MAXIMAL REGULARITY WITH TEMPORAL WEIGHTS FOR PARABOLIC PROBLEMS WITH INHOMOGENEOUS BOUNDARY CONDITIONS

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Abstract. We develop a maximal regularity approach in temporally weighted L_p-spaces for vector-valued parabolic initial-boundary value problems with inhomogeneous boundary conditions, both of static and of relaxation type. Normal ellipticity and conditions of Lopatinskii-Shapiro type are the basic structural assumptions. The weighted framework allows to reduce the initial regularity and to avoid compatibility conditions at the boundary, and it provides an inherent smoothing effect of the solutions. Our main tools are interpolation and trace theory for anisotropic Slobodetskii spaces with temporal weights, operator-valued functional calculus, as well as localization and perturbation arguments.

1. Introduction

In recent years parabolic equations with fully nonlinear boundary conditions have attracted a lot of interest since they arise in the analysis of free boundary value problems such as the Stefan problem with surface tension, see e.g. [7], [11] and [18]. These papers use an L_p-approach to such problems which yields strong solutions in maximal regularity classes. In this framework the boundary conditions are attained in a classical sense up to initial time, and not just weakly. This approach is based on linearization and on a sharp L_p-regularity theory for linear inhomogeneous initial-boundary value problems, as established in [4], [5] and [6] by Denk, Hieber, Prüss and Zacher. Besides the usual static boundary conditions, one also has to treat dynamical boundary conditions of relaxation type which arise in the context of the Stefan problem with surface tension and in related problems.

However, this approach requires regularities of the initial values (and hence of the nonlinear phase spaces) which are stronger than the norms one can control by standard a priori estimates for the nonlinear problems. In related situations it is known that one can reduce the required initial regularity by means of temporal weights. In the L_p-setting, it is natural to work in

$$L_{p,\mu}(J;X) := \big\{u : J \to X \ : \ t^{1-\mu}u \in L_p(J;X)\big\} \tag{1.1}$$

endowed with its natural norm, where $p \in (1,\infty)$, $\mu \in (1/p, 1]$, $T \in (0,\infty]$, $J := (0,T)$, and $t^{1-\mu}u$ denotes the function $t \mapsto t^{1-\mu}u(t)$ on J. The corresponding weighted Sobolev spaces are defined by

$$W^k_{p,\mu}(J;X) := \big\{u : J \to X \ : \ u, u', ..., u^{(k)} \in L_{p,\mu}(J;X)\big\} \tag{1.2}$$

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for $k \in \mathbb{N}$. These spaces and the corresponding anisotropic spaces like $\mathbb{E}_{u,\mu}(J)$ defined below are studied by the authors in detail in [16]. To a large extent they enjoy analogous properties as the corresponding unweighted spaces.

To see the effect of the weight, we consider a generator $-A$ of an analytic semigroup on a Banach space X. Then the orbit $u(t) = e^{-tA}u_0$ belongs to the 'maximal regularity space'

$$W^1_{p,\mu}(\mathbb{R}_+; X) \cap L_{p,\mu}(\mathbb{R}_+; D(A))$$

if and only if the initial value u_0 belongs to the real interpolation space

$$(X, D(A))_{\mu-1/p,p}\,,$$

see e.g. Theorem 1.14.5 in [21]. Recall that one often fixes a large $p \in (1, \infty)$ to treat non-linearities. Hence, in the unweighted case $\mu = 1$ the resulting initial regularity is close to $D(A)$. On the other hand, taking μ near $1/p$ one almost reaches the base space X. Further, for Banach spaces X of class $\mathcal{HT}$ (see Section 2), Prüss and Simonett have proved in [17] that the inhomogeneous evolution equation

$$u'(t) + Au(t) = f(t), \quad t > 0, \qquad u(0) = 0, \tag{1.3}$$

has a unique solution in $W^1_{p,\mu}(\mathbb{R}_+; X) \cap L_{p,\mu}(\mathbb{R}_+; D(A))$ for each $f \in L_{p,\mu}(J; X)$ if and only if this fact holds for the unweighted case $\mu = 1$. Since the unweighted case is well understood, see e.g. [13], the $L_{p,\mu}$-approach is quite convenient for parabolic problems covered by (1.3).

Unfortunately, it seems that a sharp regularity theory for inhomogenous boundary value problems is not possible within the abstract framework of an evolution equation like (1.3). Instead one has to restrict to a PDE setting. So we investigate vector-valued linear parabolic systems with inhomogeneous boundary conditions, such of static type, i.e.,

$$\begin{aligned} \partial_t u + \mathcal{A}(t,x,D)u &= f(t,x), && x \in \Omega, && t \in J, \\ \mathcal{B}_j(t,x,D)u &= g_j(t,x), && x \in \Gamma, && t \in J, \qquad j = 1, ..., m, \\ u(0,x) &= u_0(x), && x \in \Omega, \end{aligned} \tag{1.4}$$

as well as such of relaxation (or dynamic) type, i.e.,

$$\begin{aligned} \partial_t u + \mathcal{A}(t,x,D)u &= f(t,x), && x \in \Omega, && t \in J, \\ \partial_t \rho + \mathcal{B}_0(t,x,D)u + \mathcal{C}_0(t,x,D_\Gamma)\rho &= g_0(t,x), && x \in \Gamma, && t \in J, \\ \mathcal{B}_j(t,x,D)u + \mathcal{C}_j(t,x,D_\Gamma)\rho &= g_j(t,x), && x \in \Gamma, && t \in J, \qquad j = 1, ..., m, \\ u(0,x) &= u_0(x), && x \in \Omega, \\ \rho(0,x) &= \rho_0(x), && x \in \Gamma. \end{aligned} \tag{1.5}$$

It is assumed that $\Omega \subset \mathbb{R}^n$ is a (inner or outer) domain with compact smooth boundary $\Gamma = \partial\Omega$. In (1.4) and (1.5) the unknown $u = u(t,x)$ takes values in a Banach space E, and in (1.5) the additional unknown $\rho = \rho(t,x)$, which only lives on the boundary Γ, takes values in another Banach space F. Throughout we assume that E and F are of class $\mathcal{HT}$; for instance E and F can be finite dimensional leading to usual parabolic systems. The differential operator $\mathcal{A}$ is of order $2m$, where $m \in \mathbb{N}$, and $\mathcal{B}_j$ are corresponding boundary operators of order m_j not larger than $2m - 1$. In (1.5) the differential operators $\mathcal{C}_j$ contain tangential derivatives of any order up to $k_j \in \mathbb{N}_0$. We assume certain ellipticity and Lopatinskii-Shapiro type conditions and impose regularity conditions on the coefficients that are appropriate for the applications to quasilinear problems, see e.g. [14], [15]. The details are described in Section 2.

We look for strong solutions u of (1.4), resp. (u,ρ) of (1.5), which satisfy the respective equations pointwise almost everywhere. In particular, u shall belong to

$$\mathbb{E}_{u,\mu}(J) := W^1_{p,\mu}\big(J; L_p(\Omega;E)\big) \cap L_{p,\mu}\big(J; W^{2m}_p(\Omega;E)\big).$$

The space $\mathbb{E}_{\rho,\mu}(J)$ for ρ is chosen in accordance to the structure of (1.5) and to the trace theorems established in our paper [16], see Section 2. In our main results Theorems 2.1 and 2.2 we show the maximal $L_{p,\mu}$-regularity for (1.4) and (1.5) on J. This means that there are data spaces $\mathcal{D}_{\text{stat}}(J)$ and $\mathcal{D}_{\text{rel}}(J)$ such that (1.4) and (1.5) have unique solutions $u \in \mathbb{E}_{u,\mu}(J)$ and $(u,\rho) \in \mathbb{E}_{u,\mu}(J) \times \mathbb{E}_{\rho,\mu}(J)$, respectively, if and only if the data satisfies

$$(f, g_1, ..., g_m, u_0) \in \mathcal{D}_{\text{stat}}(J) \quad \text{and} \quad (f, g_0, g_1, ..., g_m, u_0, \rho_0) \in \mathcal{D}_{\text{rel}}(J),$$

respectively. The data spaces contain the necessary regularities for the data and their compatibility conditions at $t = 0$ enforced by the static and dynamical boundary equations in (1.4) and (1.5). The precise formulations of these spaces is suggested by the space-time trace theorems from [16]. In the unweighted case $\mu = 1$ and with essentially the same assumptions, the maximal L_p-regularity for (1.4) and (1.5) has been shown by Denk, Hieber & Prüss [5] and Denk, Prüss & Zacher [6], respectively.

We note that the problem (1.5) is more involved in the several respects. Clearly, it contains a second variable and a second evolutionary equation. Moreover, the operators $\mathcal{C}_j$ can make the main parts of the equations at the boundary highly non homogeneous which then leads to a rather sophisticated solution space $\mathbb{E}_{\rho,\mu}(J)$ and to a complicated analysis. It further can happen that $\partial_t\rho$ is continuous in t up to $t = 0$ so that the dynamical equation for ρ leads to an additional compatibility condition on the regularity of $\mathcal{B}_0(0,\cdot,D)u_0 + \mathcal{C}_0(0,\cdot,D_\Gamma)\rho_0 - g_0(0,\cdot)$.

The main feature of the weighted approach is the flexibility for the regularity of the initial values as μ varies in $(1/p, 1]$. We describe these properties in more detail at the end of Section 2, indicating here the basic points. We can solve (1.4) and (1.5) with the Besov regularity $u_0 \in B^{2(\mu-1/p)}_{p,p}(\Omega;E)$ which approaches $L_p(\Omega;E)$ as μ tends to $1/p$. Moreover, if the initial regularity is sufficently low we loose the compatibility conditions such as $\mathcal{B}_j(0,\cdot,D)u_0 + \mathcal{C}_j(0,\cdot,D_\Gamma)\rho_0 = g_j(0,\cdot)$. Since the weight has an influence only at $t = 0$, our approach yields an inherent smoothing effect for the solutions. In particular, for (1.4) one can control the norm of $u(t)$ in $B^{2(1-1/p)}_{p,p}(\Omega;E)$ by the much lower norm of u_0 in $B^{2(\mu-1/p)}_{p,p}(\Omega;E)$. For bounded Ω, this fact gives the important compactness of the semiflow solving the related nonlinear problems. Also for unbounded Ω one can thus 'upgrade' the usual a priori estimates in low norms up to $B^{2(1-1/p)}_{p,p}(\Omega;E)$ if one is able to handle the involved nonlinearities. See [14], [15] and also [10] in the framework of [17], as well as [11]. In these papers the weighted approach was used to establish convergence to equilibria and the existence of global attractors in high norms.

In Sections 3 and 4 we prove Theorems 2.1 and 2.2. We first consider model problems with homogeneous constant coefficient operators on the full-space $\mathbb{R}^n$ and on the half-space $\mathbb{R}^n_+$ in Section 3. The full-space case, where boundary conditions are not involved, is treated by means of [17]. For the half-space case with boundary conditions we apply the Fourier transform with respect to time and space. The solution operators for the resulting ordinary initial value problems have been analysed in [5] and [6] for the unweighted case. We now use a recent operator-valued Fourier multiplier theorem in the $L_{p,\mu}$-spaces due to Girardi & Weis [8] and several isomorphisms acting on a scale of weighted anisotropic fractional order spaces which are investigated in [16]. It is then possible to invert the Fourier transforms and to solve the half-space problem in the required norms. The case of a general domain is then a consequence of perturbation and localization arguments, and it is considered in Section 4.

Finally we discuss several important special cases of (1.4) and (1.5) arising as linearizations of various quasilinear parabolic problems with nonlinear static or dynamic boundary conditions. For instance, the linearization of a reaction-diffusion system with nonlinear Robin boundary conditions is of the form (1.4) for

$$E = \mathbb{R}^N, \qquad m = 1, \qquad \mathcal{A}(D) = -\Delta, \qquad \mathcal{B}_1(x, D) = \partial_\nu := \nu \cdot \mathrm{tr}_\Omega \nabla,$$

where Δ is the Laplacian and ν denotes the outer unit normal field on Γ. The linearization of Cahn-Hilliard phase-field models leads to similar problems of order 4 (i.e., $m = 2$). If we take

$$\mathcal{B}_1 = \mathrm{tr}_\Omega, \qquad \mathcal{C}_1 = -1,$$

the static boundary condition for $j = 1$ in (1.5) reads $u|_\Gamma = \rho$, which leads to inhomogeneous dynamic boundary conditions. Hence ρ is simply the spatial trace of u in this case. Now one can take $\mathcal{C}_0(x, D_\Gamma) = -\Delta_\Gamma$, the Laplace-Beltrami operator on Γ, to obtain boundary conditions describing surface diffusion, i.e.,

$$\partial_t u|_\Gamma + \partial_\nu u - \Delta_\Gamma u|_\Gamma = g_0 \qquad \text{on } \Gamma, \qquad t \in J.$$

If we choose

$$u|_\Gamma + \Delta_\Gamma \rho = g_1 \qquad \text{on } \Gamma, \qquad t \in J,$$

as the first static boundary condition in (1.5), we arrive at the linearization of the Stefan problem with surface tension as studied in [7]. Here the graph of $\rho(t, \cdot)$ is related to the unknown boundary at time t. We refer to Section 3 of [6] for further interesting problems that may be written in the form (1.5).

Notations. We write $a \lesssim b$ for some quantities a, b if there is a generic positive constant C with $a \leq Cb$. If A is a sectorial operator on a Banach space E, $\theta \in (0, 1)$ and $q \in [1, \infty]$, then we set $D_L(\theta, p) := \big(E, D(L)\big)_{\theta,q}$ for the real interpolation scale between E and $D(L)$. If X, Y are Banach spaces we denote by $\mathcal{B}(X, Y)$ the space of bounded linear operators between them, with $\mathcal{B}(X) := \mathcal{B}(X, X)$.

2. The assumptions and the results

Throughout we assume that the Banach spaces E, F are of class $\mathcal{HT}$ (or, equivalently, are UMD spaces). This means that the Hilbert transform is bounded on $L_2(\mathbb{R}; X)$ which holds, e.g., in Hilbert spaces X or if X is a reflexive Lebesgue or (fractional) Sobolev space; see Sections III.4.3-4.5 of [1]. We first describe the differential operators in (1.4) and (1.5) in detail. For both problems the operator $\mathcal{A}$ is given by

$$\mathcal{A}(t, x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(t, x) D^\alpha, \qquad x \in \Omega, \qquad t \in J,$$

where $m \in \mathbb{N}$ and $D = -\mathrm{i}\nabla$ with the euclidian gradient $\nabla = (\partial_{x_1}, ..., \partial_{x_n})$ on $\mathbb{R}^n$. Hence the order of $\mathcal{A}$ is $2m$. The coefficients take values in the bounded linear operators on E, i.e., $a_\alpha(t, x) \in \mathcal{B}(E)$. Also for both problems the boundary operators $\mathcal{B}_j$ are of the form

$$\mathcal{B}_j(t, x, D) = \sum_{|\beta| \leq m_j} b_{j\beta}(t, x) \mathrm{tr}_\Omega D^\beta, \qquad x \in \Gamma, \qquad t \in J, \qquad j = 0, ..., m,$$

where $m_j \in \{0, ..., 2m-1\}$ is the order of $\mathcal{B}_j$ and the coefficients satisfy $b_{0\beta}(t,x) \in \mathcal{B}(E,F)$ and $b_{j\beta}(t,x) \in \mathcal{B}(E)$ for $j = 1, ..., m$. We note that $\mathcal{B}_j$ acts on u by applying first the euclidian derivatives and then the spatial trace tr_Ω. We assume that each of these operators is nontrivial, i.e., $\mathcal{B}_j \neq 0$ for all j.

In problem (1.5), the boundary conditions of relaxation type involve another set of operators $\mathcal{C}_0, ..., \mathcal{C}_m$, which act only on ρ in the following way. For $t \in J$ it is assumed that $\mathcal{C}_j(t, \cdot, D_\Gamma)$ is a linear map

$$C^\infty(\Gamma; F) \to L_1(\Gamma; F)$$

such that for all $j = 0, ..., m$, all local coordinates g for Γ and all $\rho \in C^\infty(\Gamma; F)$ it holds

$$\big(\mathcal{C}_j(t, \cdot, D_\Gamma)\rho\big) \circ \mathrm{g}(x) = \sum_{|\gamma| \le k_j} c^{\mathrm{g}}_{j\gamma}(t,x) D^\gamma_{n-1}(\rho \circ \mathrm{g})(x), \qquad x \in \mathrm{g}^{-1}(\Gamma \cap U), \qquad t \in J,$$

where $U \subset \mathbb{R}^n$ is the domain of the chart corresponding to g. Here we have $D_{n-1} = -\mathrm{i}\nabla_{n-1}$ with the euclidian gradient ∇_{n-1} on $\mathbb{R}^{n-1}$. The order $k_j \in \mathbb{N}_0$ of $\mathcal{C}_j$ is given independently of the orders of $\mathcal{A}$ and the $\mathcal{B}_j$. The local coefficients $c^{\mathrm{g}}_{j\gamma}$, that may depend on the coordinates g, are assumed to satisfy $c^{\mathrm{g}}_{0\gamma}(t,x) \in \mathcal{B}(F)$ and $c^{\mathrm{g}}_{j\gamma}(t,x) \in \mathcal{B}(F,E)$ for $j = 1, ..., m$. It is assumed that at least one operator $\mathcal{C}_j$ is nontrivial. If an operator $\mathcal{C}_j$ is trivial, i.e., $\mathcal{C}_j \equiv 0$, then we set $k_j := -\infty$ as its order. Note that we do not assume that an operator $\mathcal{C}_j$ has global coefficients, in the sense that there are functions $c_{j\gamma}$ on Γ satisfying $c^{\mathrm{g}}_{j\gamma} = c_{j\gamma} \circ \mathrm{g}$ in all coordinates g. In contrast to that, the coefficients of the $\mathcal{B}_j$ are globally defined on Γ. The standard examples for such an operator $\mathcal{C}_j$ are the Laplace-Beltrami operator Δ_Γ and a convection term $\mathcal{V} \cdot \nabla_\Gamma$, where $\mathcal{V}$ is a tangential vector field and ∇_Γ is the surface gradient on Γ. Throughout we let

$$p \in (1, \infty) \quad \text{and} \quad \mu \in (1/p, 1].$$

We look for solutions u of (1.4) and (u, ρ) of (1.5) such that

$$u \in \mathbb{E}_{u,\mu} = W^1_{p,\mu}\big(J; L_p(\Omega; E)\big) \cap L_{p,\mu}\big(J; W^{2m}_p(\Omega; E)\big).$$

The weighted vector-valued $L_{p,\mu}$-spaces and the corresponding Sobolev spaces spaces $W^1_{p,\mu}$ are defined in (1.1) and (1.2), respectively, and $W^{2m}_p(\Omega; E)$ is the E-valued Sobolev space of order $2m$ over Ω. For such solutions u the differential equation on the domain Ω holds for a.e. (t,x). The regularity of u yields

$$f \in \mathbb{E}_{0,\mu} := L_{p,\mu}(J; L_p(\Omega; E)).$$

From the mapping properties of the temporal trace described in Theorem 4.2 of [16], we deduce

$$u|_{t=0} = u_0 \in X_{u,\mu} := B^{2m(\mu - 1/p)}_{p,p}(\Omega; E).$$

Here $B^s_{p,p}(\Omega; E)$ denotes the E-valued Besov spaces over Ω. We refer to [2], [19] or [24] for a definition and the properties of these spaces. Further, Lemma 3.4 of [16] shows that the operator D^β maps $\mathbb{E}_{u,\mu}$ continuously into

$$H^{1-m_j/2m}_{p,\mu}\big(J; L_p(\Omega; E)\big) \cap L_{p,\mu}\big(J; W^{2m-m_j}_p(\Omega; E)\big) \tag{2.1}$$

for $|\beta| \le m_j \le 2m-1$. Due to Theorem 4.6 of [16], the spatial trace tr_Ω maps the space in (2.1) continuously into

$$W^{\kappa_j}_{p,\mu}\big(J; L_p(\Gamma; E)\big) \cap L_{p,\mu}\big(J; W^{2m\kappa_j}_p(\Gamma; E)\big), \qquad j = 0, ..., m,$$

where the number κ_j is defined by

$$\kappa_j := 1 - \frac{m_j}{2m} - \frac{1}{2mp}, \qquad j = 0, ..., m.$$

Below we assume that $\kappa_j \neq 1 - \mu + 1/p$ for all $j = 0, ..., m$. The weighted Sobolev spaces $H^s_{p,\mu}$ and the Slobodetskii spaces $W^s_{p,\mu}$ of fractional order $s \geq 0$ are defined by complex and real interpolation, respectively. The properties of $W^s_{p,\mu}$ and $H^s_{p,\mu}$ are studied in [16]. Moreover, $W^s_p(\Gamma; E)$ is the E-valued Sobolev-Slobodetskii space of order s, where $W^s_p = B^s_{p,p}$ for $s \notin \mathbb{N}_0$. Since the dynamic boundary equation in (1.5) takes place in F and the static boundary equations in (1.4) and (1.5) take place in E, these considerations suggest that we should look at boundary data

$$g_0 \in \mathbb{F}_{0,\mu} := W^{\kappa_0}_{p,\mu}\big(J; L_p(\Gamma; F)\big) \cap L_{p,\mu}\big(J; W^{2m\kappa_0}_p(\Gamma; F)\big),$$
$$g_j \in \mathbb{F}_{j,\mu} := W^{\kappa_j}_{p,\mu}\big(J; L_p(\Gamma; E)\big) \cap L_{p,\mu}\big(J; W^{2m\kappa_j}_p(\Gamma; E)\big), \qquad j = 1, ..., m.$$

For convenience we write

$$\mathbb{F}_\mu := \mathbb{F}_{0,\mu} \times ... \times \mathbb{F}_{m,\mu}, \qquad g = (g_0, ..., g_m) \in \mathbb{F}_\mu,$$

and similiarly $\widetilde{\mathbb{F}}_\mu := \mathbb{F}_{1,\mu} \times ... \times \mathbb{F}_{m,\mu}$ and $\widetilde{g} = (g_1, ..., g_m) \in \widetilde{\mathbb{F}}_\mu$.

We now determine the regularity of ρ and ρ_0 in (1.5). Assuming sufficient smoothness of the coefficients of the operators, we look for a space $\mathbb{E}_{\rho,\mu}$ for ρ such that each term in (1.5) involving ρ acts continuously from $\mathbb{E}_{\rho,\mu}$ to the space $\mathbb{F}_{j,\mu}$ where the respective equation takes place. It can be seen as in Section 2 of [6] that

$$\begin{aligned}\mathbb{E}_{\rho,\mu} = &W^{1+\kappa_0}_{p,\mu}\big(J; L_p(\Gamma; F)\big) \cap L_{p,\mu}\big(J; W^{l+2m\kappa_0}_p(\Gamma; F)\big)\\ &\cap W^1_{p,\mu}\big(J; W^{2m\kappa_0}_p(\Gamma; F)\big) \cap \bigcap\nolimits_{j\in\widetilde{\mathcal{J}}} W^{\kappa_j}_{p,\mu}\big(J; W^{k_j}_p(\Gamma; F)\big)\end{aligned} \tag{2.2}$$

satisfies these requirements, where we put

$$\widetilde{\mathcal{J}} := \big\{ j \in \{0, ..., m\} \ : \ k_j \neq -\infty \big\}, \qquad l_j := k_j - m_j + m_0, \qquad l := \max_{j=0,...,m} l_j.$$

It is important to note that

$$2m\kappa_j + k_j = 2m\kappa_0 + l_j, \qquad j = 0, \ldots, m. \tag{2.3}$$

We represent $\mathbb{E}_{\rho,\mu}$ by the points $(0, 1 + \kappa_0)$, $(l + 2m\kappa_0, 0)$, $(2m\kappa_0, 1)$ and (k_j, κ_j), $j \in \widetilde{\mathcal{J}}$, corresponding to the space-time differentiability of the spaces Z_i on the right-hand side of (2.2). The *Newton polygon* $\mathcal{NP}$ for $\mathbb{E}_{\rho,\mu}$ is then defined as the convex hull of these points together with $(0, 0)$. The *leading part* of $\mathcal{NP}$ is the polygonal part of its boundary connecting $(0, 1 + \kappa_0)$ to $(l + 2m\kappa_0, 0)$ anti-clockwise.

Let Z_i and Z_j be two different spaces on the right-hand side of (2.2). It is shown in Proposition 3.2 of [16] that $Z_i \cap Z_j$ embeds into all spaces whose space-time regularity corresponds to the line segment connecting the two points that represent Z_i and Z_j in $\mathcal{NP}$. Consequently, the description of $\mathbb{E}_{\rho,\mu}$ given in (2.2) contains redundant spaces, in general. We derive a nonredundant description of $\mathbb{E}_{\rho,\mu}$ as in the case $\mu = 1$ presented in [6]. Here one has to distinguish three cases. In each case, a direct application Theorem 4.2 of [16] further yields the temporal trace space of ρ at $t = 0$, denoted by

$$X_{\rho,\mu} := \mathrm{tr}_{t=0}\mathbb{E}_{\rho,\mu}.$$

In the same way we also obtain that the temporal derivative $\partial_t \rho$ has a trace at $t = 0$ if $\kappa_0 > 1 - \mu + 1/p$. We denote the resulting trace space by

$$X_{\partial_t \rho,\mu} := \mathrm{tr}_{t=0} \partial_t \mathbb{E}_{\rho,\mu} \qquad \text{if } \kappa_0 > 1 - \mu + 1/p.$$

We remark that Theorem 4.2 of [16] means that these trace spaces are given by $B^{\sigma}_{p,p}(\Gamma; F)$ for the numbers $\sigma > 0$ such that $(\sigma, k + 1 - \mu + 1/p)$ belongs to leading part of $\mathcal{NP}$ for $k = 0$ and $k = 1$, respectively. We can now give the nonredundant description of the spaces $\mathbb{E}_{\rho,\mu}$, $X_{\rho,\mu}$ and $X_{\partial_t \rho,\mu}$.

Case 1: $l = 2m$. One has

$$\mathbb{E}_{\rho,\mu} = W^{1+\kappa_0}_{p,\mu}\big(J; L_p(\Gamma; F)\big) \cap L_{p,\mu}\big(J; W^{2m(1+\kappa_0)}_p(\Gamma; F)\big)$$

since all other spaces in (2.2) correspond to points on or below the straight line $s = 1 + \kappa_0 - r/2m$ from $(0, 1 + \kappa_0)$ to $(2m + 2m\kappa_0, 0)$ due to (2.3). It follows that

$$X_{\rho,\mu} = B^{2m(\kappa_0+\mu-1/p)}_{p,p}(\Gamma; F), \qquad X_{\partial_t\rho,\mu} = B^{2m(\kappa_0-(1-\mu+1/p))}_{p,p}(\Gamma; F) \quad \text{if } \kappa_0 > 1 - \mu + 1/p.$$

Case 2: $l < 2m$. One has

$$\mathbb{E}_{\rho,\mu} = W^{1+\kappa_0}_{p,\mu}\big(J; L_p(\Gamma; F)\big) \cap L_{p,\mu}\big(J; W^{l+2m\kappa_0}_p(\Gamma; F)\big) \cap W^1_{p,\mu}\big(J; W^{2m\kappa_0}_p(\Gamma; F)\big)$$

since $(1, 2m\kappa_0)$ lies above the line segment $s = 1 + \kappa_0 - r(1 + \kappa_0)/(l + 2m\kappa_0)$ from $(0, 1 + \kappa_0)$ to $(2m + 2m\kappa_0, 0)$ and all points (κ_j, k_j) are below the line $s = 1 + (2m\kappa_0 - r)/l$ connecting $(1, 2m\kappa_0)$ and $(0, l + 2m\kappa_0)$. We then obtain the trace spaces

$$X_{\rho,\mu} = B^{2m\kappa_0+l(\mu-1/p)}_{p,p}(\Gamma; F), \qquad X_{\partial_t\rho,\mu} = B^{2m(\kappa_0-(1-\mu+1/p))}_{p,p}(\Gamma; F) \quad \text{if } \kappa_0 > 1 - \mu + 1/p.$$

Case 3: $l > 2m$. Now $(1, 2m\kappa_0)$ belongs to the interior of $\mathcal{NP}$ and it holds

$$\mathbb{E}_{\rho,\mu} = W^{1+\kappa_0}_{p,\mu}\big(J; L_p(\Gamma; F)\big) \cap L_{p,\mu}\big(J; W^{l+2m\kappa_0}_p(\Gamma; F)\big) \cap \bigcap\nolimits_{j\in\mathcal{J}} W^{\kappa_j}_{p,\mu}\big(J; W^{k_j}_p(\Gamma; F)\big),$$

where $\mathcal{J} = \{j_1, ..., j_{q_{\max}}\} \subset \widetilde{\mathcal{J}}$, $q_{\max} \in \mathbb{N}$, contains those indices $j \in \widetilde{\mathcal{J}}$ so that (k_j, κ_j) belongs to the leading part of the Newton polygon, i.e., the points

$$P_0 = (0, 1 + \kappa_0), \;\; P_1 = (k_{j_1}, \kappa_{j_1}), \;\; \dots, \;\; P_{q_{\max}} = (k_{j_{q_{\max}}}, \kappa_{j_{q_{\max}}}), \;\; P_{q_{\max}+1} = (l + 2m\kappa_0, 0)$$

are the vertices of the leading part. It is assumed that $\mathcal{J}$ is ordered so that $k_{j_{q_1}} < k_{j_{q_2}}$ and $\kappa_{j_{q_1}} > \kappa_{j_{q_2}}$ for $q_1 < q_2$. We also define $k_{-1} := 0$, $\kappa_{-1} := 1 + \kappa_0$, $m_{-1} := m_0 - 2m$ and $l_{-1} := 2m$. We further denote the edge in the Newton polygon connecting the points P_q and P_{q+1} by $\mathcal{NP}_q$, $q = 0, ..., q_{\max}$, and set

$$\mathcal{J}_{2q} := \big\{j \in \mathcal{J} \cup \{-1\} \;:\; (k_j, \kappa_j) = P_q\big\}, \qquad q = 0, ..., q_{\max},$$
$$\mathcal{J}_{2q+1} := \big\{j \in \mathcal{J} \cup \{-1\} \;:\; (k_j, \kappa_j) \in \mathcal{NP}_q\big\}, \qquad q = 0, ..., q_{\max}.$$

The temporal trace space of $\partial_t \rho$ is obtained by Theorem 4.2 of [16] from the spaces corresponding to P_0 and P_1. We thus deduce

$$X_{\partial_t\rho,\mu} = B^{k_{j_1}(\kappa_0-(1-\mu+1/p))/(1+\kappa_0-\kappa_{j_1})}_{p,p}(\Gamma; F) \quad \text{if } \kappa_0 > 1 - \mu + 1/p.$$

For $X_{\rho,\mu}$ one has to distinguish three more cases.

Case 3(i): If $\kappa_j > 1 - \mu + 1/p$ for all $j \in \mathcal{J}$ then

$$X_{\rho,\mu} = B^{l+2m(\kappa_0-(1-\mu+1/p))}_{p,p}(\Gamma; F).$$

Here we apply Theorem 4.2 of [16] to the spaces corresponding to $P_{q_{\max}}$ and $P_{q_{\max}+1}$.

Case 3(ii): Denote by $j_{q_1} \in \mathcal{J}$ be the smallest index with $\kappa_{j_{q_1}} > 1-\mu+1/p$, and by $j_{q_2} \in \mathcal{J}$ the largest index with $\kappa_{j_{q_2}} < 1-\mu+1/p$. Using the spaces corresponding to these indices, we conclude that

$$X_{\rho,\mu} = B_{p,p}^{k_{j_{q_1}}+(\kappa_{j_{q_1}}-(1-\mu+1/p))\frac{k_{j_{q_2}}-k_{j_{q_1}}}{\kappa_{j_{q_2}}-\kappa_{j_{q_1}}}}(\Gamma;F).$$

Case 3(iii): If $\kappa_j < 1-\mu+1/p$ for all $j \in \mathcal{J}$, then we employ the spaces related to P_0 and P_1 to derive

$$X_{\rho,\mu} = B_{p,p}^{k_{j_1}(\kappa_0+\mu-1/p)/(1+\kappa_0-\kappa_{j_1})}(\Gamma;F).$$

For later purpose we finally note that in Case 3 the embedding

$$\mathbb{E}_{\rho,\mu} \hookrightarrow W^1_{p,\mu}\big(J;W_p^s(\Gamma;F)\big), \qquad s := \frac{k_{j_1}\kappa_0}{1+\kappa_0-\kappa_{j_1}}, \tag{2.4}$$

follows from Proposition 3.2 of [16] by interpolating of the spaces corresponding to P_0 and P_1.

We next consider the assumptions on the coefficients of the operators. For a Banach space X of class $\mathcal{HT}$, we write

$$\mathbb{F}_{j,\mu}\big(J\times\Gamma;X\big) := W^{\kappa_j}_{p,\mu}\big(J;L_p(\Gamma;X)\big) \cap L_{p,\mu}\big(J;W_p^{2m\kappa_j}(\Gamma;X)\big), \qquad j \in \{0,...,m\}.$$

The following assumptions shall guarantee that each summand of the operators in (1.4) and (1.5) maps continuously between the relevant spaces described above. In view of the mapping properties of the traces and the derivatives, the multiplication with the coefficients has to be a continuous map on $\mathbb{F}_{j,\mu}\big(J\times\Gamma;X\big)$ for the relevant X. Moreover, to perform localization and perturbation, we require that the top order coefficients of the operators are bounded and uniformly continuous.

- **(SD)** For $|\alpha| < 2m$ we have either $2m(\mu-1/p) > 2m-1+n/p$ and $a_\alpha \in \mathbb{E}_{0,\mu}(J\times\Omega;\mathcal{B}(E))$, or $a_\alpha \in L_\infty\big(J\times\Omega;\mathcal{B}(E)\big)$. For $|\alpha| = 2m$ it holds $a_\alpha \in BUC(\overline{J}\times\overline{\Omega};\mathcal{B}(E))$. If Ω is unbounded then for $|\alpha| = 2m$ in addition the limits $a_\alpha(t,\infty) := \lim_{|x|\to\infty} a_\alpha(t,x)$ exist uniformly in $t \in \overline{J}$.
- **(SB)** Let $\mathcal{E}_0 = \mathcal{B}(E,F)$ and $\mathcal{E} = \mathcal{B}(E)$. For $j = 0,...,m$ and $|\beta| \le m_j$ it holds either $b_{j\beta} \in C^{\tau_j,2m\tau_j}\big(\overline{J}\times\Gamma;\mathcal{E}\big)$ with some $\tau_j > \kappa_j$, or $b_{j\beta} \in \mathbb{F}_{j,\mu}\big(J\times\Gamma;\mathcal{E}\big)$ and $\kappa_j > 1-\mu+1/p+\frac{n-1}{2mp}$.
- **(SC)** Let $\mathcal{F}_0 = \mathcal{B}(F)$ and $\mathcal{F} = \mathcal{B}(F,E)$ and let $\mathrm{g}: V \to \Gamma$ be any coordinates for Γ. For $j = 0,...,m$ and $|\gamma| \le k_j$ it holds either $c^{\mathrm{g}}_{j\gamma} \in C^{\tau_j,2m\tau_j}\big(\overline{J}\times\Gamma;\mathcal{E}\big)$ with some $\tau_j > \kappa_j$, or $c^{\mathrm{g}}_{j\gamma} \in \mathbb{F}_{j,\mu}\big(J\times V;\mathcal{F}\big)$ and $\kappa_j > 1-\mu+1/p+\frac{n-1}{2mp}$.

We discuss these assumptions. First, in (SD) one can relax the boundedness assumptions for $|\alpha| < 2m$, see e.g. [14]. The fact that

$$C^{\tau_j,2m\tau_j}\big(\overline{J}\times\Gamma;\mathcal{B}(X)\big)\cdot\mathbb{F}_{j,\mu}\big(J\times\Gamma;X\big) \hookrightarrow \mathbb{F}_{j,\mu}\big(J\times\Gamma;X\big)$$

for $\tau_j > \kappa_j$ can be seen using the intrinsic norm for $W^{\kappa_j}_{p,\mu}$ and $W_p^{2m\kappa_j}$ given in equation (2.6) in [16] and Section 1 of [2], respectively. If $\kappa_j > 1-\mu+1/p+\frac{n-1}{2mp}$, then Theorem 4.2 of [16] and Sobolev's embeddings show that

$$\mathbb{F}_{j,\mu}\big(J\times\Gamma;X\big) \hookrightarrow BUC\big(\overline{J}\times\Gamma;X\big).$$

Using this fact and again the intrinsic norms of $W^{\kappa_j}_{p,\mu}$ and $W_p^{2m\kappa_j}$, we then derive

$$\mathbb{F}_{j,\mu}\big(J\times\Gamma;\mathcal{B}(X)\big)\cdot\mathbb{F}_{j,\mu}\big(J\times\Gamma;X\big) \hookrightarrow \mathbb{F}_{j,\mu}\big(J\times\Gamma;X\big).$$

The assumption $\kappa_j > 1 - \mu + 1/p + \frac{n-1}{2mp}$ is only valid if p and $\mu > 1/p$ are sufficiently large. In fact, the assumption is equivalent to $p(2m\mu - m_j) > n + 2m$. The conditions in (SB) and (SC) are not optimal. For all $p \in (1,\infty)$, one can determine weaker Sobolev regularities for the coefficients than the ones given here which still meet the requirements described above, see [5], [6] and Section 1.3.4 of [14]. On the other hand, (SB) and (SC) are already sufficient for the applications to quasilinear problems, see e.g. [14] and [15].

We next state the structural assumptions on the operators, which are the same as in [5] and [6]. In the sequel, the subscript $\sharp$ denotes the principle part of a differential operator, with an important exception for the $\mathcal{C}_j$ where we put

$$\mathcal{C}_{j\sharp} := 0 \qquad \text{if } j \notin \mathcal{J}.$$

We thus consider only the principle parts of the operators $\mathcal{C}_j$ corresponding to a point on the leading part of the Newton polygon for $\mathbb{E}_{\rho,\mu}$. First, we assume that $\mathcal{A}$ is normally elliptic:

(E) For all $t \in \overline{J}$, $x \in \overline{\Omega}$ and $|\xi| = 1$, it holds $\sigma\big(\mathcal{A}_\sharp(t,x,\xi)\big) \subset \mathbb{C}_+ := \{\operatorname{Re} z > 0\}$. If Ω is unbounded, then it holds in addition $\sigma\big(\mathcal{A}_\sharp(t,\infty,\xi)\big) \subset \mathbb{C}_+$ for all $t \in \overline{J}$ and $|\xi| = 1$.

We further need conditions of Lopatinskii-Shapiro type. In their formulation, local coordinates g for the boundary Γ are called *associated* to $x \in \Gamma$ if the corresponding chart (U,φ) satisfies

$$\varphi(x) = 0, \qquad \varphi'(x) = \mathcal{O}_{\nu(x)}, \qquad \varphi(U \cap \Omega) \subset \mathbb{R}^n_+, \qquad \varphi(U \cap \Gamma) \subset \mathbb{R}^{n-1} \times \{0\},$$

where $\mathcal{O}_{\nu(x)}$ is a fixed orthogonal matrix that rotates the outer normal $\nu(x)$ of Γ at x to $(0,...,0,-1) \in \mathbb{R}^n$. It is easy to see that such a chart (U,φ) always exists. For coordinates g associated to $x \in \Gamma$, we define the rotated operators $\mathcal{A}^\nu$ and $\mathcal{B}_j^\nu$ by

$$\mathcal{A}^\nu(t,x,D) := \mathcal{A}\big(t,x,\mathcal{O}^T_{\nu(x)}D\big), \qquad \mathcal{B}^\nu_j(t,x,D) := \mathcal{B}_j\big(t,x,\mathcal{O}^T_{\nu(x)}D\big), \quad j = 0,...,m.$$

Moreover, we introduce the local operators $\mathcal{C}^g_j$ with respect to g by setting

$$\mathcal{C}^{\mathrm{g}}_j(t,x,D_{n-1}) := \sum_{|\gamma| \le k_j} c^{\mathrm{g}}_{j\gamma}(t,\mathrm{g}^{-1}(x))D^\gamma_{n-1}, \qquad j = 0,...,m,$$

where $c^{\mathrm{g}}_{j\gamma}$ are the local coefficients from the definition of $\mathcal{C}_j$. We continue with the second structural assumption concerning (1.4).

(LS$_{\text{stat}}$) For each fixed $t \in \overline{J}$ and $x \in \Gamma$, for each $\lambda \in \overline{\mathbb{C}_+}$ and $\xi' \in \mathbb{R}^{n-1}$ with $|\lambda| + |\xi'| \neq 0$ and each $h \in E^m$ the ordinary initial value problem

$$\begin{aligned} \lambda v(y) + \mathcal{A}^\nu_\sharp(t,\xi',D_y)v(y) &= 0, \qquad y > 0, \\ \mathcal{B}^\nu_{j\sharp}(t,\xi',D_y)v|_{y=0} &= h_j, \qquad j = 1,...,m, \end{aligned}$$

has a unique solution $v \in C_0([0,\infty);E)$.

Here the space $C_0([0,\infty);E)$ consists of the continuous E-valued functions on $[0,\infty)$ vanishing at ∞. For the problem (1.5) with relaxation type boundary conditions we need two assumptions of Lopatinskii-Shapiro type in the Cases 2 and 3. First, in each case we require a natural analogue of (LS$_{\text{stat}}$).

(LS$_{\mathbf{rel}}$) For each fixed $x \in \Gamma$, choose coordinates g associated to x. Then for every $t \in \overline{J}$, $\lambda \in \overline{\mathbb{C}_+}$ and $\xi' \in \mathbb{R}^{n-1}$ with $|\lambda| + |\xi'| \neq 0$, $h_0 \in F$ and $h_j \in E$, $j = 1, ..., m$, the ordinary initial value problem

$$\begin{aligned} \big(\lambda + \mathcal{A}^{\nu}_{\sharp}(t,x,\xi',D_y)\big)v(y) &= 0, \qquad y>0,\\ \mathcal{B}^{\nu}_{0\sharp}(t,x,\xi',D_y)v|_{y=0} + \big(\lambda + \mathcal{C}^{\mathrm{g}}_{0\sharp}(t,x,\xi')\big)\sigma &= h_0,\\ \mathcal{B}^{\nu}_{j\sharp}(t,x,\xi',D_y)v|_{y=0} + \mathcal{C}^{\mathrm{g}}_{j\sharp}(t,x,\xi')\sigma &= h_j, \qquad j=1,...,m, \end{aligned}$$

has a unique solution $(v, \sigma) \in C_0([0,\infty); E) \times F$.

In the Cases 2 and 3, the following additional 'asymptotic' conditions are required, respectively.

(LS$_{\infty}^{-}$) Let $l < 2m$. For each fixed $x \in \Gamma$, choose coordinates g associated to x. Then for every $t \in \overline{J}$, $h_0 \in F$, $h_j \in E$, $j = 1, ..., m$, and each $\lambda \in \overline{\mathbb{C}_+}$, $\xi' \in \mathbb{R}^{n-1}$ with $|\lambda| + |\xi'| \neq 0$, the ordinary initial value problem

$$\begin{aligned} \big(\lambda + \mathcal{A}^{\nu}_{\sharp}(t,x,\xi',D_y)\big)v(y) &= 0, \qquad y>0,\\ \mathcal{B}^{\nu}_{j\sharp}(t,x,\xi',D_y)v|_{y=0} &= h_j, \qquad j=1,...,m, \end{aligned}$$

and for all $\lambda \in \overline{\mathbb{C}_+}$ and $|\xi'| = 1$ the problem

$$\begin{aligned} \mathcal{A}^{\nu}_{\sharp}(t,x,\xi',D_y)v(y) &= 0, \qquad y>0,\\ \mathcal{B}^{\nu}_{0\sharp}(t,x,\xi',D_y)v|_{y=0} + \big(\lambda + \mathcal{C}^{\mathrm{g}}_{0\sharp}(t,x,\xi')\big)\sigma &= h_0,\\ \mathcal{B}^{\nu}_{j\sharp}(t,x,\xi',D_y)v|_{y=0} + \mathcal{C}^{\mathrm{g}}_{j\sharp}(t,x,\xi')\sigma &= h_j, \qquad j=1,...,m, \end{aligned}$$

has a unique solution $(v, \sigma) \in C_0([0,\infty); E) \times F$, respectively.

(LS$_{\infty}^{+}$) Let $l > 2m$. For each fixed $x \in \Gamma$, choose coordinates g associated to x. Then for each $t \in \overline{J}$, $h_0 \in F$, $h_j \in E$, $j = 1, ..., m$, and each $\lambda \in \overline{\mathbb{C}_+}$ and $\xi \in \mathbb{R}^{n-1}\backslash\{0\}$, the ordinary initial value problem

$$\begin{aligned} \big(\lambda + \mathcal{A}^{\nu}_{\sharp}(t,x,\xi',D_y)\big)v(y) &= 0, \qquad y>0,\\ \mathcal{B}^{\nu}_{j\sharp}(t,x,\xi',D_y)v|_{y=0} + \delta_{j,\,\mathcal{J}_{2q_{\max}+1}}\mathcal{C}^{\mathrm{g}}_{j\sharp}(t,x,\xi')\sigma &= h_j, \qquad j=0,...,m, \end{aligned}$$

and further for all $\lambda \in \overline{\mathbb{C}_+}\backslash\{0\}$, $|\xi'| = 1$ and $q = 1, ..., 2q_{\max}$, the problem

$$\begin{aligned} \big(\lambda + \mathcal{A}^{\nu}_{\sharp}(t,x,0,D_y)\big)v(y) &= 0, \qquad y>0,\\ \mathcal{B}^{\nu}_{0\sharp}(t,x,0,D_y)v|_{y=0} + \delta_{-1,\,\mathcal{J}_q}\lambda\sigma + \delta_{0,\,\mathcal{J}_q}\mathcal{C}^{\mathrm{g}}_{0\sharp}(t,x,\xi')\sigma &= h_0,\\ \mathcal{B}^{\nu}_{j\sharp}(t,x,0,D_y)v|_{y=0} + \delta_{j,\,\mathcal{J}_q}\mathcal{C}^{\mathrm{g}}_{j\sharp}(t,x,\xi')\sigma &= h_j, \qquad j=1,...,m, \end{aligned}$$

has a unique solution $(v, \sigma) \in C_0([0,\infty); E) \times F$, respectively. Here $\delta_{j,\,\mathcal{J}_q} = 1$ if $j \in \mathcal{J}_q$ and $\delta_{j,\,\mathcal{J}_q} = 0$ otherwise.

In [5] and [6] it is shown that these conditions are necessary for maximal L_p-regularity of (1.5) on finite intervals. In Section 3 of [6] they are verified for a variety of concrete problems from the applications, see also [14] and [15]. If E and F are finite dimensional, it suffices to consider $h_0 = h_j = 0$ in the above conditions.

We can now state our maximal $L_{p,\mu}$-regularity results. We start with the one for (1.4).

Theorem 2.1. *Let E be a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$ and $\mu \in (1/p, 1]$. Let $J = (0,T)$ be a finite interval, and let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary*

$\Gamma = \partial\Omega$. *Assume that* (E), (LS$_{\mathrm{stat}}$), (SD) *and* (SB) *hold true and that* $\kappa_j \neq 1-\mu+1/p$ *for* $j = 1, ..., m$. *Then the problem*

$$\begin{aligned} \partial_t u + \mathcal{A}(t,x,D)u &= f(t,x), && x\in\Omega, && t\in J, \\ \mathcal{B}_j(t,x,D)u &= g_j(t,x), && x\in\Gamma, && t\in J, && j=1,...,m, \\ u(0,x) &= u_0(x), && x\in\Omega, \end{aligned}$$

has a unique solution $u = \mathcal{L}_{\mathrm{stat}}(f,\widetilde{g},u_0) \in \mathbb{E}_{u,\mu}$ *if and only if* $(f,\widetilde{g},u_0)\in\mathcal{D}_{\mathrm{stat}}$, *where*

$$\mathcal{D}_{\mathrm{stat}} := \big\{(f,\widetilde{g},u_0)\in\mathbb{E}_{0,\mu}\times\widetilde{\mathbb{F}}_\mu\times X_{u,\mu} \ :\ \text{for } j=1,...,m \text{ it holds} \\ \mathcal{B}_j(0,\cdot,D)u_0 = g_j(0,\cdot) \text{ on } \Gamma \text{ if } \kappa_j > 1-\mu-1/p\big\}.$$

The corresponding solution operator $\mathcal{L}_{\mathrm{stat}}:\mathcal{D}_{\mathrm{stat}}\to\mathbb{E}_{u,\mu}$ *is continuous. If* $\mathcal{L}_{\mathrm{stat}}$ *is restricted to*

$$\mathcal{D}^0_{\mathrm{stat}} := \big\{(f,\widetilde{g},u_0)\in\mathcal{D}_{\mathrm{stat}} \ :\ g_j|_{t=0}=0 \text{ if } \kappa_j>1-\mu-1/p,\ \ j=1,...,m\big\},$$

for any given $T_0>0$ *the operator norm of the restriction is uniformly bounded for* $T\in(0,T_0]$.

In the situation of the theorem, it is clear that if the coefficients

$$(-\mathrm{i})^{|\alpha|}a_\alpha,\quad |\alpha|\le 2m,\qquad (-\mathrm{i})^{|\beta|}b_{j\beta},\quad |\beta|\le m_j,\quad j=1,...,m,$$

and the data are real-valued, then also the solution u is real-valued. We next state the maximal regularity result for (1.5).

Theorem 2.2. *Let* E *and* F *be Banach spaces of class* $\mathcal{HT}$, $p\in(1,\infty)$ *and* $\mu\in(1/p,1]$. *Let* $J=(0,T)$ *be a finite interval, and let* $\Omega\subset\mathbb{R}^n$ *be a domain with compact smooth boundary* $\Gamma=\partial\Omega$. *Assume that* (E), (LS$_{\mathrm{rel}}$), (SD), (SB) *and* (SC) *are valid and that, in addition, if* $l<2m$ *condition* (LS$^-_\infty$) *holds and if* $l>2m$ *condition* (LS$^+_\infty$) *holds. Assume further that* $\kappa_j\neq 1-\mu+1/p$ *for* $j=0,...,m$. *Then the problem*

$$\begin{aligned} \partial_t u + \mathcal{A}(t,x,D)u &= f(t,x), && x\in\Omega, && t\in J, \\ \partial_t\rho + \mathcal{B}_0(t,x,D)u + \mathcal{C}_0(t,x,D_\Gamma)\rho &= g_0(t,x), && x\in\Gamma, && t\in J, \\ \mathcal{B}_j(t,x,D)u + \mathcal{C}_j(t,x,D_\Gamma)\rho &= g_j(t,x), && x\in\Gamma, && t\in J, && j=1,...,m, \\ u(0,x) &= u_0(x), && x\in\Omega, \\ \rho(0,x) &= \rho_0(x), && x\in\Gamma, \end{aligned}$$

has a unique solution $(u,\rho)\in\mathbb{E}_{u,\mu}\times\mathbb{E}_{\rho,\mu}$ *if and only if* $(f,g,u_0,\rho_0)\in\mathcal{D}_{\mathrm{rel}}$, *where*

$$\mathcal{D}_{\mathrm{rel}} := \big\{(f,g,u_0,\rho_0)\in\mathbb{E}_{0,\mu}\times\mathbb{F}_\mu\times X_{u,\mu}\times X_{\rho,\mu} \ :\ \text{for } j=1,...,m \text{ it holds} \\ \mathcal{B}_j(0,\cdot,D)u_0+\mathcal{C}_j(0,\cdot,D_\Gamma)\rho_0 = g_j(0,\cdot) \text{ on } \Gamma \text{ if } \kappa_j>1-\mu+1/p; \\ g_0(0,\cdot)-\mathcal{B}_0(0,\cdot,D)u_0-\mathcal{C}_0(0,\cdot,D_\Gamma)\rho_0\in X_{\partial_t\rho,\mu} \text{ if } \kappa_0>1-\mu+1/p\big\}.$$

The corresponding solution operator $\mathcal{L}_{\mathrm{rel}}:\mathcal{D}_{\mathrm{rel}}\to\mathbb{E}_{u,\mu}\times\mathbb{E}_{\rho,\mu}$ *is continuous. If* $\mathcal{L}_{\mathrm{rel}}$ *is restricted to*

$$\mathcal{D}^0_{\mathrm{rel}} := \big\{(f,g,u_0,\rho_0)\in\mathcal{D}_{\mathrm{rel}} \ :\ g_j|_{t=0}=0 \text{ if } \kappa_j>1-\mu-1/p,\ \ j=0,...,m\big\},$$

for any given $T_0>0$ *the operator norm of the restriction is uniformly bounded for* $T\in(0,T_0]$.

A similiar statement as above holds for real-valued solutions. In the theorems, the spaces $\mathcal{D}_{\text{stat}}$ and $\mathcal{D}_{\text{rel}}$ are considered as closed subspaces of $\mathbb{E}_{0,\mu}\times\widetilde{\mathbb{F}}_{\mu}\times X_{u,\mu}$ and $\mathbb{E}_{0,\mu}\times\mathbb{F}_{\mu}\times X_{u,\mu}\times X_{\rho,\mu}$, respectively. They contain the compatibility conditions of the boundary inhomogeneities and the initial values at $t=0$, which are necessary for the solvability of (1.4) and (1.5), respectively.

One needs the spaces $\mathcal{D}^0_{\text{stat}}$ and $\mathcal{D}^0_{\text{rel}}$ with vanishing initial values since the resulting uniform estimates are crucial for fixed point arguments arising in the context of quasilinear problems. They are considered as closed subspaces of $\mathbb{E}_{0,\mu}\times{}_0\widetilde{\mathbb{F}}_{\mu}\times X_{u,\mu}$ and $\mathbb{E}_{0,\mu}\times{}_0\mathbb{F}_{\mu}\times X_{u,\mu}\times X_{\rho,\mu}$, respectively, where ${}_0\widetilde{\mathbb{F}}_{\mu}$ and ${}_0\mathbb{F}_{\mu}$ are defined as follows. For a Banach space X of class $\mathcal{HT}$ and $s=[s]+s_*$ with $[s]\in\mathbb{N}_0$, $s_*\in[0,1)$ we set

$$ {}_0W^s_{p,\mu}(J;X) := \big({}_0W^{[s]}_{p,\mu}(J;X),{}_0W^{[s]+1}_{p,\mu}(J;X)\big)_{s_*,p}, $$

where ${}_0W^k_{p,\mu}(J;X) := \big\{u\in W^k_{p,\mu}(J;X) : u(0),...,u^{(k-1)}(0)=0\big\}$ is considered as a closed subspace of $W^k_{p,\mu}(J;X)$, and then

$$ {}_0\mathbb{F}_{j,\mu} := {}_0W^{\kappa_j}_{p,\mu}\big(J;L_p(\Gamma;E)\big)\cap L_{p,\mu}\big(J;W^{2m\kappa_j}_p(\Gamma;E)\big), \qquad j=1,...,m. $$

Analogously we define the spaces ${}_0\mathbb{F}_{0,\mu}$, ${}_0\widetilde{\mathbb{F}}_{\mu}$, ${}_0\mathbb{F}_{\mu}$, ${}_0\mathbb{E}_{u,\mu}$ and ${}_0\mathbb{E}_{\rho,\mu}$. It is shown in Proposition 2.10 of [16] that ${}_0W^s_{p,\mu}=W^s_{p,\mu}$ if $0<s<1-\mu+1/p$ and

$$ {}_0W^s_{p,\mu}=\big\{u\in W^s_{p,\mu} \;:\; u^{(l)}(0)=0,\;\; l\in\{0,...,k\}\big\} $$

if $k+1-\mu+1/p<s<k+2-\mu+1/p$ for $k\in\mathbb{N}_0$, with equivalent norms, respectively. In other words, the trace at $t=0$ of a derivative of $u\in{}_0W^s_{p,\mu}$ vanishes if it exists.

Compared to the unweighted approach, the maximal $L_{p,\mu}$-regularity approach has the following advantages, where we restrict to the setting of (1.4). Analogous statements are valid for (1.5).

- *Flexible initial regularity:* We obtain solutions for initial values in $B^s_{p,p}(\Omega;E)$, where $s\in(0,2m(1-1/p)]$.
- *Inherent smoothing effect:* Away from the initial time, $\tau\in(0,T)$, the solutions belong to
$$ W^1_p\big(\tau,T;L_p(\Omega;E)\big)\cap L_p\big(\tau,T;W^{2m}_p(\Omega;E)\big)\hookrightarrow C\big(\overline{J};B^{2m(1-1/p)}_{p,p}(\Omega;E)\big). $$
- *Control solutions in a strong norm at a later time by a weaker norm at an earlier time and the data*: For $s=2m(\mu-1/p)\in(0,2m(1-1/p)]$ it holds
$$ |u(T)|_{B^{2m(1-1/p)}_{pp}(\Omega;E)} \le C(T)(|f|_{\mathbb{E}_{0,\mu}}+|\widetilde{g}|_{\widetilde{\mathbb{F}}_{\mu}}+|u_0|_{B^s_{p,p}(\Omega;E)}). $$
- *Avoid compatibility conditions:* Given $p\in(1,\infty)$, if μ is sufficiently close to $1/p$, we have $\kappa_j<1-\mu+1/p$ for all j and thus obtain a unique solution $u\in\mathbb{E}_{u,\mu}$ for arbitrary data in $\mathbb{E}_{0,\mu}\times\widetilde{\mathbb{F}}_{\mu}\times X_{u,\mu}$.

The rest of the paper is concerned with the proofs of the Theorems 2.1 and 2.2.

3. The model problems

We first consider the full-space case $\Omega=\mathbb{R}^n$ without boundary conditions and assume that the coefficients of the differential operator

$$ \mathcal{A}(D)=\sum_{|\alpha|=2m} a_\alpha D^\alpha $$

are constant, i.e., $a_\alpha \in \mathcal{B}(E)$ are independent of (t,x). Observe that there are no lower order terms and that $\mathcal{A}$ is homogeneous of degree $2m$. We have the following maximal $L_{p,\mu}$-regularity result for $\mathcal{A}$ on the half-line.

Lemma 3.1. *Let E be a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$, $\mu \in (1/p,1]$, and assume that the constant coefficient operator $\mathcal{A}$ satisfies* (E). *Then there is a unique solution $u = \mathcal{S}_F(f,u_0) \in \mathbb{E}_{u,\mu}(\mathbb{R}_+ \times \mathbb{R}^n)$ of*

$$\begin{aligned} u + \partial_t u + \mathcal{A}(D)u &= f(t,x), \qquad x \in \mathbb{R}^n, \qquad t>0, \\ u(0,x) &= u_0(x), \qquad x \in \mathbb{R}^n, \end{aligned} \tag{3.1}$$

if and only if

$$f \in \mathbb{E}_{0,\mu}(\mathbb{R}_+ \times \mathbb{R}^n) \quad \text{and} \quad u_0 \in X_{u,\mu}(\mathbb{R}^n).$$

Proof. Lemma 4.2 of [5] shows that the realization of the operator $1+\mathcal{A}$ on $L_p(\mathbb{R}^n;E)$ with domain $D(1+\mathcal{A}) = W_p^{2m}(\mathbb{R}^n;E)$ admits maximal L_p-regularity on the half-line. Since

$$X_{u,\mu}(\mathbb{R}^n) = B_{p,p}^{2m(\mu-1/p)}(\mathbb{R}^n;E) = \big(L_p(\mathbb{R}^n;E), W_p^{2m}(\mathbb{R}^n;E)\big)_{\mu-1/p,p},$$

the assertion follows from Theorem 3.2 of [17]. ■

The model problems for (1.4) and (1.5) on the half-space involve boundary conditions and thus require a much greater effort. To construct the solution, one uses an operator-valued Fourier multiplier theorem in $L_{p,\mu}$. For Banach spaces X, Y and a symbol $m \in L_{1,\mathrm{loc}}\big(\mathbb{R};\mathcal{B}(X,Y)\big)$ one introduces an operator T_m by setting

$$T_m f := \mathcal{F}^{-1} m \mathcal{F} f, \qquad f \in \mathcal{F}^{-1} C_c^\infty(\mathbb{R};X),$$

where $\mathcal{F}$ denotes the Fourier transform on the real line. We can restrict T_m to functions on $\mathbb{R}_+$. Observe that T_m is densely defined on $L_{p,\mu}(\mathbb{R}_+;X)$. We also use the analogous definition on the space $L_p(\mathbb{R}^n;X)$. The next result is due to Girardi & Weis [8].

Theorem 3.2. *Let $p \in (1,\infty)$, $\mu \in (1/p,1]$, and let X,Y be Banach spaces of class $\mathcal{HT}$. Assume that $m \in C^1\big(\mathbb{R}\backslash\{0\};\mathcal{B}(X,Y)\big)$ satisfies $\mathcal{R}\big(\{m(\lambda), \lambda m'(\lambda) : \lambda \neq 0\}\big) < \infty$. Then $T_m \in \mathcal{B}\big(L_{p,\mu}(\mathbb{R}_+;X), L_{p,\mu}(\mathbb{R}_+;Y)\big)$.*

Here, the $\mathcal{R}$-bound of a family $\mathcal{T} \subset \mathcal{B}(X,Y)$ is denoted by $\mathcal{R}(\mathcal{T})$. For a definition and properties of $\mathcal{R}$-boundedness we refer to [4] or [13]. Under more restrictive assumptions on the symbol m we can give a short proof a multiplier theorem in $L_{p,\mu}$, employing a result of Krée [12] (which is also used in the proof in [8]).

Proposition 3.3. *In addition to the assumptions of Theorem 3.2, suppose that m satisfies*

$$m \in C^2\big(\mathbb{R}\backslash\{0\};\mathcal{B}(X,Y)\big), \qquad |m''(\lambda)|_{\mathcal{B}(X,Y)} \lesssim |\lambda|^{-2} \;\text{ for }\; \lambda \neq 0.$$

Then $T_m \in \mathcal{B}\big(L_{p,\mu}(\mathbb{R}_+;X), L_{p,\mu}(\mathbb{R}_+;Y)\big)$.

Proof. The operator-valued multiplier theorem for the unweighted case $\mu = 1$ shows that T_m extends to a bounded operator from $L_p(\mathbb{R}_+;X)$ to $L_p(\mathbb{R}_+;Y)$; see Theorem 3.4 of [22]. Moreover, following the lines of the proof of Lemma VI.4.4.2 of [20], the assumptions on m imply that T_m may be represented as a convolution operator with a kernel $k \in C\big(\mathbb{R}\backslash\{0\};\mathcal{B}(X,Y)\big)$ satisfying $|k(t)|_{\mathcal{B}(X,Y)} \lesssim |t|^{-1}$. It now follows from Théorème 2 of [12] that T_m is also bounded from $L_{p,\mu}(\mathbb{R}_+;X)$ to $L_{p,\mu}(\mathbb{R}_+;Y)$, for all $\mu \in (1/p,1]$. ■

We next treat the half-space model problem corresponding to (1.5), where we proceed similarly as in Section 4 of [6]. On $\Omega = \mathbb{R}^n_+$ with boundary $\Gamma = \mathbb{R}^{n-1}$ we consider the homogeneous differential operator

$$\mathcal{A}(D) = \sum_{|\alpha|=2m} a_\alpha D^\alpha$$

and the homogeneous boundary operators

$$\mathcal{B}_j(D) = \sum_{|\beta|=m_j} b_{j\beta} \mathrm{tr}_{\mathbb{R}^n_+} D^\beta, \qquad \mathcal{C}_j(D_{n-1}) = \sum_{|\gamma|=k_j} c_{j\gamma} D^\gamma_{n-1} \qquad j = 0, ..., m.$$

The coefficients of the operators

$$a_\alpha, b_{j\beta} \in \mathcal{B}(E), \quad c_{j\gamma} \in \mathcal{B}(F, E), \quad j = 1, ..., m, \quad b_{0\beta} \in \mathcal{B}(E, F), \quad c_{0\gamma} \in \mathcal{B}(F)$$

are assumed to be independent of t and x. If nothing else is indicated, now all spaces have to be understood over $\mathbb{R}_+ \times \mathbb{R}^n_+$ and over $\mathbb{R}_+ \times \mathbb{R}^{n-1}$.

Lemma 3.4. *Let E and F be Banach spaces of class $\mathcal{HT}$, $p \in (1, \infty)$, and $\mu \in (1/p, 1]$. We assume that* (E) *and* (LS$_{\mathrm{rel}}$) *are valid and that condition* (LS$^-_\infty$) *holds if $l < 2m$ and condition* (LS$^+_\infty$) *holds if $l > 2m$. Let $(f, g, u_0, \rho_0) \in \mathcal{D}_{\mathrm{rel}}$. Then there is a unique solution $(u, \rho) \in \mathbb{E}_{u,\mu} \times \mathbb{E}_{\rho,\mu}$ of*

$$\begin{aligned} u + \partial_t u + \mathcal{A}(D)u &= f(t, x), && x \in \mathbb{R}^n_+, && t > 0, \\ \rho + \partial_t \rho + \mathcal{B}_0(D)u + \mathcal{C}_0(D_{n-1})\rho &= g_0(t, x), && x \in \mathbb{R}^{n-1}, && t > 0, \\ \mathcal{B}_j(D)u + \mathcal{C}_j(D_{n-1})\rho &= g_j(t, x), && x \in \mathbb{R}^{n-1}, && t > 0, \quad j = 1, ..., m, \\ u(0, x) &= u_0(x), && x \in \mathbb{R}^n_+, \\ \rho(0, x) &= \rho_0(x), && x \in \mathbb{R}^{n-1}. \end{aligned} \tag{3.2}$$

Proof. (I) We first show uniqueness for (3.2). We use the space $Z := L_p(\mathbb{R}^n_+; E) \times W^s_p(\mathbb{R}^{n-1}; F)$ with $s = 2m\kappa_0$ in the Cases 1 and 2 as well as $s = k_{j_1}\kappa_0/(1 + \kappa_0 - \kappa_{j_1})$ in Case 3. On Z, we introduce the operator A defined by

$$A(u, \rho) := \big((1 + \mathcal{A})u, \mathcal{B}_0 u + (1 + \mathcal{C}_0)\rho\big), \qquad (u, \rho) \in D(A),$$

with domain

$$\begin{aligned} D(A) := \big\{(u, \rho) \in W^{2m}_p(\mathbb{R}^n_+; E) \times W^{l+2m\kappa_0}_p(\mathbb{R}^{n-1}; F) \ : \\ \mathcal{B}_j u + (1 + \mathcal{C}_j)\rho = 0, \ \ j = 1, ..., m; \quad \mathcal{B}_0 u + \mathcal{C}_0 \rho \in W^s_p(\mathbb{R}^{n-1}; F)\big\}. \end{aligned}$$

By (the proof of) Theorem 2.2 of [6], A generates an analytic C_0-semigroup on Z. Due to (2.2) and (2.4), the space $\mathbb{E}_{u,\mu} \times \mathbb{E}_{\rho,\mu}$ embeds into

$$\mathbb{G} := \mathbb{E}_{u,\mu}(\mathbb{R}_+) \times \big(W^1_{p,\mu}(\mathbb{R}_+; W^s_p(\mathbb{R}^{n-1}; F)) \cap L_{p,\mu}(\mathbb{R}_+; W^{l+2m\kappa_0}_p(\mathbb{R}^{n-1}; F))\big).$$

Let $u \in \mathbb{G}$ be a solution of (3.2) with $u_0 = 0$, $\rho_0 = 0$, $f = 0$ and $g_0 = \cdots = g_m = 0$. Since $L_{p,\mu}(J; Z) \hookrightarrow L_1(J; Z)$, it follows that u is a mild solution of the inhomogeneous evolution equation for A on Z with trivial data, and thus $u = 0$.

(II) The rest of the proof is concerned with the existence of solutions of (3.2). We write $x = (x', y) \in \mathbb{R}^n_+$ with $x' \in \mathbb{R}^{n-1}$ and $y > 0$, as well as $\mathcal{F}_{x'}$ and $\mathcal{F}_t$ for the partial Fourier transform with respect to x' and $t \in \mathbb{R}$, with covariable $\xi' \in \mathbb{R}^{n-1}$ and $\theta \in \mathbb{R}$, respectively. In order to apply $\mathcal{F}_t$, we extend a function with compact support in $\mathbb{R}_+$ by 0 to $\mathbb{R}$. In the same way as in Section 4.1 of [6] one can see that it sufficies to consider the case

$$f = 0, \qquad g = (g_0, ..., g_m) \in {}_0\mathbb{F}_\mu, \qquad u_0 = 0, \qquad \rho_0 = 0.$$

(See Lemma 3.2.2 and Proposition 3.2.3 of [14].) Moreover we first assume that

$$g \in \mathcal{D} := C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^{n-1}; F \times E^m).$$

It can be seen as in Lemma 1.3.14 of [14] that $\mathcal{D}$ is dense in ${}_0\mathbb{F}_\mu$. For such data the problem (3.2) was solved in the proof of Theorem 2.1 of [6]. In the following we estimate the norm of the solution (u, ρ) in the weighted solution space ${}_0\mathbb{E}_{u,\mu} \times {}_0\mathbb{E}_{\rho,\mu}$ by the norm of g in ${}_0\mathbb{F}_\mu$. For this estimate, we have to derive an appropriate representation of (u, ρ). We apply $\mathcal{F}_{x'}\mathcal{F}_t$ to (3.2) and arrive for any $\theta \in \mathbb{R}$ and $\xi' \in \mathbb{R}^{n-1}$ at the ordinary initial value problem

$$\begin{aligned} (1 + \mathrm{i}\theta)v + \mathcal{A}(\xi', D_y)v &= 0, & y > 0,\\ (1 + \mathrm{i}\theta)\sigma + \mathcal{B}_0(\xi', D_y)v|_{y=0} + \mathcal{C}_0(\xi')\sigma &= \big(\mathcal{F}_{x'}\mathcal{F}_t g_0\big)(\theta, \xi'), & \\ \mathcal{B}_j(\xi', D_y)v|_{y=0} + \mathcal{C}_j(\xi')\sigma &= \big(\mathcal{F}_{x'}\mathcal{F}_t g_j\big)(\theta, \xi'), & j = 1, ..., m. \end{aligned} \tag{3.3}$$

In Section 4.3 of [6] it is shown that (3.3) possesses for all θ and ξ' a unique solution $\big(v(\theta, \xi', \cdot), \sigma(\theta, \xi')\big)$ which may be represented as follows. We define the symbols

$$\vartheta := (1 + \mathrm{i}\theta + |\xi'|^{2m})^{1/2m}, \qquad b := \frac{|\xi'|}{\vartheta}, \qquad \zeta := \frac{\xi'}{|\xi'|}, \qquad a := \frac{1 + \mathrm{i}\theta}{\vartheta^{2m}},$$

and the so-called boundary symbol $s(\theta, \xi')$ by

$$\begin{aligned} s(\theta, \xi') &:= 1 + \mathrm{i}\theta + |\xi'|^l \qquad \text{in the Cases 1 and 2},\\ s(\theta, \xi') &:= 1 + \mathrm{i}\theta + \sum_{j\in\mathcal{J}} |\xi'|^{k_j}\vartheta^{m_0 - m_j} \qquad \text{in Case 3}. \end{aligned}$$

Then it holds

$$\begin{aligned} v(\theta, \xi', y) &= \text{ first component of } e^{\vartheta \mathrm{i} A_0(b\zeta, a)y} P_s(b\zeta, a) M_u^0(b, \zeta, \vartheta)\big(\vartheta^{-m_j}\mathcal{F}_{x'}\mathcal{F}_t g_j(\theta, \xi')\big)_{j=0,\dots,m},\\ \sigma(\theta, \xi') &= s(\theta, \xi')^{-1}\vartheta^{m_0} M_\rho^0(b, \zeta, \vartheta)\big(\vartheta^{-m_j}\mathcal{F}_{x'}\mathcal{F}_t g_j(\theta, \xi')\big)_{j=0,\dots,m}. \end{aligned}$$

Here we have used holomorphic functions

$$A_0 : \mathbb{C}^{n-1} \times \mathbb{C} \to \mathcal{B}(E^{2m}), \qquad P_s : \mathbb{C}^{n-1} \times \mathbb{C} \to \mathcal{B}(E^{2m}),$$
$$M_u^0 : D_b \times D_\zeta \times \Sigma \to \mathcal{B}(F \times E^m, E^{2m}), \qquad M_\rho^0 : D_b \times D_\zeta \times \Sigma \to \mathcal{B}(F \times E^m, F),$$

where $D_b \subset \mathbb{C}$ and $D_\zeta \subset \mathbb{C}^{n-1} \setminus \{0\}$ are bounded open sets satisfying

$$(\overline{B}_{1/2}(1/2))^{1/2m} \subset D_b, \qquad \{\zeta \in \mathbb{R}^{n-1} \,:\, |\zeta| = 1\} \subset D_\zeta,$$

and $\Sigma = \Sigma_\phi = \big\{z \in \mathbb{C}\backslash\{0\} : |\mathrm{arg} z| < \phi\big\}$ is a sector with $\phi \in (\frac{\pi}{4m}, \pi)$. The spectrum of $\mathrm{i}A_0(b\zeta, a)$ has a gap at the imaginary axis, and $P_s(b\zeta, a)$ is the spectral projection corresponding to the stable part of the spectrum. The maps M_u^0 and M_ρ^0 have the crucial property that

$$\big\{|\xi'|^{|\alpha'|} D_{\xi'}^{\alpha'} M_u^0(\widetilde{b}, \xi'|\xi'|^{-1}, \widetilde{\vartheta}) \,:\, \alpha' \in \{0, 1\}^{n-1},\;\; \xi' \neq 0,\;\; \widetilde{b} \in D_b,\;\; \widetilde{\vartheta} \in \Sigma\big\} \tag{3.4}$$

is an $\mathcal{R}$-bounded set of operators in $\mathcal{B}(F \times E^m, E^{2m})$, and that

$$\big\{|\xi'|^{|\alpha'|} D_{\xi'}^{\alpha'} M_\rho^0(\widetilde{b}, \xi'|\xi'|^{-1}, \widetilde{\vartheta}) \,:\, \alpha' \in \{0, 1\}^{n-1},\;\; \xi' \neq 0,\;\; \widetilde{b} \in D_b,\;\; \widetilde{\vartheta} \in \Sigma\big\} \tag{3.5}$$

is an $\mathcal{R}$-bounded set of operators in $\mathcal{B}(F \times E^m, F)$. For the solvability and the representation of the solution of (3.3) in [6] only the condition ($\mathrm{LS}_{\mathrm{rel}}$) is needed. In the Cases 2 and 3 the asymptotic Lopatinskii-Shapiro conditions (LS_∞^-) and (LS_∞^+) are required to show the $\mathcal{R}$-boundedness of the sets in (3.4) and (3.5), because of the unboundedness of ϑ. In Case 1 the symbols M_u^0 and M_ρ^0 do not depend on ϑ, so that in this case additional conditions are not necessary.

Since $\mathcal{F}_{x'}\mathcal{F}_t g$ belongs for $g \in \mathcal{D}$ to the Schwartz class and all derivatives of the terms involved in the representation of the solution grow at most polynomially, we can apply the inverse Fourier transforms and obtain that

$$u = \text{ first component of } \mathcal{F}_t^{-1}\mathcal{F}_{x'}^{-1} e^{\vartheta \mathrm{i} A_0(b\zeta,a)y} P_s(b\zeta,a) M_u^0(b,\zeta,\vartheta)\big(\vartheta^{-m_j}\mathcal{F}_{x'}\mathcal{F}_t g_j\big)_{j=0,\dots,m},$$
$$\rho = \mathcal{F}_t^{-1}\mathcal{F}_{x'}^{-1} s(\theta,\xi')^{-1}\vartheta^{m_0} M_\rho^0(b,\zeta,\vartheta)\big(\vartheta^{-m_j}\mathcal{F}_{x'}\mathcal{F}_t g_j\big)_{j=0,\dots,m}$$

is the unique solution of (3.2) with $f = 0$, $u_0 = 0$, $\rho_0 = 0$ and $g \in \mathcal{D}$.

(III) We derive another representation of the solution by identifying the Fourier multipliers with operators. For a function $h \in \mathcal{S}(\mathbb{R}^{n-1}; E^{2m})$ and fixed $(x', y) \in \mathbb{R}^n_+$ we calculate

$$\begin{aligned}
\big(\mathcal{F}_{x'}^{-1} e^{\mathrm{i}\vartheta A_0 y} P_s h\big)(x') &= \big(\mathcal{F}_{x'}^{-1} e^{\mathrm{i}\vartheta A_0 (y+\widetilde{y})} P_s e^{-\widetilde{y}\vartheta} h\big)(x')|_{\widetilde{y}=0} \\
&= -\int_0^\infty \partial_{\widetilde{y}}\big(\mathcal{F}_{x'}^{-1} e^{\mathrm{i}\vartheta A_0 (y+\widetilde{y})} P_s e^{-\widetilde{y}\vartheta} h\big)(x')\,\mathrm{d}\widetilde{y} \\
&= \int_0^\infty \big(\mathcal{F}_{x'}^{-1} e^{\mathrm{i}\vartheta A_0 (y+\widetilde{y})} P_s \frac{1-\mathrm{i}A_0}{\vartheta^{2m-1}} \vartheta^{2m} e^{-\widetilde{y}\vartheta} h\big)(x')\,\mathrm{d}\widetilde{y} \\
&= \int_0^\infty \big(\mathcal{F}_{x'}^{-1} e^{\mathrm{i}\vartheta A_0 (y+\widetilde{y})} P_s \frac{1-\mathrm{i}A_0}{\vartheta^{2m-1}}\big) * \big((L_\theta \mathcal{E}_\theta \mathcal{F}_{x'}^{-1} h)(\cdot,\widetilde{y})\big)(x')\,\mathrm{d}\widetilde{y},
\end{aligned} \tag{3.6}$$

neglecting the arguments of A_0 and P_s. Here the operator L_θ is defined by

$$L_\theta := 1 + \mathrm{i}\theta + (-\Delta_{n-1})^m = \mathcal{F}_{x'}^{-1}\vartheta^{2m}\mathcal{F}_{x'},$$

where the last equality holds, e.g., on Schwartz functions. We observe that for a bounded holomorphic scalar function φ on a sector Σ_τ with $\tau \in (0,\pi)$ the operator $\varphi(-\Delta_{n-1})$ defined via the $\mathcal{H}^\infty$-calculus for $-\Delta_{n-1}$ on $L_p(\mathbb{R}^{n-1}; E)$ coincides with the Fourier multiplier $\mathcal{F}_{x'}^{-1}\varphi(|\cdot|^2)\mathcal{F}_{x'}$, see Example 10.2 of [13]. Moreover, the $\mathcal{H}^\infty$-calculus extends the usual Dunford type calculus for sectorial operators, see Remark 9.9 of [13]. Therefore, the extension operator $\mathcal{E}_\theta$, which corresponds to $y \mapsto e^{-y\vartheta}$, is given by

$$(\mathcal{E}_\theta f)(x', y) := e^{-y L_\theta^{1/2m}} f(x'), \qquad x' \in \mathbb{R}^{n-1}, \qquad y > 0,$$

for $f \in L_p(\mathbb{R}^{n-1}; E)$. We also obtain the equality

$$\mathcal{F}_{x'}^{-1}\vartheta^{2m} e^{-\cdot\vartheta} h = L_\theta\,\mathcal{E}_\theta\,\mathcal{F}_{x'}^{-1} h, \qquad h \in \mathcal{S}(\mathbb{R}^{n-1}; E^m),$$

which we have used in the last line of (3.6). For $\theta \in \mathbb{R}$ and $f \in L_p(\mathbb{R}^n_+; E^{2m})$ we thus define the operator $\mathcal{T}(\theta)$ by

$$(\mathcal{T}(\theta) f)(x', y) := \text{ first component of } \int_0^\infty \big(\mathcal{F}_{x'}^{-1} e^{\mathrm{i}\vartheta A_0 (y+\widetilde{y})} P_s \frac{1-\mathrm{i}A_0}{\vartheta^{2m-1}}\big) * f(\cdot,\widetilde{y})(x')\,\mathrm{d}\widetilde{y}.$$

The proofs of Lemmas 4.3 and 4.4 in [5] show that $\mathcal{T} \in C^1\big(\mathbb{R}; \mathcal{B}(L_p(\mathbb{R}^n_+; E^{2m}), W_p^{2m}(\mathbb{R}^n_+; E))\big)$ and that

$$\big\{D^\alpha \mathcal{T}(\theta),\ \theta\frac{\partial}{\partial\theta} D^\alpha\mathcal{T}(\theta)\ :\ \theta \in \mathbb{R},\ \ |\alpha| \le 2m\big\} \tag{3.7}$$

is an $\mathcal{R}$-bounded set of operators in $\mathcal{B}\big(L_p(\mathbb{R}^n_+;E^{2m}),L_p(\mathbb{R}^n_+;E)\big)$. Further, as above one can see that $\vartheta^{-m_j}\mathcal{F}_{x'}=\mathcal{F}_{x'}L_\theta^{-m_j/2m}$ on Schwartz functions, for $j=0,...,m$. This fact leads to

$$u=\mathcal{F}_t^{-1}\mathcal{T}(\theta)L_\theta\mathcal{E}_\theta\mathcal{F}_{x'}^{-1}M_u(b,\zeta,\vartheta)\mathcal{F}_{x'}\big(L_\theta^{-m_j/2m}\mathcal{F}_tg_j\big)_{j=0,...,m}.$$

The Dunford type calculus for sectorial operators yields for $\theta\in\mathbb{R}$ and $y>0$ the representation

$$L_\theta e^{-yL_\theta^{1/2m}}=\frac{1}{2\pi\mathrm{i}}\int_\Xi ze^{-yz^{1/2m}}(z-L_\theta)^{-1}\,\mathrm{d}z,$$

where $\Xi=(\infty,\delta]e^{\mathrm{i}3\pi/2}\cup\delta e^{\mathrm{i}[3\pi/2,-3\pi/2]}\cup[\delta,\infty)e^{-\mathrm{i}3\pi/2}$ for some sufficiently small $\delta>0$. Hence for each $y>0$ the $\mathcal{B}\big(L_p(\mathbb{R}^{n-1};E)\big)$-valued function $\theta\mapsto L_\theta e^{-yL_\theta^{1/2m}}$ is smooth and all of its derivatives are bounded. So we can apply the inverse Fourier transform with respect to t and obtain that

$$L_\theta e^{-yL_\theta^{1/2m}}=\mathcal{F}_tLe^{-yL^{1/2m}}\mathcal{F}_t^{-1}$$

on Schwartz functions, where $L:=1+\partial_t+(-\Delta_{n-1})^m$ and $\mathcal{E}:=e^{-\cdot L^{1/2m}}$. Here, for $X\in\{E,F\}$ we consider L as an operator on $L_{p,\mu}\big(\mathbb{R}_+;L_p(\mathbb{R}^{n-1};X)\big)$ with the domain

$$D(L)=D(\partial_t)+D((-\Delta_{n-1})^m)={}_0W^1_{p,\mu}\big(\mathbb{R}_+;L_p(\mathbb{R}^{n-1};X)\big)\cap L_{p,\mu}\big(\mathbb{R}_+;W^{2m}_p(\mathbb{R}^{n-1};X)\big).$$

In Lemma 3.1 of [16] we have established that L is invertible and sectorial with angle not larger than $\pi/2$. Similarly one can treat fractional powers and derive $L_\theta^{-m_j/2m}=\mathcal{F}_tL^{-m_j/2m}\mathcal{F}_t^{-1}$. We arrive at

$$u=\mathcal{L}_ug:=\big(\mathcal{F}_t^{-1}\mathcal{T}(\theta)\mathcal{F}_t\big)\,L\,\mathcal{E}\,\big(\mathcal{F}_t^{-1}\mathcal{F}_{x'}^{-1}M_u^0(b,\zeta,\vartheta)\mathcal{F}_{x'}\mathcal{F}_t\big)\,\big(L^{-m_j/2m}g_j\big)_{j=0,...,m},$$

Analogous arguments show that the second component ρ can be represented by

$$\rho=\mathcal{L}_\rho g:=S^{-1}L^{m_0/2m}\,\big(\mathcal{F}_t^{-1}\mathcal{F}_{x'}^{-1}M_\rho^0(b,\zeta,\vartheta)\mathcal{F}_{x'}\mathcal{F}_t\big)\big(L^{-m_j/2m}g_j\big)_{j=0,...,m},$$

with the operator

$$S:=1+\partial_t+(-\Delta_{n-1})^{l/2}\qquad\text{in the Cases 1 and 2,}$$
$$S:=1+\partial_t+\sum_{j\in\mathcal{J}}(-\Delta_{n-1})^{k_j/2}L^{(m_0-m_j)/2m}\qquad\text{in Case 3.}$$

Using the properties of L proved in Lemma 3.1 of [16], it can be shown as in Section 4.2 of [6] that S is an isomorphism between ${}_0\mathbb{E}_{\rho,\mu}$ and ${}_0\mathbb{F}_{0,\mu}$. Because $\mathcal{D}$ is a dense subset of ${}_0\mathbb{F}_\mu$, it now remains to prove the estimate

$$|\mathcal{L}_ug|_{\mathbb{E}_{u,\mu}}+|\mathcal{L}_\rho g|_{\mathbb{E}_{\rho,\mu}}\lesssim|g|_{{}_0\mathbb{F}_\mu},\qquad g\in\mathcal{D}.\tag{3.8}$$

If (3.8) has been verified then the solution operator $\mathcal{L}:=(\mathcal{L}_u,\mathcal{L}_\rho)$ extends continuously to an operator from ${}_0\mathbb{F}_\mu$ to ${}_0\mathbb{E}_{u,\mu}\times{}_0\mathbb{E}_{\rho,\mu}$, and this extension yields the solution of (3.2).

(IV) Lemma 3.1 of [16] says that for $s\in(0,1]$ we have

$$D_L(s,p)={}_0W^s_{p,\mu}\big(\mathbb{R}_+;L_p(\mathbb{R}^{n-1};X)\big)\cap L_{p,\mu}\big(\mathbb{R}_+;W^{2ms}_p(\mathbb{R}^{n-1};X)\big).$$

Therefore, for $j=1,...,m$ the operator $L^{-m_j/2m}$ maps the space ${}_0\mathbb{F}_{j,\mu}=D_L(\kappa_j,p)$ continuously into

$${}_0\mathbb{Y}_E:=D_L(1-1/2mp,p)={}_0W^{1-1/2mp}_{p,\mu}\big(\mathbb{R}_+;L_p(\mathbb{R}^{n-1};E)\big)\cap L_{p,\mu}\big(\mathbb{R}_+;W^{2m-1/p}_p(\mathbb{R}^{n-1};E)\big).$$

The same arguments yield that $L^{-m_0/2m}$ maps ${}_0\mathbb{F}_{0,\mu}$ continuously into ${}_0\mathbb{Y}_F$, which is defined as ${}_0\mathbb{Y}_E$ with E replaced by F. We next prove that the operator

$$\mathcal{M}^0 := \mathcal{F}_t^{-1}\mathcal{F}_{x'}^{-1}M^0(b,\zeta,\vartheta)\mathcal{F}_{x'}\mathcal{F}_t$$

on $\mathcal{D}$ with the symbol $M^0 : D_b \times D_\zeta \times \Sigma \to \mathcal{B}(F \times E^m, E^{2m} \times F)$ given by

$$M^0(b,\zeta,\vartheta) := \big(M_u^0(b,\zeta,\vartheta), M_\rho^0(b,\zeta,\vartheta)\big),$$

extends continuously to an element of $\mathcal{B}\big({}_0\mathbb{Y}_F \times {}_0\mathbb{Y}_E^m, {}_0\mathbb{Y}_E^{2m} \times {}_0\mathbb{Y}_F\big)$. To this end, we consider the approximating operators

$$\mathcal{M}^{0,\varepsilon} := \mathcal{F}_t^{-1}\mathcal{F}_{x'}^{-1}M^0(b,\zeta,\vartheta)(1+\vartheta)^{-\varepsilon}\mathcal{F}_{x'}\mathcal{F}_t, \qquad \varepsilon \in (0,1).$$

Observe that $\mathcal{M}^{0,\varepsilon}(1+L^{1/2m})^\varepsilon = \mathcal{M}^0$ on $\mathcal{D}$. Cauchy's formula yields the representation

$$\mathcal{M}^{0,\varepsilon} = -\frac{1}{4\pi^2}\int_{\Xi_\vartheta}\int_{\Xi_b} \mathcal{F}_t^{-1}\mathcal{F}_{x'}^{-1}M^0(\widetilde{b},\zeta,\widetilde{\vartheta})(1+\widetilde{\vartheta})^{-\varepsilon}(\widetilde{b}-b)^{-1}(\widetilde{\vartheta}-\vartheta)^{-1}\mathcal{F}_{x'}\mathcal{F}_t \,\mathrm{d}\widetilde{b}\,\mathrm{d}\widetilde{\vartheta},$$

with $\Xi_\vartheta = (-\infty,0]e^{-\mathrm{i}\phi_*} \cup [0,\infty)e^{\mathrm{i}\phi_*}$ for some $\phi_* \in (\pi/4m, \phi)$, and where Ξ_b is a closed curve in D_b surrounding $(\overline{B}_{1/2}(1/2))^{1/2m}$. Since $\zeta = \frac{\xi'}{|\xi'|}$ is independent of θ, we may rewrite the above equality as

$$\mathcal{M}^{0,\varepsilon} = -\frac{1}{4\pi^2}\int_{\Xi_\vartheta}\int_{\Xi_b} \mathcal{F}_{x'}^{-1}M^0(\widetilde{b},\zeta,\widetilde{\vartheta})\mathcal{F}_{x'}(1+\widetilde{\vartheta})^{-\varepsilon}(\widetilde{b}-B)^{-1}(\widetilde{\vartheta}-L^{1/2m})^{-1}\,\mathrm{d}\widetilde{b}\,\mathrm{d}\widetilde{\vartheta},$$

where $B := (-\Delta_{n-1})^{1/2}L^{-1/2m}$ corresponds to the symbol $b = \frac{|\xi'|}{\vartheta}$. The realization of B on $L_{p,\mu}\big(\mathbb{R}_+; L_p(\mathbb{R}^{n-1};E)\big)$ is a bounded operator, and its spectrum is contained in the set $(\overline{B}_{1/2}(1/2))^{1/2m}$. This can be seen using the joint functional calculus for ∂_t and $(-\Delta_{n-1})^m$ on $L_{p,\mu}\big(\mathbb{R}_+; L_p(\mathbb{R}^{n-1};E)\big)$, see Theorem 4.5 of [9].

Due to the $\mathcal{R}$-boundedness of the sets (3.4) and (3.5), the operator-valued Fourier-multiplier theorem in $\mathbb{R}^{n-1}$ (Theorem 3.25 of [4], see also Theorem 4.13 of [13]) and real interpolation imply that that the operators

$$M^1(\widetilde{b},\widetilde{\vartheta}) := \mathcal{F}_{x'}^{-1}M^0(\widetilde{b},\cdot,\widetilde{\vartheta})\mathcal{F}_{x'}, \qquad \widetilde{b} \in D_b, \qquad \widetilde{\vartheta} \in \Sigma,$$

extend continuously to elements of $\mathcal{B}\big(W_p^s(\mathbb{R}^{n-1}; F \times E^m), W_p^s(\mathbb{R}^{n-1}; E^{2m} \times F)\big)$, $s \geq 0$, with uniformly bounded operators norms. Since M^0 is holomorphic, also M^1 is holomorphic in its arguments. By canonical pointwise extension we thus obtain that

$$M^1 : D_b \times \Sigma \to \mathcal{B}\big({}_0\mathbb{Y}_F \times {}_0\mathbb{Y}_E^m, {}_0\mathbb{Y}_E^{2m} \times {}_0\mathbb{Y}_F\big)$$

is bounded and holomorphic. Using L as an isomorphism $D(L) \to L_{p,\mu}\big(\mathbb{R}_+; L_p(\mathbb{R}^{n-1};E)\big)$ that commutes with B, we see that the spectrum of the realization of B on $D(L)$ is also contained in $(\overline{B}_{1/2}(1/2))^{1/2m}$. By interpolation, the same holds on ${}_0\mathbb{Y}_F \times {}_0\mathbb{Y}_E^m$. Hence, we may rewrite $\mathcal{M}^{0,\varepsilon}$ as

$$\mathcal{M}^{0,\varepsilon} = -\frac{1}{4\pi^2}\int_{\Xi_\vartheta}\int_{\Xi_b} M^1(\widetilde{b},\widetilde{\vartheta})(1+\widetilde{\vartheta})^{-\varepsilon}(\widetilde{b}-B)^{-1}(\widetilde{\vartheta}-L^{1/2m})^{-1}\,\mathrm{d}\widetilde{b}\,\mathrm{d}\widetilde{\vartheta},$$

where the curve integrals are now defined in $\mathcal{B}\big({}_0\mathbb{Y}_F \times {}_0\mathbb{Y}_E^m, {}_0\mathbb{Y}_E^{2m} \times {}_0\mathbb{Y}_F\big)$. We thus obtain

$$\mathcal{M}^{0,\varepsilon} = \frac{1}{2\pi\mathrm{i}}\int_{\Xi_\vartheta} M^2(\widetilde{\vartheta})(1+\widetilde{\vartheta})^{-\varepsilon}(\widetilde{\vartheta}-L^{1/2m})^{-1}\,\mathrm{d}\widetilde{\vartheta}$$

for a bounded holomorphic map

$$M^2 : \Sigma \to \mathcal{B}\big({}_0\mathbb{Y}_F \times {}_0\mathbb{Y}_E^m, {}_0\mathbb{Y}_E^{2m} \times {}_0\mathbb{Y}_F\big).$$

Since the realization of $L^{1/2m}$ on $L_{p,\mu}\big(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; E)\big)$ is sectorial with angle not larger than $\pi/4m$, it follows from Corollary 1 of [3] that $L^{1/2m}$ admits a bounded operator-valued $\mathcal{H}^\infty$-calculus with $\mathcal{H}^\infty$-angle not larger than $\pi/4m$ on the real interpolation spaces ${}_0\mathbb{Y}_E^m$ and ${}_0\mathbb{Y}_F$, respectively. From this fact and the boundedness of M^2 on Σ we infer

$$|\mathcal{M}^{0,\varepsilon}|_{\mathcal{B}({}_0\mathbb{Y}_F\times{}_0\mathbb{Y}_E^m,{}_0\mathbb{Y}_E^{2m}\times{}_0\mathbb{Y}_F)} \lesssim \sup_{\widetilde{\vartheta}\in\Sigma} |M^2(\widetilde{\vartheta})(1+\widetilde{\vartheta})^{-\varepsilon}|_{\mathcal{B}({}_0\mathbb{Y}_F\times{}_0\mathbb{Y}_E^m,{}_0\mathbb{Y}_E^{2m}\times{}_0\mathbb{Y}_F)} \le C, \tag{3.9}$$

where C does not depend on $\varepsilon \in (0,1)$. Due to Proposition 2.2 of [4], for $h \in D(L^2)$ the map $\varepsilon \mapsto (1 + L^{1/2m})^\varepsilon h$ is continuous with values in $D_L(1 - 1/2mp, p)$. Together with (3.9), this fact yields

$$|\mathcal{M}^0 h|_{{}_0\mathbb{Y}_E^{2m}\times{}_0\mathbb{Y}_F} \lesssim \limsup_{\varepsilon\to 0} |\mathcal{M}^{0,\varepsilon}|_{\mathcal{B}({}_0\mathbb{Y}_F\times{}_0\mathbb{Y}_E^m,{}_0\mathbb{Y}_E^{2m}\times{}_0\mathbb{Y}_F)}\, |(1 + L^{1/2m})^\varepsilon h|_{{}_0\mathbb{Y}_F\times{}_0\mathbb{Y}_E^m} \lesssim |h|_{{}_0\mathbb{Y}_F\times{}_0\mathbb{Y}_E^m}.$$

Since $D(L^2)$ is dense in $D_L(1 - 1/2mp, p)$, we obtain that $\mathcal{M}^0$ extends to an element of $\mathcal{B}\big({}_0\mathbb{Y}_F \times {}_0\mathbb{Y}_E^m, {}_0\mathbb{Y}_E^{2m} \times {}_0\mathbb{Y}_F\big)$, as asserted.

(V) Now we can show the required estimate for $\mathcal{L}_u$, i.e.,

$$|\mathcal{L}_u g|_{\mathbb{E}_{u,\mu}} \lesssim |g|_{{}_0\mathbb{F}_\mu}, \qquad g \in \mathcal{D}. \tag{3.10}$$

The extension operator $\mathcal{E} = e^{-\cdot L^{1/2m}}$ maps continuously

$$D_L(1 - 1/2mp, p) = D_{L^{1/2m}}(2m - 1/p, p) \to L_p\big(\mathbb{R}_+; D(L)\big),$$

and L maps the space $L_p\big(\mathbb{R}_+; D(L)\big)$ continuously into

$$L_p\big(\mathbb{R}_+; L_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; E))\big) = L_{p,\mu}\big(\mathbb{R}_+; L_p(\mathbb{R}^n_+; E)\big).$$

Of course, here E may be replaced by F. Thus $L\,\mathcal{E}$ maps continuously

$${}_0\mathbb{Y}_E^{2m} \times {}_0\mathbb{Y}_F \to L_{p,\mu}\big(\mathbb{R}_+; L_p(\mathbb{R}^n_+; E^{2m} \times F)\big).$$

Theorem 3.2 and the $\mathcal{R}$-boundedness of (3.7) imply that $\mathcal{F}_t^{-1}\mathcal{T}(\cdot)\mathcal{F}_t$ extends to a continuous operator

$$L_{p,\mu}\big(\mathbb{R}_+; L_p(\mathbb{R}^n_+; E^{2m})\big) \to L_{p,\mu}\big(\mathbb{R}_+; W_p^{2m}(\mathbb{R}^n_+; E)\big).$$

Alternatively, this fact follows from Proposition 3.3 since one can show that the operator family

$$\big\{\theta^2 \frac{\partial}{\partial\theta^2} D^\alpha \mathcal{T}(\theta) \;:\; \theta \in \mathbb{R},\;\; |\alpha| \le 2m\big\}$$

is bounded in $\mathcal{B}\big(L_p(\mathbb{R}^n_+; E^{2m}), L_p(\mathbb{R}^n_+; E)\big)$ arguing as in the proof of Lemma 4.4 of [5]. The equation for u shows that its $\mathbb{E}_{u,\mu}$-norm can be controlled by its $L_{p,\mu}\big(\mathbb{R}_+; W_p^{2m}(\mathbb{R}^n_+; E)\big)$-norm. So we have established (3.10). We finally consider the required estimate for $\mathcal{L}_\rho$. As above we obtain that $L^{m_0/2m}$ maps continuously

$${}_0\mathbb{Y}_F = D_L(1 - 1/p, p) \to D_L(\kappa_0, p) = {}_0\mathbb{F}_{0,\mu}.$$

Since S^{-1} is an isomorphism from ${}_0\mathbb{F}_{0,\mu}$ to ${}_0\mathbb{E}_{\rho,\mu}$, this gives the estimate for $\mathcal{L}_\rho$. ■

The analogous half-space result for (1.4) reads as follows.

Lemma 3.5. *Let E be a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$, $\mu \in (1/p, 1]$, and assume that* (E) *and* (LS) *are valid. Then for $(f, \widetilde{g}, u_0) \in \mathcal{D}_{\mathrm{stat}}$ there is a unique solution $u \in \mathbb{E}_{u,\mu}$ of*

$$\begin{aligned} u + \partial_t u + \mathcal{A}(D)u &= f(t,x), && x \in \mathbb{R}^n_+, && t > 0, \\ \mathcal{B}_j(D)u &= g_j(t,x), && x \in \mathbb{R}^{n-1}, && t > 0, \qquad j = 1, ..., m, \\ u(0,x) &= u_0(x), && x \in \mathbb{R}^n_+. \end{aligned} \tag{3.11}$$

■

We refrain from giving a detailed proof of this result, which is similar to the one of Lemma 3.4 and also less sophisticated. (See Section 2 of [14] for the details.) Again we may restrict to the case $f = 0$, $\widetilde{g} \in {}_0\widetilde{\mathbb{F}}_\mu$ and $u_0 = 0$. Applying the partial Fourier transforms with respect to t and x' to (3.11) we arrive at an ordinary initial value problem, whose solution operator is for regular data $(g_1, ..., g_m)$ given by

$$\widetilde{\mathcal{L}} = \widetilde{\mathcal{T}}\big(L^{1-m_j/2m}\mathcal{E}g_j\big)_{j=1,...,m},$$

due to Lemma 4.3 of [5]. Here $\widetilde{\mathcal{T}}$ has the same properties as $\mathcal{T}$ and L, $\mathcal{E}$ are given as in the proof of Lemma 3.4. The arguments given in the Steps IV and V of the proof above yield that $\widetilde{\mathcal{L}} \in \mathcal{B}({}_0\widetilde{\mathbb{F}}_\mu, {}_0\mathbb{E}_{u,\mu})$, which implies the solvability of (3.11) as asserted.

4. The general problem on a domain

Theorems 2.1 and 2.2 are now a consequence of the above results for the model problems and a perturbation and localization procedure, analogous to the one in e.g. Section 4.5 of [6]. We only sketch the proof below since the full procedure is rather lengthy and tedious. The arguments are worked out in great detail in Sections 2.3, 2.4, 3.2.2 and 3.3 of [14]. Moreover, we concentrate on (1.5) since the proof for (1.4) is similar and a bit simpler.

Proof of Theorem 2.2. (I) Let us first consider the necessary conditions on the data. The considerations in Section 2 and the assumptions (SD), (SB) and (SC) yield that $\mathcal{A} \in \mathcal{B}\big(\mathbb{E}_{u,\mu}, \mathbb{E}_{0,\mu}\big)$ and $\mathcal{B}_j \in \mathcal{B}\big(\mathbb{E}_{u,\mu}, \mathbb{F}_{j,\mu}\big)$, $\mathcal{C}_j \in \mathcal{B}\big(\mathbb{E}_{\rho,\mu}, \mathbb{F}_{j,\mu}\big)$ for $j = 0, ..., m$. Moreover, we have

$$W^{\kappa_j}_{p,\mu}\big(J; L_p(\Gamma; E)\big) \hookrightarrow BUC\big(\overline{J}; L_p(\Gamma; E)\big) \qquad \text{if } \kappa_j > 1 - \mu + 1/p$$

for $j = 1, ..., m$, due to Proposition 2.10 of [16]. Thus in this case the j-th boundary equation in (1.5) must hold up to $t = 0$ by continuity, which explains the compatibility conditions in $\mathcal{D}_{\mathrm{rel}}$ for this case. Similiarly, for $j = 0$ the regularity compatibility at the boundary is needed if $\kappa_0 > 1 - \mu + 1/p$, i.e., if $\partial_t\rho$ has a trace at $t = 0$. For the existence of a solution $(u, \rho) \in \mathbb{E}_{u,\mu} \times \mathbb{E}_{\rho,\mu}$, it is therefore necessary that the data in (1.5) belong to $\mathcal{D}_{\mathrm{rel}}$.

(II) Let us show that $(f, g, u_0, \rho_0) \in \mathcal{D}_{\mathrm{rel}}$ is also sufficient for the existence of a unique solution $(u, \rho) \in \mathbb{E}_{u,\mu} \times \mathbb{E}_{\rho,\mu}$ of (1.5). Uniqueness follows as in Step I of the proof of Lemma 3.4. For the existence of the solution (u, ρ), note that it suffices to consider small $T > 0$ by a standard compactness argument. For simplicity, we assume that Ω is bounded. The case of unbounded Ω requires minor modifications.

We cover $\overline{\Omega}$ by a finite number of open balls B_i such that $B_i \cap \Gamma = \emptyset$ for $i = 1, ..., N_F$ and $B_i \cap \Gamma \neq \emptyset$ for $i = N_H + 1, ..., N_F$, where $N_F, N_H \in \mathbb{N}$. We further take a smooth partition of unity ψ_i for $\overline{\Omega}$ subordinate to this cover. Let $(u, \rho) \in \mathbb{E}_{u,\mu} \times \mathbb{E}_{\rho,\mu}$. Now, (u, ρ) solves (1.5) if and only if $(u_i, \rho_i) = (\psi_i u, \psi_i \rho)$ satisfies

$$\begin{aligned}
\partial_t u_i + \mathcal{A} u_i &= \psi_i f + [\mathcal{A}, \psi_i] u, && \text{in } \Omega \cap B_i, \quad t \in J, \\
\partial_t \rho_i + \mathcal{B}_0 u_i + \mathcal{C}_0 \rho_i &= \psi_i g_0 + [\mathcal{B}_0, \psi_i] u + [\mathcal{C}_0, \psi_i]\rho, && \text{on } \Gamma \cap B_i, \quad t \in J, \\
\mathcal{B}_j u_i + \mathcal{C}_j \rho_i &= \psi_i g_j + [\mathcal{B}_0, \psi_i] u + [\mathcal{C}_0, \psi_i]\rho, && \text{on } \Gamma \cap B_i, \quad t \in J, \quad j = 1, ..., m, \\
u_i|_{t=0} &= \psi_i u_0, && \text{in } \Omega \cap B_i, \\
\rho_i|_{t=0} &= \psi_i \rho_0, && \text{on } \Gamma \cap B_i,
\end{aligned} \tag{4.1}$$

for all $i = 1, ..., N_H$, where $[\mathcal{A}, \psi_i]u = \mathcal{A}(\psi_i u) - \psi_i \mathcal{A} u$. For $i = 1, ..., N_F$ no boundary conditions are involved in (4.1). We extend the coefficients of $\mathcal{A}$ outside B_i to $\mathbb{R}^n$ such that (SD) is still valid, and denote the operator with extended coefficients by $\mathcal{A}^i$. Then u_i solves (4.1) for $i = 1, ..., N_F$ if and only if it solves

$$\begin{aligned}
\partial_t u_i + \mathcal{A}^i u_i &= \psi_i f + [\mathcal{A}, \psi_i] u, && \text{in } \mathbb{R}^n, \quad t \in J, \\
u_i|_{t=0} &= \psi_i u_0, && \text{in } \mathbb{R}^n.
\end{aligned} \tag{4.2}$$

Due to the continuity of the top order coefficients of $\mathcal{A}$, the top order part of the operator $\mathcal{A}^i$ is a small perturbation of a homogeneous constant coefficient operator satisfying (E) if the extension of the coefficients is appropriate, provided T and the radius of B_i are sufficiently small. Poincaré's inequality in the $L_{p,\mu}$-spaces (Lemma 2.12 in [16]) allows to estimate lower order terms with constants decreasing to 0 as $T \to 0$, see Lemma 1.3.13 of [14]. Using Lemma 3.1, we can now solve (4.2) by a straightforward fixed point argument. We thus obtain a continuous solution operator $\mathcal{L}_F^i : \mathbb{E}_{0,\mu}(J \times \mathbb{R}^n) \times X_{u,\mu}(\mathbb{R}^n) \to \mathbb{E}_{u,\mu}(J \times \mathbb{R}^n)$ for (4.2). It follows that

$$u_i = \mathcal{L}_F^i \big(\psi_i f + [\mathcal{A}, \psi_i] u, \psi_i u_0\big), \qquad i = 1, ..., N_F.$$

Observe that the commutator terms are of lower order. For $i = N_F + 1, ..., N_H$ the boundary conditions in (4.1) are present. We choose the B_i so small that we have a chart φ_i for Γ with domain B_i associated to some $x_i \in \Gamma$. Denoting by Φ_i the corresponding push-forward operator, i.e., $\Phi_i v = v \circ \varphi_i^{-1}$, we obtain that (u_i, ρ_i) solves (4.1) if and only if $(v_i, \sigma_i) = (\Phi_i u_i, \Phi_i \rho_i)$ solves

$$\begin{aligned}
\partial_t v_i + \big(\Phi_i \mathcal{A} \Phi_i^{-1}\big) v_i &= \Phi_i \big(\psi_i f + [\mathcal{A}, \psi_i] u\big), && \text{in } \mathbb{R}^n_+ \cap \varphi_i(B_i), \\
\partial_t \sigma_i + \big(\Phi_i \mathcal{B}_0 \Phi_i^{-1}\big) v_i + \mathcal{C}_0^{\mathrm{g}_i} \sigma_i &= \Phi_i \big(\psi_i g_0 + [\mathcal{B}_0, \psi_i] u + [\mathcal{C}_0, \psi_i]\rho\big), && \text{on } \mathbb{R}^{n-1} \cap \varphi_i(B_i), \\
\big(\Phi_i \mathcal{B}_j \Phi_i^{-1}\big) v_i + \mathcal{C}_j^{\mathrm{g}_i} \sigma_i &= \Phi_i \big(\psi_i g_j + [\mathcal{B}_j, \psi_i] u + [\mathcal{C}_j, \psi_i]\rho\big), && \text{on } \mathbb{R}^{n-1} \cap \varphi_i(B_i), \\
v_i|_{t=0} &= \Phi_i \psi_i u_0, && \text{in } \mathbb{R}^n_+ \cap \varphi_i(B_i), \\
\sigma_i|_{t=0} &= \Phi_i \psi_i \rho_0, && \text{on } \mathbb{R}^{n-1} \cap \varphi_i(B_i),
\end{aligned}$$

for $t \in J$ and $j = 1, ..., m$. Recall that $\mathcal{C}_j^{\mathrm{g}_i}$ denotes the local representation of $\mathcal{C}_j$ with respect to the coordinates g_i corresponding to φ_i. According to Theorem 10.3 of [23], at $t \in \overline{J}$ and x_i the principal parts of the operators $\Phi_i \mathcal{A} \Phi_i^{-1}$ and $\Phi_i \mathcal{B}_j \Phi_i^{-1}$ are given by

$$\mathcal{A}_\sharp\big(t, x_i, \mathcal{O}_{\nu(x_i)}^T D\big), \qquad \mathcal{B}_{j\sharp}\big(t, x_i, \mathcal{O}_{\nu(x_i)}^T D\big),$$

respectively. Extending now the coefficients of the transformed operators $\Phi_i \mathcal{A} \Phi_i^{-1}$, $\Phi_i \mathcal{B}_j \Phi_i^{-1}$ and $\mathcal{C}_j^{\mathrm{g}_i}$ such that (SD), (SB) and (SC) remain valid, we obtain that $(\Phi_i u_i, \Phi_i \rho_i)$ solves a half-space problem with operators that are either of lower order or small perturbations of constant coefficient operators satisfying the conditions of Lemma 3.4. As for the full-space case, if T

and B_i are sufficiently small, then a continuous solution operator $\mathcal{L}_H^i$ exists for this half-space problem, which maps the relevant data space continuously into $\mathbb{E}_{\rho,\mu}(J \times \mathbb{R}^n_+) \times \mathbb{E}_{\rho,\mu}(J \times \mathbb{R}^n_+)$. For $i = N_F + 1, ..., N_H$ we thus obtain

$$(u_i, \rho_i) = \Phi_i^{-1} R_i \mathcal{L}_H^i \big(\Phi_i(\psi_i f + [\mathcal{A}, \psi_i]u), \Phi_i(\psi_i g + [\mathcal{B}, \psi_i]u + [\mathcal{C}, \psi_i]\rho), \Phi_i\psi_i u_0, \Phi_i \psi_i \rho_0\big).$$

Here R_i is the restriction to $\mathbb{R}^n_+ \cap \varphi_i(B_i)$ and we have set $\mathcal{B} = (\mathcal{B}_0, ..., \mathcal{B}_m)$ and $\mathcal{C} = (\mathcal{C}_0, ..., \mathcal{C}_m)$ for simplicity. Again, the commutator terms are of lower order.

(III) We next choose smooth functions ϕ_i, $i = 1, ..., N_H$, satisfying $\phi_i \equiv 1$ on $\operatorname{supp} \psi_i$ and $\operatorname{supp} \phi_i \subset B_i$. The above considerations show that if (u, ρ) solves (1.5) then it is a fixed point of the map $\mathcal{G}_{f,g,u_0,\rho_0}(u, \rho) := \sum_i \phi_i(u_i, \rho_i)$ on the complete metric space

$$Z_{u_0,\rho_0} := \big\{(u, \rho) \in \mathbb{E}_{u,\mu} \times \mathbb{E}_{\rho,\mu} \; : \; u|_{t=0} = u_0, \;\; \rho|_{t=0} = \rho_0\big\}.$$

We remark that Z_{u_0,ρ_0} is nonempty by Lemma 4.4 of [16] and Lemma 3.2.2 of [14]. Since the operators in the arguments of $\mathcal{L}_F^i$ and $\mathcal{L}_H^i$ are of lower order, one can show that for all data $(f^*, g^*, u_0^*, \rho_0^*) \in \mathcal{D}_{\mathrm{rel}}$ the map $\mathcal{G}_{f^*,g^*,u_0^*,\rho_0^*}$ has indeed a unique fixed point on $Z_{u_0^*,\rho_0^*}$, making T and B_i once more smaller if necessary. Another fixed point argument yields for given data $(f, g, u_0, \rho_0) \in \mathcal{D}_{\mathrm{rel}}$ the appropriate auxiliary data $(f^*, g^*, u_0^*, \rho_0^*) \in \mathcal{D}_{\mathrm{rel}}$ such that the fixed point of $\mathcal{G}_{f^*,g^*,u_0^*,\rho_0^*}$ is the solution of (1.5).

(IV) To finish the proof, note that the continuity of the resulting solution operator $\mathcal{L}_{\mathrm{rel}}$ for (1.5) is a consequence of the open mapping theorem. Moreover, the norm of $\mathcal{L}_{\mathrm{rel}}$ restricted to $\mathcal{D}^0_{\mathrm{rel}}$ is uniform in T due to an extension argument. It uses the extension operator from Lemma 2.5 of [16] for the ${}_0W^s_{p,\mu}$-spaces over J to the half-line, whose norm is independent of the length of J. ■

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