STRICHARTZ ESTIMATES FOR MAXWELL EQUATIONS IN MEDIA: THE STRUCTURED TWO DIMENSIONAL CASE

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Abstract. We prove Strichartz estimates for two dimensional Maxwell equations with diagonal Lipschitz permittivity of special structure. These estimates have no loss in regularity that occurs in general for C^1-coefficients. In particular, in the charge free case we recover Strichartz estimates of wave equations with C^2-coefficients in two dimensions up to endpoints.

1. Introduction and main result

The Maxwell equations are the foundation of electro-magnetic theory. Despite its importance, dispersive properties of the linear Maxwell system in media have only recently be studied systematically at least on full space, see [4], [6], [7], [8], [9], as well as [1], [3] for earlier contributions. For the two dimensional situation (1.1), in [8] we have obtained results comparable to the case of the scalar wave equation, cf. [12], [13]. It is known that for Lipschitz coefficients one has a loss of derivatrives in these Strichartz estimates compared to C^2-coefficients, in general, see [10] for the wave and [8] for the 2D Maxwell case. However, in the recent work [2] it was discovered that this loss does not appear for the wave equation under certain structural assumptions on the coefficients, see (1.6). In this note, we show an analogous result for the 2D Maxwell system for structured Lipschitz coefficients.

We investigate the two-dimensional Maxwell system

$$\begin{cases} \partial_t \mathcal{D} &= \nabla_\perp \mathcal{H} - \mathcal{J}, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^2, \\ \partial_t \mathcal{B} &= -\nabla \times \mathcal{E}, \end{cases} \tag{1.1}$$

for the electric $\mathcal{D}, \mathcal{E} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ and the magnetic fields $\mathcal{B}, \mathcal{H} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, and the current density $\mathcal{J} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$. Here we set $\nabla_\perp = (\partial_2, -\partial_1)^\top$ and $\nabla \times v = \partial_1 v_2 - \partial_2 v_1$. These equations are equipped with the instantaneous linear material laws

$$\mathcal{D} = \varepsilon(x)\mathcal{E}, \qquad \mathcal{B} = \mu(x)\mathcal{H},$$

for the permittivity $\varepsilon : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ and the permeability $\mu : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$. It is assumed that ε is symmetric and strictly positive definite. To focus on the main difficulties, we let $\mu = 1$ for simplicity, which is also a usual assumption

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in optics (after normalizing the vacuum permittivity ε_0 to 1), see [5]. However, our results easily generalize to strictly positive functions μ as in Theorem 1.1.

The system (1.1) arises as a restriction of the usual three dimensional Maxwell system (with $\mu = 1$) if the initial values $\mathcal{D}_0$ and $\mathcal{B}_0 = \mathcal{H}_0$ only depend on $(x, y) \in \mathbb{R}^2$ and if their components $\mathcal{E}_{03}$, $\mathcal{B}_{01}$, $\mathcal{B}_{02}$, as well as $\mathcal{J}_3$ vanish. Moreover, in the 3D permittivity tensor the components $\varepsilon_{3j} = \varepsilon_{j3}$ have to be zero for $j \in \{1, 2\}$. These restrictions on the fields are then conserved by the evolution equations.

In our recent paper [8] we have shown sharp Strichartz estimates for permittivities $\varepsilon \in C^s(\mathbb{R}^3, \mathbb{R}^{2\times 2})$ with $0 \le s \le 2$. To formulate them, we let $u = (\mathcal{D}, \mathcal{B})$ be the state, denote the (electric) charges by $\rho_e = \nabla \cdot \mathcal{D}$, and write

$$P = \begin{pmatrix} \partial_t & 0 & -\partial_2 \\ 0 & \partial_t & \partial_1 \\ \partial_1(\varepsilon^{21}\cdot) - \partial_2(\varepsilon^{11}\cdot) & \partial_1(\varepsilon^{22}\cdot) - \partial_2(\varepsilon^{12}\cdot) & \partial_t \end{pmatrix}. \tag{1.2}$$

where (ε^{ij}) is the inverse matrix of $\varepsilon = (\varepsilon_{ij})$. (Here we change notation compared to [8].) We call exponents (wave) admissible Strichatz pairs in spatial dimension d if

$$\frac{2}{p} + \frac{d-1}{q} \le \frac{d-1}{2}, \qquad 2 \le p, q \le \infty, \qquad \rho = \frac{d}{2} - \frac{d}{q} - \frac{1}{p}, \tag{1.3}$$

where $q < \infty$ if $d = 3$. If the first inequality is an equality, (p, q) are called sharp. (Note that $\rho \ge 0$ and $\rho = 0$ for the pair $p = \infty$, $q = 2$ corresponding to the energy estimate (1.11).) For admissible pairs with $d = 2$, C^s-coefficients and the loss parameter $\sigma = \frac{2-s}{2+s}$, we have established

$$\||D|^{-\rho-\frac{\sigma}{2}} u\|_{L^pL^q} \lesssim \|u\|_{L^2} + \|Pu\|_{\dot{H}^{-\sigma}} + \||D|^{-\frac{1}{2}-\frac{\sigma}{2}}\rho_e\|_{L^2} \tag{1.4}$$

in Theorem 1.2 of [8]. (If $q = \infty$, one has to replace L^∞ by a Besov space and analogously in (1.7) below.) Here we let $L^pL^q = L^p_{x_0}L^q_{x'} = L^p(\mathbb{R}, L^q(\mathbb{R}^2))$, $L^p = L^p_x = L^pL^p$, and $|D|^\alpha = \mathcal{F}^{-1}|\xi|^\alpha\mathcal{F}$ for the space-time Fourier transform. We also write $L^p_TL^q = L^p_TL^q_{x'} = L^p(0, T; L^q(\mathbb{R}^2))$ for $T > 0$. Throughout, $x = (x_0, x') = (t, x') \in \mathbb{R} \times \mathbb{R}^2$ are the space-time variables and $\xi = (\xi_0, \xi') = (\tau, \xi') \in \mathbb{R} \times \mathbb{R}^2$ the Fourier variables. Accordingly, spatial fractional derivatives are denoted by $|D'|^\alpha = \mathcal{F}^{-1}_{x'}|\xi'|^\alpha\mathcal{F}_{x'}$.

In (1.4) the regularity loss $\frac{\sigma}{2}$ compared to C^2-coefficients is sharp in general, as we have seen by a counter-example in [8] that is inspired by [10]. Except for the charge term, the estimate (1.4) corresponds to the results for the wave equation in Tataru's paper [12], which also have the loss $\frac{\sigma}{2}$ for C^s-coefficients (being sharp, in general, see [10]). The charge term in (1.4) compensates the degeneracy of the main symbol of P, which is a fundamental difference between the Maxwell and wave case, tied to the system character of (1.1).

However, recently the first author proved with Frey in [2] that Strichartz estimates *without loss* hold for wave equations with Lipschitz coefficients under certain structural assumptions. We state the results of [2] for the 2D case only. There coefficients $a_1, a_2 \in C^{0,1}(\mathbb{R})$ were considered under the ellipticity assumption

$$\exists\, \overline{\kappa}, \underline{\kappa} > 0: \quad \forall\, x \in \mathbb{R}: \quad \underline{\kappa} \le a_i(x) \le \overline{\kappa}. \tag{1.5}$$

For the wave operator

$$Q = \partial_{tt} - \big(\partial_1(a_1(x_1)\partial_1 + \partial_2(a_2(x_2)\partial_2)\big) \tag{1.6}$$

and sharp admissible pairs (p, q), the Strichartz estimates without loss

$$\| |D'|^{1-\rho} v \|_{L_T^p L^q} \lesssim_T \| \nabla u \|_{L_T^\infty L^2} + \| Q u \|_{L_T^1 L^2} \tag{1.7}$$

were proven in Corollary 4.5 of [2]. Hence, for the wave operator (1.6) with $C^{0,1}$-coefficients we have the same Strichartz estimate (1.7) as for the wave equation with general (elliptic) C^2-coefficients, see e.g. [12].

In this note we revisit our approach from [8] and show a loss-less Strichartz estimate for solutions to (1.1) after frequency localization under the structural conditions

$$\varepsilon(x) = \operatorname{diag}(\varepsilon_1(x_2), \varepsilon_2(x_1)), \quad \text{where } \varepsilon_i \in C^{0,1}(\mathbb{R}) \quad \text{satisfy (1.5)}, \tag{1.8}$$

on the permittivities.

Theorem 1.1. *Assume that (p, q, ρ) satisfy (1.3) for $d = 2$ and ε fulfills (1.8). Let P be given by (1.2), $u = (\mathcal{D}, \mathcal{B})$, $\rho_e = \nabla \cdot \mathcal{D}$, and $T \geq 1$. We then obtain the Strichartz estimates*

$$\begin{aligned}\sup_{\lambda \in 2^{\mathbb{N}_0} \cup \{0\}} (1+\lambda)^{-\rho} \| S'_\lambda u \|_{L_T^p L_{x'}^q} \lesssim_T \| u(0) \|_{L_{x'}^2} + \| P u \|_{L_T^1 L_{x'}^2} + \| |D'|^{-1/2} \rho_e(0) \|_{L_{x'}^2} \\ + \| |D'|^{-1/2} \partial_t \rho_e \|_{L_T^1 L_{x'}^2}. \end{aligned} \tag{1.9}$$

Let also $\varepsilon \in B^1_{\infty,2}(\mathbb{R}^2)$. Then we have

$$\begin{aligned}\| |D'|^{-\rho} u \|_{L_T^p L_{x'}^q} \lesssim_T \| u(0) \|_{L_{x'}^2} + \| P u \|_{L_T^1 L_{x'}^2} + \| |D'|^{-1/2} \rho_e(0) \|_{L_{x'}^2} \\ + \| |D'|^{-1/2} \partial_t \rho_e \|_{L_T^1 L_{x'}^2} \end{aligned} \tag{1.10}$$

for $q < \infty$. If $q = \infty$, one has to replace the left-hand side by $\| u \|_{L_T^p \dot{B}^{-\rho}_{\infty,2}}$.

The theorem is proved in the next section. Here we first discuss the result and its proof a bit.

Above we use a spatial Littlewood–Paley decomposition $(S'_\lambda)_{\lambda \in 2^{\mathbb{N}_0}}$, see (2.1). For (1.10), the slightly improved first-order regularity of ε is needed to sum the Littlewood–Paley pieces in a commutator argument, see (2.10). We note that (1.8) excludes the counter-examples to (1.10) from Section 7 of [8].

We next explain the differences between the right-hand sides of (1.4) with $\sigma = 0$ compared to (1.9) and (1.10). Differentiating the energy $\frac{1}{2} \int (\varepsilon \mathcal{E}(t) \cdot \mathcal{E}(t) + |\mathcal{H}(t)|^2) \, dx'$ in time, one obtains

$$\| u(t) \|_{L_{x'}^2} \lesssim_{\underline{\kappa}, \overline{\kappa}} \| u(0) \|_{L_{x'}^2} + \| P u \|_{L^1(0,t;L^2)}. \tag{1.11}$$

(For time-varying coefficients one would need here $\partial_t \varepsilon \in L_T^1 L^\infty$.) Hence it is enough to show (1.9) and (1.10) with $\| u \|_{L^2}$ instead of $\| u(0) \|_{L_{x'}^2}$ on the right-hand side. In step 1) of the proof we will also see how one can pass from $\| P u \|_{L^2}$ to $\| P u \|_{L^1(0,t;L^2)}$ by means of Duhamel's formula, though with a T-depending constant. This argument also modifies the charge term.

We state the above result with spatial regularity only. But, as seen in the proof, the low frequency part of u and the frequency ranges $|\tau| \gg |\xi'|$ can be handled directly (without involving ρ_e) so that one could replace $|D'|$ by $|D|$. Observe that Sobolev's embedding already gives

$$\||D|^{-\rho}u\|_{L^pL^q} \lesssim \||D|^{\frac{1}{2}}u\|_{L^2},$$

so that we have to gain half a derivative to derive (1.10). In particular, if we only know $\||D'|^{-1/2}\rho_e\|_{L^2} \sim \||D'|^{\frac{1}{2}}\mathcal{D}\|_{L^2}$ for the charge, then (1.10) would not improve on Sobolev's embedding. On the other hand, (1.1) implies

$$\rho_e(t) = \nabla \cdot \mathcal{D}(0) - \int_0^t \nabla \cdot \mathcal{J}(s)\,\mathrm{d}s \tag{1.12}$$

so that the charge is given by the data. Moreover, we have $\rho_e(0) = \nabla \cdot \mathcal{D}(0)$ and $\partial_t \rho_e = -\nabla \cdot \mathcal{J}$ in (1.9) and (1.10).

In three spatial dimensions, dispersive estimates for the Maxwell system depend very much on the behavior of the eigenvalues of $\varepsilon(x)$ and $\mu(x)$ since these heavily influence the characteristic surface S of the problem (the null set of the principal symbol of P), see our recent contributions [4], [6], [9], and the references therein. Only in the isotropic case of scalar ε and μ, Strichartz estimates with admissible exponents (1.3) for $d = 3$ as for the wave equation are known so far, see [6]. For smooth coefficents and vanishing charges this was already shown in [1], which is the only other reference on Strichartz estimates for the Maxwell system with non-constant coefficients we are aware of.

Already for constant diagonal coefficents $\varepsilon = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and $\mu = \mathrm{diag}(\mu_1, \mu_2, \mu_3)$ in the fully anisotropic case $\varepsilon_i/\mu_i \neq \varepsilon_j/\mu_j$ for $i \neq j$, the admissible range of exponents for the Strichartz estimate is reduced to $\frac{2}{p} + \frac{1}{q} \leq \frac{1}{2}$ as in 2D instead of $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$ as in 3D for the wave equations. This is caused by a loss of curvature for S in this case, compared to $\partial_{tt} w = \Delta w$ where S is the light cone $\{\tau = \pm|\xi'|\}$. Moreover, the slices S_τ of S for fixed $\tau \neq 0$ have four conical singularities in the above fully anisotropic case. See [3, 4, 6, 9] for a detailed discussion. So it is worthwhile to study the influence of structured coefficients to dispersive properties of the Maxwell system first in the two dimensional case.

In our proof we follow the general strategy from [8]. However, there we used C^2-coefficients in most of the relevant arguments, so that we have to argue differently at various points. As in [12] we first reduce to functions u which are localized in the space-time unit cube $[0,1]^3$ and in Fourier space near a large dyadic frequency $\lambda \in 2^{\mathbb{N}_0}$. The frequency localization is more demanding in the present situation since the relevant commutator $[P, S'_\lambda]u$ is uniformly bounded in L^2, but not square summable, for Lipschitz coefficients. (There is no problem if they belong to in C^s if $s > 1$.) In (2.10) we manage to sum in λ using the assumption $\varepsilon \in B^1_{\infty,2}$, which is only needed here. Then the coefficients are truncated to frequencies less or equal λ. We next diagonalize the principal symbol p as in [8]. Using also the FBI transform and results from [11], we can treat the frequency range $|\tau| \gg |\xi'|$ by an elliptic estimate and the degenerate range $|\xi'| \gg |\tau|$ by means of the charges.

The remaining part $|\tau| \sim |\xi'|$ near the light cone is handled by means of the wave estimate (1.7) from [2], after passsing to a second-order formulation of the Maxwell system. Only here we use the special structure of ε from (1.8). For

C^2-coefficients in [8] we had employed results from Tataru's paper [12] instead. We note that C^s-coefficients were treated in [8] (and in [12]) employing further frequency cut-offs of the coefficients, which led to the loss of regularity in (1.4).

2. Proof of Theorem 1.1

As noted above we use some arguments from [8]. In the sequel we focus on the differences to [8]. We proceed in five steps using the following dyadic frequency decomposition. Let $\chi \in C_c^\infty(\mathbb{R};\mathbb{R}_{\geq 0})$ radially decrease with $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. We set

$$S'_\lambda = \mathcal{F}_{x'}^{-1}(\chi(|\xi'|/\lambda) - \chi(|\xi'|/2\lambda))\mathcal{F}_{x'}$$

for $\lambda \in 2^{\mathbb{N}_0}$. Moreover, we write

$$S'_0 = I - \sum_{\lambda \in 2^{\mathbb{N}_0}} S'_\lambda, \qquad S'_{\geq \lambda} = \sum_{\mu \geq \lambda} S'_\mu, \qquad \tilde{S}'_\lambda = \sum_{\mu = \lambda/8}^{8\lambda} S'_\mu, \tag{2.1}$$

with sums over dyadic numbers. We write S^τ_λ etc. for the corresponding operators in 1D (giving a temporal decomposition), and S_λ for the full 3D version in ξ. The Besov space $B^s_{p,q}(\mathbb{R}^d)$ for $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $1 \leq q < \infty$ contains those $f \in \mathcal{S}'(\mathbb{R}^d)$ with finite norm

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d)} = \Big(\sum_{\lambda \in 2^{\mathbb{N}_0} \cup \{0\}} (1+\lambda)^{qs} \|S'_\lambda f\|^q_{L^p(\mathbb{R}^d)} \Big)^{1/q},$$

$B^s_{p,\infty}(\mathbb{R}^d)$ is defined by an obvious modification. Note that it is enough to prove Theorem 1.1 for sharp pairs with $\frac{2}{p} + \frac{1}{q} = \frac{1}{2}$ because of Sobolev's embedding.

1) Reduction to L^2 on the left. To establish (1.9), it suffices to show

$$\sup_{\lambda \in 2^{\mathbb{N}_0} \cup \{0\}} (1+\lambda)^{-\rho} \|S'_\lambda u\|_{L^p_T L^q_{x'}} \lesssim_T \|u(0)\|_{L^2_{x'}} + \|u\|_{L^2_x} + \|Pu\|_{L^2_x} + \||D'|^{-1/2} \rho_e\|_{L^2_x}. \tag{2.2}$$

Similarly, (1.10) follows from

$$\||D'|^{-\rho} u\|_{L^p_T L^q} \lesssim_T \|u(0)\|_{L^2_{x'}} + \|u\|_{L^2_x} + \|Pu\|_{L^2_x} + \||D'|^{-1/2} \rho_e\|_{L^2_x}. \tag{2.3}$$

We check this only for (2.2), as (2.3) is treated in the same way.

Once (2.2) is proved, we can derive (1.9) by localization in time and the energy estimate (1.11). To this end, we extend u to $(-T, 2T)$ by reflection and cut-off such that $\mathrm{supp}(\tilde{u}) \subseteq (-T, 2T)$. An application of (2.2) to $\tilde{u}$ yields

$$\begin{aligned} \sup_{\lambda \in 2^{\mathbb{N}_0} \cup \{0\}} (1+\lambda)^{-\rho} \|S'_\lambda u\|_{L^p_T L^q_{x'}} &= \sup_{\lambda \in 2^{\mathbb{N}_0} \cup \{0\}} (1+\lambda)^{-\rho} \|S'_\lambda \tilde{u}\|_{L^p_T L^q_{x'}} \\ &\lesssim \|u(0)\|_{L^2_{x'}} + \|\tilde{u}\|_{L^2} + \|P\tilde{u}\|_{L^2} + \||D'|^{-\frac{1}{2}} \tilde{\rho}_e\|_{L^2} \\ &\lesssim \|u(0)\|_{L^2_{x'}} + \|u\|_{L^2_T L^2} + \|Pu\|_{L^2_T L^2} + \||D'|^{-\frac{1}{2}} \rho_e\|_{L^2} \\ &\lesssim_T \|u(0)\|_{L^2_{x'}} + \|Pu\|_{L^1_T L^2} + \|Pu\|_{L^2_T L^2} + \||D'|^{-\frac{1}{2}} \rho_e\|_{L^2}. \end{aligned} \tag{2.4}$$

At this point, we use Duhamel's formula

$$u(t) = U(t)u(0) + \int_0^t U(t-s)Pu(s)\,\mathrm{d}s$$

for the C_0-group $U(\cdot)$ solving (1.1), and the estimate (2.4) for the homogeneous problem with initial values $u(0)$, respectively $Pu(s)$. Taking into account $\rho_e(0) = \nabla \cdot \mathcal{D}(0)$ and $\partial_t \rho_e = -\nabla \cdot \mathcal{J}$ from (1.12), we deduce (1.9).

2) Localization and frequency truncation. We carry out a dyadic frequency localization and frequency-truncate the coefficients accordingly.

In the first step, we observe that Bernstein's inequality, (1.11), and Hölder's inequality yield

$$\| |D'|^{-\rho} S_0' u \|_{L_T^p L^q} \lesssim \| |D'|^{\frac{1}{p}} S_0' u \|_{L_T^p L^2} \lesssim_T \| u(0) \|_{L^2_{x'}} + \| Pu \|_{L^2}.$$

In particular, we can replace $|D'|^{-\rho}$ by $\langle D' \rangle^{-\rho} = \mathcal{F}_{x'}^{-1} \langle \xi' \rangle^{\alpha} \mathcal{F}_{x'}$ with $\langle \xi' \rangle^2 = 1 + |\xi'|^2$. As in Section 3.2 of [8] we restrict to u that are supported in $[0,1]^3$ by means of a partition of unity.

We truncate the frequencies of ε at $\lambda/8$ and denote the resulting coefficients by $\varepsilon_{\lesssim \lambda}$, and the corresponding operator by P_λ, cf. (1.2). Moreover, $P_{\sim \lambda}$ is the operator with coefficients $\varepsilon_{\sim \lambda}$ that are frequency-localized near λ. Since $\| \varepsilon_{\gtrsim \lambda} \|_{L^\infty} \lesssim \lambda^{-1} \| \varepsilon \|_{C^{0,1}}$, we can fix a frequency $\lambda_0 \geq 1$ such that the lower bound (1.8) is true for $\varepsilon_{\lesssim \lambda}$ if $\lambda \geq \lambda_0$.

We next deduce (2.2) from the frequency localized bound

$$\lambda^{-\rho} \| S_\lambda' u \|_{L_T^p L^q} \lesssim_T \| S_\lambda' u(0) \|_{L^2_{x'}} + \| S_\lambda' u \|_{L^2} + \| P_\lambda S_\lambda' u \|_{L^2} + \lambda^{-\frac{1}{2}} \| S_\lambda' \rho_e \|_{L^2}. \quad (2.5)$$

for $\lambda \gg 1$. To pass from (2.5) to (2.2), one has to bound $\| P_\lambda S_\lambda' u \|_{L^2}$ by $\| S_\lambda' P u \|_{L^2}$ plus terms like $\| \tilde{S}_\lambda' u \|_{L^2}$. We use fixed-time commutator arguments to this end. We note that

$$\begin{aligned} \| P_\lambda S_\lambda' u \|_{L^2} &= \| \tilde{S}_\lambda' P_\lambda S_\lambda' u \|_{L^2} \leq \| \tilde{S}_\lambda' P S_\lambda' u \|_{L^2} + \| \tilde{S}_\lambda' P_{\sim \lambda} S_\lambda' u \|_{L^2} \\ &\leq \| S_\lambda' P u \|_{L^2} + \| \tilde{S}_\lambda' [P, S_\lambda'] u \|_{L^2} + \| \tilde{S}_\lambda' P_{\sim \lambda} S_\lambda' u \|_{L^2}. \end{aligned} \quad (2.6)$$

Write $[P, S_\lambda'] = [P, S_\lambda'] \tilde{S}_\lambda' + S_\lambda' P (1 - \tilde{S}_\lambda')$. In the second term we can replace the coefficients ε of P with $\varepsilon_{\gtrsim \lambda} = S_{\gtrsim \lambda}' \varepsilon$ as the low frequencies of ε do not appear in the frequency interaction:

$$S_\lambda' P (1 - \tilde{S}_\lambda') u = S_\lambda' P_{\gtrsim \lambda} (1 - \tilde{S}_\lambda') u. \quad (2.7)$$

Since P is in divergence form, standard properties of Lipschitz functions yield

$$\begin{aligned} \| \tilde{S}_\lambda' [P, S_\lambda'] u \|_{L^2} &\lesssim \lambda \| \tilde{S}_\lambda' [\varepsilon, S_\lambda'] \tilde{S}_\lambda' u \|_{L^2} + \lambda \| \varepsilon_{\gtrsim \lambda} \|_{L^\infty} \| (1 - \tilde{S}_\lambda) u \|_{L^2} \quad (2.8) \\ &\lesssim \| \tilde{S}_\lambda u \|_{L^2} + \| \varepsilon \|_{C^{0,1}} \| u \|_{L^2}, \\ \| \tilde{S}_\lambda' P_{\sim \lambda} S_\lambda' u \|_{L^2} &\lesssim \lambda \| \tilde{S}_\lambda(\varepsilon) S_\lambda' u \|_{L^2} \lesssim \| \varepsilon \|_{C^{0,1}} \| S_\lambda' u \|_{L^2}. \quad (2.9) \end{aligned}$$

Hence, (2.2) follows from (2.5).

To reduce (2.3) to (2.5), we use the square function estimate in $L^q(\mathbb{R}^2)$ for $2 \leq q < \infty$ and Minkowski's inequality (note that $p, q \geq 2$), obtaining

$$\| |D'|^{-\rho} S_{\gtrsim 1}' u \|_{L_T^p L^q} \lesssim \Big(\sum_{\lambda \gtrsim 1} \lambda^{-2\rho} \| S_\lambda' u \|_{L_T^p L^q}^2 \Big)^{\frac{1}{2}}.$$

For $q = \infty$, we employ the definition of Besov spaces instead of the Littlewood–Paley theorem. Invoking (2.5), we need to show that

$$\Big(\sum_{\lambda \gtrsim 1} \| P_\lambda S'_\lambda u \|_{L^2}^2 \Big)^{\frac{1}{2}} \lesssim \| u \|_{L^2} + \| P u \|_{L^2}.$$

In (2.6) the first and third term sum up due to (2.9), already for Lipschitz coefficients. It remains to verify

$$\sum_{\lambda \gtrsim 1} \| \tilde{S}'_\lambda [P, S'_\lambda] u \|_{L^2}^2 \lesssim \| u \|_{L^2}^2.$$

The second term in (2.8) is not square summable. To use the extra Besov regularity of ε, we go back to (2.7) and write

$$\begin{aligned} \| S'_\lambda P (1 - \tilde{S}'_\lambda) u \|_{L^2_{x'}} &\lesssim \lambda \| \varepsilon_{\sim \lambda} S'_{\ll \lambda} \|_{L^2_{x'}} + \lambda \| \tilde{S}'_\lambda \varepsilon_{\gtrsim \lambda} S'_{\gtrsim \lambda} u \|_{L^2_{x'}} \\ &\lesssim \lambda \| \varepsilon_{\sim \lambda} \|_{L^\infty_{x'}} \| u \|_{L^2_{x'}} + \lambda \sum_{\mu \gtrsim \lambda} \| \varepsilon_{\sim \mu} \|_{L^\infty_{x'}} \| S'_\mu u \|_{L^2_{x'}}. \end{aligned}$$

Square summing the first term in the last line yields

$$\sum_{\lambda \gtrsim 1} \lambda^2 \| \varepsilon_{\sim \lambda} \|_{L^\infty_{x'}}^2 \| u \|_{L^2_{x'}}^2 \lesssim \| \varepsilon \|_{B^1_{\infty,2}}^2 \| u \|_{L^2_{x'}}^2.$$

By means of Hölder's inequality and Fubini's theorem, we estimate the square sum of second term by

$$\begin{aligned} \sum_{\lambda \gtrsim 1} \lambda^2 \Big(\sum_{\mu \gtrsim \lambda} \| \varepsilon_{\sim \mu} \|_{L^\infty_{x'}} \| S'_\mu u \|_{L^2_{x'}} \Big)^2 &\lesssim \sum_{\lambda \geq 1} \lambda^2 \sum_{\mu \gtrsim \lambda} \| \varepsilon_{\sim \mu} \|_{L^\infty_{x'}}^2 \sum_{\mu \gtrsim \lambda} \| S'_\mu u \|_{L^2_{x'}}^2 \\ &\lesssim \sum_{\mu \geq 1} \| \varepsilon_{\sim \mu} \|_{L^\infty_{x'}}^2 \sum_{\lambda \lesssim \mu} \lambda^2 \; \| u \|_{L^2_{x'}}^2 \\ &\lesssim \sum_{\mu \geq 1} \mu^2 \| \varepsilon_{\sim \mu} \|_{L^\infty_{x'}}^2 \; \| u \|_{L^2_{x'}}^2 = \| \varepsilon \|_{B^1_{\infty,2}}^2 \| u \|_{L^2_{x'}}^2. \end{aligned} \tag{2.10}$$

As a result, (2.5) also implies (2.3) if $\varepsilon \in B^1_{\infty,2}$.

3) Diagonalization. We diagonalize the main symbol of P as in Section 3.1 [8], obtaining

$$\begin{aligned} p(x,\xi) &= \mathrm{i} \begin{pmatrix} \xi_0 & 0 & -\xi_2 \\ 0 & \xi_0 & \xi_1 \\ -\xi_2 \varepsilon^{11} & \xi_1 \varepsilon^{22} & \xi_0 \end{pmatrix} = m(x,\xi) d(x,\xi) m(x,\xi)^{-1} \\ &= \begin{pmatrix} -\xi_1^* \varepsilon^{22}(x) & \xi_2^* & -\xi_2^* \\ -\xi_2^* \varepsilon^{11}(x) & -\xi_1^* & \xi_1^* \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{i}\xi_0 & 0 & 0 \\ 0 & \mathrm{i}(\xi_0 - |\xi'|_{\tilde{\varepsilon}}) & 0 \\ 0 & 0 & \mathrm{i}(\xi_0 + |\xi'|_{\tilde{\varepsilon}}) \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} -\xi_1^* & -\xi_2^* & 0 \\ \frac{1}{2} \xi_2^* \varepsilon^{11}(x) & -\frac{1}{2} \xi_1^* \varepsilon^{22}(x) & \frac{1}{2} \\ -\frac{1}{2} \xi_2^* \varepsilon^{11}(x) & \frac{1}{2} \xi_1^* \varepsilon^{22}(x) & \frac{1}{2} \end{pmatrix} \end{aligned}$$

with $|\xi'|^2_{\tilde{\varepsilon}} = \langle \xi', \tilde{\varepsilon}(x)\xi'\rangle$, $\tilde{\varepsilon}(x) = \operatorname{adj}(\varepsilon^{-1}(x)) = \operatorname{diag}(\varepsilon^{22}(x), \varepsilon^{11}(x))$, and $\xi_i^* = \xi_i/|\xi'|_{\tilde{\varepsilon}}$ for $i = 1, 2$. See also [7]. Here we use that ε is diagonal in our case, though this is not needed in this and the next step.

Strictly speaking, the symbols in the diagonalization depend on λ, but we suppress the dependence in the following to lighten the notation.

4) Estimate off the light cone. We use the diagonalization to localize also the temporal frequencies μ of u to the spatial frequency λ. in the next step. To this end, in this step we treat the case that μ differs much from λ.

a) Let $\mu \gg \lambda$. Here the operator P_λ is elliptic and gains one derivative. More precisely, Bernstein's inequality yields

$$\lambda^{-\rho}\|S^\tau_\mu S'_\lambda u\|_{L^pL^q} \lesssim \lambda^{-\rho}\lambda^{1-\frac{2}{q}}\mu^{\frac{1}{2}-\frac{1}{p}}\|S^\tau_\mu S'_\lambda u\|_{L^2} \le \mu^{\frac{1}{2}}\|S^\tau_\mu S'_\lambda u\|_{L^2}$$

Now we use the FBI transform

$$T_\mu f(z) = C\mu^{\frac{9}{4}} \int_{\mathbb{R}^3} \mathrm{e}^{-\frac{\mu}{2}(z-y)^2} f(y)\,\mathrm{d}y, \qquad z = x - \mathrm{i}\xi \in T^*\mathbb{R}^3 \simeq \mathbb{R}^6,$$

see [11], and set $v_\mu = T_\mu S^\tau_\mu S'_\lambda u$. We recall that $T_\mu : L^2(\mathbb{R}^3) \to L^2_\Phi(\mathbb{R}^6)$ is an isometry, where the range space has the weight $\Phi(z) = \mathrm{e}^{-\mu\xi^2}$. Using (15) in [12], one can check that v_μ is essentially supported in $B(0,2)\times\{1 \sim |\xi_0| \gg |\xi'|\} =: U$ and $\|v_\mu\|_{L^2(U^c)} \lesssim_N \mu^{-N}\|S^\tau_\mu S'_\lambda u\|_{L^2}$. So it remains to estimate $\|v_\mu\|_{L^2(U)}$.

Since p is strictly positive on U, Theorem 1 of [11] implies

$$\begin{aligned}\|v_\mu\|_{L^2(U)} &\lesssim \|p(x,\xi)v_\mu\|_{L^2(U)} \lesssim \|p(x,\xi)v_\mu\|_{L^2_\Phi} \\ &\lesssim \mu^{-1}\|P_\lambda(x,D)S^\tau_\mu S'_\lambda u\|_{L^2} + \mu^{-\frac{1}{2}}\|S^\tau_\mu S'_\lambda u\|_{L^2}. \qquad (2.11)\end{aligned}$$

This suffices for summation over $\mu \gg \lambda$, and we have thus shown

$$\lambda^{-\rho}\|S^\tau_{\gg\lambda} S'_\lambda u\|_{L^p_T L^q} \lesssim_T \|S'_\lambda u\|_{L^2} + \|P_\lambda S'_\lambda u\|_{L^2}. \qquad (2.12)$$

b) Let $\mu \ll \lambda$. Here we see that the non-degenerate components of $d(x,\xi)$ are elliptic and the degenerate first component is estimated by the charges. As above, Bernstein's inequality yields

$$\lambda^{-\rho}\|S^\tau_\mu S'_\lambda u\|_{L^pL^q} \lesssim \lambda^{\frac{1}{p}}\mu^{\frac{1}{2}-\frac{1}{p}}\|S^\tau_\mu S'_\lambda u\|_{L^2}.$$

We let $T_\lambda S^\tau_\mu S'_\lambda u = v_\lambda$, which is now essentially supported in $\{1 \sim |\xi'| \gg |\xi_0|\}$ and obtain

$$\|v_\lambda\|_{L^2_\Phi} = \|m(x,\xi)m^{-1}(x,\xi)v_\lambda\|_{L^2_\Phi} \lesssim \|m^{-1}(x,\xi)v_\lambda\|_{L^2_\Phi}.$$

Using Theorem 1 of [11], the component $[m^{-1}(x,\xi)v_\lambda]_1$ is estimated by

$$\|[m(x,\xi)^{-1}v_\lambda]_1\|_{L^2_\Phi} \lesssim \lambda^{-1}\|\nabla\cdot S'_\lambda\mathcal{D}\|_{L^2_x} + \lambda^{-\frac{1}{2}}\|S'_\lambda\mathcal{D}\|_{L^2_x}.$$

By the essential support property, the components d_2 and d_3 are strictly prositive. For $i = 2, 3$ we thus obtain

$$\begin{aligned}\|[m(x,\xi)^{-1}v_\lambda]_i\|_{L^2_\Phi} &\lesssim \|[d(x,\xi)m(x,\xi)^{-1}v_\lambda]_i\|_{L^2_\Phi} \lesssim \|m(x,\xi)d(x,\xi)m(x,\xi)^{-1}v_\lambda\|_{L^2_\Phi} \\ &= \|p(x,\xi)v_\lambda\|_{L^2_\Phi}.\end{aligned}$$

This fact allows to gain derivatives as in (2.11) and leads to

$$\lambda^{-\rho}\|S_\mu^\tau S'_\lambda u\|_{L^pL^q} \lesssim \mu^{\frac{1}{2}-\frac{1}{p}}\lambda^{\frac{1}{p}-\frac{1}{2}}\||D'|^{-\frac{1}{2}}\nabla\cdot S'_\lambda\mathcal{D}\|_{L^2_x} + \mu^{\frac{1}{2}-\frac{1}{p}}\lambda^{\frac{1}{p}-\frac{1}{2}}\|S'_\lambda u\|_{L^2} + \mu^{\frac{1}{2}-\frac{1}{p}}\lambda^{\frac{1}{p}-1}\|P_\lambda S'_\lambda u\|_{L^2}.$$

Summing over $\mu \leq \lambda$, we derive

$$\lambda^{-\rho}\|S^\tau_{\ll\lambda}S'_\lambda u\|_{L^p_TL^q} \lesssim_T \|S'_\lambda u\|_{L^2} + \|P_\lambda S'_\lambda u\|_{L^2} + \||D'|^{-\frac{1}{2}}S'_\lambda\rho_e\|_{L^2_x}. \tag{2.13}$$

5) Estimate near the light cone. In view of (2.12) and (2.13), for (2.5) it remains to estimate the space-time region $\{|\tau|\sim|\xi'|\sim\lambda\}$. Set $(\mathcal{D}_\lambda,\mathcal{H}_\lambda) = S^\tau_{\sim\lambda}S'_\lambda u$ and $\mathcal{J}_\lambda = P_\lambda S^\tau_{\sim\lambda}S'_\lambda u = S^\tau_{\sim\lambda}P_\lambda S'_\lambda u$. To estimate $(\mathcal{D}_\lambda,\mathcal{H}_\lambda)$, we pass to the second order equation starting from

$$\begin{cases} \partial_t\mathcal{D}_{1\lambda} = \partial_2\mathcal{H}_\lambda + \mathcal{J}_{1\lambda}, \\ \partial_t\mathcal{D}_{2\lambda} = -\partial_1\mathcal{H}_\lambda + \mathcal{J}_{2\lambda}, \\ \partial_t\mathcal{H}_\lambda = \partial_2(\varepsilon_{1\lambda}^{-1}\mathcal{D}_{1\lambda}) - \partial_1(\varepsilon_{2\lambda}^{-1}\mathcal{D}_{2\lambda}) + \mathcal{J}_{3\lambda}. \end{cases} \tag{2.14}$$

Taking another time derivative in the third equation, we find

$$\partial_t^2\mathcal{H}_\lambda = \partial_2(\varepsilon_{1\lambda}^{-1}\partial_2\mathcal{H}_\lambda) + \partial_1(\varepsilon_{2\lambda}^{-1}\partial_1\mathcal{H}_\lambda) + \partial_2(\varepsilon_{1\lambda}^{-1}\mathcal{J}_{1\lambda}) - \partial_1(\varepsilon_{2\lambda}^{-1}\mathcal{J}_{2\lambda}) + \partial_t\mathcal{J}_{3\lambda}.$$

Setting $f = \partial_2(\varepsilon_{1\lambda}^{-1}\mathcal{J}_{1\lambda}) - \partial_1(\varepsilon_{2\lambda}^{-1}\mathcal{J}_{2\lambda}) + \partial_t\mathcal{J}_{3\lambda}$, the standard energy estimate and (2.14) imply

$$\|\nabla\mathcal{H}_\lambda(t)\|_{L^2_{x'}} \lesssim \|\nabla_x\mathcal{H}_\lambda(0)\|_{L^2_{x'}} + \|f\|_{L^1_TL^2} \lesssim_T \lambda\|S'_\lambda u(0)\|_{L^2_{x'}} + \lambda\|\mathcal{J}_\lambda\|_{L^2}.$$

We now use (1.7) taken from [2] and obtain

$$\lambda^{1-\rho}\|\mathcal{H}_\lambda\|_{L^p_TL^q} \lesssim_T \|\nabla\mathcal{H}_\lambda\|_{L^\infty_TL^2} + \|f\|_{L^1_TL^2} \lesssim_T \lambda\|S'_\lambda u(0)\|_{L^2_{x'}} + \lambda\|\mathcal{J}_\lambda\|_{L^2}. \tag{2.15}$$

Furthermore, the first and second equation in (2.14) give

$$\lambda^{-\rho}\|\mathcal{D}_{i\lambda}\|_{L^pL^q} \lesssim \lambda^{-\rho}\lambda^{-1}\|\partial_t\mathcal{D}_{i\lambda}\|_{L^pL^q} \lesssim \lambda^{-\rho}(\lambda^{-1}\|\partial_j\mathcal{H}_\lambda\|_{L^pL^q} + \lambda^{-1}\|\mathcal{J}_{i\lambda}\|_{L^pL^q}).$$

with $j \neq i$ in $\{1,2\}$. The first term has been bounded by $\|S'_\lambda u(0)\|_{L^2_{x'}} + \|\mathcal{J}_\lambda\|_{L^2}$ in (2.15). Due to Sobolev's embedding, the second term can be estimated by

$$\lambda^{-\rho}\lambda^{-1}\|\mathcal{J}_\lambda\|_{L^pL^q} \lesssim \lambda^{-\frac{1}{2}}\|\mathcal{J}_\lambda\|_{L^2}.$$

Hence, (2.5) is shown and the proof of Theorem 1.1 is complete. □

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