0,\infty)$ with support in $\overline{B}(0,\frac1n)$ and functions $\chi_n\in C^1(\mathbb{R}^2,\mathbb{R}^2)$ with range in $B(0,n+1)$,

$\chi_n(\zeta) = \zeta$ for $|\zeta| \le n$, $|\chi_n'(\zeta)| \le c$, and $|\chi_n(\zeta)| \le c|\zeta|$ for a constant $c \ge 1$. We set

$$\tilde{\varphi}_n(t,x,\zeta) = \varphi(t,x,\chi_n(\zeta)), \qquad \varphi_n(t,x,\cdot) = \rho_n * \tilde{\varphi}_n(t,x,\cdot),$$
$$\tilde{f}_n(t,x,\zeta) = \chi_n'(\zeta)^\top f(t,x,\chi_n(\zeta)), \qquad f_n(t,x,\cdot) = \rho_n * \tilde{f}_n(t,x,\cdot).$$

It is straightforward to check that $\nabla_\zeta \varphi_n = f_n$ and $\varphi_n \ge 0$ for all n. Moreover, for each $n \in \mathbb{N}$, the maps $\tilde{f}_n$, f_n, $\partial_\zeta f_n$, $\partial_t f_n$ are bounded on $[-t_0,t_0] \times \mathcal{M} \times \mathbb{C}$ (in general not uniformly in n) and $|\partial_\zeta^k \varphi_n(t,x,\zeta)| \le c_{n,k}(1+|\zeta|)$ for $(t,x,\zeta) \in [-t_0,t_0] \times \mathcal{M} \times \mathbb{C}$ and $k \in \mathbb{N}_0$. Finally, f_n and φ_n satisfy (8.2) respectively (8.12) with constants uniform in n, as well as

$$|\partial_t f_n(t,x,\zeta)| \le c\ell(t_0)(1+|\zeta|) \quad \text{and} \quad |\partial_t \varphi_n(t,x,\zeta)| \le c\ell(t_0)(1+|\zeta|^2)$$

for $n \in \mathbb{N}$, a.e. $t \in [-t_0,t_0]$, and some constants $c \ge 0$. Set $K_n(w) = (0, -f_n(t,\cdot,u))$. Observe that $K_n : [-t_0,t_0] \times X \to X$ is (globally) Lipschitz for each $n \in \mathbb{N}$ by the properties of f_n.

Let $w_0 = (u_0,u_1) \in X$ and take a sequence $(w_{0,n} = (u_{0,n},u_{1,n})) \in D(A_+)$ converging to w_0 in X. Due to [54, Theorem 6.1.6], problem (8.4) for K_n and $w_{0,n}$ has a (unique) solution $w_n \in C^1([0,\infty),X) \cap C([0,\infty),D(A_+))$, where $D(A_+)$ is endowed with the graph norm. It is now easy to see that

$$u_n \in C^2([0,\infty),L^2(\mathcal{M})) \cap C^1([0,\infty),V) \cap C([0,\infty),D(A_+))$$

solves (8.1) for f_n and $w_{0,n}$.

We show that the solutions w_n are bounded in X uniformly in $n \in \mathbb{N}$ and then pass to the limit as $n \to \infty$. To this aim, we look at the energies

$$E^0(w) = \frac{1}{2}\int_{\mathcal{M}} (|\nabla u|^2 + |v|^2)dx, \qquad E_n(t,w) = E^0(w) + \int_{\mathcal{M}} \varphi_n(t,\cdot,u)dx.$$

The above observations imply that $E_0(w) \le E_n(t,w) \le c(1+\|w\|_X^2)$, $E_n(t,\cdot), E^0 \in C^1(X,\mathbb{R})$, and $E_n(\cdot,w) \in W^{1,\infty}_{loc}(\mathbb{R})$, where we also use that $\alpha+1 < 2d/(d-2)_+$. For a.e. $t \in [0,t_0]$, we compute

$$\begin{aligned}
\tfrac{d}{dt}E_n(t,w_n(t)) &= \int_{\mathcal{M}} \big[\partial_t u_n(t)[\partial_{tt}u_n(t) - \Delta u_n(t) + f_n(t,\cdot,u_n(t))] \\
&\qquad + (\partial_t\varphi_n)(t,\cdot,u_n(t))\big]dx \\
&= \int_{\mathcal{M}} \big[b(\partial_t u_n(t))^2 + (\partial_t\varphi_n)(t,\cdot,u_n(t))\big]dx \\
&\le c(\|(u_n(t),\partial_t u_n(t))\|_{L^2}^2 + 1), \\
E_n(t,w_n(t)) &\le E_n(0,w_{0,n}) + c + c\int_0^t \big(\|u_n(s)\|_{L^2(\mathcal{M})}^2 + E_n(s,w_n(s))\big)ds
\end{aligned}$$

for constants $c = c(t_0)$ independent of n. We can bound the L^2-norm of $u_n(t)$ by means of $u_n(s) = u_{0,n} + \int_0^s \partial_t u_n(\tau)\,d\tau$, so that

$$E_n(t,w_n(t)) \le cE_n(0,w_{0,n}) + c + c\int_0^t E_n(s,w_n(s))ds.$$

Gronwall's inequality then yields

$$E^0(t,w_n(t)) \le E_n(t,w_n(t)) \le c(\|w_{0,n}\|_X^2+1)e^{c(t_0)t} \le C(t_0)(\|w_0\|_X^2+1),$$

and hence

$$\|w_n(t)\|_X^2 = \|u_n(t)\|_{L^2}^2 + 2E^0(t,w_n(t)) \le C(t_0)(\|w_0\|_X^2+1) \tag{8.14}$$

for $0 \le t \le t_0$ and possibly different constants not depending on n and w_0.

Using (8.14), we can find a subsequence of (w_n), denoted by the same symbol, such that u_n and $\partial_t u_n$ tend to functions u and v weakly* in each $L^\infty([0,t_0],V)$ respectively $L^\infty([0,t_0],L^2(\mathcal{M}))$, and $w=(u,v)$ satisfies (8.14) for a.e. $t \in [0,t_0]$. It is straightforward to check that $v=\partial_t u$, and hence $u \in L^\infty([0,t_0],V) \cap W^{1,\infty}([0,t_0],L^2(\mathcal{M})) =: F$. Moreover, F is compactly embedded into $C([0,t_0],L^{2\alpha}(\mathcal{M}))$ by [63, Corollary 4], for instance. There thus exists a subsequence (u_{n_k}) of (u_n) such that $u_{n_k} \to u$ in $L^{2\alpha}([0,t_0]\times\mathcal{M})$ and pointwise a.e., as well as $|u_{n_k}| \le g$ a.e. for a function $g \in L^{2\alpha}([0,t_0]\times\mathcal{M})$.

To show convergence of $(f_{n_k}(t,\cdot,u_{n_k}(t)))$, let $\varepsilon > 0$ and take $t \in [0,t_0]$ such that $(u_{n_k}(t,\cdot))_k$ tends to $u(t,\cdot)$ a.e. and is bounded a.e. by $g(t,\cdot) \in L^{2\alpha}(\mathcal{M})$. Choose $x \in \mathcal{M}$ outside this null set. For sufficiently large n we have $|u(t,x)| \le n-1$. There is a number $\delta \in (0,1)$ such that $|f(t,x,\zeta) - f(t,x,u(t,x))| \le \varepsilon$ if $|\zeta - u(t,x)| \le \delta$ since f is continuous in ζ. So there exists an index k_ε such that for all $k \ge k_\varepsilon$ we have

$$\begin{aligned}|\tilde f_{n_k}(t,x,u_{n_k}(t,x)-\eta) &- f(t,x,u(t,x))| \\ &= |f(t,x,u_{n_k}(t,x)-\eta) - f(t,x,u(t,x))| \le \varepsilon\end{aligned}$$

if $|\eta| \le 1/n_k$. It follows that $|f_{n_k}(t,x,u_{n_k}(t,x)) - f(t,x,u(t,x))| \le \varepsilon$, and hence $f_{n_k}(t,\cdot,u_{n_k}(t))$ tends pointwise a.e. to $f(t,\cdot,u(t))$ as $k \to \infty$. Since f_n satisfies (8.2) uniformly in n, we have $|f_{n_k}(t,\cdot,u_{n_k}(t))| \le c\kappa(t)(1+g(t,\cdot)^\alpha)$ and thus $f_{n_k}(t,\cdot,u_{n_k}(t)) \to f(t,\cdot,u(t))$ in $L^2(\mathcal{M})$ for a.e. t. Moreover, by (8.2) and (8.14) the map $t \mapsto \|f_n(t,\cdot,u_n(t))\|_{L^2(\mathcal{M})}$ is locally bounded independent of n. Therefore the right-hand side in the mild formula

$$w_{n_k}(t) = T_+(t)w_{0,n_k} + \int_0^t T_+(t-s)K_{n_k}(s,w_{n_k}(s))ds$$

converges in $C([0,\infty),X)$ as $k \to \infty$. As seen above, the left-hand side tends to $w=(u,\partial_t u)$ weakly* in $L^\infty_{loc}([0,\infty),X)$ so that

$$w(t) = T_+(t)w_0 + \int_0^t T_+(t-s)K(s,w(s))ds$$

holds for a.e. $t \ge 0$. Since the right-hand side is continuous, w is a mild solution of (8.4) and satisfies (8.14) for all $t \ge 0$. □

Combined with the above analysis and known spectral properties, Theorem 7.6 now shows growth of orbits and thus global instability in concrete examples. One could also use Theorem 7.4, adding more assumptions on f to satisfy (A1'). We avoid doing so since the modfications are easy. Note that in view of (8.9), with an appropriate choice of signs for b and K, we can switch freely between the two equations (8.4) and (8.8). So we concentrate just on (8.8).

As the first (toy) example, we improve Theorem III in [71] in various respects: We can treat time-depending f, reduce the regularity requirements in ξ from C^2 to continuity, and obtain exponential growth instead of mere unboundedness. In all examples we assume that f satisfies (8.2) and either (8.10) or (8.13).

Example 8.3. Let $b < 0$ be constant and $\mathcal{M}$ be a manifold as above. It is straightforward to check that the (point) spectrum of A_- is given by an unbounded sequence of eigenvalues on $-\frac{b}{2} + i\mathbb{R}$ and at most finitely many points in $(-b, 0)$. It follows $s_R(-A_-) \geq -\frac{b}{2}$. By Corollary 7.10 and Propositions 8.1 and 8.2, given a function $a : [0,\infty) \to [0,\infty)$ decreasing to 0, there are a dense set of initial values $w_0 \in X$ and $t_0 \geq 0$ such that the corresponding solutions $w(t, w_0)$ of backward damped wave equation (8.8) admit the lower bound $\|w(t, w_0)\| \geq a(t)e^{-bt/2}$ for $t \geq t_0$.

The next example generalizes the preceding one by allowing the damping b to be non-stationary and far from being smooth. At the same time, it concerns only the case $d = 1$.

Example 8.4. Let now $\mathcal{M} = [0, 1]$ and $b \in \mathrm{BV}([0, 1])$ with $b \geq 0$. (Recall that we then impose Dirichlet boundary conditions.) It was proved in [19, Theorem 5.3] that there is a sequence of eigenvalues $(\lambda_n)_{n\geq1} \subset \sigma(A_-)$ with $\operatorname{Re}\lambda_n \to \frac{\beta}{2}$ as $n \to \infty$ where $\beta := \int_0^1 b(s)ds$. Therefore, $s_R(-A_-) \geq \frac{\beta}{2}$. As above, using Corollary 7.10 along with Propositions 8.1 and 8.2, for a given function $a : [0,\infty) \to [0,\infty)$ decreasing to 0 we find a dense set of initial values w_0 such the solutions of (8.8) satisfy $\|w(t, w_0)\| \geq a(t)e^{-\beta t/2}$ for $t \geq t_0$ and some $t_0 \geq 0$. Moreover, by [26, Theorem 3.4], the same result holds without assuming $b \leq 0$ if $\|b\|_\infty$ is sufficiently small.

We proceed with more involved frameworks, relying on quite subtle results from the spectral theory of damped wave equations.

Example 8.5. To explain our next example, we need to introduce several auxiliary notions pertaining to dynamics of the geodesic flow $(g^t)_{t\in\mathbb{R}}$ on a Riemannian cosphere bundle $\mathcal{S}^*\mathcal{M}$ over $\mathcal{M}$. A relevant discussion of geodesic flows can be found in [43, Appendix B]. For $\mathcal{M}$ as above, without boundary, write

$$\rho_t = (x_t, \xi_t) = g^t(\rho_0), \qquad \rho_0 = (x_0, \xi_0) \in S^*\mathcal{M},\ t \in \mathbb{R},$$

and let $\pi : S^*\mathcal{M} \to \mathcal{M}$ be a canonical projection. (One may also consider manifolds with boundary and the corresponding *generalised* geodesic flows, but this setting leads to technical complications, and is thus omitted for simplicity.) Given a damping $b \in C^\infty(\mathcal{M})$ with $b \geq 0$, define its Birkhoff ergodic average of the geodesic curve in $S^*\mathcal{M}$ as $\langle b\rangle_t(\rho_0) := \frac{1}{t}\int_0^t (b \circ \pi \circ g^s)(\rho_0)\, ds$, and let

$$b_- := \sup_{t>0} \inf_{\rho_0\in S^*\mathcal{M}} \langle b\rangle_t(\rho_0) = \lim_{t\to\infty} \inf_{\rho_0\in S^*\mathcal{M}} \langle b\rangle_t(\rho_0),$$

$$b_+ := \inf_{t>0} \sup_{\rho_0 \in S^*\mathcal{M}} \langle b \rangle_t(\rho_0) = \lim_{t\to\infty} \sup_{\rho_0 \in S^*\mathcal{M}} \langle b \rangle_t(\rho_0).$$

Note that $\langle b \rangle_\infty(\rho_0) = \lim_{t\to\infty} \langle b \rangle_t(\rho_0)$ exists almost everywhere with respect to the flow invariant Liouville measure, and we have

$$b_- \le \operatorname{ess\,inf} \langle b \rangle_\infty \le \operatorname{ess\,sup} \langle b \rangle_\infty \le b_+,$$

where all of the inequalities can in general be strict. If the geodesic flow is ergodic, then one has

$$\operatorname{ess\,inf} \langle b \rangle_\infty = \operatorname{ess\,sup} \langle b \rangle_\infty = \frac{1}{\operatorname{vol}(\mathcal{M})} \int_{\mathcal{M}} a(\rho)\, d\rho := b_\infty^*.$$

It was proved in [46] that for every $\epsilon > 0$ there are at most finite number of the eigenvalues of A_- outside the strip $[b_- + \epsilon, b_+ - \epsilon] + i\mathbb{R}$ (where, in particular $\mathcal{M}$ may have a boundary). So that the eigenvalues cluster at $\beta + i\mathbb{R}$ for some $\beta \in [b_- + \epsilon, b_+ - \epsilon] + i\mathbb{R}$, and then arguing as in Example 8.4 one may take $s_R(-A_-) = -b_-\epsilon$ for an appropriate $\epsilon > 0$. We also refer to [65] for comments on the generality of this result and an alternative proof. The result was improved in [65] by showing that an infinite number of the eigenvalues of A_- belong to the strip $[\operatorname{ess\,inf} b_\infty, \operatorname{ess\,sup} b_\infty] + i\mathbb{R}$, Moreover, if g^t is ergodic, then most of the eigenvalues cluster near $b_\infty^* + i\mathbb{R}$. (Note also that a relevant spectral localization is provided in [38] for so-called Zoll manifolds, i.e., manifolds where all geodesics are closed.) Thus, by taking s_R equal either $-\operatorname{ess\,inf} b_\infty$ or $-b_\infty^*$, we get as above an exponential lower bound for $w(t, w_0)$ given by (8.8). Clearly, all of the choices for s_R considered in this example may produce $s_R(-A_-) > 0$.

So far, our examples depended on the properties of the spectrum of A_-. However, there are interesting situations when one has to invoke the resolvent of A_- and thus to use a full strength of our Corollary 7.10.

Example 8.6. There are many examples in the literature where $(\dot{T}_-(t))_{t\ge 0}$ satisfies $\|\dot{T}_-(t)R(\mu_0, \dot{A})^\alpha\| \le M_\alpha e^{-\omega_\alpha t}$ for $t \ge 0$ and some $\alpha, \omega_\alpha, M_\alpha > 0$ as well as, at the same time, $\omega_0(\dot{T}_-) \ge 0$. In other words, $(\dot{T}_-(t))_{t\ge 0}$ decays exponentially in the operator norm as a map from $D((-\dot{A}_-)^\alpha)$ for some $\alpha > 0$ to X, but not in $B(X)$. By interpolation, it then follows that $(\dot{T}_-(t))_{t\ge 0}$ enjoys such a decay for *all* $\alpha > 0$, and for us it suffices to fix $\alpha = 1$.

In view of e.g. [76, Theorem 1.4], the resolvent $R(\lambda, \dot{A}_-)$ then extends analytically to $\{\lambda : \operatorname{Re}\lambda \ge -a\}$ for some $a > 0$, and is norm bounded there by $c(1 + |\lambda|)$. Note that $R(\lambda, \dot{A}_-)$ is unbounded on $i\mathbb{R}$. Indeed, if this was wrong, using the Neumann's series expansion for $R(is, \dot{A}_-)$, $s \in \mathbb{R}$, and the resolvent estimate $\|R(\lambda, \dot{A}_-)\| \le (\operatorname{Re}\lambda)^{-1}$ for $\operatorname{Re}\lambda > 0$, we would infer that $s_0(\dot{A}_-) < 0$, and then $\omega_0(\dot{T}_-) < 0$ by (2.3). By a standard application of Phragmen–Lindelöf's theorem, $R(\lambda, \dot{A}_-)$ also has to be unbounded on a vertical line $-\beta + i\mathbb{R}$ for some $\beta \in (0, a]$. As a result, the resolvent of $B = -A_-$ is unbounded on $\beta + i\mathbb{R}$, so that $s_R(B) \ge \beta > 0$. Hence, as above for a dense set of initial values we infer that solutions of the backward damped wave equation (8.8) possess an exponential lower bound and thus show a global instability result.

To describe concrete situations when such an effect can happen recall that a geodesic flow on a negatively curved $\mathcal{M}$ admits an abundance of periodic orbits. It was revealed in [61, Theorem 1] that each such an orbit gives rise to a smooth damping b such that $(\dot{T}_-(t))_{t\geq 0}$ does not decay exponentially, while its orbits $\dot{T}_-(t)x$ decay exponentially for sufficiently smooth initial data x. More precisely, it was shown in [61] that, for any periodic geodesic γ in $\mathcal{M}$ and $b_0 \in C^\infty(\mathcal{M})$, there exists $\epsilon > 0$ such that if b_0 vanishes in an ϵ-neighborhood of γ and is positive everywhere else, then such a decay takes place for $b = cb_0$ for all sufficiently large $c > 0$. If $s > d/2$, then its exponential rate can be made explicit, and it is defined in terms of the dynamical properties of g^t. We omit a precise description here.

If $\mathcal{M}$ is a hyperbolic surface with constant negative curvature, then as proved in [39, Theorem 1.1], the decay takes place for *all* smooth dampings b. This property can in fact be generalized to all surfaces $\mathcal{M}$ whose geodesic flow has the so-called Anosov property, see [23, Theorem 6], though such a generalisation is very deep and demanding. Other instances of the exponential decay for only smooth enough orbits of $(\dot{T}_-(t))_{t\geq 0}$, sometimes with explicit rates, can also be found in [13, Section 4], [53] and [60, Theorem 3 and the subsequent Remark]. We avoid their discussion to keep our exposition within reasonable limits.

Note that the assumption $b \geq 0$ was chosen just to fit in the framework of the existing work, and it can be avoided in many cases (e.g. in Examples 8.4, 8.3, and 8.5). In this case, s_R provides just a lower bound, not necessarily growing exponentially.

Finally, we show non-stabilizability of certain nonlinear infinite-dimensional control systems. In this way, we generalize the corresponding results in [34] or [73], for instance, where the operators B, F, and C defined below are linear and bounded. The literature on stabilization of control systems is enormous (and, thus, we skipped a discussion of asymptotics for damped wave equations as a stabilization problem, see e.g. [44], [45] concerning that). We just refer to [42] and [66] as sample works on nonlinear stabilization and to [16] for general concepts of nonlinear control. For basics of linear theory one may consult [24, Section VI.8].

Example 8.7. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X with generator A, and let the control map $B : U \to X$, the feedback $F : Y \to U$ and the observation map $C : X \to Y$ be all (possibly) nonlinear and continuous with linear growth (i.e., satisfy (6.3) with a constant c), where U and Y are Banach spaces. Combined with Corollary 6.6 and Proposition 6.3, Theorems 4.6 and 7.6 imply the following results-.

If $s_R(A) = 0$, then the system $x' = Ax + BFCx$ is not exponentially stabilizable by a compact nonlinear feedback F, i.e., it will always have solutions not decaying exponentially. If $s_R(A) > 0$, the system is not strongly stabilizable by a compact nonlinear feedback F, i.e., some of its solutions will not converge to zero. (In fact, they will grow exponentially.)

The analogous statement holds for time-discrete feedback systems $x_{n+1} = Ax_n + BFCx_n$ for bounded A, continuous B, F, and C mapping bounded sets into bounded sets, if one replaces $s_R(A)$ with $r_e(A)$.

9. Appendix

In this section we prove several results on geometric properties of Banach spaces and fine spectral theory of semigroups and their generators which are crucial for our lower estimates. Some of them, e.g. Theorem 2.2, are of independent interest. The exposition here follows [48, Section 4] with appropriate changes and improvements, which warrant an independent treatment. The section makes the paper essentially self-contained.

First, we note a geometric statement from Banach space theory. Its versions are often used in iterative constructions arising in the study of orbits of linear operators, and it are important in our studies too. To make our presentation self-contained and to provide a better understanding of our constructions, we give its proof below. Let X be a Banach space. Recall that a subset $\Lambda \subset X^*$ is norming if

$$\|x\| = \sup\Big\{\frac{|\langle x, y\rangle|}{\|y\|} : y \in \Lambda, y \neq 0\Big\}$$

for all $x \in X$. The next statement is a variation upon [47, Lemma V.36.7], and given with a full proof in view of importance for operator-theoretical constructions given in this paper.

Lemma 9.1. *Let E be a finite-dimensional subspace of a Banach space X and $\varepsilon > 0$.*

(a) *There exists a closed subspace $Y \subset X$ of finite codimension such that*

$$\|e + y\| \geq (1 - \varepsilon) \max\big\{\|e\|, \|y\|/2\big\} \tag{9.1}$$

for all $e \in E$ and $y \in Y$.

(b) *If F is a finite-dimensional subspace such that $E \subset F$, then there exists $Z \subset Y$ of finite codimension such that* (9.1) *holds for $e \in F$ and $f \in Z$.*

(c) *If $\Lambda \subset X^*$ is a norming set, then there exists a subspace $Y \subset X$ satisfying* (9.1) *such that $Y = \bigcap_{j=1}^k \operatorname{Ker} f_j$ for some $f_1, \dots, f_k \in \Lambda$.*

Proof. We can assume that $\varepsilon < 1$. The unit sphere in E is compact, therefore there exists a finite subset $D \subset \{e \in E : \|e\| = 1\}$ satisfying $\operatorname{dist}\{e, D\} \leq \varepsilon/2$ for all $e \in E$ with $\|e\| = 1$. Let $\Lambda \subset X^*$ be norming. For each $d \in D$ there is a functional $f_d \in \Lambda$ such that $\big|\big\langle d, f_d/\|f_d\|\big\rangle\big| > 1 - \varepsilon/2$. Set $Y = \bigcap_{d\in D} \operatorname{Ker} f_d$. Clearly, Y is closed with finite codimension.

To prove the required inequality, let $e \in E$ and $y \in Y$. We can assume that $\|e\| \neq 0$ since the assertion is clear for $e = 0$. Choose $d \in D$ with

$\big\|d - e/\|e\|\big\| \leq \frac{\varepsilon}{2}$. We then estimate

$$\begin{aligned}\|e+y\| &\geq \Big|\Big\langle e+y, \frac{f_d}{\|f_d\|}\Big\rangle\Big| = \Big|\Big\langle e - \|e\|d, \frac{f_d}{\|f_d\|}\Big\rangle + \Big\langle \|e\|d, \frac{f_d}{\|f_d\|}\Big\rangle\Big| \\ &\geq \|e\|(1-\varepsilon/2) - \big\|e - \|e\|d\big\| \geq \|e\|(1-\varepsilon).\end{aligned}$$

This inequality further implies

$$\begin{aligned}\|e+y\| &\geq \frac{1}{2}(1-\varepsilon)\frac{2-\varepsilon}{1-\varepsilon}\|e+y\| = \frac{1}{2}(1-\varepsilon)\Big(\|e+y\| + \frac{1}{1-\varepsilon}\|e+y\|\Big) \\ &\geq \frac{1}{2}(1-\varepsilon)\big(\|y\| - \|e\| + \|e\|\big) = \frac{1}{2}(1-\varepsilon)\|y\|.\end{aligned}$$

The claims in (b) and (c) are direct consequences of the construction above. □

It is instructive to note the following dual version of the lemma. For each $M \subset X$ with $\operatorname{codim} M < \infty$, there exists a finite dimensional subspace F of X such that each unit vector $x \in X$ can be written as $x = m + f$ for "small" m and f in the sense that $\|f\| < 1 + \varepsilon$ and $\|m\| < 2 + \varepsilon$. If X is a Hilbert space, we can take $Y = E^{\perp}$ in the lemma. Thus the subspace $Y \subset X$ constructed above plays a role of the orthogonal complement of a finite-dimensional subspace for general Banach spaces.

Next we turn to the spectral theory of semigroups related to construction of approximate eigenvectors with some additional geometric properties. First we prove a property of the boundary essential spectrum of unbounded operators, well-known in the bounded case.

Lemma 9.2. *Let A be a closed, densely defined operator on a Banach space X, such that $\rho(A) \neq \emptyset$, and $\lambda \in \partial\sigma_e(A)$. Let $M \subset X$ be a subspace of finite codimension and $\epsilon > 0$. Then there exists a unit vector $x \in M \cap D(A)$ such that $\|(A-\lambda)x\| < \epsilon$.*

Proof. We will rely on the fact, that the statement is true if A is bounded, see e.g. [47, Proposition III.19.1 and Theorem III.16.8]. Without loss of generality we may assume that $\lambda = 0$.

Let $\mu \in \rho(A)$ and let $T := A(A-\mu)^{-1} = I + \mu(A-\mu)^{-1}$. Then T is bounded. Moreover, $\sigma_e(T) \setminus \{1\} = \{1 + \frac{\mu}{z-\mu} : z \in \sigma_e(A)\}$ by (2.1). So $0 \in \partial\sigma_e(T)$. Let $M' = \operatorname{Im}((A-\mu)\restriction_M)$. Then $\operatorname{codim} M' < \infty$, and as $T \in B(X)$ there exists a sequence $(x_n)_{n\geq 1} \subset M'$ such that $\|x_n\| = 1$ for all n and $Tx_n \to 0$ as $n \to \infty$. Set $y_n = (A-\mu)^{-1}x_n$ for $n \in \mathbb{N}$. Then $y_n \in M \cap D(A)$ for all n and $Ay_n \to 0$ as $n \to \infty$. Moreover, since

$$Tx_n = x_n + \mu(A-\mu)^{-1}x_n = x_n + \mu y_n,$$

we have $\liminf_{n\to\infty} \|y_n\| = 1/|\mu| > 0$. I t remains to choose ay_n for appropriate $a > 0$ and n. □

The proof of Theorem 2.2 is based on the next lemma.

Lemma 9.3. *Let A generate the C_0-semigroup $(T(t))_{t\geq 0}$ on the Banach space X. Let either $\omega > s_e(A)$ or $\omega = s_e(A)$ and $(\omega + i\mathbb{R}) \cap \sigma_e(A) = \emptyset$. Assume that either*

$$\sigma_p(A) \cap (\omega + i\mathbb{R}) \text{ is infinite}$$

or

$$\sigma_p(A) \cap (\omega + i\mathbb{R}) \quad \text{is at most finite} \qquad \text{and} \quad \limsup_{|b|\to\infty} \|R(\omega + ib, A)\| = \infty.$$

Then there exist sequences $\{\mu_n : n \geq 1\} \subset \omega + i\mathbb{R}$ and $(u_n)_{n\geq 1} \subset D(A)$ such that $\|u_n\| = 1$ for all $n \in \mathbb{N}$, $\|(\mu_n - A)u_n\| \to 0$ as $n \to \infty$, and for every $y^* \in D(A^*)$ we have

$$\langle u_n, y^* \rangle \to 0, \quad n \to \infty.$$

In particular, if X is reflexive, then (u_n) tends weakly to 0 (since then $D(A^*)$ is dense in X^*).

Proof. By our assumptions, there exist a sequence $(b_n)_{n\geq 1}$ with $|b_n| \to \infty$ and unit vectors $u_n \in D(A)$ for $n \geq 1$ such that $\|(\omega + ib_n - A)u_n\| \to 0$ as $n \to \infty$. Set $\mu_n = \omega + ib_n$. We show that

$$\langle u_n, y^* \rangle \to 0, \quad n \to \infty,$$

for each $y^* \in D(A^*)$. Let $y^* \in D(A^*) \subset X^*$ have norm 1. Pick a vector $y \in D(A)$ with $\langle y, y^* \rangle > \frac{1}{2}$. Let $M = \operatorname{Ker} y^*$. Write $u_n = m_n + \alpha_n y$ for some $m_n \in M$ and $\alpha_n \in \mathbb{C}$. Then the sequences $(m_n)_{n\geq 1}$ and $(\alpha_n)_{n\geq 1}$ are bounded. Furthermore,

$$\langle (\mu_n - A)u_n, y^* \rangle \to 0, \qquad n \to \infty,$$

and

$$\begin{aligned}\langle (\mu_n - A)u_n, y^* \rangle &= \langle (\mu_n - A)m_n, y^* \rangle + \alpha_n \langle (\mu_n - A)y, y^* \rangle \\ &= \alpha_n \mu_n \langle y, y^* \rangle - \langle m_n, A^* y^* \rangle - \alpha_n \langle Ay, y^* \rangle.\end{aligned}$$

Since the last two terms are uniformly bounded and $|\mu_n| \to \infty$, we have $\alpha_n \to 0$ as $n \to \infty$. It follows $\langle u_n, y^* \rangle = \alpha_n \langle y, y^* \rangle \to 0$. ☐

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X, and set

$$\|x\|' := \sup_{x^* \in D(A^*), \|x^*\| \leq 1} |x^*(x)|$$

for $x \in X$.

Then $\| \cdot \|'$ is an equivalent norm satisfying

$$\|x\|' \leq \|x\| \leq \alpha \|x\|', \qquad \text{where} \quad \alpha = \limsup_{t\to 0} \|T(t)\|, \tag{9.2}$$

for all $x \in X$. See e.g. [51, Theorems 1.3.1 and 1.3.5]. Hence, renorming X, we can make $D(A^*)$ a norming set for the Banach space $(X, \| \cdot \|')$.

Recall the definitions of the resolvent bound s_R and of the notion of admissibility given in Section 2. Now we are ready to prove Theorem 2.2 stated there and describing one of admissible ω in resolvent terms. More precisely, we show that if A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X, then the number $s_R(A)$ given by (2.4) is admissible.

Proof of Theorem 2.2. Let M be a subspace of finite codimension in X given by $M = \bigcap_{j=1}^k \operatorname{Ker} f_j$ for some functionals $f_1, \dots, f_k \in D(A^*)$. Let $\varepsilon > 0$ and $t_0 > 0$ be fixed, and set $K := \sup\{\|T(t)\| : 0 \leq t \leq t_0\}$. Since the admissibility does not depend on equivalent renormings of the underlying space, in view of (9.2), we can assume that $D(A^*)$ is a norming set for X.

Let $s_R(A) = s_e(A)$ and $\mu \in \sigma_e(A) \cap (s_R(A) + i\mathbb{R})$. Lemma 9.2 then yields $x \in D(A) \cap M$ with $\|x\| = 1$ and $\|(A - \mu)x\| \leq \varepsilon$. Hence

$$\|T(t)x - e^{\mu t}x\| = \left\| \int_0^t e^{\mu(t-s)} T(s)(\mu - A)x ds \right\| \leq \varepsilon t_0 e^{s_R(A)t_0} K$$

for all $t \in [0, t_0]$.

Let $s_R(A) \geq s_e(A)$ and $\sigma_e(A) \cap (s_R(A) + i\mathbb{R}) = \emptyset$. Employing Lemma 9.3, we find sequences $(\mu_n)_{n\geq 1} \subset \mathbb{C}$ with $\mu_n = s_R(A) + ib_n$ for $n \in \mathbb{N}$ and $(u_n)_{n\geq 1} \subset D(A)$ with $\|u_n\| = 1$ such that

$$\langle u_n, y^* \rangle \to 0 \quad \text{for every } y^* \in D(A^*) \quad \text{and} \qquad \|(\mu_n - A)u_n\| \to 0, \quad n \to \infty.$$

By [59, Lemma 7.4], there exists a finite-dimensional subspace $F \subset D(A)$ such that $X = M \oplus F$. Let P be the projection onto F with $\operatorname{Ker} P = M$. By the choice of u_n we have $\|Pu_n\| \to 0$ so that $\|(I - P)u_n\| \to 1$ as $n \to \infty$ and

$$\left\| u_n - \frac{u_n - Pu_n}{\|u_n - Pu_n\|} \right\| \to 0, \qquad n \to \infty.$$

Choose $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \left\| u_{n_0} - \frac{u_{n_0} - Pu_{n_0}}{\|u_{n_0} - Pu_{n_0}\|} \right\| &\leq \min\left\{ \frac{\varepsilon}{4K}, \frac{\varepsilon}{4e^{s_R(A)t_0}} \right\}, \\ \|(\mu_{n_0} - A)u_{n_0}\| &\leq \frac{\varepsilon}{4t_0 K \max\{1, e^{t_0 s_R(A)}\}}. \end{aligned}$$

Set

$$\mu = \mu_{n_0} \quad \text{and} \quad x = \frac{u_{n_0} - Pu_{n_0}}{\|u_{n_0} - Pu_{n_0}\|}.$$

For every $0 \leq t \leq t_0$, we have

$$\|T(t)u_{n_0} - e^{\mu t}u_{n_0}\| = \left\| \int_0^t e^{\mu(t-s)} T(s)(\mu_{n_0} - A)u_{n_0} ds \right\| \leq \varepsilon/4$$

and

$$\begin{aligned} \|T(t)x - e^{\mu t}x\| &\leq \|T(t)x - T(t)u_{n_0}\| + \|T(t)u_{n_0} - e^{\mu t}u_{n_0}\| + \|e^{\mu t}u_{n_0} - e^{\mu t}x\| \\ &\leq K\|x - u_{n_0}\| + \varepsilon/4 + e^{ts_R(A)}\|x - u_{n_0}\| \\ &< \varepsilon. \end{aligned}$$

This finishes the proof. □

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