# EXPONENTIAL DICHOTOMY AND MILD SOLUTIONS OF NONAUTONOMOUS EQUATIONS IN BANACH SPACES

Y. LATUSHKIN, T. RANDOLPH, AND R. SCHNAUBELT

ABSTRACT. We prove that the exponential dichotomy of a strongly continuous evolution family on a Banach space is equivalent to the existence and uniqueness of continuous bounded mild solutions of the corresponding inhomogeneous equation. This result addresses nonautonomous abstract Cauchy problems with unbounded coefficients. The technique used involves evolution semigroups. Some applications are given to evolution families on scales of Banach spaces arising in center manifold theory.

## 0. INTRODUCTION

In this paper we prove that a strongly continuous exponentially bounded evolution family $\{U(t,s)\}_{t\geq s}$ on a Banach space $X$ has exponential dichotomy if (and only if) for each bounded continuous $X$-valued function $f \in C_b(\mathbb{R}, X)$ there exists a unique solution $u \in C_b(\mathbb{R}, X)$ to the following inhomogeneous equation:

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)\,d\tau, \quad t \geq s. \tag{0.1}$$

Our work continues the line of research that characterizes dichotomy in terms of "Perron-type" theorems.

As an example, consider a nonautonomous abstract Cauchy problem,

$$x'(t) = A(t)x(t), \quad x(s) = x_s, \quad x_s \in D(A(s)), \quad t \geq s, \quad t, s \in \mathbb{R}, \tag{0.2}$$

on the Banach space $X$. Assume, for a moment, that (0.2) is well-posed in the sense that there exists an evolution (solving) family of operators $\{U(t,s)\}_{t\geq s}$ which gives a differentiable solution $x(\cdot)$. This means that $x(\cdot) : t \mapsto U(t,s)x(s)$, $t \geq s$, is differentiable for any given initial conditions $x(s) = x_s \in D(A(s))$, $x(t) \in D(A(t))$,

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and (0.2) holds. Now let $f$ be a locally integrable $X$-valued function on $\mathbb{R}$ and consider the inhomogeneous equation:

$$x'(t) = A(t)x(t) + f(t), \qquad t \in \mathbb{R}. \tag{0.3}$$

A function $u(\cdot)$ is called a *mild* solution of (0.3) if $u(\cdot)$ satisfies (0.1). Thus, our result shows, in particular, that the exponential dichotomic behavior of solutions to a nonautonomous abstract Cauchy problem (0.2) is equivalent to the existence and uniqueness of a bounded continuous *mild* solution to the *inhomogeneous* equation (0.3) for any $f \in C_b(\mathbb{R}, X)$. Let $R$ denote the operator that recovers the solution $u = Rf$ to (0.1) for a given $f$. We note that if $f$ is sufficiently smooth, then the function $u = Rf$ defined by (0.1) will be a differentiable solution to (0.3) (cf. Pazy (1983), Sections 5.5 and 5.7).

The characterization of dichotomy for (0.2) in terms of the solutions of the inhomogeneous equation (0.3) has a fairly long history that goes back to Perron (1930). Classical theorems of this type concerning differential equations with bounded operators $A(t)$ can be found in Daleckij and Krein (1974) (see also historical comments there) and Massera and Schaeffer (1966). For unbounded $A(t)$ a result of this type concerning *classical* solutions of (0.3), obtained by a completely different method, and applications of this result are contained in the book by Levitan and Zhikov (1982), Chapters 10, 11. Recent results for the finite dimensional case are obtained in Palmer (1988) and Ben-Artzi and Gohberg (1992) (see also the literature cited therein). "$L_p$-theorems" of this type can be found in Dore (1993). Results of this type for nonautonomous equations on the semiaxis $\mathbb{R}_+$ are considered (under some additional assumptions) in Rodrigues and Ruas-Filho (1995), and a certain class of nonautonomous parabolic equations on the semiaxis is considered in Zhang (1995).

In the autonomous case, when $A(t) \equiv A$ is an unbounded operator which generates a strongly continuous semigroup on $X$, this result is proven in Prüss (1984), Theorem 4; our paper is a direct generalization of his result. In the current paper, we address the general nonautonomous abstract Cauchy problem (0.2) on $\mathbb{R}$ where the operators $A(t)$ (with domains $D(A(t))$) are time-dependent and unbounded.

Our main result is not restricted to well-posed abstract Cauchy problems but addresses, more generally, any problem (0.2) whose solution is associated with a strongly continuous evolution family $\{U(t,s)\}_{t \geq s}$. It is often the case that an equation (0.2) does not have a differentiable solution yet does have a solution, in a mild sense, which is associated with an evolution family. For example, consider $A(t) := A_0 + A_1(t)$ where

$A_0$ generates a strongly continuous semigroup, $\{e^{tA_0}\}_{t\geq 0}$, and $t \mapsto A_1(t)$ is continuous from $\mathbb{R}$ to $B(X)$, the set of bounded linear operators on $X$. One can show that there exists an evolution family $\{U(t,s)\}_{t\geq s}$ which gives a mild solution, $x(t) = U(t,s)x(s)$, for $x(s) = x_s \in D(A(s)) = D(A_0)$, in the sense that it satisfies the integral equation

$$x(t) = e^{(t-s)A_0}x_s + \int_s^t e^{(t-\tau)A_0}A_1(\tau)x(\tau)\,d\tau.$$

However, it is not necessarily the case that $U(t,s)x$ is differentiable in $t$ for all $x \in D(A)$; see, for example, Curtain and Pritchard (1978), Chapters 2 and 9.

Therefore, we do not study only well-posed abstract Cauchy problems (0.2). Instead, we take as our starting point any strongly continuous evolution family. We assume, in addition, that $\{U(t,s)\}_{t\geq s}$ is exponentially bounded. This replaces in a natural way the classical assumption of bounded growth (see e.g. Zhang (1995), Lemma 5).

We prove that an evolution family $\{U(t,s)\}_{t\geq s}$ has exponential dichotomy if and only if the following property holds:

(M) *For every* $f \in C_b(\mathbb{R}, X)$ *there exists a unique solution* $u \in C_b(\mathbb{R}, X)$ *to* (0.1).

This result is applied in Section 2 to show that a standing hypothesis in the paper Vanderbauwhede and Iooss (1992) on central manifold theory is, in fact, equivalent to the existence of the exponential dichotomy for $\{U(t,s)\}_{t\geq s}$.

The main result is proven in Section 1 using the relatively new technique of so-called evolution semigroups; see Howland (1974), Latushkin and Montgomery-Smith (1994,1995), Latushkin et al. (1996), Latushkin and Randolph (1995), Nagel (1995), Räbiger and Schnaubelt (1994, 1996), Rau (1994a, 1994b), Nguyen Van Minh (1994), van Neerven (1996), and also Johnson (1980) and Chicone and Swanson (1981) in a different context. For a given evolution family $\{U(t,s)\}_{t\geq s}$ the evolution semigroup $\{T^t\}_{t\geq 0}$ is defined on a "super-space" $E$ of functions $f : \mathbb{R} \to X$ by the rule

$$(T^t f)(\tau) = U(\tau, \tau - t)f(\tau - t), \quad \tau \in \mathbb{R},\ t \geq 0. \tag{0.4}$$

If $E = L_p(\mathbb{R}, X)$, $1 \leq p < \infty$, or $E = C_0(\mathbb{R}, X)$, the space of continuous functions vanishing at infinity, then $\{T^t\}_{t\geq 0}$ is a strongly continuous semigroup; we will denote its generator by $\Gamma$.

The role of the evolution semigroup arises from the following facts. If $A$ generates a strongly continuous semigroup, it is well known (see, e.g., Nagel (1984)) that the asymptotic behavior of solutions to the differential equation $x'(t) = Ax(t)$ is determined in many important cases by the location of the spectrum of $A$. There are, however, examples for which this property fails to hold, and for nonautonomous Cauchy problems (0.2), even for finite-dimensional $X$, the asymptotic behavior of a solution is most likely not determined by the spectra of the operators $A(t)$. However, *it is determined* by the spectrum of $T^t$ or $\Gamma$. Namely, the following facts hold (see Latushkin and Montgomery-Smith (1994, 1995), Latushkin et al. (1996), Latushkin and Randolph (1995), Räbiger and Schnaubelt (1994,1996), Rau (1994a, 1994b)). A strongly continuous evolution family $\{U(t,s)\}_{t\geq s}$ has exponential dichotomy on $X$ if and only if the spectrum, $\sigma(T^t)$, $t>0$, does not intersect the unit circle or, equivalently, the operator $\Gamma^{-1}$ is bounded on $C_0(\mathbb{R},X)$ or $L_p(\mathbb{R},X)$. Moreover, one can associate with $T=T^1$ on $C_0(\mathbb{R},X)$ a family of weighted shift operators, $\pi_s(T)$, $s\in\mathbb{R}$, acting on $c_0(\mathbb{Z},X)$, the space of $X$-valued sequences vanishing at infinity. The exponential dichotomy of $\{U(t,s)\}_{t\geq s}$ is equivalent to the fact that the operators $(I-\pi_s(T))^{-1}$ are bounded uniformly for $s\in\mathbb{R}$. In the main step of our proof we show that condition (M) implies this fact.

In Section 2 we discuss several modifications of the condition (M). First, we replace the space $C_b(\mathbb{R},X)$ in (M) by $C_0(\mathbb{R},X)$ and show that this new condition ($\mathrm{M}_{C_0}$) is also equivalent to the dichotomy of the evolution family $\{U(t,s)\}_{t\geq s}$. Since $\{T^t\}_{t\geq 0}$ is a strongly continuous semigroup on $C_0(\mathbb{R},X)$ with the generator $\Gamma$, we are able to prove that the operator $R$, defined by ($\mathrm{M}_{C_0}$) as $u=Rf$, is, in fact, equal to $-\Gamma^{-1}$. Thus, ($\mathrm{M}_{C_0}$) is a "mild version" of the above-mentioned condition, $\Gamma^{-1}\in B(C_0(\mathbb{R},X))$, for the dichotomy of $\{U(t,s)\}_{t\geq s}$. Similar remarks hold when $C_0(\mathbb{R},X)$ is replaced by $L_p(\mathbb{R},X)$.

Next, we consider the condition ($\mathrm{M}_{E_\alpha}$) where $C_b(\mathbb{R},X)$ in (M) is replaced by a space $E_\alpha$ of $X$-valued functions growing on $\mathbb{R}$ not faster than $e^{\alpha|t|}$, $\alpha\geq 0$. This scale of Banach spaces plays a crucial role in the theory of infinite-dimensional center manifolds; see, e.g., Vanderbauwhede and Iooss (1992). We prove that ($\mathrm{M}_{E_\alpha}$) for $0\leq\alpha<\beta$, for some $\beta>0$, is also equivalent to the exponential dichotomy of $\{U(t,s)\}_{t\geq s}$, and that $R$ is an extension of $-\Gamma^{-1}$ to $E_\alpha$. This fact clarifies the standing hypotheses in Vanderbauwhede and Iooss (1992) (which addresses the autonomous case; see conditions (H), ($\Sigma$), (S) and Theorem 3.1 in that article) at once for any evolution family $\{U(t,s)\}_{t\geq s}$.

We conclude this Introduction with a brief discussion of related questions on exponential dichotomy for linear skew-product flows. We do not address the case of linear skew-product flows in the current paper, but only pose here an open problem. Let $\Theta$ be a locally compact metric space and $\{\varphi^t\}_{t\in\mathbb{R}}$ be a continuous flow on $\Theta$. Consider a strongly continuous exponentially bounded cocycle $\{\Phi^t\}_{t\in\mathbb{R}_+}$ over $\varphi^t$. This means that the function $(t,\theta)\mapsto\Phi^t(\theta)x$ from $\mathbb{R}_+\times\Theta$ to $X$ is continuous for each $x\in X$, there exist constants $C>0$ and $\omega$ such that $\|\Phi^t(\theta)\|\le Ce^{\omega t}$, and the identities $\Phi^{t+s}(\theta)=\Phi^t(\varphi^s\theta)\Phi^s(\theta)$ and $\Phi^0(\theta)=I$ hold for all $\theta\in\Theta$ and $t,s\in\mathbb{R}_+$. Define the corresponding linear skew-product flow $\hat{\varphi}^t:(\theta,x)\mapsto(\varphi^t\theta,\Phi^t(\theta)x)$, $(\theta,x)\in\Theta\times X$.

Let $C_b(\Theta,X)$ denote the space of continuous bounded functions $f:\Theta\to X$ with the sup-norm. We say that condition $(M_\Theta)$ holds if for each $f\in C_b(\Theta,X)$ there exists a unique solution $u\in C_b(\Theta,X)$ to the inhomogeneous equation

$$u(\varphi^t\theta)=\Phi^t(\theta)u(\theta)+\int_0^t\Phi^{t-s}(\varphi^s\theta)f(\varphi^s\theta)\,ds,\quad t\ge0,\quad\theta\in\Theta. \tag{0.5}$$

We pose the following question: *Does condition* $(M_\Theta)$ *imply that the linear skew-product flow* $\hat{\varphi}^t$ *has exponential dichotomy on* $\Theta$*?*

Observe, that every exponentially bounded strongly continuous evolution family $\{U(t,s)\}_{t\ge s}$ induces a linear skew-product flow over the base $\Theta=\mathbb{R}$ by setting $\varphi^t\theta=\theta+t$ for $t\in\mathbb{R}$ and

$$\Phi^t(\theta)=U(\theta+t,\theta),\ \hat{\varphi}^t:(\theta,x)\mapsto(\theta+t,U(\theta+t,\theta)x),\ (\theta,x)\in\mathbb{R}\times X \tag{0.6}$$

for $t\ge0$; see Chow and Leiva (1995). In this case, condition $(M_\Theta)$ is just condition (M), and our Theorem 1.1 gives a positive answer to the question above. We do not know the answer for general locally compact $\Theta$ and $\hat{\varphi}^t$.

We stress that usually one considers linear skew-product flows over a *compact* base $\Theta$. The exponential dichotomy of linear skew-product flows over compact $\Theta$ was studied in the classical papers by Sacker and Sell (1974,1976) and Selgrade (1975) for finite dimensional $X$ (see also Johnson, Palmer and Sell (1987) and the bibliography there). The infinite dimensional case was recently considered by Sacker and Sell (1994) and by Chow and Leiva (1994, 1995). Exponential dichotomy of linear skew-product flows was used by Shen and Yi (1995, 1996) and Yi (to appear) in their study of almost periodic and almost automorphic dynamics. We refer to the bibliography in the above mentioned papers and in Sell (1995); a review of the subject would go far beyond the scope of the current paper.

The linear skew-product flows over compact base arise in the context of linearizations of an (autonomous) evolution equation over an invariant compact set or when one considers the hull of a nonautonomous evolution equation (see the above references and in particular Sacker and Sell (1974, 1976). In the latter approach, the given equation leads to a linear skew–product flow over a "large" base space which is then studied by the technique of topological dynamics. For the existence of a compact hull one has to require some additional regularity for the coefficients of the given equation. Another approach, that goes back to Chicone and Swanson (1981), Johnson (1980) and Mather (1968), is to relate to the linear skew-product flow an evolution semigroup, similar to (0.4), see Latushkin et al. (1996) and references therein. This allows one to apply the technique of semigroup theory. As we have seen, the evolution semigroup can be defined with no regularity assumptions for the coefficient. Indeed, it can be defined even without assuming well-posedness of the given equation, in which case there is "no coefficient" for which to define the hull. The open question above is related to this semigroup approach.

Let us briefly discuss the relationship of this open question to known results for the dichotomy over a *compact* base $\Theta$. Assume, for a moment, that the "uniqueness" part of the condition $(M_\Theta)$ holds. This means that for $u \in C_b(\Theta, X)$ the homogeneous equation $u(\varphi^t\theta) = \Phi^t(\theta)u(\theta)$, $t \geq 0$, $\theta \in \Theta$, has only the trivial solution. Assume, in addition, that the cocycle is uniformly $\alpha$-contracting. In the terminology of Sacker and Sell (1994), the linear skew-product flow with these two properties is called *weakly hyperbolic*. In Sacker and Sell (1994), Theorem B, in particular, the following sufficient condition for the exponential dichotomy is given: A weakly hyperbolic linear skew-product flow has exponential dichotomy provided the stable sets have a fixed finite codimension (otherwise the behavior of $\hat{\varphi}^t$ is more complicated; see the Alternative Theorem E in Sacker and Sell (1994)). We suspect that the "existence" part of the condition $(M_\Theta)$, that is, that the solutions $u$ exist *for all* $f \in C_b(\Theta, X)$, gives yet another sufficient condition for the exponential dichotomy. Thus, an affirmative answer to the question above would supplement the Sacker-Sell theory from a "Perron-type" point of view.

We emphasize that the compactness of $\Theta$ is, of course, essential for Theorem B of Sacker and Sell (1994). For a simple counterexample on $\Theta = \mathbb{R}$, consider the linear skew-product as in (0.6) for $X = \mathbb{C}$ and $U(t,s) = \exp\int_s^t a(\tau)\,d\tau$, where $a \in C_b(\mathbb{R})$ with $a(\tau) = 1$ for $\tau \geq 1$ and $a(\tau) = -1$ for $\tau \leq -1$. Clearly, the homogeneous equation has only the trivial bounded solution; nevertheless, $\{U(t,s)\}_{t\geq s}$ does not have exponential dichotomy.

## 1. Main Result

We begin with several definitions. An *evolution family* is a family of bounded operators $\{U(t,s)\}_{t\geq s}$ on $X$ which satisfy:

i) $U(t,s) = U(t,r)U(r,s)$ and $U(t,t) = I$ for all $t \geq r \geq s$;
ii) for each $x \in X$ the function $(t,s) \mapsto U(t,s)x$ is continuous for $t \geq s$.

If, in addition, there exist constants $C \geq 1$, $\gamma \in \mathbb{R}$, such that

$$\|U(t,s)\| \leq Ce^{\gamma(t-s)}, \quad t \geq s,$$

then $\{U(t,s)\}_{t\geq s}$ is *exponentially bounded*. An evolution family $\{U(t,s)\}_{t\geq s}$ is said to *solve* the abstract Cauchy problem (0.2) if $x(\cdot) = U(\cdot,s)x_s$ is differentiable, $x(t) \in D(A(t))$ for $t \geq s$, and (0.2) holds.

Since $\{U(t,s)\}_{t\geq s}$ is assumed to be a *strongly* continuous family, the operators $A(t)$ in (0.2) might be unbounded. If $A$ is the generator of a strongly continuous semigroup, $\{e^{tA}\}_{t\geq 0}$, on $X$, then $U(t,s) := e^{(t-s)A}$, $t \geq s$, defines an exponentially bounded evolution family that solves (0.2) in the autonomous case $A(t) \equiv A$.

We now give the definition of exponential dichotomy for a general evolution family. For various definitions and the history of this notion see Chow and Leiva (1994, 1995), Daleckij and Krein (1974), Henry (1981), Massera and Schaffer (1966), Nagel (1995), Sacker and Sell (1974, 1978, 1994). For any projection-valued function $P : \mathbb{R} \to B(X)$, the complementary projection will be denoted by $Q(t) = I - P(t)$, $t \in \mathbb{R}$. If the property $P(t)U(t,s) = U(t,s)P(s)$ is satisfied for $t \geq s$, we will use the notation

$$U_P(t,s) = P(t)U(t,s)P(s), \quad U_Q(t,s) = Q(t)U(t,s)Q(s)$$

to denote the restrictions of the operator $U(t,s)$. We stress that $U_P(t,s)$ is an operator from $\operatorname{Im} P(s)$ to $\operatorname{Im} P(t)$, and $U_Q(t,s)$ acts from $\operatorname{Im} Q(s)$ to $\operatorname{Im} Q(t)$.

**Definition 1.1.** An evolution family $\{U(t,s)\}_{t\geq s}$ is said to have *exponential dichotomy* (with a constant $\beta > 0$) if there exists a projection-valued function $P : \mathbb{R} \to B(X)$ such that the function $t \mapsto P(t)x$ is continuous and bounded for each $x \in X$, and for some constant $M = M(\beta) > 0$ and all $t \geq s$ the following hold:

i) $P(t)U(t,s) = U(t,s)P(s)$;
ii) $U_Q(t,s)$ is invertible as an operator from $\operatorname{Im} Q(s)$ to $\operatorname{Im} Q(t)$;
iii) $\|U_P(t,s)\| \leq Me^{-\beta(t-s)}$;
iv) $\|U_Q(t,s)^{-1}\| \leq Me^{-\beta(t-s)}$.

If an evolution family $\{U(t,s)\}_{t\geq s}$ has exponential dichotomy, the number $\delta = \delta(\{U(t,s)\}_{t\geq s})$ defined as

$$\delta = \sup\{\beta > 0 : \{U(t,s)\}_{t\geq s} \text{ has exponential dichotomy with constant } \beta\},$$

will be called the *dichotomy bound* for $\{U(t,s)\}_{t\geq s}$.

The main result of the current paper is formulated as follows.

**Theorem 1.1.** *Assume $\{U(t,s)\}_{t\geq s}$ is an exponentially bounded evolution family. Then $\{U(t,s)\}_{t\geq s}$ has exponential dichotomy if and only if the following condition holds:*

(M) *For every $f \in C_b(\mathbb{R}, X)$ there exists a unique solution $u \in C_b(\mathbb{R}, X)$ to* (0.1).

The necessity of condition (M) is proven in the following proposition which involves the Green's function for $\{U(t,s)\}_{t\geq s}$. Note that in this proposition, we do not assume exponential boundedness of $\{U(t,s)\}_{t\geq s}$. Sufficiency in Theorem 1.1 will use the spectral mapping theorem for evolution semigroups and will be proven with a series of three lemmas.

**Proposition 1.2.** *Let $\{U(t,s)\}_{t\geq s}$ be an evolution family. If $\{U(t,s)\}_{t\geq s}$ has exponential dichotomy, then* (M) *holds.*

*Proof.* Let $P$ be a projection which satisfies the properties of Definition 1.1. Set

$$G(t,s) = P(t)U(t,s)P(s), \text{ for } t > s, \quad G(t,s) = -[Q(s)U(s,t)Q(t)]^{-1}, \quad \text{for } t < s.$$

Define the operator $\hat{G}$ by

$$\hat{G}f(t) = \int_{-\infty}^{\infty} G(t,s)f(s)\,ds, \qquad f \in C_b(\mathbb{R}, X). \tag{1.1}$$

Then $\hat{G}$ is bounded on $C_b(\mathbb{R}, X)$. Indeed, for $f \in C_b(\mathbb{R}, X)$, $\hat{G}f$ is continuous and for $t \in \mathbb{R}$,

$$\begin{aligned}
\|\hat{G}f(t)\| &\leq \|f\|_\infty \left(\int_t^\infty \|[U_Q(\tau,t)]^{-1}\|\,d\tau + \int_{-\infty}^t \|U_P(t,\tau)\|\,d\tau\right) \\
&\leq M\|f\|_\infty \left(\int_t^\infty e^{-\beta(\tau-t)}d\tau + \int_{-\infty}^t e^{-\beta(t-\tau)}d\tau\right) \\
&= \frac{2M}{\beta}\|f\|_\infty.
\end{aligned}$$

Now let $f \in C_b(\mathbb{R}, X)$ and set $u = \hat{G}f$. Then for $t \geq s$,

$$
\begin{aligned}
u(t) - U(t,s)u(s) &= \hat{G}f(t) - U(t,s)\hat{G}f(s)\\
&= \int_{-\infty}^{t} P(t)U(t,\tau)P(\tau)f(\tau)\,d\tau - U(t,s)\int_{-\infty}^{s} P(s)U(s,\tau)P(\tau)f(\tau)\,d\tau\\
&\quad - \int_{t}^{\infty}[U_Q(\tau,t)]^{-1}Q(\tau)f(\tau)\,d\tau + U(t,s)\int_{s}^{\infty}[U_Q(\tau,s)]^{-1}Q(\tau)f(\tau)\,d\tau\\
&= \int_{s}^{t} U(t,\tau)P(\tau)f(\tau)\,d\tau - \int_{t}^{\infty}[U_Q(\tau,t)]^{-1}Q(\tau)f(\tau)\,d\tau\\
&\quad + U_Q(t,s)\int_{t}^{\infty}[U_Q(\tau,t)U_Q(t,s)]^{-1}Q(\tau)f(\tau)\,d\tau\\
&\quad + \int_{s}^{t} U_Q(t,\tau)U_Q(\tau,s)[U_Q(\tau,s)]^{-1}Q(\tau)f(\tau)\,d\tau\\
&= \int_{s}^{t} U(t,\tau)P(\tau)f(\tau)\,d\tau + \int_{s}^{t} U(t,\tau)Q(\tau)f(\tau)\,d\tau\\
&= \int_{s}^{t} U(t,\tau)f(\tau)\,d\tau\,.
\end{aligned}
$$

To prove the uniqueness, take $f = 0$ and suppose there is a $u \in C_b(\mathbb{R}, X)$ so that $u(t) = U(t,s)u(s)$ for all $t \geq s \in \mathbb{R}$. Since $\{U(t,s)\}_{t\geq s}$ has exponential dichotomy,

$$P(t)u(t) = U_P(t,s)P(s)u(s), \quad Q(s)u(s) = U_Q(t,s)^{-1}Q(t)u(t),$$

and the estimates

$$\|P(t)u(t)\| \leq Me^{-\beta(t-s)}\|P(\cdot)u(\cdot)\|_\infty\,, \quad \|Q(s)u(s)\| \leq Me^{-\beta(t-s)}\|Q(\cdot)u(\cdot)\|_\infty\,,$$

yield $u = 0$. □

*Remark* 1.3. Condition (M) says that for each $f \in C_b(\mathbb{R}, X)$ there exists a unique $u \in C_b(\mathbb{R}, X)$ satisfying (0.1), so defining $Rf = u$ gives an operator, $R$, on $C_b(\mathbb{R}, X)$. The Closed Graph Theorem shows that this operator is bounded: for $\{f_n\}_{n\in\mathbb{N}} \subseteq C_b(\mathbb{R}, X)$, and $f, u \in C_b(\mathbb{R}, X)$ such that $f_n \to f$ and $u_n := Rf_n \to u$, then

$$
\begin{aligned}
u(t) &= \lim_{n\to\infty} u_n(t)\\
&= \lim_{n\to\infty}\left(U(t,s)u_n(s) + \int_s^t U(t,\tau)f_n(\tau)\,d\tau\right)\\
&= U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)\,d\tau, \quad \text{for all } t \geq s \text{ in } \mathbb{R}.
\end{aligned}
$$

We further note that the proof of Proposition 1.2 shows that if $\{U(t,s)\}_{t\geq s}$ has exponential dichotomy, then $R$ is the Green's operator, $R = \hat{G}$.

In Section 2 we will make use of the additional fact that the Green's operator defined by (1.1) is a also bounded operator on $C_0(\mathbb{R}, X)$ and $L_p(\mathbb{R}, X)$, and $\hat{G} =$

$-\Gamma^{-1}$ where $\Gamma$ denotes the generator of the evolution semigroup (0.4) on $C_0(\mathbb{R}, X)$ or $L_p(\mathbb{R}, X)$, respectively (see Latushkin et al. (1996), Theorem 5.5, for the $C_0$–case and Latushkin and Randolph (1995), Theorem 4.5, for the $L_p$–case).

To prove sufficiency of the condition (M) in the theorem, we will need some facts about evolution semigroups (see Latushkin and Montgomery-Smith (1994, 1995), Latushkin et al. (1996), Latushkin and Randolph (1995), Räbiger and Schnaubelt (1994, 1996), Rau (1994a, 1994b)). Given an exponentially bounded evolution family $\{U(t,s)\}_{t\geq s}$ on $X$, one can define a strongly continuous semigroup $\{T^t\}_{t\geq 0}$ on $E = C_0(\mathbb{R}, X)$ or $L_p(\mathbb{R}, X)$, $1 \leq p < \infty$ as in (0.4). Note that in the autonomous case (0.4) becomes: $(T^t f)(\tau) = e^{tA} f(\tau - t)$.

One interesting fact about the evolution semigroup $\{T^t\}_{t\geq 0}$ on $C_0(\mathbb{R}, X)$ is that the following spectral mapping theorem holds (see Latushkin and Montgomery-Smith (1995), Räbiger and Schnaubelt (1996)). It is this theorem which allows the property of exponential dichotomy of solutions of (0.2) to be characterized by the property that $\Gamma$ is invertible; this is described in the subsequent theorem.

**Theorem 1.4.** *The spectrum $\sigma(\Gamma)$ is invariant under translations along $i\mathbb{R}$, the spectrum $\sigma(T^t)$ is invariant under rotations about origin, and the spectral mapping property holds:*

$$e^{t\sigma(\Gamma)} = \sigma(T^t) \setminus \{0\}, \quad t \geq 0. \tag{1.2}$$

*Moreover, the spectra of $\Gamma$ and $T^t$ are independent of the choice of the spaces $C_0(\mathbb{R}, X)$ or $L_p(\mathbb{R}, X)$.*

Note that Theorem 1.4 shows that $\sigma(T^t)$, $t \geq 0$, can be described using the single operator $T = T^1$. The operator $T$ is a so-called weighted *translation* operator on $C_0(\mathbb{R}, X)$ (see Latushkin and Stepin (1992) for further references). One can associate with $T$ a family of weighted *shift* operators, $\pi_s(T)$, $s \in \mathbb{R}$, acting on the sequence spaces $c_0(\mathbb{Z}, X)$ or $\ell_\infty(\mathbb{Z}, X)$.

Fix $s \in \mathbb{R}$. Define the bounded operator $\pi_s(T)$ on $c_0(\mathbb{Z}, X)$ and $\ell_\infty(\mathbb{Z}, X)$ as follows: for $v = \{x_n\}_{n\in\mathbb{Z}}$,

$$(\pi_s(T))(v) = \{U(s+n, s+n-1)x_{n-1}\}_{n\in\mathbb{Z}}.$$

Theorem 1.5 below is taken from Latushkin et al. (to appear) (see Theorem 4.1, Proposition 4.2 and Corollary 5.1). This theorem relates the spectra $\sigma(T)$ and $\sigma(\Gamma)$ on $C_0(\mathbb{R}, X)$ to the spectra $\sigma(\pi_s(T))$, $s \in \mathbb{R}$, on $c_0(\mathbb{Z}, X)$.

**Theorem 1.5.** *The following assertions are equivalent for an exponentially bounded evolution family* $\{U(t,s)\}_{t\geq s}$ *on* $X$.

*i)* $\{U(t,s)\}_{t\geq s}$ *has exponential dichotomy;*

*ii)* $I-\pi_s(T)$ *is invertible on* $c_0(\mathbb{Z},X)$, *and there exists a constant* $K>0$, *such that*

$$\|[I-\pi_s(T)]^{-1}\|_{B(c_0(\mathbb{Z},X))}\leq K \quad \text{for all } s\in\mathbb{R}; \tag{1.3}$$

*iii)* $I-T$ *is invertible in* $C_0(\mathbb{R},X)$;

*iv)* $\Gamma$ *is invertible on* $C_0(\mathbb{R},X)$.

The strategy for the rest of the proof of Theorem 1.1 is as follows. Lemmas 1.6–1.8 consist of showing that if (M) holds, then $(I-\pi_s(T))^{-1}\in B(c_0(\mathbb{Z},X))$ and (1.3) is satisfied. We stress that $\{T^t\}_{t\geq 0}$ as in (0.4) is *not* a strongly continuous semigroup on $C_b(\mathbb{R},X)$. The operators $\pi_s(T)$ are still defined on $\ell_\infty(\mathbb{Z},X)$. Condition (M) is an "$\ell_\infty$-type" (versus "$c_0$-type") condition. We will use (M) to show that the operators $(I-\pi_s(T))^{-1}$, $s\in\mathbb{R}$, exist and are uniformly bounded on $\ell_\infty(\mathbb{Z},X)$ and derive (1.3) from this fact. Finally, by Theorem 1.5, $\{U(t,s)\}_{t\geq s}$ has exponential dichotomy provided (M) is satisfied.

In the following lemma, we use the notation $\|A\|^\bullet_Z:=\inf\{\|Az\|_Z:\|z\|_Z=1\}$ for a linear operator $A$ on a Banach space $Z$.

**Lemma 1.6.** *If* (M) *holds, then* $I-\pi_s(T)$ *is bounded from below, uniformly in* $s\in\mathbb{R}$, *on* $\ell_\infty(\mathbb{Z},X)$, *hence also on* $c_0(\mathbb{Z},X)$.

*Proof.* Suppose that

$$\inf_{s\in\mathbb{R}}\|I-\pi_s(T)\|^\bullet_{\ell_\infty(\mathbb{Z},X)}=0.$$

Then for each $\epsilon>0$ there are $s\in\mathbb{R}$ and $v=\in\ell_\infty(\mathbb{Z},X)$ such that $\|v\|_\infty=1$ and $\|(I-\pi_s(T))v\|_\infty<\epsilon$. Fix $n\in\mathbb{N}$. Then there exists $s=s(n)\in\mathbb{R}$ and $v=v(n)=\{x_m\}_{m\in\mathbb{Z}}\in\ell_\infty(\mathbb{Z},X)$ such that

$$\frac{1}{2}<\|(\pi_s(T))^kv\|_\infty\leq 2, \quad \text{for } k=0,1,\dots,2n.$$

Now, $(\pi_s(T))^kv=\{U(s+m,s+m-k)x_{m-k}\}_{m\in\mathbb{Z}}$ and so we may choose $l\in\mathbb{Z}$ such that

$$\frac{1}{2}\leq\|U(s+l,s+l-n)x_{l-n}\|_X.$$

Set $s_n:=s+l$ and $z_n:=x_{l-n}$. Choose a function $\alpha_n\in C^1(\mathbb{R})$ with the properties

i) $\text{supp}(\alpha_n)\subseteq(s_n-n,s_n+n)$;

ii) $0 \leq \alpha_n \leq 1$, with $\alpha_n(s_n) = 1$;

iii) $\|\alpha_n'\|_\infty \leq \frac{2}{n}$.

Define $f_n \in C_0(\mathbb{R}, X)$ by

$$f_n(t) := \begin{cases} \alpha_n(t)U(t, s_n - n)z_n, & t \geq s_n - n, \\ 0, & t < s_n - n. \end{cases}$$

Since $f_n \in C_0(\mathbb{R}, X)$, we can use the strongly continuous evolution semigroup defined in (0.4). Note that

$$\begin{aligned}(T^h f_n)(t) &= U(t, t-h)f(t-h) \\ &= U(t, t-h)\alpha_n(t-h)U(t-h, s_n - n)z_n \\ &= \alpha_n(t-h)U(t, s_n - n)z_n,\end{aligned}$$

for $t - h \geq s_n - n$, and hence $f_n \in D(\Gamma)$ and

$$\Gamma f_n(t) = \begin{cases} -\alpha_n'(t)U(t, s_n - n)z_n, & t \geq s_n - n. \\ 0, & t < s_n - n. \end{cases}$$

Also, $\|f_n\|_\infty \geq \|f_n(s_n)\|_X \geq \frac{1}{2}$, and

$$\|\Gamma f_n\|_\infty \leq \frac{2C}{n} \sup_{0 \leq k \leq 2n} \|U(k + s_n - n, s_n - n)z_n\|_X \leq \frac{4C}{n}. \tag{1.4}$$

Further, note that (M) implies $f_n = -R\Gamma f_n$. Indeed, for $t \geq s \geq s_n - n$

$$\begin{aligned}f_n(t) &= \alpha_n(t)U(t, s_n - n)z_n \\ &= \alpha_n(s)U(t, s_n - n)z_n + \alpha_n(t)U(t, s_n - n)z_n - \alpha_n(s)U(t, s_n - n)z_n \\ &= U(t, s)(\alpha_n(s)U(s, s_n - n)z_n) + U(t, s_n - n)z_n \int_s^t \alpha_n'(\tau)\, d\tau \\ &= U(t, s)(\alpha_n(s)U(s, s_n - n)z_n) + \int_s^t U(t, \tau)\alpha_n'(\tau)U(\tau, s_n - n)z_n\, d\tau \\ &= U(t, s)f_n(s) - \int_s^t U(t, \tau)\Gamma f_n(\tau)\, d\tau,\end{aligned}$$

and similarly for the other cases. By (1.4) and the fact that $R \in B(C_b(\mathbb{R}, X))$ (see Remark 1.3), it follows that $\lim_{n\to\infty} f_n = 0$. This contradicts $\|f_n\| \geq \frac{1}{2}$. □

**Lemma 1.7.** *If* (M) *holds, then for each* $s \in \mathbb{R}$*, the operator* $I - \pi_s(T)$ *from* $\ell_\infty(\mathbb{Z}, X)$ *to* $\ell_\infty(\mathbb{Z}, X)$ *is surjective.*

*Proof.* Let $v = \{x_n\}_{n\in\mathbb{Z}} \in \ell_\infty(\mathbb{Z}, X)$. Fix $s \in \mathbb{R}$. It suffices to find $w = \{y_n\}_{n\in\mathbb{Z}} \in \ell_\infty(\mathbb{Z}, X)$ such that

$$y_n - U(s+n, s+n-1)y_{n-1} = U(s+n, s+n-1)x_{n-1} \tag{1.5}$$

holds for all $n \in \mathbb{Z}$; i.e., $(I - \pi_s(T))(w) = \pi_s(T)(v)$. For then, $(I - \pi_s(T))(w + v) = \pi_s(T)(v) + v - \pi_s(T)(v) = v$. To obtain $w$ such that (1.5) holds, we claim there exists $f \in C_b(\mathbb{R}, X)$ such that

$$U(s+n, s+n-1)x_{n-1} = \int_{s+n-1}^{s+n} U(s+n,\tau)f(\tau)\,d\tau, \quad n \in \mathbb{Z}. \tag{1.6}$$

Since (M) holds, it then follows that there exists a unique $g \in C_b(\mathbb{R}, X)$ such that

$$g(s+n) = U(s+n, s+n-1)g(s+n-1) + \int_{s+n-1}^{s+n} U(s+n,\tau)f(\tau)\,d\tau, \quad n \in \mathbb{Z}. \tag{1.7}$$

Setting $y_n = g(s+n)$ in (1.7) then provides a $w = \{y_n\}_{n\in\mathbb{Z}} \in \ell_\infty(\mathbb{Z}, X)$ which satisfies (1.5), hence proving the lemma.

To construct $f$, choose $\alpha \in C([0, 1])$ so that $\alpha(0) = 0$, $\alpha(1) = 1$, and $\int_0^1 \alpha(\tau)\,d\tau = 1$. Then for $t \in [n+s-1, n+s)$, define

$$\begin{aligned} f(t) &= \alpha(t-n-s+1)U(t, n+s-1)x_{n-1} \\ &\qquad + (1-\alpha(t-n-s+1))U(t, n+s-2)x_{n-2}. \end{aligned}$$

This gives

$$\begin{aligned} \int_{s+n-1}^{s+n} U(s+n,\tau)f(\tau)\,d\tau &= \int_0^1 \alpha(\tau)\,d\tau \cdot U(s+n, s+n-1)x_{n-1} \\ &\qquad + \left(1 - \int_0^1 \alpha(\tau)\,d\tau\right) \cdot U(s+n, s+n-2)x_{n-2} \end{aligned}$$

which is (1.6) □

Now if (M) holds, Lemmas 1.6 and 1.7 show that $(I - \pi_s(T))^{-1}$ is bounded as an operator on $\ell_\infty(\mathbb{Z}, X)$. In fact, as the following lemma shows, $(I - \pi_s(T))^{-1}$ is a bounded operator on $c_0(\mathbb{Z}, X)$. The argument is similar to that of Lemma 1.6 in Latushkin and Randolph (1995) which is an adaptation of techniques in Kurbatov (1990). The left shift operator on $\ell_\infty(\mathbb{Z}, X)$ will be denoted by $S$: $\{x_n\}_{n\in\mathbb{Z}} \mapsto \{x_{n-1}\}_{n\in\mathbb{Z}}$.

**Lemma 1.8.** *Assume* (M) *holds and* $s \in \mathbb{R}$. *Then the bounded operator* $(I - \pi_s(T))^{-1}$ *on* $\ell_\infty(\mathbb{Z}, X)$ *leaves* $c_0(\mathbb{Z}, X)$ *invariant and hence* $(I - \pi_s(T))^{-1} \in B(c_0(\mathbb{Z}, X))$.

*Proof.* Fix $s \in \mathbb{R}$ and set $C := (I - \pi_s(T))^{-1} \in B(\ell_\infty(\mathbb{Z}, X))$. We show that $Cv \in c_0(\mathbb{Z}, X)$ for each $v \in c_0(\mathbb{Z}, X)$.

First consider sequences in $c_0(\mathbb{Z}, X)$ of the form $v = e_l \otimes x$ $(x \in X)$ whose $k^{th}$ term is $(v)_k = \delta_{kl} x$. For $x \in X$, $C_{kl} x := (C(e_l \otimes x))_k$ defines an operator $C_{kl} \in B(X)$.

Now let $v = \{x_n\}_{n \in \mathbb{Z}} \in c_0(\mathbb{Z}, X)$. To show that $Cv \in c_0(\mathbb{Z}, X)$, we note (by writing $v = \sum_l e_l \otimes x_l$) that it suffices to show

(1.8)
$$s_k := \sum_{l \in \mathbb{Z}} \|C_{kl}\|_{B(X)} \text{ exists, } \sup_{k \in \mathbb{Z}} s_k < \infty \text{ and } \lim_{|k| \to \infty} \|C_{kl}\| = 0 \text{ for each } l \in \mathbb{Z}.$$

Set $D := I - \pi_s(T)$. For $\gamma \geq 0$, define the space

$$\ell_\infty^\gamma = \{\{x_k\}_{k \in \mathbb{Z}} \in \ell_\infty(\mathbb{Z}, X) : \{e^{\gamma |k|} x_k\}_{k \in \mathbb{Z}} \in \ell_\infty(\mathbb{Z}, X)\}.$$

Define $J_\gamma : \ell_\infty(\mathbb{Z}, X) \to \ell_\infty^\gamma$ by $J_\gamma(\{x_k\}_{k \in \mathbb{Z}}) = \{e^{-\gamma |k|} x_k\}_{k \in \mathbb{Z}}$. Note that $D$ maps $\ell_\infty^\gamma$ into $\ell_\infty^\gamma$ and hence we can define a bounded operator $D(\gamma)$ on $\ell_\infty(\mathbb{Z}, X)$ by

$$D(\gamma) = J_\gamma^{-1} D J_\gamma.$$

For $v = \{x_n\}_{n \in \mathbb{Z}} \in \ell_\infty(\mathbb{Z}, X)$, we have

$$\begin{aligned} (D(\gamma) - D)v &= [J_\gamma^{-1}(\pi_s(T)) J_\gamma - \pi_s(T)]v \\ &= \left\{(e^{\gamma(|k| - |k-1|)} - 1) U(s+k, s+k-1) x_{k-1}\right\}_{k \in \mathbb{Z}}. \end{aligned}$$

Therefore, $\|D(\gamma) - D\|_{B(\ell_\infty(\mathbb{Z}, X))} \leq M(e^{\pm\gamma} - 1) \to 0$ as $\gamma \to 0$. Thus we can fix $\gamma > 0$ such that $D(\gamma)$ has a bounded inverse on $\ell_\infty(\mathbb{Z}, X)$. Set $\|D(\gamma)^{-1}\|_{B(\ell_\infty(\mathbb{Z}, X))} =: \delta$.

Now let $x \in X$, and consider $v = e_0 \otimes x$. We have

$$D(\gamma)^{-1} v = J_\gamma^{-1} C J_\gamma v = J_\gamma^{-1} C(e_0 \otimes x) = J_\gamma^{-1}(\{C_{k0} x\}_{k \in \mathbb{Z}}) = \{e^{\gamma |k|} C_{k0} x\}_{k \in \mathbb{Z}}.$$

Hence, $\|e^{\gamma |k|} C_{k0} x\| \leq \|D(\gamma)^{-1} v\|_{\ell_\infty(\mathbb{Z}, X)} \leq \delta \|e_0 \otimes x\|_{\ell_\infty(\mathbb{Z}, X)} = \delta \|x\|$, and thus

$$\|C_{k0}\|_{B(X)} \leq \delta e^{-\gamma |k|}. \tag{1.9}$$

Fix $j \in \mathbb{Z}$ and set $\tilde{D} := S^{-j} D S^j$ and $\tilde{C} := \tilde{D}^{-1}$. For a finitely supported sequence $v = \{x_k\}_{k \in \mathbb{Z}} = \sum_l e_l \otimes x_l$, we have

$$\tilde{C} v = S^{-j} C(\{x_{k-j}\}_{k \in \mathbb{Z}}) = S^{-j} \left\{\sum_{l \in \mathbb{Z}} C_{kl} x_{l-j}\right\}_{k \in \mathbb{Z}} = \left\{\sum_{l \in \mathbb{Z}} C_{k+j, l+j} x_l\right\}_{k \in \mathbb{Z}}.$$

Therefore, $\tilde{C}_{kl} = C_{k+j, l+j}$, or $C_{kj} = \tilde{C}_{k-j, 0}$. Applying (1.9) to $\tilde{C}$ yields

$$\|C_{kl}\|_{B(X)} \leq \delta e^{-\gamma |k-l|},$$

and so $s_k = \sum_l \|C_{kl}\| \leq \delta_1$ for some constant $\delta_1$. This proves (1.8). □

The proof of Theorem 1.1 now follows. Indeed, if (M) holds, Lemmas 1.6–1.8 show that $I - \pi_s(T)$ is invertible in $B(c_0(\mathbb{Z}, X))$. Moreover, by Lemma 1.6, there exists $K > 0$ such that

$$\sup_{s\in\mathbb{R}} \|(I - \pi_s(T))^{-1}\|_{c_0(\mathbb{Z},X)} = \frac{1}{\inf_{s\in\mathbb{R}} \|I - \pi_s(T)\|^{\bullet}_{c_0(\mathbb{Z},X)}} \le K.$$

Hence, Theorem 1.1 follows from Theorem 1.5.

## 2. Weighted function spaces

In this section we will consider several modifications of condition (M), where $\{U(t,s)\}_{t\ge s}$ is assumed to be exponentially bounded. First, we clarify the relationship between the operator $R$, as defined via (M), and $\Gamma^{-1}$ where $\Gamma$ is the generator of the evolution semigroup (0.4) on $C_0(\mathbb{R}, X)$ or $L_p(\mathbb{R}, X)$. For this, consider the corresponding properties:

($\mathrm{M}_{C_0}$) *For every $f \in C_0(\mathbb{R}, X)$ there exists a unique solution $u \in C_0(\mathbb{R}, X)$ to* (0.1)*;*

($\mathrm{M}_{L_p}$) *For every $f \in L_p(\mathbb{R}, X)$ there exists a unique continuous solution $u \in L_p(\mathbb{R}, X)$ to* (0.1).

We have the following extension of Theorem 1.1.

**Theorem 2.1.** *The following assertions are equivalent for an exponentially bounded evolution family $\{U(t,s)\}_{t\ge s}$ on $X$.*

*i) $\{U(t,s)\}_{t\ge s}$ has exponential dichotomy;*
*ii) property* (M) *holds;*
*iii) property* ($\mathrm{M}_{C_0}$) *holds;*
*iv) property* ($\mathrm{M}_{L_p}$) *holds.*

*The operator $R$ defined by* (M) ($\mathrm{M}_{C_0}$)*, or* ($\mathrm{M}_{L_p}$)*, is equal to the Green's operator $\hat{G}$ as defined in* (1.1)*. Moreover, on $C_0(\mathbb{R}, X)$, respectively, $L_p(\mathbb{R}, X)$, we have $R = -\Gamma^{-1}$, where $\Gamma$ denotes the generator of the evolution semigroup $\{T^t\}_{t\ge 0}$ on $C_0(\mathbb{R}, X)$, respectively, $L_p(\mathbb{R}, X)$.*

*Proof.* If $\{U(t,s)\}_{t\ge s}$ has exponential dichotomy then by Remark 1.3 the Green's operator $\hat{G}$ is defined on $C_0(\mathbb{R}, X)$ and $L_p(\mathbb{R}, X)$, and $\hat{G} = -\Gamma^{-1}$. Then by Räbiger and Schnaubelt (1996), Proposition 3.3, $\hat{G}$ maps $L_p(\mathbb{R}, X)$ into $C_0(\mathbb{R}, X)$. Moreover, it can be seen as in the proof of Proposition 1.2 that ($\mathrm{M}_{C_0}$) and ($\mathrm{M}_{L_p}$) hold, and $R = \hat{G}$.

By Theorem 1.1 and Remark 1.3 it remains to show that ($M_{C_0}$), resp. ($M_{L_p}$), yields exponential dichotomy for $\{U(t,s)\}_{t\geq s}$. Define $R$ by ($M_{C_0}$), resp. ($M_{L_p}$). In view of Theorems 1.4 and 1.5 it is enough to show that $\Gamma$ is invertible. In fact, let $f \in C_0(\mathbb{R}, X)$ or $L_p(\mathbb{R}, X)$. Then for $u = Rf$, $h \geq 0$, $t - h \geq s$,

$$\begin{aligned}(T^h u)(t) &= U(t,t-h)u(t-h)\\ &= U(t,t-h)\left[U(t-h,s)u(s) + \int_s^{t-h} U(t-h,\tau)f(\tau)\,d\tau\right]\\ &= U(t,s)u(s) + \int_s^{t-h} U(t,\tau)f(\tau)\,d\tau,\end{aligned}$$

and so

$$\begin{aligned}\frac{1}{h}\Big[(T^h u)(t) - u(t)\Big] &= \frac{1}{h}\left[U(t,s)u(s) + \int_s^{t-h} U(t,\tau)f(\tau)\,d\tau\right.\\ &\qquad\left. - \left(U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)\,d\tau\right)\right]\\ &= -\frac{1}{h}\int_{t-h}^t U(t,\tau)f(\tau)\,d\tau\\ &= -\frac{1}{h}\int_0^h U(t,t-\tau)f(t-\tau)\,d\tau.\end{aligned}$$

Consequently,

$$\frac{1}{h}\,(T^h u - u) = -\frac{1}{h}\int_0^h T^\tau f\,d\tau,$$

where we have used Neidhardt (1981), Theorem 4.2, in the case of $L_p(\mathbb{R}, X)$. Hence, $Rf = u \in D(\Gamma)$ and $\Gamma R f = -f$. On the other hand, suppose $u \in D(\Gamma)$ and $\Gamma u = 0$. Then for $t > 0$ and $n \in \mathbb{N}$ large enough, we have $(\frac{n}{t} - \Gamma)u = \frac{n}{t}u$, or $u = \frac{n}{t}\left(\frac{n}{t} - \Gamma\right)^{-1} u$, and so

$$u = \lim_{n\to\infty}\left[\frac{n}{t}\left(\frac{n}{t} - \Gamma\right)^{-1}\right]^n u = T^t u, \quad t > 0.$$

Hence, $u(t) = U(t,s)u(s)$ for $t \geq s$. Therefore, (0.1) holds for $f = 0$ and so $u = R(0) = 0$. That is, $\Gamma$ is bijective. □

Our next goal is to replace $C_b(\mathbb{R}, X)$ in (M) by a space $E_\alpha$ of $X$-valued functions that grow not faster than exponentially on $\mathbb{R}$. We consider functions $w : \mathbb{R} \to \mathbb{R}_+$ of the form $e^{-\alpha|t|}$, $\alpha \geq 0$, and define the weighted spaces $E_\alpha = \{f \in C(\mathbb{R}, X) : e^{-\alpha|\cdot|}f \in C_b(\mathbb{R}, X)\}$. Consider the following condition:

($M_{E_\alpha}$) *For every $f \in E_\alpha$ there exists a unique solution $u \in E_\alpha$ to* (0.1).

As we will see, $(\mathrm{M}_{E_\alpha})$ holds for $\{U(t,s)\}_{t\geq s}$ and $0 \leq \alpha < \beta$ for some $\beta > 0$ if and only if $\{U(t,s)\}_{t\geq s}$ has exponential dichotomy. Before proceeding, we pause to motivate the ideas as they relate to center manifold theory (see, e.g., Daleckij and Krein (1974) and Vanderbauwhede and Iooss (1992)).

In their study of center manifold theory in infinite dimensions, Vanderbauwhede and Iooss (1992) consider a semilinear differential equation of the form $y' = By + g(y)$ on a Banach space $Y$. A standing hypothesis in their work assumes the existence of a $B$-invariant decomposition $Y = Z \oplus X$ with restrictions $A = B|_X$ and $C = B|_Z$. Here, $Z$ represents a "central" part in the sense that $\sigma(C) \subset i\mathbb{R}$ (or, as in Daleckij and Krein (1974) or Yi(1993), one can assume that $\sigma(C)$ lies in a strip around $i\mathbb{R}$). Further, $X$ represents a "hyperbolic" part. The hyperbolicity condition of Vanderbauwhede and Iooss (1992) (see their Condition (H)) is given in terms of the inhomogeneous equation (0.3) with $A(t) \equiv A$ and can be reformulated as follows:

(H) *There exists $\beta > 0$ such that for each $\alpha \in [0,\beta)$ the following condition holds: For every $f \in E_\alpha$ there exists a unique solution $x \in E_\alpha$ to* (0.3) *with $A(t) \equiv A$. The solution is given by $x = Rf$ for some $R \in B(E_\alpha)$ with the property that $\|R\|_{B(E_\alpha)} \leq \gamma(\alpha)$ for some continuous function $\gamma : [0,\beta) \to \mathbb{R}_+$.*

We will prove, in particular, that hypothesis (H) is equivalent to the assumption that the evolution family $\{e^{(t-s)A}\}_{t\geq s}$ has exponential dichotomy. This is contained in the next theorem which is valid for nonautonomous equations and allows for merely mild solutions; we note that Vanderbauwhede and Iooss (1992) uses, in fact, only mild solutions to (0.3).

The proof of the theorem uses the following observation about the evolution family $\{U_\alpha(t,s)\}_{t\geq s}$ with operators $U_\alpha(t,s) = \frac{e^{-\alpha|t|}}{e^{-\alpha|s|}} U(t,s)$, $t \geq s$.

**Lemma 2.2.** $(\mathrm{M}_{E_\alpha})$ *holds for $\{U(t,s)\}_{t\geq s}$ if and only if* (M) *holds for $\{U_\alpha(t,s)\}_{t\geq s}$.*

*Proof.* First note that condition (M) for $\{U_\alpha(t,s)\}_{t\geq s}$ defines a bounded operator $R_\alpha$ on $C_b(\mathbb{R},X)$: $R_\alpha f = u$. Now consider $\Omega_\alpha : E_\alpha \to C_b(\mathbb{R},X)$ where $\Omega_\alpha(f) = e^{-\alpha|\cdot|}f$. The condition $(\mathrm{M}_{E_\alpha})$ for $\{U(t,s)\}_{t\geq s}$ defines a bounded operator $R \in B(E_\alpha)$. We have $R_\alpha = \Omega_\alpha R \Omega_\alpha^{-1}$. Therefore, (M) holds for $\{U_\alpha(t,s)\}_{t\geq s} \iff R_\alpha \in B(C_b(\mathbb{R},X))$ $\iff R \in B(E_\alpha) \iff (\mathrm{M}_{E_\alpha})$ holds for $\{U(t,s)\}_{t\geq s}$. □

**Theorem 2.3.** *Let $\{U(t,s)\}_{t\geq s}$ be an exponentially bounded evolution family. The following are equivalent:*

*i)* $\{U(t,s)\}_{t\geq s}$ *has exponential dichotomy;*

*ii) there exists* $\beta > 0$ *such that for each* $\alpha \in [0,\beta)$ *condition* $(\mathrm{M}_{E_\alpha})$ *holds for* $\{U(t,s)\}_{t\geq s}$.

*Moreover, if either one of (i) or (ii) holds, then the dichotomy bound* $\delta$ *for* $\{U(t,s)\}_{t\geq s}$ *can be estimated as follows:*

$$\delta \leq \sup\{\beta > 0: \text{ for each } \alpha \in [0,\beta) \text{ condition } (M_{E_\alpha}) \text{ holds for } \{U(t,s)\}_{t\geq s}\}. \tag{2.1}$$

*In addition, for each* $\alpha > 0$ *and each* $f \in E_\alpha$ *the solution to* (0.1) *is given by* $u = Rf$*, where the operator* $R \in B(E_\alpha)$ *is equal to the Green's operator* $\hat{G}$ *on* $E_\alpha$ *as defined in* (1.1)*, and has the property that* $\|R\|_{B(E_\alpha)} \leq \gamma(\alpha)$ *for some continuous function* $\gamma : [0,\delta) \to \mathbb{R}_+$.

*Proof.* Clearly, (ii) implies (i) by Theorem 1.1.

To prove that (i) implies (ii), we suppose that $\{U(t,s)\}_{t\geq s}$ has exponential dichotomy with some constants $\beta \in (0,\delta)$ and $M = M(\beta)$. Then for any $\alpha \in (0,\beta)$, using Definition 1.1, $\{U_\alpha(t,s)\}_{t\geq s}$ has exponential dichotomy with constants $\beta - \alpha$ and $M$. By Theorem 1.1, condition (M) holds for $\{U_\alpha(t,s)\}_{t\geq s}$. By Lemma 2.2, condition $(\mathrm{M}_{E_\alpha})$ holds for $\{U(t,s)\}_{t\geq s}$. This also shows (2.1).

As for the rest of the theorem, suppose that $\{U(t,s)\}_{t\geq s}$ has exponential dichotomy with some constants $M(\beta)$ and $\beta < \delta$. Then for $\alpha < \beta$, $\{U_\alpha(t,s)\}_{t\geq s}$ has exponential dichotomy with constants $M(\beta)$ and $\beta - \alpha$. By Theorem 1.1, (M) holds for $\{U_\alpha(t,s)\}_{t\geq s}$ and so $(\mathrm{M}_{E_\alpha})$ holds for $\{U(t,s)\}_{t\geq s}$. Now consider $R \in B(E_\alpha)$ defined as $Rf = u$ by condition $(\mathrm{M}_{E_\alpha})$. Let $R_\alpha = \Omega_\alpha R \Omega_\alpha^{-1}$ as in the proof of the above lemma. Further, $R_\alpha$ equals the operator $\hat{G}_\alpha$ corresponding to $\{U_\alpha(t,s)\}_{t\geq s}$ defined as in (1.1). We have

$$\|R\|_{B(E_\alpha)} = \|R_\alpha\|_{B(C_b(\mathbb{R},X))} = \|\hat{G}_\alpha\|_{B(C_b(\mathbb{R},X))} \leq \frac{2M(\beta)}{\beta - \alpha}.$$

It is now easy to find the required function $\gamma$. □

*Remark* 2.4. It is possible that $(\mathrm{M}_{E_\alpha})$ holds for all $\alpha \in (0,\beta)$, but $\{U(t,s)\}_{t\geq s}$ has no exponential dichotomy. Moreover, the inequality in (2.1) may be strict.

*Proof.* Let $X = \mathbb{C}$ and $0 \leq \epsilon < 1$. By setting

$$U^{(\epsilon)}(t,s) = \begin{cases} e^{\epsilon(t-s)}, & 0 > t \geq s, \\ e^t\, e^{-\epsilon s}, & t \geq 0 \geq s, \\ e^{t-s}, & t \geq s > 0, \end{cases}$$

we define a strongly continuous evolution family. For $\alpha \geq 0$ we have

$$U_\alpha^{(\epsilon)}(t,s) = e^{-\alpha|t|}\, e^{\alpha|s|}\, U^{(\epsilon)}(t,s) = \begin{cases} e^{(\alpha+\epsilon)(t-s)}, & 0 > t \geq s, \\ e^{(1-\alpha)t}\, e^{-(\epsilon+\alpha)s}, & t \geq 0 \geq s, \\ e^{(1-\alpha)(t-s)}, & t \geq s > 0. \end{cases}$$

Consequently, $\{U^{(0)}(t,s)\}_{t\geq s}$ has no exponential dichotomy, but $(\mathrm{M}_{E_\alpha})$ holds for $\{U^{(0)}(t,s)\}_{t\geq s}$ and each $0 < \alpha < 1$. Moreover, for $\epsilon > 0$ we obtain $\delta(\{U^{(\epsilon)}(t,s)\}_{t\geq s}) = \epsilon$, but $(\mathrm{M}_{E_\alpha})$ holds for $\{U^{(\epsilon)}(t,s)\}_{t\geq s}$ and each $0 < \alpha < 1$. □

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Department of Mathematics, University of Missouri, Columbia, MO 65211
*E-mail address*: yuri@math.mizzou1.missouri.edu

Department of Mathematics, University of Missouri, Rolla, MO 65409
*E-mail address*: randolph@umr.edu

Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany
*E-mail address*: rosc@michelangelo.mathematik.uni-tuebingen.de