Lecture Notes
Nonlinear Evolution Equations

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These lecture notes are based on my course from summer semester 2019, though there are small corrections and improvements, as well as minor changes in the numbering. Typically, the proofs and calculations in the notes are a bit shorter than those given in class. The drawings and many additional oral remarks from the lectures are omitted here. On the other hand, the notes contain a few proofs (mostly of peripheral statements) which have not been presented in the lectures. I partly use basic notation, definitions and facts contained in my other lecture notes without further notice.

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Contents

CHAPTER 1

Semilinear evolution equations: the basic case

Throughout $X, Y \neq \{0\}$ are Banach spaces with norm $\|\cdot\|$. By the same symbol we denote the operator norm on the space of continuous linear operators $\mathcal{B}(X,Y)$, where we put $\mathcal{B}(X) := \mathcal{B}(X,X)$. Sometimes we indicate the spaces as a subscript in norms. The spaces below have complex scalars, unless we explicitly require them to be real. For unexplained basic notation we refer to the lectures notes [**FA**], [**ST**] and [**EE**].

Chapters 1 to 4 are devoted to *semilinear evolution equations* of the form

$$u'(t) = Au(t) + F(u(t)), \quad t \in J, \qquad u(0) = u_0, \tag{1.1}$$

where A generates a C_0–semigroup $T(\cdot)$ on X and the nonlinearity F is subordinate to or of 'lower order' than A. In the present chapter we treat the simplest case that $F : X \to X$ is Lipschitz on bounded sets. As we see in the first section, one can extend the Picard–Lindelöf Theorem 4.6 of [**Ana4**] to this setting modulo certain regularity issues. In the second section we apply the developed theory to the cubic semilinear wave equation on subsets of $\mathbb{R}^3$ and discuss some basic features of its longtime behavior. In later chapters we refine and extend these results and methods to treat semilinear and quasilinear parabolic problems like reaction-diffusion systems and the nonlinear Schrödinger equation on $\mathbb{R}^m$, where we allow for powers up to 5 in the case $m = 3$.

1.1. Local wellposedness

We study equation (1.1) under the assumptions

$$\begin{aligned}
&A \text{ generates the } C_0\text{–semigroup } T(\cdot) \text{ on } X, \quad M_0 := \sup\nolimits_{t\in[0,1]} \|T(t)\|.\\
&J \subseteq \mathbb{R} \text{ is an interval with } \min J = 0. \quad F : X \to X \text{ satisfies}\\
&\forall\, r > 0 \;\; \exists\, L(r) > 0 \;\; \forall\, x, y \in \overline{B}(0,r) : \quad \|F(x) - F(y)\| \le L(r)\, \|x - y\|,\\
&\text{where the map } \quad r \mapsto L(r) \;\text{ is non-decreasing.}
\end{aligned} \tag{1.2}$$

If the estimate for F in (1.2) is true for some constants $\tilde{L}(r)$, we can replace them by the larger numbers $L(r) = \sup_{0\le s\le r} \tilde{L}(s)$ which do not decrease in r.

In essentially the same way one can also treat F which are only defined on an open subset $D \subseteq X$ or explicitely depend on t. Moreover, if $T(\cdot)$ is even a C_0-group one can also consider general time intervals J containing 0, cf. Chapter 4. We first note that (1.2) is a rather strong condition for substitution operators on L^p spaces with $p \in [1, \infty)$.

EXAMPLE 1.1. Let $X = L^p(\mu)$ for a measure space $(S, \mathcal{A}, \mu)$, $p \in [1, \infty)$, and $f : \mathbb{C} \to \mathbb{C}$ be Lipschitz with constant L. Moreover, let either $f(0) = 0$ or

$\mu(S) < \infty$. Set $(F(v))(s) = f(v(s))$ for $v \in X$ and $s \in S$. Then the map $F : X \to X$ is (globally) Lipschitz.

Indeed, let $v, w \in X$ and $s \in S$. We first note that $F(v)$ belongs to X since

$$|F(v)(s)| \le |f(v(s)) - f(0)| + |f(0)| \le L\,|v(s)| + |f(0)|.$$

We further estimate

$$|F(v)(s) - F(w)(s)| = |f(v(s)) - f(w(s))| \le L\,|v(s) - w(s)|,$$

and then the claim follows by taking the p-norm.

Here one cannot allow for locally Lipschitz f, in general. Take $S = (0,1)$, $\mu = \lambda$, and $f(z) = |z|^{\alpha-1}z$ for some $\alpha > 1$ as an example. Fix $\beta = 1/(\alpha p)$. The map $v(s) = s^{-\beta}$ then belongs to X, but $|f(v(s))| = s^{-1/p}$ does not.[1] ♢

We stress that in (1.1) the existence interval J is part of the problem. Finite time blowup already occurs for the simple ordinary differential equation

$$u'(t) = u(t)^2, \quad t \ge 0, \qquad u(0) = u_0 > 0,$$

whose solution $u(t) = (u_0^{-1} - t)^{-1}$ only exists up to time $1/u_0$.

We first state a natural solution concept. There are several variants in the literature (and we also introduce another notion below), so that we occasionally add the adjective 'classical'. Note that in some areas this word refers to somewhat different solution concepts.

Definition 1.2. *Let* (1.2) *be true and* $u_0 \in X$*. A* (classical) solution u *of* (1.1) *on* J *is a map* $u \in C^1(J, X)$ *satisfying* $u(t) \in \mathrm{D}(A)$ *and* (1.1) *for all* $t \in J$.

We state a few simple properties of solutions. The fixed point equation (1.3) is the starting point of our approach to semilinear evolution equations.

Remark 1.3. Let (1.2) be true, $u_0 \in X$, and u solve (1.1).

a) The initial value u_0 then must belong to $\mathrm{D}(A)$. The assumptions imply that $F \circ u : J_0 \to X$ is Lipschitz for all compact intervals $J_0 \subseteq J$. Moreover, the solution u is contained in $C(J, [\mathrm{D}(A)])$ since (1.1) yields $Au = u' - F(u)$.

b) From Duhamel's formula in Proposition 2.6 in [**EE**] we deduce that

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(u(s))\,\mathrm{d}s, \qquad t \in J. \tag{1.3}$$

c) Let $v \in C(J, X)$. Then also $F \circ v$ is an element of $C(J, X)$. ♢

In view of Remark 1.3 c), equation (1.3) makes sense for any continuous function u and can thus serve as a weaker solution concept for (1.1).

Definition 1.4. *Let* (1.2) *be true and* $u_0 \in X$*. A* mild solution *of* (1.1) *on* J *is a function* $u \in C(J, X)$ *satisfying* (1.3).

Observe that a mild solution fulfills $u(0) = u_0$. Notions and results from the lecture Evolution Equations, see Section 2.1 of [**EE**], allow us to interpret mild solutions as classical solutions in the (larger) *extrapolation space* X_{-1} of A. We recall that X_{-1} is the completion of X for the norm given by $\|x\|_{-1} =$

[1] In the lectures only a special case was discussed.

$\|R(\omega, A)x\|_X$ for some $\omega \in \rho(A)$, where X is considered as a linear subspace of X_{-1}. The operator A has the extension $A_{-1} \in \mathcal{B}(X, X_{-1})$ which generates the C_0-semigroup $T_{-1}(\cdot)$ on X_{-1} given by extensions of $T(t)$.

REMARK 1.5. Let (1.2) be true, $u_0 \in X$, and u be mild solution of (1.1). Since $F(u) \in C(J, X)$, Proposition 2.15 of [**EE**] implies that u belongs to $C^1(J, X_{-1})$ and that it is the classical solution of the evolution equation

$$u'(t) = A_{-1}u(t) + F(u(t)), \quad t \in J, \qquad u(0) = u_0,$$

in X_{-1}. ◊

As a first step we solve (1.3) on $[0, b]$ for small times $b > 0$ (only depending $\|u_0\|$, M_0, and F) and uniquely in certain balls of $C([0, b], X)$. Here we proceed as for ordinary differential equations, but now using semigroup theory to separate the linear main part given by $T(\cdot)$. In view of more complicated problems, we stress that one should be careful with the constants here. They must be under control as b tends to 0, and one should specify how they depend on u_0.

LEMMA 1.6. *Let (1.2) be true. Take any $\rho > 0$. Then there is a number $b_0(\rho) > 0$ (given by (1.8) below) such that for each $u_0 \in \overline{B}(0, \rho)$ there is a unique mild solution $u \in C([0, b_0(\rho)], X)$ of (1.1) on $[0, b_0(\rho)]$ satisfying $\|u(t)\| \leq 1 + M_0\rho =: r$ for all $0 \leq t \leq b_0(\rho)$. For each $b \in (0, b_0(\rho)]$, the restriction $u|_{[0,b]}$ is also the unique solution of (1.1) on $[0, b]$ with sup-norm less or equal than r. Finally, the function b_0 is non-decreasing.*

PROOF. Let $\rho > 0$ and take $u_0 \in X$ with $\|u_0\| \leq \rho$. Fix $r = 1 + M_0\rho$. For $b \in (0, 1]$ to be specified below, we define the closed ball

$$E(b) = E(b, r) = \{v \in C([0, b], X) \mid \|v\|_\infty \leq r\}. \tag{1.4}$$

It is a complete metric space for the metric induced by the sup-norm $\|\cdot\|_\infty$ on $C([0, b], X)$. To solve (1.3), we further introduce the map

$$[\Phi_{u_0}(v)](t) = \Phi(v)(t) := T(t)u_0 + \int_0^t T(t - s)F(v(s))\,\mathrm{d}s \tag{1.5}$$

for $t \in [0, b]$ and $v \in E(b)$. The function $\Phi(v)$ belongs to $C([0, b], X)$ by Remark 1.3 c) and, e.g., dominated convergence as stated in Remark 1.16 of [**EE**]. Each mild solution $u \in E(b)$ of (1.1) is a fixed point of Φ in $E(b)$, and vice versa.

Let $v, w \in E(b)$ and $t \in [0, b] \subseteq [0, 1]$. Using (1.2) and that $v(s), w(s) \in \overline{B}(0, r)$, we estimate

$$\begin{aligned}
\|\Phi(v)(t)\| &\leq M_0\|u_0\| + \int_0^t M_0\big(\|F(v(s)) - F(0)\| + \|F(0)\|\big)\,\mathrm{d}s \\
&\leq M_0\rho + tM_0\,(L(r) \max_{s\in[0,t]} |v(s)|_\infty + \|F(0)\|) \\
&\leq M_0\rho + bM_0\,(L(r)r + \|F(0)\|), \qquad (1.6)\\
\|\Phi(v)(t) - \Phi(w)(t)\| &\leq \int_0^t M_0\,\|F(v(s)) - F(w(s))\|\,\mathrm{d}s \\
&\leq bM_0L(r)\,\|v - w\|_\infty\,. \qquad (1.7)
\end{aligned}$$

We choose a final time $b \in (0, b_0(\rho)]$, setting

$$b_0(\rho) = \min\left\{1, \frac{1}{M_0(L(r)r + \|F(0)\|)}, \frac{1}{2M_0L(r)}\right\} \in (0,1]. \tag{1.8}$$

It follows that $\Phi(v) \in E(b)$ and that $\Phi : E(b) \to E(b)$ is Lipschitz with constant $\frac{1}{2}$. Banach's theorem then gives a unique fixed point $u_b = \Phi(u_b) \in E(b)$ for each $b \in (0, b_0(\rho)]$. Since $u := u_{b_0(\rho)}$ solves (1.3) also on $[0, b]$ and the norm in $E(b)$ does not exceed that in $E(b_0(\rho))$, uniqueness implies that $u|_{[0,b]} = u_b$. □

We note that the above lemma only gives a *conditional* uniqueness result among functions belonging to a certain ball. To derive unconditional uniqueness (and further properties), we show a simple result allowing us to glue and shift solutions. We work here on the level of mild solutions though it would be simpler to treat them as classical ones in X_{-1}, cf. Remark 1.5. However, the arguments below are of also of interest in other situations.

LEMMA 1.7. *Let* (1.2) *be true and* $u_0 \in X$. *Assume that* $u \in C([0, b_1], X)$ *is a mild solution of* (1.1) *on* $[0, b_1]$. *Then the following assertions hold.*

a) Let $v \in C([0, b_2], X)$ *be a mild solution of* (1.1) *on* $[0, b_2]$ *with the initial value* $u(b_1)$. *Then the concatenated function* $w \in C([0, b_1 + b_2], X)$ *given by*

$$w(t) = \begin{cases} u(t), & 0 \le t \le b_1, \\ v(t - b_1), & b_1 < t \le b_1 + b_2, \end{cases}$$

solves (1.1) *mildly on* $[0, b_1 + b_2]$ *with the initial value* u_0.

b) Let $\beta \in (0, b_1)$. *Then the shifted function* $u(\cdot + \beta) \in C([0, b_1 - \beta], X)$ *is a mild solution of* (1.1) *with the initial value* $u(\beta)$.

PROOF. a) By its definition, w is continuous and it is a mild solution of (1.1) for $t \in [0, b_1]$. Let $t \in (b_1, b_1 + b_2]$. We use the definition of w, (1.3) for u and v, and the semigroup property of $T(\cdot)$. Also substituting $r = b_1 + s$, we then calculate

$$\begin{aligned} w(t) = v(t - b_1) &= T(t - b_1)u(b_1) + \int_0^{t-b_1} T(t - b_1 - s)F(v(s))\,\mathrm{d}s \\ &= T(t-b_1)\Big[T(b_1)u_0 + \int_0^{b_1} T(b_1 - s)F(u(s))\,\mathrm{d}s\Big] + \int_{b_1}^t T(t - r)F(v(r - b_1))\,\mathrm{d}r \\ &= T(t)u_0 + \int_0^t T(t - s)F(w(s))\,\mathrm{d}s. \end{aligned}$$

b) Set $\tilde{u}(t) = u(t + \beta)$ for $t \in [0, b_1 - \beta]$. As above, we obtain

$$\begin{aligned} \tilde{u}(t) = u(t + \beta) &= T(t + \beta)u_0 + \int_0^{t+\beta} T(t + \beta - s)F(u(s))\,\mathrm{d}s \\ &= T(t)\Big[T(\beta)u_0 + \int_0^{\beta} T(\beta - s)F(u(s))\,\mathrm{d}s\Big] + \int_0^t T(t - r)F(u(r + \beta))\,\mathrm{d}r \\ &= T(t)u(\beta) + \int_0^t T(t - s)F(\tilde{u}(s))\,\mathrm{d}s. \end{aligned}$$

□

We mainly have to use shifted functions since we state and use problems like (1.1) only for the initial time 0.

In a second step we now show *unconditional uniqueness* of all mild solutions to (1.1). We deduce it from the uniqueness statement of Lemma 1.6; a more direct variant is indicated below.

LEMMA 1.8. *Let (1.2) be true and $u_0 \in X$. Assume that u and v are mild solutions of (1.1) on J_u respectively J_v. Then $u = v$ on $J_u \cap J_v$.*

PROOF. Set $J = J_u \cap J_v$. Since $u(0) = v(0)$, the number

$$\tau := \sup\{b \in J \,|\, \forall\, t \in [0, b]: \ u(t) = v(t)\}$$

belongs to $[0, \sup J]$. We assume that $u \neq v$ on J. By continuity, it follows $\tau < \sup J$ and $u(\tau) = v(\tau) =: u_1$. Hence, there are times $t_n \in J$ with $t_n \to \tau^+$ and $u(t_n) \neq v(t_n)$. Fix $\beta_0 > 0$ with $\tau + \beta_0 \in J$. For every $\beta \in (0, \beta_0]$, Lemma 1.7 shows that the functions $\tilde{u} = u(\,\cdot\, + \tau)$ and $\tilde{v} = v(\,\cdot\, + \tau)$ are mild solutions of (1.1) on $[0, \beta]$ with initial value u_1.

We now set $\rho = \|u_1\|$ and $r = 1 + M_0 \rho$, and use the number $b_0(\rho)$ from (1.8). For sufficiently small times $0 < \beta \le \min\{b_0(\rho), \beta_0\}$, the continuous maps $\tilde{u}$ and $\tilde{v}$ have sup-norms less or equal r on $[0, \beta]$ because of $\tilde{u}(0) = \tilde{v}(0) = u_1$. The uniqueness statement of Lemma 1.6 then shows that $\tilde{u}(t) = \tilde{v}(t)$ for $t \in [0, \beta]$, which contradicts the inequality $u(t_n) \neq v(t_n)$ for large n. □

In the above proof, after having fixed τ and β_0 one can also proceed differently. Let r_1 be the maximum of the sup-norms of u and v on $[\tau, \tau + \beta_1]$, where $\beta_1 := \min\{1, \beta_0\}$. Lemma 1.7 implies that

$$u(t) - v(t) = \int_\tau^t T(t - s)[F(u(s)) - F(v(s))] \, \mathrm{d}s$$

for all $t \in [\tau, \tau + \beta_1]$. As in (1.7), assumption (1.2) then leads to

$$\|u(t) - v(t)\| \le M_0 L(r_1) \int_\tau^t \|u(s) - v(s)\| \, \mathrm{d}s, \tag{1.9}$$

so that $u = v$ on $[\tau, \tau + \beta_1]$ by Gronwall's inequality, see Lemma 4.5 in [**Ana4**].[2]

In a third step we can now introduce a unique maximal solution.

DEFINITION 1.9. *Let (1.2) be true. For each initial value $u_0 \in X$ we define its* maximal existence time

$$t^+(u_0) = \sup\{b > 0 \,|\, \exists \text{ mild solution } u \text{ of } (1.1) \text{ on } [0, b]\}.$$

The maximal existence interval is $J^+(u_0) = [0, t^+(u_0))$. A mild solution u of (1.1) on $J^+(u_0)$ is called maximal solution.

The above lemmas easily imply that there is a unique maximal solution.

REMARK 1.10. Let (1.2) be true and $u_0 \in X$.

a) Lemma 1.6 provides a mild solution u of (1.1) on $[0, b_0(\|u_0\|)]$. We can also use this lemma to solve (1.1) with initial value $u(b_0(\|u_0\|))$. Lemma 1.7 then

[2]This argument was not given in the lectures.

yields a concatenated solution of (1.1) on an interval larger than $[0, b_0(\|u_0\|)]$, so that $t^+(u_0)$ belongs to $(b_0(\|u_0\|), \infty]$.

b) Let $b \in (0, t^+(u_0))$. By Definition 1.9, there is a mild solution u_b of (1.1) on $[0, b]$. Lemma 1.8 says that $u_b = u_{b'}$ on $[0, b']$ for $0 < b' < b < t^+(u_0)$. We can thus define a maximal solution of (1.1) by setting $u(t) = u_b(t)$ for $t \in [0, b] \subseteq (0, t^+(u_0))$. It is uniquely determined because of Lemma 1.8.

c) We note that the existence interval of this solution has to be right-open due to Theorem 1.11 b) below. $\Diamond$

Local wellposedness means that one has for all (or sufficiently many) initial values unique solutions of (1.1) that continuously depend on the initial values. These properties are necessary to make a prediction of the future behavior of the system that is robust under errors in the initial data and could thus be tested by an experiment. In a fourth step, we now show the continuity of $u_0 \mapsto u(t)$ near u_0 for any compact subinterval of $J^+(u_0)$. This fact is also needed if an argument only works for a dense subset of 'better' initial values, cf. Proposition 1.20. One should also prove continuous dependence on F or A. At least for F we will do this in Proposition 3.5 in a somehat different situation.

THEOREM 1.11. *Let (1.2) be true, $u_0 \in X$, and $b_0 = b_0(\|u_0\|) > 0$ be given by (1.8). Then the following assertions hold.*

a) There is a unique maximal mild solution $u = \varphi(\,\cdot\,, u_0) \in C([0, t^+(u_0)), X)$ of (1.1), where $t^+(u_0) \in (b_0(\|u_0\|), \infty]$.

b) If $t^+(u_0) < \infty$, then $\lim_{t \to t^+(u_0)^-} \|u(t)\| = \infty$.

c) Take any $b \in (0, t^+(u_0))$. Then there exists a radius $\delta = \delta(u_0, b) > 0$ such that $t^+(v_0) > b$ for all $v_0 \in \overline{B}(u_0, \delta)$. Moreover, the map

$$\overline{B}(u_0, \delta) \to C([0, b], X); \quad v_0 \mapsto \varphi(\,\cdot\,, v_0),$$

is Lipschitz continuous.

PROOF. Assertion a) was shown in Remark 1.10. To establish b), let $t^+(u_0) < \infty$. Assume that there were times $t_n \to t^+(u_0)$ for $n \to \infty$ with $t_n \in [0, t^+(u_0))$ and $C := \sup_{n \in \mathbb{N}} \|u(t_n)\| < \infty$. We choose an index $m \in \mathbb{N}$ such that $t_m + b_0(C) > t^+(u_0)$, where $b_0(C) > 0$ is given by (1.8). Lemma 1.6 yields a mild solution of (1.1) on $[0, b_0(C)]$ with initial value $u(t_m)$. By means of Lemma 1.7, we thus obtain a mild solution of (1.1) on $[0, t_m + b_0(C)]$ which contradicts the definition of $t^+(u_0)$. So claim b) is shown. We prove part c) by a basic step plus an induction argument in three more steps.

1) Let $b \in (0, t^+(u_0))$ and $u = \varphi(\,\cdot\,, u_0)$. We fix a number $b' \in (b, t^+(u_0))$ and use the radii $\overline{\rho} := 1 + \max_{0 \le t \le b'} \|u(t)\|$ and $\overline{r} := 1 + M_0 \overline{\rho}$. Let the time $\overline{b} := b_0(\overline{\rho}) \in (0, 1]$ be given by (1.8) and the operator Φ_{u_0} by (1.5). Take $v_0, w_0 \in \overline{B}(0, \overline{\rho})$. Lemma 1.6 and part a) provide mild solutions $v = \Phi_{v_0}(v) = \varphi(\,\cdot\,, v_0)$ and $w = \Phi_{w_0}(w) = \varphi(\,\cdot\,, w_0)$ of (1.1) on $[0, \overline{b}]$ with the initial values v_0 respectively w_0, where v and w are contained in the space $E(\overline{b}, \overline{r})$ from (1.4) endowed with the sup-norm $\|\cdot\|_\infty$ on $[0, \overline{b}]$. Formulas (1.7), (1.8) and (1.5) lead to the estimate

$$\|v - w\|_\infty \le \|\Phi_{v_0}(v) - \Phi_{v_0}(w)\|_\infty + \|\Phi_{v_0}(w) - \Phi_{w_0}(w)\|_\infty$$

$$\leq \tfrac{1}{2}\|v-w\|_\infty + \|T(\cdot)(v_0-w_0)\|_\infty \leq \tfrac{1}{2}\|v-w\|_\infty + M_0\,\|v_0-w_0\|,$$

$$\|v-w\|_\infty \leq 2M_0\,\|v_0-w_0\|. \tag{1.10}$$

2) We next show $t^+(v_0) > b$ inductively. For $j \in \mathbb{N}_0$ we set $b_j = j\overline{b}$. There exists a minimal index $N \in \mathbb{N}$ with $b_N > b$. If $b_N > b'$ we redefine $b_N := b' \in (b, t^+(u_0))$. We choose $\delta = (2M_0)^{-N} \in (0,1]$. We inductively show that for every $v_0 \in \overline{B}(u_0,\delta)$ and $j \in \{0,\dots,N-1\}$ the maximal mild solution $v = \varphi(\,\cdot\,; v_0)$ exists at least on $[0, b_{j+1}]$ and that $v(t)$ is an element of the ball $\overline{B}(u(t),(2M_0)^{j+1-N})$ for $t \in [b_j, b_{j+1}]$, which belongs to $\overline{B}(0,\overline{\rho})$ because of $\overline{\rho} \geq 1 + \|u\|_\infty$. This claim then yields $t^+(v_0) > b$.

3) We prove the claim. First let $j = 0$. Since $\delta \leq 1$, the vector v_0 is contained in $\overline{B}(0,\overline{\rho})$. From estimate (1.10) with $w = u$ we deduce

$$\|v(t)-u(t)\| \leq 2M_0\,\|v_0-u_0\| \leq 2M_0\delta = (2M_0)^{1-N}$$

for all $t \in [0, b_1]$, as asserted for $j = 0$.

Second, assume that the claim has been established for all $k \in \{0,\dots,j-1\}$ and some $j \in \{1,\dots,N-1\}$. It follows $\|v(b_j)\| \leq \overline{\rho}$. Lemmas 1.6 and 1.7 thus show that v exists at least on $[0, b_{j+1}]$. Moreover, the inequality (1.10) can applied to $v(t+b_j) = \varphi(t, v(b_j))$ and $u(t+b_j) = \varphi(t, u(b_j))$ for $t \in [0,\overline{b}]$. Using also the induction hypothesis, we infer the bound

$$\|v(t+b_j)-u(t+b_j)\| \leq 2M_0\,\|v(b_j)-u(b_j)\| \leq (2M_0)^{j+1-N}$$

for $t \in [0,\overline{b}]$. So the claim is true.

4) It remains to prove the Lipschitz continuity asserted in c). Let $j \in \{0,\dots,N-1\}$ and $t \in [b_j, b_{j+1}]$. By the claim in 2), the vectors $v(b_j)$ and $w(b_j)$ belong to $\overline{B}(0,\overline{\rho})$. As in step 3), inequality (1.10) implies

$$\|v(t+b_j)-w(t+b_j)\| = \|\varphi(t,v(b_j))-\varphi(t,w(b_j))\| \leq 2M_0\,\|v(b_j)-w(b_j)\|$$
$$\leq \dots \leq (2M_0)^{j+1}\|v_0-w_0\| \leq (2M_0)^N\|v_0-w_0\|. \qquad \square$$

We add a simple example for Theorem 1.11. In Section 1.2 we discuss a more sophisticated application.

Example 1.12. Let $X = L^2(\mathbb{R}^m)$ and $A = \mathrm{i}\Delta$ with $\mathrm{D}(A) = W^{2,2}(\mathbb{R}^m)$. This operator generates a unitary C_0-group on X by Example 3.9 in [**EE**]. Let $f : \mathbb{C} \to \mathbb{C}$ be (globally) Lipschitz and $f(0) = 0$. Example 1.1 says that $F(v) = f \circ v$ defines a Lipschitz map $F : X \to X$. From Theorem 1.11 we then deduce that for each $u_0 \in L^2(\mathbb{R}^m)$ the nonlinear Schrödinger equation

$$u'(t) = \mathrm{i}\Delta u(t) + \mathrm{i}F(u(t)), \quad t \in J, \qquad u(0) = u_0,$$

has unique mild local solution u, which is locally Lipschitz continuous in u_0. We will improve these statements in Chapter 4 considerably. $\Diamond$

The easiest general condition for global existence is linear growth of F as we show next. A more refined condition is given in the exercises, and we discuss the long-time behavior in some detail for a specific problem in the next section. These deeper results need the regularity theory presented below.

Corollary 1.13. *Let* (1.2) *be true. Assume that there is a constant* $c > 0$ *such that* $\|F(x)\| \leq c\,(1 + \|x\|)$ *for all* $x \in X$. *We then obtain* $t^+(u_0) = \infty$ *for every* $u_0 \in X$.

Proof. One usually shows such results via contradiction to reduce the argument to a bounded time interval. So assume that $\tau := t^+(u_0) < \infty$ for some $u_0 \in X$. Then also the number $K := \sup_{t\in[0,\tau]} \|T(t)\|$ is finite. From (1.3) and the assumption, we infer the inequality

$$\|u(t)\| \leq \|T(t)u_0\| + \int_0^t \|T(t-s)F(u(s))\|\,\mathrm{d}s \leq K\,\|u_0\| + \int_0^t Kc\,(1 + \|u(s)\|)\,\mathrm{d}s$$
$$\leq K(\|u_0\| + c\tau) + cK\int_0^t \|u(s)\|\,\mathrm{d}s$$

for all $t \in [0,\tau)$. Gronwall's Lemma 4.5 in [**Ana4**] now yields the uniform bound $\|u(t)\| \leq K(\|u_0\| + c\tau)\mathrm{e}^{cK\tau}$ which contradicts Theorem 1.11 b). □

In a fifth step we will show that the mild solution is actually a classical one on the *same* maximal existence interval provided that $u_0 \in \mathrm{D}(A)$ and F is a bit more regular. To this aim, we need some preparations. For the linear case $F = 0$ the next result is clear since then $u' = Au = T(\cdot)Au_0$ is locally bounded.

Lemma 1.14. *Let* (1.2) *be true and* $u_0 \in \mathrm{D}(A)$. *Then the maximal mild solution* $u = \varphi(\,\cdot\,, u_0) : [0, t^+(u_0)) \to X$ *of* (1.1) *is locally Lipschitz continuous.*

Proof. Take $b \in [0, t^+(u_0))$ and $0 \leq t \leq t+h \leq b$. Equation (1.3) leads to

$$\begin{aligned} u(t+h) - u(t) &= T(t)(T(h)u_0 - u_0) + \int_0^h T(t+h-s)F(u(s))\,\mathrm{d}s \\ &\quad + \int_h^{t+h} T(t+h-s)F(u(s))\,\mathrm{d}s - \int_0^t T(t-\tau)F(u(\tau))\,\mathrm{d}\tau \\ &= \int_0^h T(t+s)Au_0\,\mathrm{d}s + \int_0^h T(t+h-s)F(u(s))\,\mathrm{d}s \\ &\quad + \int_0^t T(t-\tau)(F(u(\tau+h)) - F(u(\tau)))\,\mathrm{d}\tau, \end{aligned} \tag{1.11}$$

where we have used Lemma 1.19 of [**EE**] and subtituted $\tau = s - h$. The quantities $r = \sup_{0\leq s\leq b} \|u(s)\|$, $K = \sup_{0\leq s\leq b} \|T(s)\|$, and $C = \sup_{0\leq s\leq b} \|F(u(s))\|$ are finite. Formula (1.11) combined with (1.2) yields

$$\|u(t+h) - u(t)\| \leq K\|Au_0\|h + KCh + KL(r)\int_0^t \|u(s+h) - u(s)\|\,\mathrm{d}s.$$

Gronwall's inequality then implies that

$$\|u(t+h) - u(t)\| \leq K(\|Au_0\| + C)\mathrm{e}^{KL(r)b}h. \qquad \square$$

In our regularity theorem we will require that F is 'real continuously differentiable.' To that purpose, we define

$$\mathcal{B}_{\mathbb{R}}(X,Y) := \Big\{T : X \to Y \,\Big|\, T \text{ is } \mathbb{R}\text{-linear and } \|T\|_{\mathcal{B}_{\mathbb{R}}(X,Y)} := \sup_{\|x\|\leq 1} \|Tx\| < \infty\Big\},$$

recalling that the Banach spaces X and Y are complex. As for $\mathcal{B}(X,Y)$ one shows that $\mathcal{B}_{\mathbb{R}}(X,Y)$ is a Banach space when endowed with $\|\cdot\|_{\mathcal{B}_{\mathbb{R}}(X,Y)}$. Each map T in $\mathcal{B}_{\mathbb{R}}(X,Y)$ is Lipschitz continuous. We clearly have $\mathcal{B}(X,Y)\subset\mathcal{B}_{\mathbb{R}}(X,Y)$, but the converse inclusion is false even for $X=Y=\mathbb{C}$. As usual we write $\mathcal{B}_{\mathbb{R}}(X):=\mathcal{B}_{\mathbb{R}}(X,X)$.

Let $\emptyset\neq D\subseteq X$ be open. A map $F:D\to Y$ is called *real differentiable* at $x_0\in D$ if there is an operator $L\in\mathcal{B}_{\mathbb{R}}(X,Y)$ such that the limit

$$\lim_{\substack{h\to0,\,h\neq0,\\ x_0+h\in D}}\frac{1}{\|h\|}\|F(x_0+h)-F(x_0)-Lh\|=0$$

exists. We then set $F'(x_0):=L$ and call $F'(x_0)$ the *derivative* of F at x_0. We say that F is *real continuously differentiable* on D if F is real differentiable at each point of D and the function

$$F':D\to\mathcal{B}_{\mathbb{R}}(X,Y);\quad x\mapsto F'(x),$$

is continuous. In this case we write $F\in C^1_{\mathbb{R}}(D,Y)$. If the derivative is $\mathbb{C}$-linear, F is called *differentiable*, and we use the notation $C^1(D,Y)\subset C^1_{\mathbb{R}}(D,Y)$ if F' is also continuous. The usual rules of calculus (up to the implicit function theorem) hold in these settings with analogous proofs and straightforward modifications. See Chapter VII in [**AE2**]. We discuss examples in the next section which also show the neccessity to employ real differentiability.

If D is convex and $F\in C^1_{\mathbb{R}}(D,Y)$, the fundamental theorem of calculus (see Remark 1.16 in [**EE**]) and the chain rule yield the formula

$$F(z)-F(x)=\int_0^1\tfrac{\mathrm{d}}{\mathrm{d}t}\,F(x+t(z-x))\,\mathrm{d}t=\int_0^1F'(x+t(z-x))(z-x)\,\mathrm{d}t \quad (1.12)$$

for all $x,z\in D$. In this situation we thus obtain the inequality

$$\|F(z)-F(x)\|\le\max_{0\le t\le1}\|F'(x+t(z-x))\|\,\|z-x\| \quad (1.13)$$

for all $z,x\in D$. As a result, a function $F\in C^1_{\mathbb{R}}(X,Y)$ is Lipschitz on bounded sets provided that its derivative is bounded on bounded sets. (Observe that a continuous function on a Banach space does not need to be bounded on a closed ball.) We establish a final prerequisite.

Lemma 1.15. *Let $u\in C([a,b),X)$ be differentiable from the right with $\frac{\mathrm{d}^+}{\mathrm{d}t}v$ in $C([a,b),X)$. Then u belongs to $C^1([a,b),X)$ and $u'=v$.*

Proof. Fix $h\in(0,b-a)$ and $t\in(a+h,b)$. The Hahn-Banach theorem yields a functional $x_h^*\in X^*$ with $\|x_h^*\|=1$ and

$$\big|\big\langle\tfrac{1}{h}(u(t)-u(t-h))-v(t),x_h^*\big\rangle\big|=\big\|\tfrac{1}{h}(u(t)-u(t-h))-v(t)\big\|=:D_h(t).$$

By Corollary 2.1.2 of [**Pa**], the scalar function $\varphi_h:=\langle u,x_h^*\rangle$ is continuously differentiable on $[a,b)$, so that $\langle v,x_h^*\rangle=\frac{\mathrm{d}^+}{\mathrm{d}t}\varphi_h=\varphi_h'$. We then compute

$$\begin{aligned}D_h(t)&=\big|\tfrac{1}{h}(\varphi_h(t)-\varphi_h(t-h))-\varphi_h'(t)\big|=\Big|\frac{1}{h}\int_{t-h}^t(\varphi_h'(\tau)-\varphi_h'(t))\,\mathrm{d}\tau\Big|\\&=\Big|\frac{1}{h}\int_{t-h}^t\langle v(\tau)-v(t),x_h^*\rangle\,\mathrm{d}\tau\Big|\le\frac{h}{h}\max_{t-h\le\tau\le t}\|v(\tau)-v(t)\|.\end{aligned}$$

Since the right-hand side tends to 0 as $h \to 0$, the map u is differentiable at each $t \in [a, b)$ with the (continuous) derivative v. □

In the next proof we follow a standard strategy to prove additional regularity of a given (mild) solution. Assume for a moment that our mild solution u was in fact a solution of (1.1) in $C^1(J, X)$. Moreover, let $u_0 \in \mathrm{D}(A)$ and $F \in C^1_{\mathbb{R}}(X, X)$. One can then differentiate (1.1) with respect to t and obtain the linear (non-autonomous) evolution equation

$$v'(t) = Av(t) + F'(u(t))v(t), \quad t \in J^+(u_0), \qquad v(0) = u'(0) = Au_0 + F(u_0).$$

for $v := u'$, where the perturbations $B(t) := F'(u(t))$ belong to $\mathcal{B}_{\mathbb{R}}(X)$. We can now pass to the integrated version of this equation (see (1.14) below), which is easy to solve in our case. The resulting solution v is a candidate for the time derivative of u. To verify the differentiability of u, we then rewrite the difference quotient of u by means of (1.3) and subtract the equation (1.14) for v. A Gronwall type estimate finally yields the assertion.

Theorem 1.16. *Let (1.2) be true, $u_0 \in \mathrm{D}(A)$, $F \in C^1_{\mathbb{R}}(X, X)$, and assume that F' is bounded on bounded sets. Then the maximal mild solution $u = \varphi(\,\cdot\,; u_0)$ of (1.1) in fact solves (1.1) on $[0, t^+(u_0))$ classically.*

Proof. Let $u_0 \in \mathrm{D}(A)$ and $b \in (0, t^+(u_0))$. We have to show that u belongs to $C^1([0, b), X)$ since then $F \circ u$ is an element of $C^1([0, b), X)$ and thus the assertion will follow from Theorem 2.9 of [**EE**]. Set $K = \sup_{0 \le s \le b} \|T(s)\| < \infty$.

1) We first prove a preliminary result. The operators $B(s) := F'(u(s)) \in \mathcal{B}_{\mathbb{R}}(X)$ depend continuously on $s \in [0, b]$ and $L = \sup_{0 \le s \le b} \|B(s)\|$ is finite. We solve the ($\mathbb{R}$-linear, non-autonomous) problem

$$v(t) = T(t)(F(u_0) + Au_0) + \int_0^t T(t-s)B(s)v(s)\,\mathrm{d}s =: \Phi(v)(t) \tag{1.14}$$

as in Lemma 1.6 for $t \in [0, b]$. To this aim, put $\alpha = 2KL$. We endow $E = C([0, b], X)$ with the equivalent norm

$$\|v\|_\alpha = \max_{t \in [0,b]} \mathrm{e}^{-\alpha t} \|v(t)\|.$$

Let $v, w \in E$. The map Φv clearly belongs to E. We estimate

$$\begin{aligned}\|\Phi v - \Phi w\|_\alpha &\le \max_{t \in [0,b]} \Big\| \int_0^t \mathrm{e}^{-\alpha(t-s)} T(t-s)B(s)\mathrm{e}^{-\alpha s}(v(s) - w(s))\,\mathrm{d}s \Big\| \\ &\le KL \max_{t \in [0,b]} \int_0^t \mathrm{e}^{-\alpha(t-s)} \mathrm{d}s\, \|v - w\|_\alpha \le \frac{KL}{\alpha} \|v - w\|_\alpha = \frac{1}{2}\, \|v - w\|_\alpha,\end{aligned}$$

employing only real linearity of $B(s)$. The contraction mapping principle thus yields a unique solution $v \in C([0, b], X)$ of (1.14).

2) We now show that the function v of step 1) is the derivative of u. Let $0 \le t < t + h \le b$. Equations (1.11) and (1.14) imply that

$$\begin{aligned}w_h(t) &:= \tfrac{1}{h}(u(t+h) - u(t)) - v(t) \\ &= T(t)\tfrac{1}{h}(T(h) - I)u_0 - T(t)Au_0\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{h} T(t) \int_0^h T(h-s) F(u(s))\,\mathrm{d}s - T(t)F(u_0) \\
&+ \int_0^t T(t-s)\big[\tfrac{1}{h}(F(u(s+h)) - F(u(s))) - F'(u(s))v(s)\big]\,\mathrm{d}s \\
=:&\ S_1(h,t) + S_2(h,t) + S_3(h,t).
\end{aligned}
$$

We first observe that

$$
\begin{aligned}
\|S_1(h,t)\| &\le K \big\|\tfrac{1}{h}(T(h)-I)u_0 - Au_0\big\| =: \alpha_1(h) \ \longrightarrow\ 0, \\
\|S_2(h,t)\| &= \Big\| T(t)\frac{1}{h}\int_0^h (T(h-s)F(u(s)) - F(u_0))\,\mathrm{d}s \Big\| \\
&\le \frac{hK}{h} \sup_{0\le s\le h} \|T(h-s)F(u(s)) - F(u_0)\| =: \alpha_2(h) \ \longrightarrow\ 0
\end{aligned}
$$

as $h \to 0^+$, using $u_0 \in \mathrm{D}(A)$ in the first limit and Lemma 1.13 of [**EE**] for the second one. We then write

$$
\begin{aligned}
S_3(h,t) &= \int_0^t T(t-s)\tfrac{1}{h}[F(u(s+h)) - F(u(s)) - F'(u(s))(u(s+h)-u(s))]\,\mathrm{d}s \\
&\quad + \int_0^t T(t-s)F'(u(s))w_h(s)\,\mathrm{d}s =: S_{3,1}(h,t) + S_{3,2}(h,t).
\end{aligned}
$$

Lemma 1.14 says that the function u is Lipschitz on $[0,b]$, say with constant ℓ. By means of this fact and (1.12), we estimate $\|S_{3,1}(h,t)\|$ by

$$
\begin{aligned}
&Kb \sup_{\substack{0\le s\le b\\ 0\le s+h\le b}} \frac{1}{h}\Big\|\int_0^1 [F'(u(s)+\tau(u(s+h)-u(s))) - F'(u(s))](u(s+h)-u(s))\,\mathrm{d}\tau\Big\| \\
&\quad \le \frac{Kb\ell h}{h} \sup_{\substack{0\le s\le b\\ 0\le s+h\le b\\ 0\le\tau\le 1}} \|F'(u(s)+\tau(u(s+h)-u(s))) - F'(u(s))\| =: \alpha_3(h).
\end{aligned}
$$

Since F' is uniformly continuous on the compact set

$$
\{u(s) + \tau(u(r)-u(s)) \,|\, 0\le\tau\le 1,\ 0\le r,s\le b\},
$$

the quantity $\alpha_3(h)$ tends to 0 as $h \to 0^+$. Altogether we have shown the bound

$$
\|w_h(t)\| \le \alpha_1(h) + \alpha_2(h) + \alpha_3(h) + KL\int_0^t \|w_h(s)\|\,\mathrm{d}s.
$$

Gronwall's inequality thus yields

$$
\|w_h(t)\| \le (\alpha_1(h) + \alpha_2(h) + \alpha_3(h))\mathrm{e}^{tKL}
$$

for all $t \in [0,b]$. Letting $h \to 0^+$, we then derive that u is differentiable from the right and that the right-hand side derivative coincides with v. Since v is continuous on $[0,b]$, Lemma 1.15 implies that u belongs to $C^1([0,b),X)$. $\square$

This theorem will be used in the next section.

1.2. A semlinear wave equation

In this section we mainly study the cubic nonlinear wave equation on an open subset of $\mathbb{R}^3$. As a preparation we first show the differentiability of substitution operators $F(v) = \varphi(v)$ on L^p-spaces, which are the prototypical nonlinearities in many situations. Actually, later on we only need the case $\varphi(z) = |z|^{\alpha-1}z$.

The approach of this course relies on the lectures on evolution equations and on spectral theory, where we worked with complex Banach spaces. In addition, in Chapter 4 we investigate the nonlinear Schrödinger equation which requires complex scalars. On the other hand, our model nonlinearity $F(v) = |v|^{\alpha-1}v$ is only real, but not complex differentiable (for $\alpha > 1$). We thus identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way and just require that φ belongs to $C^1(\mathbb{R}^2, \mathbb{R}^2)$ and *not* that it is holomorphic as a function $\varphi : \mathbb{C} \to \mathbb{C}$. To deal with the resulting problems, we introduce a bit of notation.

Let $z = x + \mathrm{i}y \in \mathbb{C}$. For $\varphi = (\varphi_1, \varphi_2) : \mathbb{R}^2 \to \mathbb{R}^2$, we set

$$\varphi(z) = \varphi(\operatorname{Re} z, \operatorname{Im} z) = \varphi_1(z) + \mathrm{i}\varphi_2(z) \in \mathbb{C}.$$

For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $M = \binom{\xi}{\eta} \in \mathbb{R}^{2\times 2}$, the real scalar product and matrix-vector product on $\mathbb{R}^2$ are written as

$$\xi \cdot z = \xi_1 \operatorname{Re} z + \xi_2 \operatorname{Im} z = \operatorname{Re}((\xi_1 + \mathrm{i}\xi_2)\overline{z}) \in \mathbb{R} \quad \text{and} \quad Mz = \xi \cdot z + \mathrm{i}\eta \cdot z \in \mathbb{C}.$$

Observe that we do not have $\mathbb{C}$-linearity in general, though we use complex notation for convenience.

We treat substitution operators $F(v) = \varphi(v)$ on L^p spaces with $p \in [1, \infty)$ under certain growth conditions of φ and φ'. Depending on the growth, F maps L^p into L^q with $q < p$. As indicated by Example 1.1, such a loss of integrability cannot be avoided in the case $F(v) = |v|^{\alpha-1}v$ if $\alpha > 1$. We note that substitution operators are differentiable on C_b without any growth restrictions on $\varphi \in C^1(\mathbb{R}^2, \mathbb{R}^2)$. This much easier case $p = \infty$ is postponed to Example 3.15. The following result is also true on σ-finite measure spaces $(S, \mathcal{A}, \mu)$. We restrict ourselves to Borel sets B in $\mathbb{R}^m$ to avoid the (somewhat hidden) use of product measures.

LEMMA 1.17. *Let $\varphi \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ fulfill the bounds $|\varphi(z)| \le c_0 |z|^\alpha$ and $|\varphi'(z)| \le c_1 |z|^{\alpha-1}$ for all $z \in \mathbb{C} \cong \mathbb{R}^2$ and some constants $c_j \ge 0$ and $\alpha > 1$. Let $p \in [\alpha, \infty)$ and $B \subseteq \mathbb{R}^m$ be a Borel set endowed with the Lebesgue mesasure. The follwing assertions hold.*

a) The map $F : L^p(B) \to L^{p/\alpha}(B)$; $F(v) = \varphi(v) = \varphi_1(v) + \mathrm{i}\varphi_2(v)$, belongs to $C^1_{\mathbb{R}}(L^p(B), L^{p/\alpha}(B))$. Its derivative at $v \in L^p(B)$ is given by

$$F'(v)w = \varphi'(v)w = \nabla\varphi_1(v) \cdot w + \mathrm{i}\nabla\varphi_2(v) \cdot w,$$

and it is bounded by $\|F'(v)\|_{\mathcal{B}_{\mathbb{R}}(L^p, L^{p/\alpha})} \le c_1 \|v\|_p^{\alpha-1}$ for all $v, w \in L^p(B)$.

b) Let $\varphi = (\varphi_1, 0)$. The map $\Phi : L^\alpha(B) \to \mathbb{R}$; $\Phi(v) = \int_S \varphi_1(v)\,\mathrm{d}\mu$, belongs to $C^1_{\mathbb{R}}(L^\alpha(B), \mathbb{R})$. Its derivative at $v \in L^p(B)$ is given by

$$\Phi'(v)w = \varphi'(v)w = \int_S \nabla\varphi_1(v) \cdot w \,\mathrm{d}x,$$

and it is bounded by $\|\Phi'(v)\|_{\mathcal{B}_{\mathbb{R}}(L^\alpha, \mathbb{R})} \le c_1 \|v\|_p^{\alpha-1}$ for all $v, w \in L^\alpha(B)$.

PROOF. Let $Jw = \int w \,\mathrm{d}x$ for $w \in L^1(B)$. Since $J \in \mathcal{B}_{\mathbb{R}}(L^1(B), \mathbb{R})$ and $\Phi = J \circ F$, assertion b) follows from a) with $p = \alpha$ by the chain rule.

To show part a), take $v, w \in L^p(B)$ and set $q = p/(\alpha - 1) \in [p', \infty)$. Because of $|\varphi(v)| \le c_0 |v|^\alpha$ and $|\varphi'(v)| \le c_1 |v|^{\alpha-1}$, the functions $F(v) = \varphi(v)$ and $\varphi'(v)$ belong to $L^{p/\alpha}(B)$ resp. $L^q(B)$. The map $L^p(B) \to L^{p/\alpha}(B)$; $w \mapsto \varphi'(v)w$, is $\mathbb{R}$-linear and bounded since Hölder's inequality with $\frac{\alpha}{p} = \frac{1}{q} + \frac{1}{p}$ yields

$$\|\varphi'(v)w\|_{p/\alpha} \le c_1 \, \||v|^{\alpha-1}\|_q \, \|w\|_p = c_1 \, \|w\|_p \Big(\int_B |v|^p \,\mathrm{d}x \Big)^{\frac{\alpha-1}{p}} = c_1 \, \|v\|_p^{\alpha-1} \, \|w\|_p \, .$$

For the differentiablity we compute

$$D(w) := F(v + w) - F(v) - \varphi'(v)w = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau} \varphi(v + \tau w) \,\mathrm{d}\tau - \varphi'(v)w$$
$$= \int_0^1 \big(\varphi'(v + \tau w) - \varphi'(v)\big) w \,\mathrm{d}\tau$$

a.e. on B. Set $f(\tau, w) = \varphi'(v + \tau w) - \varphi'(v)$. It is easy to see that the integrand $(\tau, x) \mapsto f(\tau, w)(x)w(x)$ is measurable on $[0, 1] \times B$. Minkowski's inequality for integrals (cf. Satz X.6.21 in [**AE3**]) and Hölder's inequality imply the estimate

$$\Big\| \int_0^1 f(\tau, w)w \,\mathrm{d}\tau \Big\|_{p/\alpha} \le \int_0^1 \|f(\tau, w)w\|_{p/\alpha} \,\mathrm{d}\tau \le \|w\|_p \int_0^1 \|f(\tau, w)\|_q \,\mathrm{d}\tau.$$

Let $I(w)$ denote the last integral. We have to show that $I(w) \to 0$ as $w \to 0$ in $L^p(B)$. Due to a simple contradiction argument, for each null sequence (w_n) in $L^p(B)$ we have to find a subsequence $(w_{n_j})_j$ such that $I(w_{n_j})$ tends to 0 as $j \to \infty$. So let $w_n \to 0$. The Riesz-Fischer theorem then provides a subsequence $(w_{n_j})_j$ and a function $g \in L^p(B)$ such that $w_{n_j} \to 0$ a.e. as $j \to \infty$ and $|w_{n_j}| \le g$ a.e. for all $j \in \mathbb{N}$. Take $\tau \in [0, 1]$. The continuity and the growth of φ' imply that the function $f(\tau, w_{n_j})$ tends to 0 a.e. as $j \to \infty$ and has the majorant

$$|f(\tau, w_{n_j})| \le c_1 \big(|v| + g\big)^{\alpha-1} + c_1 \, |v|^{\alpha-1} =: h \in L^q(B).$$

By dominated convergence, $(f(\tau, w_{n_j}))_j$ converges to 0 in $L^q(B)$ for each fixed τ. Since $\|f(\tau, w_{n_j})\|_q \le \|h\|_q$, Lebesgue's theorem for Bochner integrals yields the limit $I(w_{n_j}) \to 0$ as $j \to \infty$. (See Remark 1.16 in [**EE**].) We have shown that $F : L^p(B) \to L^{p/\alpha}(B)$ is differentiable with derivative $F'(v)w = \varphi'(v)w$.

It remains to show the continuity of the map $L^p(B) \to \mathcal{B}_{\mathbb{R}}(L^p(B), L^{p/\alpha}(B))$; $v \mapsto F'(v)$.[3] As above, for each sequence $v_n \to v$ in $L^p(B)$ there has to a subsequence such that $F'(v_{n_j})$ tends to $F'(v)$ in $\mathcal{B}_{\mathbb{R}}(L^p(B), L^{p/\alpha}(B))$ as $j \to \infty$. Take $v_n \to v$ and w in $L^p(B)$. Proceeding as in the previous paragraph, we choose an a.e. converging subsequence v_{n_j} with a majorant g in $L^p(B)$. The functions $\varphi'(v_{n_j}) - \varphi'(v)$ tend 0 a.e. and are bounded by $c_1 \,(g^{\alpha-1} + |v|^{\alpha-1}) \in L^q(B)$, and hence tend to 0 in $L^q(B)$. Hölder's inquality then yields the estimate

$$\|F'(v_{n_j})w - F'(v)w\|_{p/\alpha} \le \|\varphi'(v_{n_j}) - \varphi'(v)\|_q \, \|w\|_p \, ,$$

which shows the asserted continuity in operator norm. □

We will mostly use the following special case of the above result.

[3] This part of the proof was omitted in the lectures.

Corollary 1.18. *Let $\alpha,\beta > 1$, $p \in [\alpha,\infty)$, and $B \subseteq \mathbb{R}^m$ be a Borel set endowed with the Lebesgue mesasure. Then the maps*

$$F : L^p(B) \to L^{p/\alpha}(B); \quad F(v) = |v|^{\alpha-1}v,$$
$$\Phi : L^\beta(B) \to \mathbb{R}; \quad \Phi(v) = \int_B |v|^\beta \,\mathrm{d}x,$$

are continuously differentiable and Lipschitz on all bounded sets. Their derivatives are given by

$$F'(v)w = |v|^{\alpha-1}v + (\alpha-1)|v|^{\alpha-3}v\operatorname{Re}(v\overline{w}) \qquad \text{for } v,w \in L^p(B),$$
$$\Phi'(v)w = \beta\int_B |v|^{\beta-1}\operatorname{Re}(v\overline{w})\,\mathrm{d}x \qquad \text{for } v,w \in L^\beta(B).$$

Proof. For F we look at $\varphi(z) = |z|^{\alpha-1}z$ and for Φ at $\psi(z) = z^\beta$, where $z \in \mathbb{C} \cong \mathbb{R}^2$. Writing $r = \operatorname{Re} z$ and $s = \operatorname{Im} z$, we compute

$$\begin{aligned}
\nabla\psi(z) &= \big(\partial_r(r^2+s^2)^{\frac{\beta}{2}}, \partial_s(r^2+s^2)^{\frac{\beta}{2}}\big) = \beta\left((r^2+s^2)^{\frac{\beta}{2}-1}r, (r^2+s^2)^{\frac{\beta}{2}-1}s\right)\\
&= \beta|z|^{\beta-2}z,\\
\varphi'(z) &= \Big(\partial_r\big((r^2+s^2)^{\frac{\alpha-1}{2}}\left(\begin{smallmatrix} r\\ s\end{smallmatrix}\right)\big), \partial_s\big((r^2+s^2)^{\frac{\alpha-1}{2}}\left(\begin{smallmatrix} r\\ s\end{smallmatrix}\right)\big)\Big)\\
&= \begin{bmatrix} (r^2+s^2)^{\frac{\alpha-1}{2}} + (\alpha-1)(r^2+s^2)^{\frac{\alpha-3}{2}}r^2 & (\alpha-1)(r^2+s^2)^{\frac{\alpha-3}{2}}sr\\ (\alpha-1)(r^2+s^2)^{\frac{\alpha-3}{2}}rs & (r^2+s^2)^{\frac{\alpha-1}{2}} + (\alpha-1)(r^2+s^2)^{\frac{\alpha-3}{2}}s^2 \end{bmatrix}\\
&= \begin{pmatrix} |z|^{\alpha-1} & 0\\ 0 & |z|^{\alpha-1}\end{pmatrix} + (\alpha-1)|z|^{\alpha-3}\begin{pmatrix} r\\ s\end{pmatrix}\begin{pmatrix} r & s\end{pmatrix}
\end{aligned}$$

for $z \neq 0$ and $\varphi'(0) = 0$. For $w \in \mathbb{C}$ with $\rho = \operatorname{Re} w$ and $\sigma = \operatorname{Im} w$, it follows

$$\begin{aligned}
\nabla\psi(z)\cdot w &= \beta|z|^{\beta-2}z\cdot w = \beta|z|^{\beta-2}\operatorname{Re}(z\overline{w}),\\
\varphi'(z)w &= |z|^{\alpha-1}w + (\alpha-1)|z|^{\alpha-3}\left(\begin{smallmatrix} r\\ s\end{smallmatrix}\right)(r\rho+s\sigma)\\
&= |z|^{\alpha-1}w + (\alpha-1)|z|^{\alpha-3}z\operatorname{Re}(z\overline{w}).
\end{aligned}$$

It is easy to see that φ' and $\nabla\psi$ are continuous and that the growth assumptions of Lemma 1.17 are satisfied. Combined with (1.13), it implies the assertions. □

We now treat a prototypical semilinear wave equation on an open and bounded subset $G \subseteq \mathbb{R}^3$ with a C^1–boundary. Let $a \in \mathbb{R}$ and J be again an interval with $\min J = 0$ and $J^\circ \neq \emptyset$. The Dirichlet–Laplacian Δ_D in $L^2(G)$ was introduced in Example 1.52 of [**EE**]: A function $v \in W_0^{1,2}(G)$ belongs to $\mathrm{D}(\Delta_D)$ if and only if

$$\exists\, f =: \Delta_D v \in L^2(G) \;\; \forall\, \varphi \in W_0^{1,2}(G): \quad (f|\varphi)_{L^2} = -\int_G \nabla v\cdot\nabla\overline{\varphi}\,\mathrm{d}x.$$

We often drop the subscript of the scalar product. The Hilbert space $W_0^{1,2}(G)$ is equipped with the equivalent norm given by $\||\nabla v|\|_2$, cf. (1.33) in [**EE**]. The operator Δ_D is invertible, dissipative and self-adjoint in $L^2(G)$. It has a bounded invertible extension $\Delta_D : W_0^{1,2}(G) \to W^{-1,2}(G)$ acting as

$$\forall\, \varphi \in W_0^{1,2}(G): \quad \langle\varphi, \Delta_D v\rangle_{W_0^{1,2}(G)} = -\int_G \nabla v\cdot\nabla\varphi\,\mathrm{d}x,$$

where $W^{-1,2}(G) = W_0^{1,2}(G)^*$. Moreover, $[\mathrm{D}(\Delta_D)]$ is continuously embedded into $W_0^{1,2}(G)$. See Example 1.52 of [**EE**]. As noted after this example, the domain of Δ_D is equal to $W^{2,2}(G) \cap W_0^{1,2}(G)$ if ∂G is of class C^2.

For given initial data $w_0 \in W_0^{1,2}(G)$ and $w_1 \in L^2(G)$, we want to solve the cubic semilinear wave equation

$$\partial_{tt} w(t) = \Delta_D w(t) - a|w(t)|^2 w(t), \quad t \in J, \qquad w(0) = w_0, \quad \partial_t w(0) = w_1. \tag{1.15}$$

We look for a *weak solution*

$$w \in C^2(J, W^{-1,2}(G)) \cap C^1(J, L^2(G)) \cap C(J, W_0^{1,2}(G))$$

of (1.15). If even $w_0 \in \mathrm{D}(\Delta_D)$ and $w_1 \in W_0^{1,2}(G)$, we expect a *classical solution*

$$w \in C^2(J, L^2(G)) \cap C^1(J, W_0^{1,2}(G)) \cap C(J, [\mathrm{D}(\Delta_D)]).$$

Recall that $w(t) \in W_0^{1,2}(G)$ means that the trace of $w(t)$ on ∂G vanishes. In this sense (1.15) has (homogeneous) Dirichlet boundary conditions.

To treat the nonlinearity in (1.15), we set $f(v) = -a|v|^2 v$. Sobolev's Theorem 3.26 of [**ST**] yields the embedding $W_0^{1,2}(G) \hookrightarrow L^6(G)$, since $1 - \frac{3}{2} = -\frac{3}{6}$. Corollar 1.18 thus shows that

$$f \in C^1_{\mathbb{R}}(W_0^{1,2}(G), L^2(G)) \qquad \text{and} \quad f' \text{ is bounded on balls.} \tag{1.16}$$

As in the linear case, see Example 2.4 of [**EE**], we pass to an equivalent problem which is of first order in time. We introduce

$$Z = W_0^{1,2}(G) \times L^2(G), \quad A = \begin{pmatrix} 0 & I \\ \Delta_D & 0 \end{pmatrix}, \quad \text{and} \quad \mathrm{D}(A) = \mathrm{D}(\Delta_D) \times W_0^{1,2}(G).$$

Example 1.53 of [**EE**] says that A is skew-adjoint and thus generates a unitary C_0–group $T(\cdot)$ on Z. The extrapolation space for A is given by $Z_{-1} \cong L^2(G) \times W^{-1,2}(G)$ and the extrapolated operator by

$$A_{-1} = \begin{pmatrix} 0 & I \\ \Delta_D & 0 \end{pmatrix} \qquad \text{with} \quad \Delta_D : W_0^{1,2}(G) \to W^{-1,2}(G),$$

see Example 2.17 of [**EE**]. We next set

$$F(\varphi, \psi) = \begin{pmatrix} 0 \\ f(\varphi) \end{pmatrix}.$$

This map belongs to $C^1_{\mathbb{R}}(Z)$ and F' is bounded on balls due to (1.16), and thus F is Lipschitz on bounded sets by (1.13), as required for the results of the previous section. We stress that for at most cubic f the nonlinearity F maps Z into itself in three space dimensions.

Arguing as in Examples 2.4, 2.10 and 2.17 of [**EE**], one can show that the problem (1.15) is equivalent to

$$u'(t) = Au(t) + F(u(t)), \quad t \in J, \qquad u(0) = u_0 := (w_0, w_1), \tag{1.17}$$

for the above maps A and F. More precisely, $u \in C^1(J, Z) \cap C(J, [\mathrm{D}(\Delta_D)])$ solves (1.17) if and only if w is a classical solution of (1.15), and $u \in C(J, Z)$ solves (1.17) mildly if and only if w is a weak solution of (1.15). In both cases, we have $u = (w, \partial_t w)$. We can now show local well-posedness of (1.15).

Proposition 1.19. *Let $G \subseteq \mathbb{R}^3$ be open and bounded with $\partial G \in C^1$ and let $a \in \mathbb{R}$. Then the following assertions are true.*

a) For each $u_0 = (w_0, w_1) \in Z = W_0^{1,2}(G) \times L^2(G)$, there is a maximal existence time $t^+(u_0) > b_0(\|u_0\|_Z) > 0$ and a unique maximal weak solution $w = \varphi(\cdot\,; u_0)$ of (1.15) on $[0, t^+(u_0))$.

b) Let w_0 and w_1 be real-valued. Then also w takes real values.

c) Let $t^+(u_0) < \infty$. Then $\lim_{t \to t^+(u_0)} \||(\nabla w(t), \partial_t w(t))|_2\|_{L^2} = \infty$.

d) Let $b \in (0, t^+(u_0))$. Then there is a radius $\delta = \delta(u_0, b) > 0$ such that for all initial data $\tilde{u}_0 = (\tilde{w}_0, \tilde{w}_1) \in \overline{B}_Z(u_0, \delta)$ we have $t^+(\tilde{u}_0) > b$ and the map $\overline{B}_Z(u_0, r) \to C([0, b], X)$; $\tilde{u}_0 \mapsto \varphi(\cdot\,; \tilde{u}_0)$, is Lipschitz.

e) Let $u_0 \in \mathrm{D}(A) = \mathrm{D}(\Delta_D) \times W_0^{1,2}(G)$. Then w is a classical solution of (1.15) on $[0, t^+(u_0))$.

Proof. By the remarks before the statement, we can apply Theorems 1.11 and 1.16 to the problem (1.17). The resulting mild and classical solutions u of (1.17) for $u_0 \in Z$, respectively $u_0 \in \mathrm{D}(A)$, yield weak and classical solutions w of the semilinear wave equation (1.17) as observed above, where we use that mild solutions in Z to (1.17) are classical solutions in $Z_{-1} \cong L^2(G) \times W^{-1,2}(G)$ because of Remark 1.5. We have thus shown assertions a), c), d), and e).

To prove part b), let $u_0 = (w_0, w_1)$ be real-valued. We first note that the set $\{v \in L^2(G) \mid \operatorname{Im} v = 0\}$ is closed in $L^2(G)$. Hence, it is enough to consider $u_0 \in \mathrm{D}(A)$ and a classical solution thanks to d), e), and the density of $\mathrm{D}(A)$ in Z. Let w solve (1.15) for $u_0 \in \mathrm{D}(A)$. The functions $g := \operatorname{Re} w$ and $v := \operatorname{Im} w$ then also belong to $C^2(J^+(u_0), L^2(G)) \cap C(J^+(u_0), [\mathrm{D}(\Delta_D)])$ and satisfy

$$\partial_{tt} v(t) = \Delta_D v(t) - a g(t)^2 v(t) - a v(t)^3, \quad t \in J^+(u_0), \qquad v(0) = 0, \quad \partial_t v(0) = 0.$$

We put $h = -ag^2 v - av^3$. Fix $b \in (0, t^+(u_0))$ and let $t \in [0, b]$. We have $[\mathrm{D}(\Delta_D)] \hookrightarrow W_0^{1,2}(G) \hookrightarrow L^6(G)$ by Sobolev's embedding, so that $c(b) := \sup_{t \in [0,b]} \|w(t)\|_6^2$ is finite. Hölder's inequality with $\frac{1}{2} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$ then shows the bounds

$$\|g(t)^2 v(t)\|_2 \le \|g(t)\|_6^2 \, \|v(t)\|_6 \le c(b) \, \|v(t)\|_6, \qquad \|v(t)\|^3 \le c(b) \, \|v(t)\|_6.$$

Observe that $\tilde{u} = (v, \partial_t v)$ is the mild solution of the problem

$$\tilde{u}'(t) = A\tilde{u}(t) + \begin{pmatrix} 0 \\ h(t) \end{pmatrix}, \quad t \in [0, b], \qquad \tilde{u}(0) = 0.$$

by Example 2.17 of [**EE**]. It follows

$$\begin{aligned} \|v(t)\|_{1,2} \le \|\tilde{u}(t)\|_Z &= \Big\| \int_0^t T(t-s) \begin{pmatrix} 0 \\ h(s) \end{pmatrix} \mathrm{d}s \Big\| \\ &\le \int_0^t \|h(s)\|_2 \, \mathrm{d}s \le 2\,|a| c(b) C_{\mathrm{Sob}} \int_0^t \|v(s)\|_{1,2} \, \mathrm{d}s \end{aligned}$$

for $t \in [0, b]$. Gornwall's lemma thus yields $v = 0$; i.e., assertion b) is true. ☐

For $a \geq 0$ we will show global existence of the weak solution w of (1.15) for all initial data $u_0 = (w_0, w_1) \in Z = W_0^{1,2}(G) \times L^2(G)$. Our reasoning relies on the *energy* given by

$$E : Z \to \mathbb{R}; \quad E(\varphi, \psi) = \int_g (\tfrac{1}{2}|\psi|^2 + \tfrac{1}{2}|\nabla\varphi|^2 + \tfrac{a}{4}|\varphi|^4)\,\mathrm{d}x,$$

$$\begin{aligned} E_w(t) &:= E(w(t), \partial_t w(t)) = \int_G (\tfrac{1}{2}|\partial_t w(t)|^2 + \tfrac{1}{2}|\nabla w(t)|^2 + \tfrac{a}{4}|w(t)|^4)\,\mathrm{d}x \qquad (1.18)\\ &= \tfrac{1}{2}\|(w(t), \partial_t w(t))\|_Z^2 + \tfrac{a}{4}\|w(t)\|_4^4 \end{aligned}$$

for $t \in J^+(u_0)$. Since $W_0^{1,2}(G) \hookrightarrow L^4(G)$ by Sobolev's embedding and $1 - \frac{3}{2} \geq -\frac{3}{4}$, Corollary 1.18 shows that E belongs to $C^1_{\mathbb{R}}(Z, \mathbb{R})$. Observe that $E(\varphi, \psi)$ controls the Z–norm of (φ, ψ) *provided* that $a \geq 0$.

We next show that E is constant along weak solutions of (1.15) so that it is a natural quantity for the nonlinear wave equation. (It corresponds to the physical energy as we have set all material parameters except a equal to 1 and also ignore physical units.) This fact leads global existence of all solutions if $a \geq 0$, due to the blow-up condition in Proposition 1.19. However, one can only derive the preservation of energy for *classical solutions* of (1.15) by a direct computation. Using the continuous dependence on data and the density of $\mathrm{D}(A)$ in Z, we can then extend the energy equality to weak solutions by approximation. Here it is crucial that a classical solution exists (in $\mathrm{D}(A)$)) until the maximal existence time $t^+(w_0, w_1)$ as a weak solution. We thus need the full power of the well-posedness theory of the previous section in the next argument, which is protoypical for many nonlinear systems.

Proposition 1.20. *Let $G \subseteq \mathbb{R}^3$ be open and bounded with $\partial G \in C^1$, $a \in \mathbb{R}$, and w be maximal weak solution of* (1.15) *for some $(w_0, w_1) \in Z = W_0^{1,2}(G) \times L^2(G)$. Then the following assertions are true.*

a) $E_w(t) = E_w(0) = \frac{1}{2}\|(w_0, w_1)\|_Z^2 + \frac{a}{4}\|w_0\|_4^4$ for $t \in [0, t^+(w_0, w_1))$.

b) Let $a \geq 0$. Then $t^+(w_0, w_1) = \infty$ for all initial values $(w_0, w_1) \in Z$.

Proof. a) We first show the equality for $u_0 \in (w_0, w_1) \in \mathrm{D}(A)$ and the corresponding classical solution w of (1.15). Let $t \in [0, t^+(u_0))$. Since the map $t \mapsto \partial_t w$ is differentiable in $L^2(G)$ and $t \mapsto w(t)$ in $W_0^{1,2}(G) \hookrightarrow L^6(G)$, Corollary 1.18 and the chain rule show that E_w has the derivative

$$E_w'(t) = \mathrm{Re} \int_G \Big(\partial_t w(t)\partial_{tt}\overline{w(t)} + \nabla w(t)\nabla\partial_t\overline{w(t)} + a|w(t)|^2 w(t)\partial_t\overline{w(t)}\Big)\mathrm{d}s.$$

In the second summand on the right hand we can use the definition of the Laplacian because of $w(t) \in \mathrm{D}(\Delta_D)$ and $\partial_t\overline{w(t)} \in W_0^{1,2}(G)$. Employing also $\mathrm{Re}\, z = \mathrm{Re}\,\overline{z}$ and the equation (1.15), we then compute

$$E_w'(t) = \mathrm{Re} \int_G \big(\partial_{tt} w(t) - \Delta w(t) + a|w(t)|^2 w(t)\big)\partial_t\overline{w(t)}\,\mathrm{d}s = 0,$$

resulting in $E(w(t), \partial_t w(t)) = E(w_0, w_1)$.

For given data $u_0 = (w_0, w_1)$ in Z we find a sequence $(u_{0,n})_n$ in $\mathrm{D}(A)$ converging to u_0 in Z. Let $b \in (0, t^+(u_0))$. Proposition 1.19 d) says that $t^+(u_{0,n}) > b$

for all sufficiently large n and that the corresponding solution $(w_n, \partial_t w_n)$ tends to $(w, \partial_t w)$ in Z uniformly in $t \in [0, b]$ as $n \to \infty$. We can apply the first part of the proof to w_n on $[0, b]$ in view of Proposition 1.19 e). Part a) then follows from the continuity of $E : Z \to \mathbb{R}$.

b) Statement a) implies that $2E_w(t) = 2E_w(0) \geq \|(w(t), \partial_t w(t))\|_Z^2$ if $a \geq 0$. The blow-up condition in Proposition 1.19 c) thus yields assertion b). □

In the case $a < 0$ the above reasoning fails since $E_w(t)$ does not control the norm of Z. We first look at the case of Neumann boundary conditions $\partial_\nu w = 0$ on ∂G. Spatially constant real functions $w(t, x) = \varphi(t)$ satisfy this condition and belong to the kernel of the Laplacian. Such a map w thus solves the Neumann-version of (1.15) if and only if $\varphi'' = |a|\,\varphi^3$. For all $\varphi(0) = c > 0$ this equation has the solution

$$\varphi_c(t) = \frac{c}{1 - c\sqrt{|a|/2}\,t}$$

which has maximal existence time $t_c^+ = \sqrt{2|a|^{-1}c^{-2}}$. Hence, an 'ODE blow-up' is present in our semilinear wave equation if $a < 0$.

To establish blow-up for Dirichlet conditions and much more initial values, we will derive a differential equality for the map $\phi(t) = \|w(t)\|_2^2/4$ which implies the desired explosion.

Proposition 1.21. *Let $G \subseteq \mathbb{R}^3$ be open and bounded with $\partial G \in C^1$ and $a < 0$. Assume that the initial data $u_0 = (w_0, w_1) \in Z$ have real values, nonpositive energy $E(w_0, w_1) \leq 0$, i.e.,*

$$\tfrac{|a|}{4}\,\|w_0\|_4^4 \geq \tfrac{1}{2}\,\||\nabla w_0|_2\|_2^2 + \tfrac{1}{2}\,\|w_1\|_2^2\,, \tag{1.19}$$

and that the estimate

$$\int_G w_0 w_1 \,\mathrm{d}x > 0 \tag{1.20}$$

is true. Set $C = 4|a|/\lambda(G)$ and $\tau = 4\sqrt{3}/(\sqrt{C}\,\|w_0\|_2)$. We then obtain

$$t^+(u_0) \leq t_0 + \tau \qquad \text{and} \qquad \|w(t)\|_2 \geq 2\big(2\,\|w(t_0)\|_2^{-1} - \sqrt{C/12}\,(t - t_0)\big)^{-1}$$

for $t \in [t_0, t_0 + \tau)$ and a time $t_0 \geq 0$ depending on w_0 and w_1, see (1.23).

Observe that the conditions (1.19) and (1.20) are satisfied if $w_1 \in W_0^{1,2}(G)$ is real and non-zero and $w_0 = \tau w_1$ for large $\tau > 0$. Moreover, if the assumptions are true for u_0 with a strict inequality in (1.19), then they even hold on a ball in Z around u_0.

Proof. Proposition 1.19 provides a maximal weak real-valued solution

$$w \in C(J^+, W_0^{1,2}(G)) \cap C^1(J^+, L^2) \cap C^2(J^+, W^{-1,2}(G))$$

of (1.15) for the data u_0 as in the assertion. Let $t \in J^+ = J^+(u_0)$. We set $\phi(t) = \frac{1}{4}\|w(t)\|_2^2 = \frac{1}{4}(w(t)|w(t))$. This function has the derivative

$$\phi'(t) = \frac{1}{2}\int_G w(t)\partial_t w(t)\,\mathrm{d}x = \frac{1}{2}\,\langle w(t), \partial_t w(t)\rangle_{W_0^{1,2}}\,.$$

Assumption (1.20) yields $\phi'(0) > 0$. We can differentiate once more, obtaining

$$\begin{aligned}
\phi''(t) &= \frac{1}{2}\int_G (|\partial_t w(t)|^2 \,\mathrm{d}x + \frac{1}{2}\langle w(t), \partial_{tt} w(t)\rangle_{W_0^{1,2}} \\
&= \frac{1}{2}\int_G |\partial_t w(t)|^2 \,\mathrm{d}x + \frac{1}{2}\langle w(t), \Delta_D w(t)\rangle_{W_0^{1,2}} - \frac{a}{2}\int_G |w(t)|^4 \,\mathrm{d}x \\
&= \frac{1}{2}\int_G \big(|\partial_t w(t)|^2 - ||\nabla w(t)|_2|_2^2 - a|w(t)|^4\big) \,\mathrm{d}x
\end{aligned}$$

where we also insert (1.15) and use the definition of the extended Laplacian. The conservation of energy from Proposition 1.21 a) and assumption (1.19) yield

$$\phi''(t) = \int_G |\partial_t w(t)|^2 \,\mathrm{d}x + \frac{|a|}{4}\int_G |w(t)|^4 \,\mathrm{d}x - E(w_0, w_1) \geq \frac{|a|}{4}\int_G |w(t)|^4 \,\mathrm{d}x.$$

On the other hand, Hölder's inequality yields the bound

$$\phi(t)^2 = \frac{1}{16}\Big(\int_G \|w(t)\|^2 \,\mathrm{d}x\Big)^2 \leq \frac{\lambda(G)}{16}\int_G |w(t)|^4 \,\mathrm{d}x.$$

Together we derive

$$\phi''(t) \geq \frac{4|a|}{\lambda(G)}\phi(t)^2 = C\phi(t)^2.$$

Integrating two times, we see that

$$\begin{aligned}
&\phi'(t) \geq \phi'(0) + C\int_0^t \phi(s)^2 \,\mathrm{d}s \geq \phi'(0) > 0, \\
&\phi(t) \geq \phi(0) + t\phi'(0),
\end{aligned} \tag{1.21}$$

and hence ϕ strictly increases. We can now estimate

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}(\phi'(t))^2 = \phi''(t)\phi'(t) \geq C\phi(t)^2\phi'(t).$$

on J^+. Two more integrations imply the inequality

$$\phi'(t)^2 \geq \phi'(0)^2 + 2C\int_0^t \phi'(s)\phi(s)^2 \,\mathrm{d}s = \phi'(0)^2 + \tfrac{2C}{3}\phi(t)^3 - \tfrac{2C}{3}\phi(0)^3. \tag{1.22}$$

We suppose that $J^+ = \mathbb{R}_{\geq 0}$. Since $\phi'(0) > 0$, we can fix a time $t_0 \geq 0$ with

$$\phi(0) + t_0\phi'(0) \geq \max\{0, (2\phi(0)^3 - 3\phi'(0)^2/C)^{\frac{1}{3}}\}. \tag{1.23}$$

Let $t \geq t_0$. The lower estimate (1.21) leads to

$$\phi(t)^3 \geq \phi(t_0)^3 \geq (\phi(0) + t_0\phi'(0))^3$$

From (1.22) it thus follows

$$\phi'(t)^2 \geq \tfrac{C}{3}\phi(t)^3 + \tfrac{C}{3}(\phi(0) + t_0\phi'(0))^3 + \phi'(0)^2 - \tfrac{2C}{3}\phi(0)^3 \geq \tfrac{C}{3}\phi(t)^3$$

by the choice to t_0. As a result, ϕ satisfies the differential inequality

$$\phi'(t) \geq \sqrt{C/3}\,\phi(t)^{\frac{3}{2}}, \quad t \geq t_0, \qquad \phi(t_0) = \|w(t_0)\|_2^2/4.$$

The corresponding equation

$$\psi'(t) = \sqrt{C/3}\,\psi(t)^{\frac{3}{2}}, \quad t \geq 0, \qquad \psi(t_0) = \|w(t_0)\|_2^2/4,$$

has the blow-up solution $\psi(t-t_0) = \big(2\,\|w(t_0)\|_2^{-1} - \sqrt{C/12}\,(t-t_0)\big)^{-2}$ for $t_0 \le t < t_0+\tau$. As in Lemma 5.10 of [**Ana4**] we can now show that $\|w(t)\|_2^2/4 = \phi(t) \ge \psi(t)$ for $t \in [t_0, t_0+\tau)$, which is the assertion. □

In the exercises we discuss further properties of the semilinear wave equation.

CHAPTER 2

Interpolation theory and regularity

Interpolation theory is an independent branch of functional analysis which has important applications in many fields of mathematics. To explain the basic idea in our context, we look at the spaces $[\mathrm{D}(A)] \hookrightarrow X$ for a generator A of an analytic semigroup $T(\cdot)$. We want to construct 'interpolation spaces' $[\mathrm{D}(A)] \hookrightarrow Y \hookrightarrow X$, which means that all operators $T \in \mathcal{B}(X)$ having a bounded restriction $T_1 : [\mathrm{D}(A)] \to [\mathrm{D}(A)]$ also leave invariant Y and the restriction $T_Y : Y \to Y$ is bounded. We use such spaces to establish norm bounds on $T(t) : X \to Y$ for $t > 0$ using the known ones for $T(t) : X \to X$ and $T(t) : X \to [\mathrm{D}(A)]$. This fact will be crucial for the treatment of a large class of reaction-diffusion equations and other semilinear 'parabolic' problems in the next chapter.

Among others, the monographs [**BL**], [**Lu2**] and [**Tr1**] give an introduction to interpolation theory. The above indicated applications to parabolic evolution equations are stressed in [**Lu1**] and [**Lu2**], for instance. We focus here on these applications and do not develop the general theory explicitely, though it is hidden in some of the proofs. In this sense the next section is similar to Section II.5 of [**EN**] (which is concerned with the spaces $D_A(\alpha,\infty)$ and $D_A(\alpha)$ in our notation), but we are closer to interpolation theory omitting certain other aspects investigated in [**EN**].

2.1. Real interpolation spaces for semigroups

In this section we always work in the following setting, sometimes adding more restrictions and assumptions.

$$\begin{aligned} &A \text{ generates the } C_0\text{–semigroup } T(\cdot) \text{ on } X, \quad M_0 := \sup\nolimits_{t\in[0,1]} \|T(t)\|. \\ &\text{Let} \quad \alpha \in (0,1) \quad \text{and} \quad p \in [1,\infty]. \end{aligned} \tag{2.1}$$

Recall that $T(\cdot)x$ is continuous for $x \in X$ and continuously differentiable for $x \in \mathrm{D}(A)$. One can define the 'real interpolation spaces' between X and $[\mathrm{D}(A)]$ by looking at $x \in X$ such that $T(\cdot)x$ is Hölder continuous (or satisfies an L^p variant of this property). To that purpose, we define

$$\varphi_{\alpha,x}(s) = s^{-\alpha}\,\|T(s)x - x\| \qquad \text{for} \;\; x \in X, \;\; s > 0.$$

If $T(\cdot)$ is a C_0–group, we set $\varphi_{\alpha,x}(s) = |s|^{-\alpha}\,\|T(s)x - x\|$ for all $s \neq 0$. We further introduce the weighted space $L^p_*(J) = L^p(J, \mathrm{d}s/|s|)$ for any Borel set $J \subseteq \mathbb{R} \setminus \{0\}$, and abbreviate $L^p_* = L^p_*((0,1])$. Observe that $L^\infty_*(J) = L^\infty(J)$ and that $L^p_*(J)$ is endowed with the norm given by

$$\|f\|^p_{L^p_*(J)} = \int_J |f(s)|^p\,\frac{\mathrm{d}s}{|s|}$$

if $p \in [1,\infty)$. Some special features of these spaces are discussed below. We can now formulate the basic notions of this chapter.

Definition 2.1. *Let* (2.1) *be true and* $x \in X$. *We define the quantities*

$$[x]_{\alpha,p} = \|\varphi_{\alpha,x}\|_{L^p_*} \in [0,\infty], \qquad \|x\|_{\alpha,p} = \|x\| + [x]_{\alpha,p}.$$

The real interpolation space *between* X *and* $[\mathrm{D}(A)]$ *of order* $\alpha \in (0,1)$ *and exponent* $p \in [1,\infty]$ *is given by*

$$D_A(\alpha,p) = \{x \in X \mid [x]_{\alpha,p} < \infty\},$$

and the continuous interpolation space *by the closure*

$$D_A(\alpha) = \overline{\mathrm{D}(A)}^{\|\cdot\|_{\alpha,\infty}}.$$

It is easy to see that $D_A(\alpha,p)$ is a vector space with norm $\|\cdot\|_{\alpha,p}$ and that $D_A(\alpha)$ is a closed subspace of $D_A(\alpha,\infty)$. Observe that $D_A(\alpha,p)$ is defined just by an estimate (and not by a limit such as the space of continuous functions). In Example 2.3 below we see that $D_A(\alpha) \neq D_A(\alpha,\infty)$, in general.

We first discuss slight variants of the above concepts, where $x \in X$. For $0 < a < b \le n \in \mathbb{N}$, an elementary estimate yields

$$\|x\|+\|\varphi_{\alpha,x}\|_{L^p_*((0,a])} \le \|x\|+\|\varphi_{\alpha,x}\|_{L^p_*((0,b])} \le \|x\|+\|\varphi_{\alpha,x}\|_{L^p_*((0,a])}+c_0\|x\|, \quad (2.2)$$

where for $p \in [1,\infty)$ the constant $c_0 = c_0(a,b,\alpha,M_0)$ is given by

$$c_0 := (M_0^n+1)\Big(\int_a^b s^{-\alpha p-1}\,\mathrm{d}s\Big)^{\frac{1}{p}} \le \frac{M_0^n+1}{\alpha^{1/p}a^\alpha} \le \frac{M_0^n+1}{\alpha a^\alpha}\,,$$

and by $c_0 = a^{-\alpha}(M_0^n+1)$ if $p=\infty$. In Definition 2.1 one can thus replace the interval $J=(0,1]$ by any $J=(0,\tau]$, just obtaining an equivalent norm.

To establish similar results for unbounded intervals, we have to impose more conditions on the semigroup. First, let $T(\cdot)$ be bounded. Setting $M := \sup_{t\ge 0}\|T(t)\| < \infty$, we infer $|\varphi_{\alpha,x}(s)| \le s^{-\alpha}(1+M)\|x\|$ and thus

$$\|\varphi_{\alpha,x}\|_{L^p_*(1,\infty)} \le \alpha^{-1/p}(1+M)\|x\| \le \alpha^{-1}(1+M)\|x\| =: c_1\|x\|.$$

This inequality yields the norm equivalence

$$\|x\|_{\alpha,p} \le \|x\| + \|\varphi_{\alpha,x}\|_{L^p_*(\mathbb{R}_+)} \le \|x\|_{\alpha,p} + (1+c_1)\|x\|. \quad (2.3)$$

Next, let $T(\cdot)$ be a C_0–group bounded by $\tilde{M}$ on $\mathbb{R}$. Using that

$$\varphi_{\alpha,x}(s) = (-s)^{-\alpha}\|T(s)(x-T(-s)x)\| \le \tilde{M}\varphi_{\alpha,x}(-s)$$

for $s<0$, we estimate

$$\|x\|_{\alpha,p} \le \|x\| + \|\varphi_{\alpha,x}\|_{L^p_*(\mathbb{R}\setminus\{0\})} \le (1+\tilde{M})\,\|x\|_{\alpha,p} + c_1(1+\tilde{M})\|x\| + \|x\|. \quad (2.4)$$

We further check that also a rescaling of the semigroup leads to an equivalent norm on the interpolation spaces. This fact will be useful in some proofs. Let $\omega \in \mathbb{R}$ and $s \in (0,1]$. Recall that $A-\omega I$ generates the C_0–semigroup $(\mathrm{e}^{-\omega t}T(t))_{t\ge 0}$ by Lemma 1.18 of [**EE**]. This 'rescaled' semigroup decays exponentially if ω is larger than the growth bound $\omega_0(A)$ of A, cf. Definition 1.5 of [**EE**]. Using the exponential series, we compute

$$\varphi_{\alpha,x}(s) \le s^{-\alpha}\|\mathrm{e}^{\omega s}(\mathrm{e}^{-\omega s}T(s)x-x)\| + s^{-1}|\mathrm{e}^{\omega s}-1|\,s^{1-\alpha}\|x\|$$

$$
\begin{aligned}
&\le \mathrm{e}^{\omega_+}\big(\|x\| + s^{-\alpha}\|\mathrm{e}^{-\omega s}T(s)x - x\|\big), \qquad (2.5)\\
s^{-\alpha}\|\mathrm{e}^{-\omega s}T(s)x - x\| &\le s^{-\alpha}\mathrm{e}^{-\omega s}\|T(s)x - x\| + s^{-1}|\mathrm{e}^{-\omega s} - 1|\, s^{1-\alpha}\|x\|\\
&\le \mathrm{e}^{\omega_-}\,(\|x\| + \varphi_{\alpha,x}(s)).
\end{aligned}
$$

In the next proposition we collect basic properties of the real interpolation spaces, which follow from their definition by elementary arguments and standard semigroup theory. Observe that they become smaller if α increases or if p decreases. Moreover, spaces with larger α are smaller independent of p. In this sense the parameter p provides a 'fine tuning.' As announced above, the space $D_A(\alpha,\infty)$ consists of the vectors x with Hölder continuous orbits $T(\cdot)x$.

Here and below, we write $a \lesssim_M b$ if $a \le cb$ for all $a, b \in \mathbb{R}$ and a constant $c = c(M) > 0$ depending on M, as well as $a \eqsim_M b$ if $a \lesssim_M b$ and $b \lesssim_M a$.

PROPOSITION 2.2. *Let* (2.1) *be true,* $0 < \alpha < \beta < 1$, $1 \le p \le q < \infty$, $b > 0$, *and* $x \in X$. *Then the following assertions hold.*

a) $D_A(\alpha,p)$ *and* $D_A(\alpha)$ *are Banach spaces for* $\|\cdot\|_{\alpha,p}$ *and* $\|\cdot\|_{\alpha,\infty}$, *respectively.*

b) $[\mathrm{D}(A)] \hookrightarrow D_A(\beta,\infty) \hookrightarrow D_A(\alpha,1) \hookrightarrow X$.

c) Let $p \in [1,\infty)$. *Then* $\mathrm{D}(A)$ *is dense in* $D_A(\alpha,p)$.

d) $D_A(\alpha,1) \hookrightarrow D_A(\alpha,p) \hookrightarrow D_A(\alpha,q) \hookrightarrow D_A(\alpha) \subseteq D_A(\alpha,\infty)$.

e) $x \in D_A(\alpha,\infty)$ *if and only if* $T(\cdot)x \in C^\alpha([0,b],X)$.

f) $x \in D_A(\alpha)$ *if and only if* $t^{-\alpha}(T(t)x - x) \longrightarrow 0$ *in* X *as* $t \to 0$.

(In the proof are bounds on the norms of the embeddings. Note $\|x\|_{\alpha,p} \le \|x\|_{\beta,p}$.*)*

PROOF. Take $s \in (0,1]$ and let α, β, p, and q be given as in the statements. We show each part of the proposition separately.

a) In view of the remarks after Definition 2.1, we only have to prove that $(D_A(\alpha,p), \|\cdot\|_{\alpha,p})$ is complete. Let (x_n) be a Cauchy sequence in $D_A(\alpha,p)$. We thus have $c_p := \sup_n \|x_n\|_{\alpha,p} < \infty$, and the vectors x_n converge to some x in X since X is a Banach space. Hence, $\varphi_{\alpha,x_n}(s)$ tends to $\varphi_{\alpha,x}(s)$ as $n \to \infty$. If $p = \infty$, it follows that $\varphi_{\alpha,x}(s) \le \limsup_n \varphi_{\alpha,x_n}(s) \le c_\infty$. For $p < \infty$, Fatou's Lemma yields

$$
\|\varphi_{\alpha,x}\|_{L^p_*}^p = \int_0^1 \lim_{n\to\infty} \varphi_{\alpha,x_n}(s)^p \frac{\mathrm{d}s}{s} \le \liminf_{n\to\infty} \int_0^1 \varphi_{\alpha,x_n}(s)^p \frac{\mathrm{d}s}{s} \le c_p^p.
$$

In both cases x belongs to $D_A(\alpha,p)$.

Let $\varepsilon > 0$. There is an index $N_\varepsilon \in \mathbb{N}$ such that $[x_m - x_n]_{\alpha,p} \le \varepsilon$ for all $n, m \ge N_\varepsilon$. Since $\varphi_{\alpha,x_m - x_n}$ tends pointwise to $\varphi_{\alpha,x-x_n}$ as $m \to \infty$, we obtain as above the bound

$$
[x - x_n]_{\alpha,p} = \|\varphi_{\alpha,x-x_n}\|_{L^p_*} \le \limsup_{m\to\infty} \|\varphi_{\alpha,x_m-x_n}\|_{L^p_*} \le \varepsilon
$$

for all $n \ge N_\varepsilon$. As a consequence, x_n converges to x in $D_A(\alpha,p)$.

b) The last embedding in part b) is clear. To see the second, for $x \in D_A(\beta,\infty)$ we simply estimate

$$
\|\varphi_{\alpha,x}\|_{L^1_*} = \int_0^1 s^{\beta-\alpha-1}\, s^{-\beta}\|T(s)x - x\|\,\mathrm{d}s \le \frac{1}{\beta-\alpha}\,\|\varphi_{\beta,x}\|_{L^\infty_*}\,.
$$

Let $x \in \mathrm{D}(A)$. Lemma 1.19 of [**EE**] then implies the inequality

$$\varphi_{\beta,x}(s) = s^{1-\beta} \Big\| \frac{1}{s} \int_0^s T(\tau)Ax \,\mathrm{d}\tau \Big\| \le M_0 \,\|Ax\|,$$

so that $[\mathrm{D}(A)] \hookrightarrow D_A(\beta,\infty)$.

c) Let $x \in D_A(\alpha,p)$ and $p \in [1,\infty)$. For $n \in \mathbb{N}$ with $n > \omega_0(A)$, we set $x_n = nR(n,A)x \in \mathrm{D}(A)$ and $c = \sup_{n>\omega_0(A)} \|nR(n,A)\| < \infty$. Observe that

$$\varphi_{\alpha,x-x_n}(s) = s^{-\alpha}\,\|(T(s)-I)(x-x_n)\| \;\longrightarrow\; 0, \qquad \text{as } \; n\to\infty,$$
$$0 \le \varphi_{\alpha,x-x_n}(s) \le \varphi_{\alpha,x}(s) + s^{-\alpha}\,\|nR(n,A)(T(s)-I)x\| \le (1+c)\,\varphi_{\alpha,x}(s).$$

By dominated convergence, the functions $\varphi_{\alpha,x-x_n}$ thus tend to 0 in L^p_* as $n\to\infty$ which yields assertion c).

d) Let $x \in D_A(\alpha,r)$ for $r\in[1,\infty)$. We compute

$$\begin{aligned}
s^{-\alpha}\|T(s)x-x\| &= \Big(2^{-\alpha r} + \alpha r \int_s^2 \tau^{-\alpha r-1}\,\mathrm{d}\tau\Big)^{\frac{1}{r}} \|T(s)x-x\| \\
&\le 2^{-\alpha}(M_0+1)\|x\| + (\alpha r)^{\frac1r}\Big(\int_s^2 \tau^{-\alpha r}\,\|T(s)x-T(\tau)x+T(\tau)x-x\|^r\,\frac{\mathrm{d}\tau}{\tau}\Big)^{\frac1r} \\
&\le (M_0+1)\|x\| + \mathrm{e}^{1/\mathrm{e}}\Big(\int_s^2 (\tau-s)^{-\alpha r}\,\|T(s)(x-T(\tau-s)x)\|^r\,\frac{\mathrm{d}\tau}{\tau-s}\Big)^{\frac1r} \\
&\qquad + \mathrm{e}^{1/\mathrm{e}}\Big(\int_s^2 \tau^{-\alpha r}\,\|T(\tau)x-x\|^r\,\frac{\mathrm{d}\tau}{\tau}\Big)^{\frac1r} \\
&\le \mathrm{e}^{1/\mathrm{e}}(M_0+1)\,\Big(\|x\| + \Big(\int_0^2 \sigma^{-\alpha r}\,\|T(\sigma)x-x\|^r\,\frac{\mathrm{d}\sigma}{\sigma}\Big)^{\frac1r}\Big) \\
&\lesssim_{M_0,\alpha} \|x\|_{\alpha,r}\,,
\end{aligned}$$

where we substituted $\sigma = \tau - s$ and used (2.2). It follows that $D_A(\alpha,r) \hookrightarrow D_A(\alpha,\infty)$. For $x\in D_A(\alpha,q)$, due to part b) there are vectors $x_n \in \mathrm{D}(A)$ converging to x in $D_A(\alpha,q)$, and hence in $D_A(\alpha,\infty)$. This means that $D_A(\alpha,q)$ is even embedded into $D_A(\alpha)$. From $D_A(\alpha,p)\hookrightarrow D_A(\alpha,\infty)$ we finally infer

$$\|\varphi_{\alpha,x}\|_q = \Big(\int_0^1 |\varphi_{\alpha,x}|^p\,|\varphi_{\alpha,x}|^{q-p}\,\frac{\mathrm{d}s}{s}\Big)^{\frac1q} \le \|\varphi_{\alpha,x}\|_p^{\frac pq}\;\|\varphi_{\alpha,x}\|_\infty^{1-\frac pq} \lesssim_{\alpha,M_0} \|\varphi_{\alpha,x}\|_p\,,$$

establishing statement d).

e) The implication '$\Leftarrow$' is clear. For the other implication, let $x\in D_A(\alpha,\infty)$, $0\le s<t\le b$, and $K := \sup_{t\in[0,b]}\|T(t)\|$, where we may assume that $b\ge 1$. If $t-s\ge 1$, one trivially has $(t-s)^{-\alpha}\,\|T(t)x-T(s)x\| \le 2K\|x\|$. For $t-s\le 1$, the semigroup property yields

$$(t-s)^{-\alpha}\,\|T(t)x-T(s)x\| \le K(t-s)^{-\alpha}\,\|T(t-s)x-x\| \le K\,[x]_{\alpha,\infty}.$$

f) For $x\in D_A(\alpha)$ and $\varepsilon>0$, there is a vector $y\in\mathrm{D}(A)$ such that $\|x-y\|_{\alpha,\infty}\le\varepsilon$. We can thus estimate

$$\begin{aligned}
\limsup_{t\to0} t^{-\alpha}\,\|T(t)x-x\| &\le \|x-y\|_{\alpha,\infty} + \limsup_{t\to0} t^{1-\alpha}\,t^{-1}\|T(t)y-y\| \\
&\le \varepsilon + 0\,\|Ay\| = \varepsilon,
\end{aligned}$$

proving the 'only if' part. Conversely, assume that $\varphi_{\alpha,x}(s)$ tends to 0 as $s \to 0$. Let $\varepsilon > 0$ and $s, t \in (0,1]$. First, there thus exists $\delta_\varepsilon \in (0,1)$ with

$$s^{-\alpha} \|(T(s)-I)(T(t)x-x)\| \le (1+M_0)s^{-\alpha} \|T(s)x-x\| \le \varepsilon$$

for all $s \in (0, \delta_\varepsilon]$ and $t \in (0,1]$. Second, we find a number $\eta \in (0,1)$ such that

$$s^{-\alpha} \|(T(s)-I)(T(t)x-x)\| \le (1+M_0)\delta_\varepsilon^{-\alpha} \|T(t)x-x\| \le \varepsilon$$

for all $s \in [\delta_\varepsilon, 1]$ and $t \in (0, \eta]$. This means that $T(t)x$ tends to x in $D_A(\alpha,\infty)$ as $t \to 0$ so that the vectors $y_n = n\int_0^{1/n} T(t)x\,\mathrm{d}t$ converge to x in $D_A(\alpha,\infty)$ as $n \to \infty$. Hence, x belongs to $D_A(\alpha)$ as $y_n \in \mathrm{D}(A)$ by Lemma 1.19 in [**EE**]. □

We now describe the interpolation spaces for the translation group.

Example 2.3. Let $X = L^q(\mathbb{R})$ for some $q \in [1,\infty)$ or $X = C_0(\mathbb{R})$ for $q = \infty$. We consider the (isometric) translation group on X given by $T(t)f = f(\cdot + t)$ for $f \in X$ and $t \in \mathbb{R}$. It has the generator $A = \mathrm{d}/\mathrm{d}s$ with domain $\mathrm{D}(A) = W^{1,q}(\mathbb{R})$ respectively $\mathrm{D}(A) = C_0^1(\mathbb{R})$, cf. Examples 1.22 and 1.43 in [**EE**]. Due to (2.4), the interpolation norms are given by

$$\begin{aligned}\|f\|_{\alpha,p} &\eqsim_\alpha \|f\|_q + \Big(\int_{\mathbb{R}} |t|^{-\alpha p-1} \|f(\cdot+t)-f\|_{L^q(\mathbb{R})}^p \,\mathrm{d}t\Big)^{\frac1p} \\ &= \|f\|_q + \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \frac{|f(s+t)-f(s)|^q}{|t|^{\alpha q+\frac{q}{p}}}\,\mathrm{d}s\Big)^{\frac{p}{q}}\,\mathrm{d}t\Big)^{\frac1p}, \\ \|f\|_{\alpha,\infty} &= \|f\|_\infty + \sup_{t\in\mathbb{R}\setminus\{0\}} \Big(\int_{\mathbb{R}} \frac{|f(s+t)-f(s)|^q}{|t|^{\alpha q}}\,\mathrm{d}s\Big)^{\frac1q}\end{aligned}$$

for $p, q \in [1,\infty)$, and for $p = q = \infty$ by the Hölder norm

$$\|f\|_{\alpha,\infty} = \|f\|_\infty + \sup_{t\in\mathbb{R}\setminus\{0\},\, s\in\mathbb{R}} \frac{|f(s+t)-f(s)|}{|t|^\alpha}\,.$$

For $q < \infty$, the space $D_A(\alpha,p)$ coincides with the *Besov space* $B^\alpha_{qp}(\mathbb{R})$, see [**Tr1**] and [**Tr2**].[1] In the special case $p = q \in [1,\infty)$, the space $B^\alpha_{pp}(\mathbb{R}) =: W^{\alpha,p}(\mathbb{R})$ is called *Slobodetskii space* (or *fractional Sobolev space*) and has the simpler norm

$$\|f\|_{\alpha,p} \eqsim \|f\|_p + \Big(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(\tau)-f(s)|^p}{|\tau-s|^{\alpha p+1}}\,\mathrm{d}\tau\,\mathrm{d}s\Big)^{\frac1p}.$$

(Here one uses Fubini's theorem and the substitution $\tau = s+t$.)

There are functions f in $C_0(\mathbb{R})$ with a finite Hölder norm $\|f\|_{\alpha,\infty}$ such that $f(s) = s^\alpha$ for $s \in [0,1]$, and thus $t^{-\alpha}\|T(t)f-f\|_\infty \ge t^{-\alpha}|f(t)-f(0)| = 1$ for all $t \in (0,1]$. Proposition 2.2 then yields $f \in D_A(\alpha,\infty)\setminus D_A(\alpha)$. ◊

We next see that the semigroup $T(\cdot)$ behaves nicely on its interpolation spaces. In general, it is not strongly continuous on $D_A(\alpha,\infty)$. Consider for instance the translation group on $C_0(\mathbb{R})$ in Example 2.3, and take a function $f \in C_0(\mathbb{R})$ such that $f(t) = (t-n)^\alpha$ on $[n, n+1/n]$. It then holds $\|T(1/n)f-f\|_{C^\alpha} \ge n^\alpha\,|f(n+1/n)-f(n)| = 1$ for all $n \in \mathbb{N}$. Nevertheless one

[1] To obtain the Besov spaces $B^\alpha_{\infty p}(\mathbb{R})$ or the Hölder spaces, one has to start with $C_b(\mathbb{R})$ instead of $C_0(\mathbb{R})$, which is not possible in our setting, but see p.13 in [**Lu2**] or [**Tr1**], [**Tr2**].

could also work on $D_A(\alpha,\infty)$, cf. Section II.5 b) in [**EN**] as well as Section 2.2 in [**Lu1**] for analytic semigroups.

Proposition 2.4. *Let (2.1) be true. Then the following assertions hold.*

a) We have $T(t)D_A(\alpha,p) \subseteq D_A(\alpha,p)$ and $T(t)D_A(\alpha) \subseteq D_A(\alpha)$ for all $t \ge 0$. The norms of the restrictions $T_{\alpha,p}(t) := T(t)|_{D_A(\alpha,p)}$ and $T_\alpha(t) := T(t)|_{D_A(\alpha)}$ are less or equal $\|T(t)\|_{\mathcal{B}(X)}$ for each $t \ge 0$. The operator families $T_{\alpha,p}(\cdot)$ and $T_\alpha(\cdot)$ are C_0–semigroups if $p < \infty$.

b) The generators of $T_{\alpha,p}(\cdot)$ and $T_\alpha(\cdot)$ are the restrictions $A_{\alpha,p}$ (with $p < \infty$) and A_α of A in the respective spaces endowed with the domains

$$\mathrm{D}(A_{\alpha,p}) = \{x \in \mathrm{D}(A) \mid Ax \in D_A(\alpha,p)\} =: D_A(1+\alpha,p),$$
$$\mathrm{D}(A_\alpha) = \{x \in \mathrm{D}(A) \mid Ax \in D_A(\alpha)\} =: D_A(1+\alpha).$$

c) Let $\lambda \in \rho(A)$. Then λ belongs to $\rho(A_{\alpha,p})$ and $\rho(A_\alpha)$, with $R(\lambda, A_{\alpha,p}) = R(\lambda, A)|_{D_A(\alpha,p)}$ and $R(\lambda, A_\alpha) = R(\lambda, A_\alpha)|_{D_A(\alpha)}$ if $p < \infty$. These resolvents have norm less or equal $\|R(\lambda, A)\|_{\mathcal{B}(X)}$.

d) We have $\sigma(A) = \sigma(A_{\alpha,p}) = \sigma(A_\alpha)$ if $p < \infty$.

Proof. Let $x \in X$, $t \ge 0$, and $s \in (0,1]$. Observe that

$$\varphi_{\alpha,T(t)x}(s) = s^{-\alpha}\,\|T(t)(T(s)x - x)\| \le \|T(t)\|\,\varphi_{\alpha,x}(s).$$

Hence, the semigroups leave invariant the interpolation spaces and their restrictions have norms less or equal $\|T(t)\|$. Of course, they are still semigroups on these spaces. Let $x \in \mathrm{D}(A)$. Proposition 2.2 yields that

$$\|T(t)x - x\|_{\alpha,p} \le c\,\|T(t)x - x\|_A \;\longrightarrow\; 0$$

as $t \to 0$. Since the restrictions are locally bounded, $T(\cdot)$ is strongly continuous on $D_A(\alpha)$ and, due to the density proved in Proposition 2.2, also on $D_A(\alpha,p)$ if $p < \infty$. We have shown assertion a).

From now on we take $p < \infty$. Let B be the generator of $T_{\alpha,p}(\cdot)$ and let $A_{\alpha,p}$ be defined as in the statement. Let $x \in \mathrm{D}(B) \subseteq D_A(\alpha,p)$. Then $\frac{1}{t}(T(t)x - x)$ converges to Bx in $D_A(\alpha,p)$, as $t \to 0$. Since $D_A(\alpha,p) \hookrightarrow X$, these vectors also tend to Bx in X. This means that x belongs to $\mathrm{D}(A)$ and that $Ax = Bx \in D_A(\alpha,p)$; i.e., $B \subseteq A_{\alpha,p}$.

Let $\lambda \in \rho(A)$. We show that λ is contained in $\rho(A_{\alpha,p})$, implying that $\rho(B)$ and $\rho(A_{\alpha,p})$ both contain a right halfplane, and hence $B = A_{\alpha,p}$ by Lemma 1.24 of [**EE**]. Let $x \in D_A(\alpha,p)$. Then $AR(\lambda,A)x = \lambda R(\lambda,A)x - x$ is also an element of $D_A(\alpha,p)$, so that $R(\lambda,A)x \in D_A(\alpha+1,p)$ and

$$(\lambda I - A_{\alpha,p})R(\lambda,A)x = (\lambda I - A)R(\lambda,A)x = x$$

because of the definition of $A_{\alpha,p}$. For $x \in D_A(\alpha+1,p)$, we further have

$$R(\lambda,A)(\lambda I - A_{\alpha,p})x = R(\lambda,A)(\lambda I - A)x = x$$

It follows that λ belongs to $\rho(A_{\alpha,p})$ and $R(\lambda, A_{\alpha,p}) = R(\lambda,A)|_{D_A(\alpha,p)}$. The estimate for $R(\lambda, A_{\alpha,p})$ is then shown as for $T_{\alpha,p}(t)$. The results in b) and c) for $D_A(\alpha)$ are proved in the same way.

Statement d) now is a direct consequence of Proposition IV.2.17 in [**EN**]. □

We extend Example 2.3 to second derivatives and several dimensions, mostly without giving proofs.

EXAMPLE 2.5. Let $p = q \in [1, \infty)$ for simplicity. We first look at $X = L^p(\mathbb{R})$, $A = \mathrm{d}/\mathrm{d}s$ and $\mathrm{D}(A) = W^{1,p}(\mathbb{R})$ as in Example 2.3. One defines the space

$$W^{1+\alpha,p}(\mathbb{R}) := \{u \in W^{1,p}(\mathbb{R}) \,|\, u' \in W^{\alpha,p}(\mathbb{R})\} = D_A(1+\alpha, p).$$

Based on somewhat deeper interpolation theory, Example 5.14 in [**Lu2**] yields the identity $W^{1+\alpha,p}(\mathbb{R}) = D_{A^2}(\frac{1}{2} + \frac{\alpha}{2}, p)$. Observe that $A^2 = \mathrm{d}^2/\mathrm{d}s^2$ is the one-dimensional Laplacian with domain $\mathrm{D}(A^2) = \{u \in W^{1,p}(\mathbb{R}) \,|\, u' \in W^{1,p}(\mathbb{R})\} = W^{2,p}(\mathbb{R})$. It generates a C_0–semigroup, cf. Example 1.49 in [**EE**].

One introduces the Slobodetskii spaces on $\mathbb{R}^m$ as for $m = 1$ by

$$W^{\alpha,p}(\mathbb{R}^m) = \left\{ u \in L^p(\mathbb{R}^m) \,\Big|\, \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{|u(y) - u(x)|^p}{|y - x|^{\alpha p + m}} \,\mathrm{d}x \,\mathrm{d}y < \infty \right\},$$
$$W^{1+\alpha,p}(\mathbb{R}^m) = \{u \in W^{1,p}(\mathbb{R}^m) \,|\, \nabla u \in W^{\alpha,p}(\mathbb{R}^m)^m\}.$$

Let $p \in (1, \infty)$. Example 2.29 in [**EE**] says that Δ with $\mathrm{D}(\Delta) = W^{2,p}(\mathbb{R}^m)$ generates an (analytic) C_0–semigroup. Again employing more interpolation theory, it can be shown that

$$W^{\alpha,p}(\mathbb{R}^m) = D_\Delta(\tfrac{\alpha}{2}, p) \quad \text{and} \quad W^{1+\alpha,p}(\mathbb{R}^m) = D_\Delta(\tfrac{1}{2} + \tfrac{\alpha}{2}, p).$$

See Examples 5.16 and 5.15 in [**Lu2**], where also the cases $p \in \{1, \infty\}$ are treated. When comparing with the examples in [**Lu2**], one has to use Proposition 5.7 in this book. $\Diamond$

Two equivalent definitions. We next characterize the interpolation spaces in terms of the resolvent of A and, for analytic semigroups, by the time derivative $\frac{\mathrm{d}}{\mathrm{d}t} T(\cdot) = AT(\cdot)$.

In the proof of the first equivalence, we need the following facts which highlight the role of the weight $1/t$ of the spaces $L^p_*(J)$. The multiplicative group $\mathbb{R}_+$ posseses the invariant measure $\mathrm{d}t/t$; i.e.,

$$\int_0^\infty f(\lambda s) \,\frac{\mathrm{d}s}{s} = \int_0^\infty f(\tau) \,\frac{\mathrm{d}\tau}{\tau} \tag{2.6}$$

holds for every non-negative measurable function f and each $\lambda > 0$, due to the substitution $\tau = \lambda s$. Similarly, one obtains

$$\int_0^\infty f(s^{-1}) \,\frac{\mathrm{d}s}{s} = \int_0^\infty f(\tau) \,\frac{\mathrm{d}\tau}{\tau}. \tag{2.7}$$

By means of these identities, as for the additive group $\mathbb{R}$ and the Lebesgue measure one can prove Young's inequality for the convolution integral

$$(f * g)(\lambda) = \int_0^\infty f(\lambda s^{-1}) g(s) \,\frac{\mathrm{d}s}{s}$$

for $f \in L^p_*(\mathbb{R}_+)$ and $g \in L^1_*(\mathbb{R}_+)$, namely

$$\|f * g\|_{L^p_*(\mathbb{R}_+)} \le \|f\|_{L^p_*(\mathbb{R}_+)} \, \|g\|_{L^1_*(\mathbb{R}_+)} \tag{2.8}$$

To use the above results directly, we take the interval $J = \mathbb{R}_+$ in the next result and thus assume that the semigroup is bounded by $M := \sup_{t \ge 0} \|T(t)\| <$

∞. This is always true if we replace the given A in (2.1) by $A - \omega I$ for some $\omega > \omega_0(A)$. The constants below would then depend on ω, too. For $x \in X$, $\alpha \in (0,1)$ and $\lambda > 0$, we introduce the function

$$\chi_{\alpha,x}(\lambda) = \|\lambda^\alpha AR(\lambda, A)x\|. \tag{2.9}$$

Observe that $\chi_{\alpha,x}$ is bounded for $\alpha = 0$ and $x \in X$ as well as for $\alpha = 1$ and $x \in \mathrm{D}(A)$ by the Hille–Yosida estimate $\|\lambda R(\lambda, A)\| \le M$ from Proposition 2.21 in [**EE**]. The result belows says that the interpolation spaces fill the gap between these two extreme cases. Note that the limit $s \to 0$ is replaced by $\lambda \to \infty$ compared to Proposition 2.2 f). Besides (2.6)–(2.8), the proofs rely on basic formulas from [**EE**] relating generators with their resolvent and semigroup.

PROPOSITION 2.6. *Let* (2.1) *hold with* $M = \sup_{t\ge 0} \|T(t)\| < \infty$. *We then have*

$$D_A(\alpha, p) = \{x \in X \mid \chi_{\alpha,x} \in L^p_*(\mathbb{R}_+)\},$$

and the norm $\|\cdot\|_{\alpha,p}$ *is equivalent to* $x \mapsto \|x\| + \|\chi_{\alpha,x}\|_{L^p_*(\mathbb{R}_+)}$. *(In the proof one finds estimates on the corresponding constants.) Moreover,* x *belongs to* $D_A(\alpha)$ *if and only if* $\chi_{\alpha,x}(\lambda) \to 0$ *as* $\lambda \to \infty$.

PROOF. Let $\alpha \in (0,1)$ and $p \in [1, \infty]$.

1) Let $x \in D_A(\alpha, p)$ and $\lambda > 0$. Proposition 1.21 of [**EE**] and (2.6) yield

$$AR(\lambda, A)x = \lambda R(\lambda, A)x - x = \int_0^\infty \lambda \mathrm{e}^{-\lambda t} t^\alpha t^{-\alpha} (T(t)x - x)\, \mathrm{d}t,$$

$$\chi_{\alpha,x}(\lambda) \le \int_0^\infty (\lambda t)^{1+\alpha} \mathrm{e}^{-\lambda t}\, \varphi_{\alpha,x}(t)\, \frac{\mathrm{d}t}{t} = \int_0^\infty s^{1+\alpha} \mathrm{e}^{-s}\, \varphi_{\alpha,x}(\lambda^{-1}s)\, \frac{\mathrm{d}s}{s}. \tag{2.10}$$

Setting $\tilde{\varphi}_{\alpha,x}(\tau) = \varphi_{\alpha,x}(\tau^{-1})$, we infer from Young's inequality (2.8) the bound

$$\|\chi_{\alpha,x}\|_{L^p_*(\mathbb{R}_+)} \le \|\tilde{\varphi}_{\alpha,x}\|_{L^p_*(\mathbb{R}_+)} \int_0^\infty s^{1+\alpha} \mathrm{e}^{-s}\, \frac{\mathrm{d}s}{s} = \Gamma(1+\alpha)\, \|\varphi_{\alpha,x}\|_{L^p_*(\mathbb{R}_+)}$$

with the classical Gamma function and also using (2.7).

Let $x \in D_A(\alpha)$ and $s > 0$. We then have $\varphi_{\alpha,x}(s\lambda^{-1}) \le c(\alpha, M)\, \|x\|_{\alpha,\infty}$ and $\varphi_{\alpha,x}(s\lambda^{-1}) \to 0$ as $\lambda \to \infty$ because of (2.3) respectively Proposition 2.2 f). By dominated convergence, estimate (2.10) leads to $\chi_{\alpha,x}(\lambda) \to 0$ as $\lambda \to \infty$.

2) For the converse, take $\chi_{\alpha,x}$ from $L^p_*(\mathbb{R}_+)$ and $s > 0$. We decompose x into a vector in $\mathrm{D}(A)$ with a weight and one in X, writing

$$x = s^{-1}R(s^{-1}, A)x - AR(s^{-1}, A)x =: x_1 - x_2.$$

Lemma 1.19 of [**EE**] and (2.9) now yield

$$\|T(s)x_1 - x_1\| \le \int_0^s \|T(\tau)Ax_1\|\, \mathrm{d}\tau \le sM\, \|Ax_1\| = s^\alpha M \chi_{\alpha,x}(s^{-1}),$$

$$\|T(s)x_2 - x_2\| \le (M+1)\|x_2\| = (1+M)s^\alpha \chi_{\alpha,x}(s^{-1}).$$

It follows that $\varphi_{\alpha,x}(s) \le (1+2M)\chi_{\alpha,x}(s^{-1})$ and hence

$$\|\varphi_{\alpha,x}\|_{L^p_*(\mathbb{R}_+)} \le (1+2M)\, \|\chi_{\alpha,x}\|_{L^p_*(\mathbb{R}_+)}$$

by (2.7). In view of estimate (2.3), the first assertion is shown.

If $\chi_{\alpha,x}(\lambda) \to 0$ as $\lambda \to \infty$, the above pointwise inequality yields the limit $\varphi_{\alpha,x}(s) \to 0$ as $s \to 0$ so that the second claim results from Propsition 2.2 f). $\square$

As a preparation for the next characterization, we prove an important estimate for power-weighted L^p–norms called *Hardy's inquality.*

LEMMA 2.7. *Let* $a \in (0,\infty]$, $\alpha > 0$, $p \in [1,\infty)$ *and* $\varphi : (0,a) \to \mathbb{R}_{\geq 0}$ *be measurable. We then have*

$$\int_0^a t^{-\alpha p}\Big(\int_0^t \varphi(s)\,\frac{\mathrm{d}s}{s}\Big)^p\,\frac{\mathrm{d}t}{t} \leq \frac{1}{\alpha^p}\int_0^t s^{-\alpha p}\varphi(s)^p\,\frac{\mathrm{d}s}{s}.$$

PROOF. We can assume that the right-hand side in the above inequality is finite. We set $f(\tau,t) = (f(\tau))(t) = t^{-\alpha}\varphi(\tau t)$ for $t \in (0,a)$ and $\tau \in (0,1]$. The substitution $\tau = s/t$ yields

$$C := \Big(\int_0^a t^{-\alpha p}\Big(\int_0^t \varphi(s)\,\frac{\mathrm{d}s}{s}\Big)^p\,\frac{\mathrm{d}t}{t}\Big)^{\frac{1}{p}} = \Big(\int_0^a \Big(\int_0^1 (f(\tau))(t)\,\frac{\mathrm{d}\tau}{\tau}\Big)^p\,\frac{\mathrm{d}t}{t}\Big)^{\frac{1}{p}}.$$

Below we show that $f : (0,1] \to L^p_*(0,a)$ is strongly measurable, compare the discussion before Lemma 4.4 in [**EE**]. We can thus rewrite the above formula invoking a Bochner integral. Substituting also $s = \tau t$, it then follows

$$\begin{aligned}
C &= \Big\| \int_0^\tau f(\tau)\,\frac{\mathrm{d}\tau}{\tau}\Big\|_{L^p_*(0,a)} \leq \int_0^1 \|f(\tau)\|_{L^p_*(0,a)}\,\frac{\mathrm{d}\tau}{\tau} = \int_0^1 \Big[\int_0^a t^{-\alpha p}\varphi(\tau t)^p\,\frac{\mathrm{d}t}{t}\Big]^{\frac{1}{p}}\,\frac{\mathrm{d}\tau}{\tau}\\
&= \int_0^1 \Big(\int_0^{a\tau} \tau^{\alpha p} s^{-\alpha p}\varphi(s)^p\,\frac{\mathrm{d}s}{s}\Big)^{\frac{1}{p}}\,\frac{\mathrm{d}\tau}{\tau} \leq \int_0^1 \tau^{\alpha}\Big(\int_0^a s^{-\alpha p}\varphi(s)^p\,\frac{\mathrm{d}s}{s}\Big)^{\frac{1}{p}}\,\frac{\mathrm{d}\tau}{\tau}\\
&= \frac{1}{\alpha}\Big(\int_0^a s^{-\alpha p}\varphi(s)^p\,\frac{\mathrm{d}s}{s}\Big)^{\frac{1}{p}}.
\end{aligned}$$

We finally indicate a proof of the claimed strong measurability.[2] Let $0 \leq g \in L^{p'}_*(0,a)$. Since the function $(\tau,t) \mapsto f(\tau,t)g(t)$ is measurable and non-negative, Fubini's theorem shows the measurability of the map

$$(0,1] \to \mathbb{R};\ \tau \mapsto \langle f(\tau), g\rangle_{L^p_*(0,a)} = \int_0^a f(\tau,t)g(t)\,\frac{\mathrm{d}t}{t}.$$

This fact is then true for all $g \in L^{p'}_*(0,a)$. Pettis' measurability theorem now yields that f is strongly measurable, see Theorem 1.1.1 in [**ABHN**]. $\square$

The next proposition describes the interpolation spaces of an analytic semigroup in terms of its time derivative $\frac{\mathrm{d}}{\mathrm{d}t}T(t) = AT(t)$. This result will be crucial for our appplications to parabolic problems. To this aim, we define the quantity

$$\psi_{\alpha,x}(s) = s^{1-\alpha}\|AT(s)x\|$$

for $x \in X$, $\alpha \in (0,1)$, and $s > 0$. Observe that it becomes bounded (for $s \in (0,1]$, say) if $x \in \mathrm{D}(A)$ and $\alpha = 1$, as well as for $x \in X$ and $\alpha = 0$ by Theorem 2.23 of [**EE**]. Again we want to interpolate between these two starting points. As in the previous result we use basic semigroup theory, but now also Hardy's inequality.

[2]This part was omitted in the lectures. The above calculation can also be justified using *Minkowski's inequality for integrals*, cf. Satz X.6.24 in [**AE3**].

PROPOSITION 2.8. *Let* (2.1) *hold and* $T(\cdot)$ *be analytic. We then have*

$$D_A(\alpha,p)=\{x\in X\mid \psi_{\alpha,x}\in L^p_*\},$$

and the norm $\|\cdot\|_{\alpha,p}$ *is equivalent to* $x\mapsto \|x\|+\|\psi_{\alpha,x}\|_{L^p_*}$. *(In the proof one finds estimates on the corresponding constants.) For* $p\in(1,\infty)$, *it follows*

$$x\in D_A(1-\tfrac{1}{p},p) \iff AT(\cdot)x\in L^p((0,1],X).$$

Moreover, x *belongs to* $D_A(\alpha)$ *if and only if* $\psi_{\alpha,x}(s)\to 0$ *as* $s\to 0$.

PROOF. Let $x\in X$ and $s\in(0,1]$.

1) Using Lemma 1.19 of [**EE**], we estimate

$$\begin{aligned}\varphi_{\alpha,x}(s)&=\lim_{\varepsilon\to 0} s^{-\alpha}\,\|T(s)x-T(\varepsilon)x\|=\lim_{\varepsilon\to 0} s^{-\alpha}\left\|\int_\varepsilon^s \tau^{\alpha-1}\tau^{1-\alpha}AT(\tau)x\,\mathrm{d}\tau\right\|\\&\le \limsup_{\varepsilon\to 0} s^{-\alpha}\int_\varepsilon^s \tau^{\alpha-1}\psi_{\alpha,x}(\tau)\,\mathrm{d}\tau\le \frac{1}{\alpha}\sup_{0<\tau\le s}\psi_{\alpha,x}(\tau).\end{aligned}$$

This inequality yields the first half of the first assertion for $p=\infty$ and of the last assertion because of Propsition 2.2 f). Let $p\in[1,\infty)$ and $\psi_{\alpha,x}\in L^p_*$. Proceeding as above, from Hardy's Lemma 2.7 we infer the bound

$$\begin{aligned}[x]^p_{\alpha,p}&\le \int_0^1 s^{-\alpha p}\Big(\int_0^s \tau\,\|AT(\tau)x\|\,\frac{\mathrm{d}\tau}{\tau}\Big)^p\,\frac{\mathrm{d}s}{s}\\&\le \frac{1}{\alpha^p}\int_0^1 \tau^{-\alpha p}\tau^p\|AT(\tau)x\|^p\,\frac{\mathrm{d}\tau}{\tau}=\alpha^{-p}\,\|\psi_{\alpha,x}\|^p_{L^p_*}.\end{aligned}$$

2) For the converse, we put $M_1:=\sup_{0<s\le 1}\|sAT(s)\|$. (See Theorem 2.23 of [**EE**].) Let $x\in D_A(\alpha,p)$. Lemma 1.19 of [**EE**] implies

$$\begin{aligned}s^{1-\alpha}AT(s)x&=s^{-\alpha}T(s)(T(s)x-x)-s^{-\alpha}AT(s)\int_0^s \tau^\alpha\tau^{-\alpha}(T(\tau)x-x)\,\mathrm{d}\tau,\\ \psi_{\alpha,x}(s)&\le M_0\varphi_{\alpha,x}(s)+M_1 s^{-1-\alpha}\int_0^s \tau^\alpha\,\varphi_{\alpha,x}(\tau)\,\mathrm{d}\tau.\end{aligned}$$

In the case $p=\infty$, we derive

$$\psi_{\alpha,x}(s)\le M_0\varphi_{\alpha,x}(s)+\frac{M_1}{1+\alpha}\sup_{\tau\in(0,s]}\varphi_{\alpha,x}(\tau),$$

and deduce the asserted equivalence for $p=\infty$ and the final assertion. Let $p\in[1,\infty)$. The previous estimate and Hardy's inequality lead to

$$\begin{aligned}\psi_{\alpha,x}(s)&\le M_0\varphi_{\alpha,x}(s)+M_1 s^{-\alpha}\int_0^s \tau^\alpha\,\varphi_{\alpha,x}(\tau)\,\frac{\mathrm{d}\tau}{\tau},\\ \|\psi_{\alpha,x}\|_{L^p_*}&\le M_0\|\varphi_{\alpha,x}\|_{L^p_*}+\frac{M_1}{\alpha}\Big[\int_0^1 \tau^{-\alpha p}\tau^{\alpha p}\varphi_{\alpha,x}(\tau)^p\,\frac{\mathrm{d}\tau}{\tau}\Big]^{\frac{1}{p}}\le (M_0+\tfrac{M_1}{\alpha})\|\varphi_{\alpha,x}\|_{L^p_*}\end{aligned}$$

concluding the proof of the first assertion.

3) Let $p\in[1,\infty)$. The second claim then follows from the formula

$$\|\psi_{1-1/p,x}\|^p_{L^p_*}=\int_0^1 s^{p(1-(1-1/p))}\,\|AT(s)x\|^p\,\frac{\mathrm{d}s}{s}=\int_0^1 \|AT(s)x\|^p\,\mathrm{d}s. \qquad \square$$

Let $b > 0$, where allow for $b = \infty$ if $T(\cdot)$ and $s \mapsto sAT(s)$ are bounded on $\mathbb{R}_+$, cf. Theorem 2.23 of [**EE**]. We set $M_0' = \sup_{0<s<b} \|T(s)\|$ and $M_1 = \sup_{0<s<b} \|sAT(s)\|$. As in the above proof one shows that

$$\|\psi_{\alpha,x}\|_{L^p_*(0,b)} \le c(\alpha, M_0', M_1')\, \|\varphi_{\alpha,x}\|_{L^p_*(0,b)}. \tag{2.11}$$

The next theorem shows that our spaces $D_A(\alpha,p)$ and $D_A(\alpha)$ are indeed *interpolation spaces* in the usual sense of this concept. For an operator $T \in \mathcal{B}(X,Y)$ mapping a subspace $X_0 \subseteq X$ into a subspace $Y_0 \subseteq Y$, we denote the restriction of T acting from X_0 to Y_0 by the same symbol. In the proof below we implicitly use the standard definition of real interpolation spaces via the 'k–functional,' see Remark 2.11c).

THEOREM 2.9. *Assume that A and B generate C_0–semigroups $T(\cdot)$ and $S(\cdot)$ on X and Y, respectively, and that the operator $V \in \mathcal{B}(X,Y)$ satisfies $V\mathrm{D}(A) \subseteq \mathrm{D}(B)$ and $V \in \mathcal{B}(X_1,Y_1)$, where $X_1 := [\mathrm{D}(A)]$ and $Y_1 := [\mathrm{D}(B)]$. Let $0<\alpha<1$ and $1 \le p \le \infty$. Then V maps $D_A(\alpha,p)$ into $D_B(\alpha,p)$ and $D_A(\alpha)$ into $D_B(\alpha)$, we have $V \in \mathcal{B}(D_A(\alpha,p), D_B(\alpha,p))$ and $V \in \mathcal{B}(D_A(\alpha), D_B(\alpha))$, and it holds*

$$\|V\|_{\mathcal{B}(D_A(\alpha,p),D_B(\alpha,p))},\ \|V\|_{\mathcal{B}(D_A(\alpha),D_B(\alpha))} \le c\,\|V\|^{1-\alpha}_{\mathcal{B}(X,Y)}\ \|V\|^{\alpha}_{\mathcal{B}(X_1,Y_1)}$$

for a constant only depending on A, B and α, as indicated in the proof.

PROOF. In view of (2.5), after rescaling if necessary, we may assume that the semigroups are exponentially stable. So let A and B be invertible and the semigroups be bounded.

Take $x \in X$, $s \in (0,1]$, and $t > 0$. Since the result is trivially true for $V = 0$, we may assume that $V \ne 0$. We set $\|V\|_0 = \|V\|_{\mathcal{B}(X,Y)}$, $\|V\|_1 = \|V\|_{\mathcal{B}(X_1,Y_1)}$, and $N_0 = \sup_{0\le s\le 1} \|S(s)\|$. Let $x = x_0 + x_1$ for some $x_0 \in X$ and $x_1 \in X_1$. As $Vx_1 \in Y_1$, Lemma 1.19 of [**EE**] yields

$$\begin{aligned}\|S(s)Vx - Vx\|_Y &\le \|S(s)Vx_0 - Vx_0\|_Y + \|S(s)Vx_1 - Vx_1\|_Y\\ &\le (N_0+1)\|Vx_0\|_Y + \int_0^s \|S(\tau)BVx_1\|_Y\,\mathrm{d}\tau\\ &\le (N_0+1)\|Vx_0\|_Y + sN_0\,\|Vx_1\|_{Y_1}\\ &\le (N_0+1)\|V\|_0\,(\|x_0\|_X + s\|V\|_1\,\|V\|_0^{-1}\,\|x_1\|_{X_1}).\end{aligned}$$

We now define the *k–functional* by

$$k(t,x) = \inf\{\,\|x_0\|_X + t\|x_1\|_{X_1} \,|\, x = x_0 + x_1,\ x_0 \in X,\ x_1 \in X_1\}. \tag{2.12}$$

Below we use the decomposition of x given in (2.13), which already appeared in the proof of Proposition 2.8. However, the proof given here also requires the infimum in (2.12) over all decompositions. Taking this infimum, we deduce

$$\varphi^B_{\alpha,Vx}(s) \le (N_0+1)\|V\|_0\, s^{-\alpha} k(s\|V\|_1\,\|V\|_0^{-1}, x).$$

Here and below we use the superscript B in an obvious way. We have the decomposition $x = x_0 + x_1$ with

$$x_0 = -AR(t^{-1},A)x \in X \quad \text{and} \quad x_1 = t^{-1}R(t^{-1},A)x \in X_1. \tag{2.13}$$

The k–functional can thus be dominated by

$$k(t,x) \le \|AR(t^{-1},A)x\|_X + t\,\|t^{-1}R(1/s,A)x\|_{X_1} \le (1+\|A^{-1}\|)\|AR(t^{-1},A)x\|_X.$$

Let $p < \infty$. The substitution $t = s\|V\|_1 \, \|V\|_0^{-1}$ and (2.7) then imply

$$
\begin{aligned}
[Vx]^B_{\alpha,p} &\le (N_0+1)\, \|V\|_0 \left[\int_0^1 s^{-\alpha p}\, k(s\|V\|_1\, \|V\|_0^{-1}, x)^p \, \frac{\mathrm{d}s}{s}\right]^{1/p} \\
&\le (N_0+1)\|V\|_0 \left[\int_0^\infty t^{-\alpha p}\, \|V\|_1^{\alpha p}\, \|V\|_0^{-\alpha p}\, k(t,x)^p \, \frac{\mathrm{d}t}{t}\right]^{1/p} \\
&\le (N_0+1)\,(1+\|A^{-1}\|)\, \|V\|_0^{1-\alpha}\, \|V\|_1^{\alpha} \left[\int_0^\infty t^{-\alpha p}\, \|AR(t^{-1},A)x\|_X^p \, \frac{\mathrm{d}t}{t}\right]^{1/p} \\
&= (N_0+1)\,(1+\|A^{-1}\|)\, \|V\|_0^{1-\alpha}\, \|V\|_1^{\alpha} \left[\int_0^\infty \lambda^{\alpha p}\, \|AR(\lambda,A)x\|_X^p \, \frac{\mathrm{d}\lambda}{\lambda}\right]^{1/p} \\
&= (N_0+1)\,(1+\|A^{-1}\|)\, \|V\|_0^{1-\alpha}\, \|V\|_1^{\alpha}\, \|\chi_{\alpha,p}\|_{L^p_*(\mathbb{R}_+)} \,.
\end{aligned}
$$

The norm of $D_B(\alpha,p)$ also contains the term $\|y\|_Y$.[3] To deal with it, let $x = x_0 + x_1$ for some $x_0 \in X$ and $x_1 \in X_1$. We estimate

$$
\begin{aligned}
\|Vx\|_Y &\le \|V\|_0 \big(\|x_0\|_X + \|V\|_1\, \|V\|_0^{-1}\, \|x_1\|_{X_1}\big) = \|V\|_0\, k(\|V\|_1\, \|V\|_0^{-1}, x) \\
&\le \|V\|_0 \sup_{t>0} t^{-\alpha}\, k(t\|V\|_1\, \|V\|_0^{-1}, x) = \|V\|_0 \sup_{s>0} (s\|V\|_1^{-1}\, \|V\|_0)^{-\alpha}\, k(s,x) \\
&= \|V\|_0^{1-\alpha}\, \|V\|_1^{\alpha} \sup_{s>0} (\alpha p)^{\frac{1}{p}} \Big(\int_s^\infty \tau^{-\alpha p-1} k(s,x)^p \, \mathrm{d}\tau\Big)^{1/p} \\
&\le \alpha \mathrm{e}^{\frac{1}{\mathrm{e}}}\, \|V\|_0^{1-\alpha}\, \|V\|_1^{\alpha} \Big(\int_0^\infty \tau^{-\alpha p} k(\tau,x)^p \, \frac{\mathrm{d}\tau}{\tau}\Big)^{1/p} \\
&\le \mathrm{e}^{\frac{1}{\mathrm{e}}} (1+\|A^{-1}\|)\, \|V\|_0^{1-\alpha}\, \|V\|_1^{\alpha}\, \|\chi_{\alpha,p}\|_{L^p_*(\mathbb{R}_+)} \,,
\end{aligned}
$$

using that $k(\cdot,x)$ is non-decreasing and the above computation at the end.

Proposition 2.6 then yields the assertion for $p < \infty$. The case $p = \infty$ can be handled in a similar, but simpler way. The remaining result then follows from $VD_A(\alpha) = V\overline{\mathrm{D}(A)} \subseteq \overline{V\mathrm{D}(A)} \subseteq \overline{\mathrm{D}(B)} = D_B(\alpha)$ with closures in the (α,∞) norms, employing the continuity of V. ☐

The above result implies the 'interpolation estimate' for the norms, which can also be proved by elementary methods in many cases.

Corollary 2.10. *Let (2.1) hold and $x \in \mathrm{D}(A)$. We then have the inequality*

$$\|x\|_{\alpha,p} \le c\, \|x\|_X^{1-\alpha}\, \|x\|_A^{\alpha}$$

for a constant $c = c(A,\alpha) > 0$.

Proof. For $x \in \mathrm{D}(A)$, we consider the operator $V_x : \mathbb{C} \to X_1$ given by $V_x\mu = \mu x$. On $\mathbb{C}$ we choose the semigroup $R(\cdot) = I$ generated by the 0 operator with domain $\mathbb{C}$. Observe that V_x has the norms $\|x\|_X$ in $\mathcal{B}(\mathbb{C},X)$, $\|x\|_{\alpha,p}$ in $\mathcal{B}(\mathbb{C},D_A(\alpha,p))$, and $\|x\|_A$ in $\mathcal{B}(\mathbb{C},X_1)$. The corollary now follows from the above theorem. ☐

Remark 2.11. a) We point out that the interpolation estimate in Corollary 2.10 does not imply the interpolation property expressed by Theorem 2.9.

[3]The following argument was not given in the lectures

b) Let $B \in \mathcal{B}(D_A(\alpha,p),X)$ for some $\alpha \in [0,1)$ and $p \in [1,\infty]$. Let $a > 0$ and $x \in \mathrm{D}(A)$. Corollary 2.10 and Young's inequality with $p = 1/\alpha$ yield

$$\begin{aligned}\|Bx\| &\le \|B\|\,\|x\|_{\alpha,p} \le c\,\|B\|(\alpha/a)^{\alpha}\|x\|_X^{1-\alpha}\,(a/\alpha)^{\alpha}\|x\|_A^{\alpha}\\ &\le a\|x\|_A + (1-\alpha)\big(c\,\|B\|(\alpha/a)^{\alpha}\big)^{1/(1-\alpha)}\,\|x\|_X.\end{aligned}$$

This estimate allows us to apply the perturbation theorems for analytic or dissipative semigroups for suitable A and B, see Section 3.1 in [**EE**].

c) Let X and X_1 be Banach spaces which are linear subspaces of vector space Z whose addition and scalar multiplication are continuous for some metric on Z. One can then define the k–functional as in (2.12). Setting $\kappa_{\alpha,x}(s) = s^{-\alpha}k(s,x)$ for $s > 0$, one then introduces the real interpolation space

$$(X,X_1)_{\alpha,p} = \{x \in X \mid \kappa_{\alpha,x} \in L^p_*(\mathbb{R}_+)\}$$

endowed with the norm $\|\kappa_{\alpha,x}\|_{L^p_*(\mathbb{R}_+)}$. Arguing as in the proof of Theorem 2.9, one sees that this space coincides with our real interpolation space with equivalent norms if X_1 is the domain of a generator, see Proposition 5.7 in [**Lu2**].

This fact tells us that $D_A(\alpha,p)$ and $D_A(\alpha)$ do *not* depend on the generator itself, but only on the Banach spaces X and $[\mathrm{D}(A)]$.

Any spaces E and F satisfying the conclusion of Theorem 2.9 are called interpolation spaces (of order α) between X and $[\mathrm{D}(A)]$ and between Y and $[\mathrm{D}(B)]$, respectively. Another important class of such spaces are the 'complex interpolation spaces' $[X,X_1]_\alpha$ of order $\alpha \in (0,1)$. It can be shown that $[L^p(\mu),L^q(\mu)]_\alpha = L^r(\mu)$ for $\frac{1}{r} = (1-\alpha)\frac{1}{p} + \alpha\frac{1}{q}$ and $1 \le p,q \le \infty$, see e.g. Example 2.11 in [**Lu2**]. In this case the assertion of Corollary 2.10 is a standard consequence of Hölder's inequality. The real interpolation spaces between $L^p(\mu)$ and $L^q(\mu)$ are the 'Lorentz spaces', see Example 1.27 in [**Lu2**]. ◊

We next give a typical application of the interpolation property to the theory of function spaces, which is the basis of our investigations in Chapter 3.

Example 2.12. Let $G \subseteq \mathbb{R}^m$ be bounded and open with $\partial G \in C^2$, $\alpha \in (0,1)$, and $p \in (1,\infty)$. On $L^p(G)$ we consider $A = \Delta$ with $\mathrm{D}(A) = W^{2,p}(G) \cap W^{1,p}_0(G)$, and on $L^p(\mathbb{R}^m)$ the operator $B = \Delta$ with $\mathrm{D}(B) = W^{2,p}(\mathbb{R}^m)$. These operators generate (analytic) C_0–semigroups and their graph norms are equaivalent to the norms of $W^{2,p}(G)$ respectively $W^{2,p}(\mathbb{R}^m)$, cf. Example 2.29 in [**EE**].

There is an (extension) operator $E \in \mathcal{B}(L^p(G),L^p(\mathbb{R}^m))$ whose restriction belongs to $\mathcal{B}(W^{2,p}(G),W^{2,p}(\mathbb{R}^m))$ such that $Ef = f$ on G for all $f \in L^p(G)$, see Theorem 3.24 in [**ST**]. Theorem 2.9 then implies that E induces a map in $\mathcal{B}(D_A(\alpha,p),D_B(\alpha,p))$. Example 5.15 in [**Lu2**] yields $D_B(\alpha,p) = W^{2\alpha,p}(\mathbb{R}^m)$ if $\alpha \ne \frac{1}{2}$, see also Example 2.5 above. Using the restriction operator $Rg = g|_G$ on $L^p(\mathbb{R}^m)$, we thus obtain the embedding

$$RE : D_A(\alpha,p) \hookrightarrow W^{2\alpha,p}(G) := \{u \in L^p(G) \mid \exists\, v \in W^{2\alpha,p}(\mathbb{R}^m) : v|_G = u\}.$$

By the same reasoning, we have $D_C(\alpha,p) \hookrightarrow W^{2\alpha,p}(G)$ for any generator C on $L^p(G)$ such that $\mathrm{D}(C) \subseteq W^{2,p}(G)$ and $\|\cdot\|_C \simeq \|\cdot\|_{2,p}$.

In the above definition, the norm of u in $W^{2\alpha,p}(G)$ is given by the infimum of all norms $\|v\|_{W^{2\alpha,p}(\mathbb{R}^m)}$ with $v|_G = u$. However, $W^{2\alpha,p}(G)$ also possesses an equivalent 'intrinsic' norm of the same type as those in Example 2.5, see Theorem 4.4.2.2 in [**Tr1**].

In the exceptional case $\alpha = 1/2$, one again needs more interpolation theory. In this case the above results are true with $W^{2\alpha,p}$ replaced by the integer Besov space B^1_{pp}, see Example 5.14 of [**Lu2**] or Theorem 4.3.1.2 and 4.4.2.2 of [**Tr1**]. This space differs from $W^{1,p}(G)$ if $p \neq 2$.

The embeddings $D_A(\alpha, p) \hookrightarrow W^{2\alpha,p}(G)$ and $D_A(\alpha, p) \hookrightarrow B^1_{pp}(G)$ are sufficient for later applications. But actually much more is known. We first note that the trace operator tr maps $W^{2\alpha,p}(G)$ and $B^1_{pp}(G)$ continuously into $L^p(\partial G)$ if $\alpha > 1/(2p)$ due to the fractional trace theorem, cf. Theorem 4.7.1 in [**Tr1**]. Since $\mathrm{D}(A)$ is dense in $D_A(\alpha, p)$ by Proposition 2.2, we infer that

$$D_A(\alpha,p) \hookrightarrow \begin{cases} W^{2\alpha,p}_0(G) := \{u \in W^{2\alpha}_p(G) \,|\, \mathrm{tr}\, u = 0\}, & \frac{1}{2p} < \alpha \le 1,\ \alpha \neq \frac{1}{2}, \\ B^1_{pp,0}(G) := \{u \in B^1_{pp}(G) \,|\, \mathrm{tr}\, u = 0\}, & \alpha = \frac{1}{2}. \end{cases}$$

The above results are of a functional analytic nature, except for the description of the domains of A and B. Tools from partial differential equations and harmonic analysis allow to establish equalities instead of embeddings, namely

$$D_A(\alpha,p) = \begin{cases} W^{2\alpha,p}(G), & 0 < \alpha < \frac{1}{2p}, \\ W^{2\alpha,p}_0(G), & \frac{1}{2p} < \alpha \le 1,\ \alpha \neq \frac{1}{2}, \\ B^1_{pp,0}(G), & \alpha = \frac{1}{2}, \end{cases}$$

see Theorem 4.3.3 in [**Tr1**] and the references therein. ♢

Finally, we state a result on compact embeddings needed in the next chapter. The proof requires more facts from interpolation theory not presented here. See Corollary 3.8.2 of [**BL**].

Proposition 2.13. *Let (2.1) be true. Set $X_0 = X$ and $X_1 = [\mathrm{D}(A)]$. For $0 < \alpha < \beta < 1$ we consider $X_\alpha \in \{D_A(\alpha, p), D_A(\alpha) \,|\, p \in [1,\infty]\}$ and $X_\beta \in \{D_A(\beta, q), D_A(\beta) \,|\, q \in [1,\infty]\}$. Assume that $I : X_1 \to X_0$ is compact. Then also the embeddings $X_\beta \hookrightarrow X_\alpha$ are compact.*

2.2. Regularity of analytic semigroups

In this section we treat basic regularity properties of linear parabolic evolution equations complementing the results established in Theorems 2.23 and 2.30 of [**EE**]. We will first look at semigroup orbits and then at the inhomogeneous problem. The term 'parabolic' means that we assume that A generates the analytic C_0–semigroup $T(\cdot)$ on X and it is motivated by the applications to diffusion-type equations. These examples will be discussed in the next chapter.

Recall that a C_0–semigroup $T(\cdot)$ is analytic if and only if it maps X into $[\mathrm{D}(A)]$ with norm less or equal c/t for $t \in (0, 1]$. This property is also equivalent to a resolvent estimate for the generator A (i.e., the sectoriality of angle $\varphi > \pi/2$ of $A - \omega I$ for some $\omega \ge 0$). See Theorem 2.25 and Remark 2.26 of [**EE**]. We call an analytic semigroup *bounded* if $\sup_{t>0}(\|T(t)\| + \|tAT(t)\|) < \infty$.

For convenience, we write X_α for any of the spaces $D_A(\alpha, p)$ or $D_A(\alpha)$ with $\alpha \in (0, 1)$ and $p \in [1,\infty)$. We further set $X_0 = X$, $X_1 = [\mathrm{D}(A)]$, and $\|x\|_\alpha = \|x\|_{X_\alpha}$ for $\alpha \in [0, 1]$. In Proposition 2.4 we have seen that the 'parts'

$A_{\alpha,p}$ and A_α of A in $D_A(\alpha,p)$ and $D_A(\alpha)$ generate C_0–semigroups in these spaces, respectively, which are the restrictions of $T(\cdot)$ to the respective space. For simplicity, we now use the symbols A and $T(t)$ for all these objects. We sometimes write $C^0(J,X)$ instead of $C(J,X)$.

We first describe the regularizing effect of an analytic semigroup in the scale of interpolation spaces. The main point is that the norm of $T(t): X \to X_\beta$ is bounded by $ct^{-\beta}$ and thus integrable on $(0,1]$ for $\beta < 1$. Integrability fails if $\beta = 1$ which was the only case studied in [**EE**]. These mapping properties allow us to show 'full' regularity of the orbits away from $t = 0$ even for initial values $x \in X$.

Theorem 2.14. *Let A generate the analytic C_0–semigroup $T(\cdot)$ on X, $\alpha,\beta \in [0,1]$, and $b \geq 1$. Then the following assertions hold.*

a) The restrictions of $T(\cdot)$ to X_α are also analytic C_0–semigroups.

b) Let $k \in \{0,1\}$, $\alpha \in [0,\beta]$ if $k = 0$, and $x \in X_\alpha$. For $t \in (0,b]$ we then obtain $\|A^kT(t)x\|_\beta \leq c(b,\alpha,\beta,A)\, t^{\alpha-\beta-k}\, \|x\|_\alpha$. If $T(\cdot)$ is bounded, the constant does not depend on $b \geq 1$.

c) Let $x \in X_\alpha$ and $\beta \leq \alpha$. Then $T(\cdot)x$ belongs to $C^{\alpha-\beta}([0,b],X_\beta)$. Let $x \in X$ and $\varepsilon \in (0,b)$. Then $T(\cdot)x$ is an element of $C^{1-\beta}([\varepsilon,b],X_\beta)$ for all $\beta \in [0,1]$.

Proof. a) Let $\alpha \in (0,1)$. As a generator of an analytic C_0-semigroup, $A - \omega I$ is sectorial of angle $\varphi > \pi/2$ on X for some $\omega \geq 0$. This resolvent estimate is transfered to X_α via Proposition 2.4c). For $\alpha = 1$ this fact simply follows from the formula $AR(\lambda,A) = R(\lambda,A)A$ for $\lambda \in \rho(A)$. The first assertion is then a consequence of Theorem 2.25 of [**EE**].

b) Let $x \in X_\alpha$ and $t \in (0,b]$. In the estimates below we omit the dependence of A. If $T(\cdot)$ is bounded, they do not depend on b. We proceed in three steps.

1) Let $\alpha = 0$. In the first main step we interpolate between the estimates for $T(t): X \to X$ and $T(t): X \to X_1$ using Corollary 2.10. Indeed, Theorem 2.25 and Remark 2.26 of [**EE**] and Corollary 2.10 yield

$$\begin{aligned}\|A^kT(t)x\|_\beta &\lesssim_\beta \|T(t/2)A^kT(t/2)x\|_1^\beta\, \|T(t/2)A^kT(t/2)x\|_0^{1-\beta}\\ &\lesssim_b (t/2)^{-\beta}\, \|A^kT(t/2)x\|_0 \lesssim t^{-\beta-k}\, \|x\|_0\,.\end{aligned}$$

2) Let $\alpha > 0$. From step 1), (2.11), (2.2) (or (2.3) for bounded $T(\cdot)$), and Proposition 2.2 we deduce

$$\begin{aligned}\|tAT(t)x\|_\beta &\leq 2(t/2)^\alpha \|T(t/2)\|_{\mathcal{B}(X,X_\beta)}\, \|(t/2)^{1-\alpha}AT(t/2)x\|_0\\ &\lesssim_{\beta,b} t^{\alpha-\beta}\|x\|_{\alpha,\infty} \lesssim_\alpha t^{\alpha-\beta}\, \|x\|_\alpha.\end{aligned}$$

3) Take $\alpha > 0$ and $k = 0$. The case $\alpha = \beta$ follows from part a). Let $0 < \alpha < \beta$. Recall that $\frac{\mathrm{d}}{\mathrm{d}t}T(t) = AT(t)$. Steps 1) and 2) as well as Proposition 2.2 imply

$$\begin{aligned}\|T(t)x\|_\beta &= \Big\|T(b)x - \int_t^b AT(s)x\,\mathrm{d}s\Big\|_\beta \leq \|T(b)x\|_\beta + \int_t^b \|AT(s)x\|_\beta\,\mathrm{d}s\\ &\lesssim_{\alpha,\beta} b^{-\beta}\, \|x\|_0 + \int_t^b s^{\alpha-\beta-1}\|x\|_\alpha\,\mathrm{d}s \lesssim_{\beta-\alpha} (b^{-\beta} - b^{\alpha-\beta} + t^{\alpha-\beta})\, \|x\|_\alpha\\ &\leq t^{\alpha-\beta}\, \|x\|_\alpha\,.\end{aligned}$$

(Here we use $b \geq 1$. If $b < 1$, one could replace b by 1 in the above computation.)

c) The first part of assertion c) follows from part a) if $\alpha = \beta$, and from Proposition 2.2 if $\beta = 0$. So let $\beta \in (0, \alpha)$, $0 \leq s < t \leq b$ and $x \in X_\alpha$. Statement b) then yields

$$\|T(t)x - T(s)x\|_\beta = \|(T(t-s) - I)T(s)x\|_\beta \leq \int_0^{t-s} \|AT(\tau)T(s)x\|_\beta \,\mathrm{d}\tau$$
$$\lesssim_{b,\alpha,\beta} \int_0^{t-s} \tau^{\alpha-\beta-1} \,\mathrm{d}\tau \; \|T(s)x\|_\alpha \lesssim_{b,\beta-\alpha} (t-s)^{\alpha-\beta} \|x\|_\alpha \,.$$

The last claim then follows from $T(t)x - T(s)x = (T(t-\varepsilon) - T(s-\varepsilon))T(\varepsilon)x$ and $T(\varepsilon)x \in \mathrm{D}(A)$. □

In Example 2.2.11 of [**Lu1**] one can find an analytic semigroup which is unbounded in $\mathcal{B}(D_A(\alpha,\infty), D_A(\alpha,p))$ if $p \in [1,\infty)$ for $t \in (0,1]$. So one also has to pay a price if one only decreases the 'fine tuning parameter' p. By induction, one can define the scale X_α to all $\alpha \geq 0$ and extend the above theorem to this setting, see e.g. Proposition 2.2.9 in [**Lu1**].

We turn our attention to the inhomogeneous problem

$$u'(t) = Au(t) + f(t), \quad t \in J, \qquad u(0) = x, \tag{2.14}$$

for $J = (0,b]$, $J' = [0,b]$, and given $x \in X$ and $f \in C(J', X)$.[4] A *(classical) solution of* (2.14) *on* J is a function $u \in C(J', X) \cap C^1(J, X)$ such that $u'(t) \in \mathrm{D}(A)$ for all $t \in J$ and (2.14) holds. It is a *solution on* J' if we can replace here J by J' throughout. If a solution of (2.14) on J exists, it is uniquely given by the *mild solution*

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)\,\mathrm{d}s =: T(t)x + v(t), \qquad t \in J', \tag{2.15}$$

see Proposition 2.6 in [**EE**]. The summand $T(\cdot)x$ has been studied above.

In the proof of Theorem 2.30 of [**EE**] we have seen that $v : J' \to X$ is Hölder continuous of any exponent less than 1 and that it is continuously differentiable if $f \in C^\alpha(J', X)$ for some $\alpha > 0$. Example 4.1.7 of [**Lu1**] shows that one cannot take $\alpha = 0$, in general. We now improve the results from [**EE**] by using interpolation spaces instead of X. By the next result, for $f \in C(J', X)$ the orders of space and time regularity of v sum up to 1, provided that none is zero. We denote by $B(M, Y)$ the space of bounded functions from a set M to Y, endowed with the supnorm $\|f\|_{\infty,Y}$.

Theorem 2.15. *Let A generate the analytic C_0–semigroup $T(\cdot)$ on X, $f \in C(J', X)$, and v be given by* (2.15). *Then the follwing assertions hold.*

a) Let either $\alpha \in (0,1)$ and $\beta \in [0,\alpha]$ or $\alpha = 1$ and $\beta \in [0,1)$. Then v belongs to $C^{1-\alpha}(J', X_\beta)$ with norm bounded by $c(\alpha, b, A)\|f\|_{\infty,X}$. The constants do not depend on b if $T(\cdot)$ is bounded.

b) Let $f \in C^\alpha([0,b], X)$ or $f \in B([0,b], X_\alpha)$ for some $\alpha \in (0,1)$. Then v solves (2.14) *on J' with $x = 0$, and the quantity $\|v'\|_{\infty,X} + \|Av\|_{\infty,X}$ is bounded by $c(\alpha, b, A)\|f\|_{C^\alpha}$ repectively $c(\alpha, b, A)\|f\|_{\infty X_\alpha}$.*

[4]This notation slightly differs from the lectures. It now fits to the next chapter.

PROOF. a) Let $\alpha \in (0,1)$. It is enough to show v belongs to $C^{1-\alpha}(J', X_\alpha)$ and the corresponding estimate since $\|x\|_{\beta,p} \le \|x\|_{\alpha,p}$ and $\|f\|_\infty \le \|f\|_{C^{1-\beta}}$.

Let $0 < s \le t \le b$. We do not indicate the dependence of the constants on A, and they are independent of b if $T(\cdot)$ is bounded. Theorem 2.14 b) implies that

$$\|v(t)\|_\alpha \lesssim_{b,\alpha} \int_0^t (t-s)^{-\alpha} \|f(s)\|_0 \,\mathrm{d}s \le \frac{1}{1-\alpha}\, t^{1-\alpha} \|f\|_{\infty,X} .$$

Using the differentiability of $T(\cdot)$ and this theorem, we further compute

$$\begin{aligned}
v(t) - v(s) &= \lim_{\varepsilon\to 0} \int_0^{s-\varepsilon} (T(t-\tau) - T(s-\tau)) f(\tau)\,\mathrm{d}\tau + \int_s^t T(t-\tau) f(\tau)\,\mathrm{d}\tau \\
&= \lim_{\varepsilon\to 0} \int_0^{s-\varepsilon} \int_{s-\tau}^{t-\tau} AT(\sigma) f(\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau + \int_s^t T(t-\tau) f(\tau)\,\mathrm{d}\tau,
\end{aligned}$$

$$\begin{aligned}
\|v(t) - v(s)\|_\alpha &\lesssim_{b,\alpha} \limsup_{\varepsilon\to 0} \int_0^{s-\varepsilon} \int_{s-\tau}^{t-\tau} \sigma^{-1-\alpha} \|f\|_\infty \,\mathrm{d}\sigma\,\mathrm{d}\tau + \int_s^t (t-\tau)^{-\alpha} \|f\|_\infty \,\mathrm{d}\tau \\
&\le \Big(\frac{1}{\alpha} \int_0^s ((s-\tau)^{-\alpha} - (t-\tau)^{-\alpha})\,\mathrm{d}\tau + \frac{1}{1-\alpha}\,(t-s)^{1-\alpha} \Big) \|f\|_{\infty,X} \\
&\le \frac{2}{\alpha(1-\alpha)}\,(t-s)^{1-\alpha}\,\|f\|_{\infty,X} .
\end{aligned}$$

b) The first part was shown in Theorem 2.30 of [**EE**]. Let $f \in B(J', X_\alpha)$ for $\alpha \in (0,1)$. Take $0 < s \le t \le b$. Due to Theorem 2.14, the function given by $\varphi(s) = \|AT(t-s)f(s)\| \lesssim_{b,\alpha} (t-s)^{\alpha-1} \|f\|_{\infty,X_\alpha}$ is integrable on $[0,t]$. Since A is closed, we deduce that $v(t) \in \mathrm{D}(A)$. As in step a), we then obtain

$$\|Av(t)\| \le \int_0^t \|T(t-s)f(s)\|_1 \,\mathrm{d}s \lesssim_{\alpha,b} \int_0^t (t-s)^{\alpha-1} \|f\|_{\infty,X_\alpha} \,\mathrm{d}s = \frac{t^\alpha}{\alpha} \|f\|_{\infty,X_\alpha},$$

$$\begin{aligned}
&\|Av(t) - Av(s)\| \\
&\quad\le \varlimsup_{\varepsilon\to 0} \int_0^{s-\varepsilon} \int_{s-\tau}^{t-\tau} \|T(\tfrac{\sigma}{2}) AT(\tfrac{\sigma}{2}) f(\tau)\|_1 \,\mathrm{d}\sigma\,\mathrm{d}\tau + \int_s^t \|T(t-\tau)f(\tau)\|_1 \,\mathrm{d}\tau \\
&\quad\lesssim_{b,\alpha} \int_0^s \int_{s-\tau}^{t-\tau} \sigma^{-1} \|AT(\tfrac{\sigma}{2}) f(\tau)\|_0 \,\mathrm{d}\sigma\,\mathrm{d}\tau + \int_s^t (t-\tau)^{\alpha-1} \|f(\tau)\|_\alpha \,\mathrm{d}\tau \\
&\quad\lesssim_{b,\alpha} \int_0^s \int_{s-\tau}^{t-\tau} \sigma^{\alpha-2} \|f(\tau)\|_\alpha \,\mathrm{d}\sigma\,\mathrm{d}\tau + \int_s^t (t-\tau)^{\alpha-1} \|f(\tau)\|_\alpha \,\mathrm{d}\tau \\
&\quad\le \Big(\frac{1}{1-\alpha} \int_0^s ((s-\tau)^{\alpha-1} - (t-\tau)^{\alpha-1})\,\mathrm{d}\tau + \frac{(t-s)^\alpha}{\alpha} \Big) \|f\|_{\infty,X_\alpha} \\
&\quad\le \frac{2}{\alpha(1-\alpha)}\,(t-s)^\alpha\,\|f\|_{\infty,X_\alpha}.
\end{aligned}$$

As a result, Av belongs to $C([0,b],X)$ noting $v(0) = 0$. The assertion follows since $v' = Av + f$. □

We note that in part b) of the above proof we have even shown that $Av \in C^\alpha([0,b],X)$. Actually, it can proved that the terms u' and Au have the same regularity as the given function f, if we work in $D_A(\alpha,\infty)$ and assume that $x \in \mathrm{D}(A)$ and also $Ax + f(0) \in D_A(\alpha,\infty)$ in the case $f \in C^\alpha(J',X)$, respectively $Ax+f(0) \in D_A(\alpha,\infty)$ in the case $f \in C(J',X) \cap B(J',D_A(\alpha,\infty))$. This property of 'maximal regularity of type C^α' is shown in Theorem 4.3.1 and Corollary 4.3.9 of [**Lu1**]. In Chapter 4 of this monograph many variants and refinements of the maximal regularity results and of Theorem 2.15 a) are established also using weighted spaces for irregular data. See also the exercises.

CHAPTER 3

Semilinear parabolic problems

In this chapter we again study semilinear evolution equations of the form

$$u'(t) = Au(t) + F(u(t)), \quad t \in J, \qquad u(0) = x, \tag{3.1}$$

but now requiring that A generates an analytic C_0-semigroup. The results of the previous chapter allow us to treat nonlinear terms f mapping an interpolation space of A into X. So they are still of 'lower order', but one is far less restricted than in the first chapter where we could only treat the case $F : X \to X$. Typical examples are reaction-diffusion systems which we study below in some detail.

The wellposedness theory of semilinear parablic equations can be developed similar as for ordinary differential equations since $T(t) : X \to X_\alpha$ is bounded by the integrable function $ct^{-\alpha}$ on $(0, 1]$ due to Theorem 2.14. In the study of the long-time behavior one has partly to assume that A has compact resolvent. On the other hand, concrete applications may depend on detailed knowledge of the properties of a differential operator A with boundary conditions which here is much harder to obtain than in the matrix case.

We state the setting of this chapter. Throughout, let $X = X_0$, $X_1 = \mathrm{D}(A)$,

$$X_\alpha \in \{D_A(\beta, p), D_A(\beta) \,|\, \beta \in (0, 1),\, p \in [1, \infty)\},$$

as well as $\|x\|_\alpha = \|x\|_{X_\alpha}$ and $\overline{B}_\alpha(x, r) = \overline{B}_{X_\alpha}(x, r)$ for $\alpha \in [0, 1]$. We assume

$$\begin{aligned}
&A \text{ generates the analytic } C_0\text{–semigroup } T(\cdot) \text{ on } X, \quad \alpha \in [0,1), \quad u_0 \in X_\alpha,\\
&M_0 := \sup\nolimits_{t\in[0,1]} \|T(t)\|, \quad M_1 := \sup\nolimits_{t\in(0,1]} \|t^\alpha T(t)\|_{\mathcal{B}(X,X_\alpha)}.\\
&\emptyset \neq J \subseteq \mathbb{R} \text{ is an interval with } \inf J = 0, \quad 0 \notin J, \quad J' := J \cup \{0\}.\\
&F : X_\alpha \to X \text{ satisfies } \ \forall\, r > 0 \ \ \exists\, L(r) > 0 \ \ \forall\, x, y \in \overline{B}_\alpha(0, r) :\\
&\|F(x) - F(y)\|_\alpha \le L(r)\, \|x - y\|, \quad \text{where } \ r \mapsto L(r) \ \text{ is non-decreasing.}
\end{aligned} \tag{3.2}$$

The numbers $M_0, M_1 \ge 1$ are finite because of Theorem 2.14. As after (1.2) one sees that the last restriction is made without loss of generality. In contrast to the first chapter, now the time interval J cannot include 0.

3.1. Local wellposedness and global existence

A function u is called a *(classical) solution of* (3.1) *on* J if it belongs to $C(J', X_\alpha) \cap C^1(J, X) \cap C(J, X_1)$ and satisfies (3.1). It is a *(classical) solution on* J' if even $u \in C^1(J', X) \cap C(J', X_1)$ and (3.1) is true for $t \in J'$. In both cases, $f = F(u) : J' \to X$ is continuous by (3.2) so that Proposition 2.6 of [**EE**] says that u is a *mild solution* of (3.1); i.e., a function $u \in C(J', X_\alpha)$ fulfilling the fixed point problem

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(u(s))\,\mathrm{d}s =: T(t)u_0 + v(t), \qquad t \in J'. \tag{3.3}$$

If the semigroup $T(\cdot)$ is not analytic, there is a significant difference between mild and classical solutions, cf. Theorem 1.16. In the parabolic case, however, the regularity results of the previous chapter show that a mild solution u immediately becomes regular; i.e., $u(t)$ belongs to X_1 for all $t > 0$ and mild solutions coincide with classical ones on J. In Chapter 7 of the monograph [**Lu1**] one finds much more refined versions of the next fundamental lemma.

Lemma 3.1. *Let (3.2) be true and $u \in C(J', X_\alpha)$ be a mild solution of (3.1). Then u is a classical solution on J, and on J' if $u_0 \in X_1$.*

Proof. Let v be given by (3.3), $b \in J$, and $\varepsilon \in (0, b)$. Since $T(\cdot)$ is analytic, the orbit $T(\cdot)u_0 \in C(\mathbb{R}_+, X_1) \cap C^1(\mathbb{R}_+, X)$ satisfies $\mathrm{d}/\mathrm{d}t\, T(t)u_0 = AT(t)u_0$ for $t > 0$. Theorems 2.14 and 2.15 yield that $T(\cdot)u_0$ is an element of $C^{1-\alpha}([\varepsilon, b], X_\alpha)$ and v of $C^{1-\alpha}([0, b], X_\alpha)$, respectively. As a result, u belongs to $C^{1-\alpha}([\varepsilon, b], X_\alpha)$, and hence $F(u)$ to $C^{1-\alpha}([\varepsilon, b], X)$ by (3.2). Again from Theorem 2.15 we infer that u is contained in $C^1([\varepsilon, b], X) \cap C([\varepsilon, b], X_1)$ and solves (3.1) on $[\varepsilon, b]$. Since $0 < \varepsilon < b$ are arbitrary, the first assertion has been shown.

Let $u_0 \in X_1$. Then $T(\cdot)u_0$ even belongs to $C^{1-\alpha}([0, b], X_\alpha)$ and we can take $\varepsilon = 0$ in the above reasoning. (See Theorem 2.14.) □

The above result easily implies that one can shift and glue mild (or classical) solutions as in Lemma 1.7, which we will use below often without further notice.

We next solve the fixed point problem (3.3) on a possibly small time interval by means of the contraction mapping principle. We follow the approach of the first chapter, but now exploiting the (linear) regularity theory established in the previous section.

Lemma 3.2. *Let (3.2) be true. Take any $\rho > 0$. Then there is a time $b_0(\rho) > 0$ such that for every $u_0 \in \overline{B}_\alpha(\rho)$ there is a unique solution $u = \varphi(\cdot\,; u_0)$ of (3.1) on $(0, b_0(\rho)]$ satisfying $\|u(t)\|_\alpha \le r := 1 + M_0\rho$ for all $t \in [0, b_0(\rho)]$. The map b_0 is defined in (3.7), and it is non-decreasing.*

Proof. Let $b \in (0, 1]$, $\rho > 0$, and $r = 1 + M_0\rho > \rho$. We introduce the space

$$E(b) = E(b, r) = \{u \in C([0, b], X_\alpha) \mid \|u\|_{\infty,\alpha} := \sup_{0 \le t \le b} \|u(t)\|_\alpha \le r\}. \tag{3.4}$$

Note that $E(b)$ is complete for the metric $\mathrm{d}(v, w) = \|v - w\|_{\infty,\alpha}$. Let $u_0 \in \overline{B}_\alpha(\rho)$ be the initial value, $v, w \in E(b)$, and $t \in [0, b]$. We define the map

$$[\Phi_{u_0}(u)](t) = [\Phi(u)](t) = T(t)u_0 + \int_0^t T(t-s)F(u(s))\,\mathrm{d}s. \tag{3.5}$$

Clearly, a fixed point $\Phi(u) = u$ is a solution of (3.3), and u thus solves (3.1) on J. We will obtain such a fixed point for sufficiently small $b > 0$.

Since $F(v) \in C([0, b], X)$ and $u_0 \in X_\alpha$, Theorems 2.14 and 2.15 imply that $\Phi(v) \in C([0, b], X_\alpha)$. Using also (3.2), we estimate

$$\|\Phi(v)(t)\|_\alpha \le M_0\|u_0\|_\alpha + M_1 \int_0^t (t-s)^{-\alpha}\, \|F(v(s)) - F(0) + F(0)\|_0\,\mathrm{d}s$$

$$\leq M_0\rho + M_1 \int_0^t (t-s)^{-\alpha} \left(L(r)\|v(s)-0\|_\alpha + \|F(0)\|_0\right) \mathrm{d}s$$
$$\leq M_0\rho + \frac{M_1}{1-\alpha}(rL(r) + \|F(0)\|_0)\, b^{1-\alpha} \leq r,$$

where we choose times $b \in (0,1]$ with

$$b \leq b_1(\rho) := \Big(\frac{1-\alpha}{M_1(rL(r) + \|F(0)\|_0)}\Big)^{\frac{1}{1-\alpha}}.$$

In the same way, we compute

$$\begin{aligned} \|\Phi(v)(t) - \Phi(w)(t)\|_\alpha &\leq M_1 \int_0^t (t-s)^{-\alpha} \|F(v(s)) - F(w(s))\|_0 \,\mathrm{d}s \\ &\leq M_1 \int_0^t (t-s)^{-\alpha} L(r) \|v(s) - w(s)\|_\alpha \,\mathrm{d}s \\ &\leq \frac{M_1 L(r)}{1-\alpha}\, b^{1-\alpha}\, \|v-w\|_{\infty,\alpha} \leq \frac{1}{2}\, \|v-w\|_{\infty,\alpha} \end{aligned} \qquad (3.6)$$

for every final time

$$0 < b \leq b_0(\rho) := \min\Big\{1, b_1(\rho), \Big(\frac{1-\alpha}{2M_1L(r)}\Big)^{\frac{1}{1-\alpha}}\Big\}. \qquad (3.7)$$

As a result, the map $\Phi : E(b) \to E(b)$ is a strict contraction and we obtain a unique solution $u = \varphi(\cdot\,; u_0) = \Phi(u)$ of (3.1) on $(0,b]$ which belongs to $E(b)$. □

Exactly as in Lemma 1.8 we next derive a uniqueness result without the condition that the functions are bounded by r. One could it also prove as in (1.9) using the singular Gronwall inequality (3.9) below.[1]

LEMMA 3.3. *Let* (3.2) *be true and u and v be solutions of* (3.1) *on J_u respectively J_v. Then $u = v$ on $J_u \cap J_v$.*

PROOF. Set $J = J_u \cap J_v$. Since $u(0) = v(0)$, the number

$$\tau := \sup\{b \in J \,|\, \forall\, t \in [0,b] :\ u(t) = v(t)\}$$

belongs to $[0, \sup J]$. We assume that $u \neq v$ on J. By continuity, it follows $\tau < \sup J$ and $u(\tau) = v(\tau) =: u_1 \in X_\alpha$. There are times $t_n \in J$ with $t_n \to \tau^+$ and $u(t_n) \neq v(t_n)$. Fix $\beta_0 > 0$ with $\tau + \beta_0 \in J$. For $\beta \in (0, \beta_0]$, the functions $\tilde{u} = u(\,\cdot\, + \tau)$ and $\tilde{v} = v(\,\cdot\, + \tau)$ solve (3.1) on $(0,\beta]$ with initial value u_1.

We now set $\rho = \|u_1\|$ and $r = 1 + M_0\rho > \rho$, and use the number $b_0(\rho)$ from (3.7). For sufficiently small times $0 < \beta \leq \min\{b_0(\rho), \beta_0\}$, the maps $\tilde{u}$ and $\tilde{v}$ have norms less or equal r in $C([0,\beta], X_\alpha)$ because of $\tilde{u}(0) = \tilde{v}(0) = u_1$. The uniqueness statement (in the proof) of Lemma 3.2 then shows that $\tilde{u}(t) = \tilde{v}(t)$ for $t \in [0,\beta]$, which contradicts the inequality $u(t_n) \neq v(t_n)$ for large n. □

Following the first chapter, we now introduce the *maximal existence time*

$$t_+(u_0) = \sup\{b > 0 \,|\, \exists \text{ a solution } u_b \text{ of (3.1) on } (0,b]\}$$

of (3.1) still assuming (3.2). The *maximal existence interval* is $J_+(u_0) = (0, t_+(u_0))$ or $J'_+(u_0) = [0, t_+(u_0))$. Lemma 3.2 yields that $t_+(u_0) \geq b_0(\|u_0\|_\alpha)$.

[1]The next proof was omitted in the lectures.

We actually have $t_+(u_0) > b_0(\|u_0\|_\alpha)$ since we can restart (3.1) with initial value $u(b_0(\|u_0\|_\alpha))$. If $b < \beta < t_+(u_0)$, then $u_b = u_\beta$ on $[0, b]$ by Lemma 3.3. This fact allows us to define a *maximal solution* of (3.1) by setting $u(t) = u_b(t)$ for $t \in [0, b] \subseteq [0, t_+(u_0))$. It is uniquely determined because of Lemma 3.3.

We can now show the local wellposedness of (3.1).[2] Continuous dependence on F is established in Proposition 3.5.

THEOREM 3.4. *Let (3.2) be true and $b_0 = b_0(\|u_0\|_\alpha) > 0$ be given by (3.7). Then the following assertions hold.*

a) There is a unique maximal solution $u = \varphi(\,\cdot\,, u_0) \in C([0, t_+(u_0)), X_\alpha) \cap C^1(J_+(u_0), X) \cap C(J_+(u_0), X_1)$ of (1.1), where $t_+(u_0) \in (b_0, \infty]$ and $J_+(u_0) = (0, t_+(u_0))$. If $u_0 \in X_1$, we have $u \in C^1(J'_+(u_0), X) \cap C(J'_+(u_0), X_1)$.

b) If $t_+(u_0) < \infty$, then $\lim_{t \to t_+(u_0)^-} \|u(t)\|_\alpha = \infty$.

c) Take any $b \in (0, t_+(u_0))$. Then there exists a radius $\delta = \delta(u_0, b) > 0$ such that $t_+(v_0) > b$ for all $v_0 \in \overline{B}(u_0, \delta)$. Moreover, the map

$$\overline{B}_\alpha(u_0, \delta) \to C([0, b], X_\alpha); \quad v_0 \mapsto \varphi(\,\cdot\,, v_0),$$

is Lipschitz continuous.

PROOF. Assertion a) was shown above and in Lemma 3.1. To establish b), let $t_+(u_0) < \infty$. Assume that there were times $t_n \to t_+(u_0)$ for $n \to \infty$ with $t_n \in [0, t_+(u_0))$ and $C := \sup_{n \in \mathbb{N}} \|u(t_n)\|_\alpha < \infty$. We choose an index $m \in \mathbb{N}$ such that $t_m + b_0(C) > t_+(u_0)$, where $b_0(C) > 0$ is given by (3.7). Lemma 3.2 yields a solution $\tilde{u}$ of (3.1) on $(0, b_0(C)]$ with initial value $u(t_m)$. Glueing u and the shifted $\tilde{u}$, we then obtain a solution of (3.1) on $(0, t_m + b_0(C)]$ which contradicts the definition of $t_+(u_0)$. So assertion b) is shown. We next prove part c) by a basic step plus an induction argument in three more steps.

1) Let $b \in (0, t_+(u_0))$ and $u = \varphi(\,\cdot\,, u_0)$. We fix a number $b' \in (b, t_+(u_0))$ and use the radii $\overline{\rho} := 1 + \max_{0 \le t \le b'} \|u(t)\|_\alpha$ and $\overline{r} := 1 + M_0 \overline{\rho}$. The uniform bound by $\overline{\rho}$ will crucially be used below. Let the time $\overline{b} := b_0(\overline{\rho}) \in (0, 1]$ be given by (3.7) and the operator Φ_{u_0} by (3.5). Take $v_0, w_0 \in \overline{B}_\alpha(0, \overline{\rho})$. Lemma 3.2 and its proof provide solutions $v = \Phi_{v_0}(v) = \varphi(\,\cdot\,, v_0)$ and $w = \Phi_{w_0}(w) = \varphi(\,\cdot\,, w_0)$ of (3.1) on $(0, \overline{b}]$ with the initial values v_0 respectively w_0, where v and w belong to the space $E(\overline{b}, \overline{r})$ from (3.4) endowed with the norm $\|\cdot\|_{\infty,\alpha}$. Formulas (3.6), (3.7) and (3.5) lead to the contraction estimate

$$\begin{aligned}
\|v - w\|_{\infty,\alpha} &\le \|\Phi_{v_0}(v) - \Phi_{v_0}(w)\|_{\infty,\alpha} + \|\Phi_{v_0}(w) - \Phi_{w_0}(w)\|_{\infty,\alpha} \\
&\le \tfrac{1}{2}\|v - w\|_{\infty,\alpha} + \|T(\cdot)(v_0 - w_0)\|_{\infty,\alpha} \\
&\le \tfrac{1}{2}\|v - w\|_{\infty,\alpha} + M_0\|v_0 - w_0\|_\alpha, \\
\|v - w\|_{\infty,\alpha} &\le 2M_0\|v_0 - w_0\|_\alpha. \qquad (3.8)
\end{aligned}$$

2) We next show $t_+(v_0) > b$ by iteration. For $j \in \mathbb{N}_0$ we set $b_j = j\overline{b}$. There exists a minimal index $N \in \mathbb{N}$ with $b_N > b$. If $b_N > b'$ we redefine $b_N := b' \in (b, t_+(u_0))$. We choose a (sufficiently small) radius $\delta = (2M_0)^{-N} \in (0, 1)$ for the initial values. We inductively show that for every $v_0 \in \overline{B}_\alpha(u_0, \delta)$ and

[2] The next proof was omitted in the lectures as it is very close to that of Theorem 1.11.

$j \in \{0, \dots, N-1\}$ the maximal solution $v = \varphi(\,\cdot\,; v_0)$ exists at least on $[0, b_{j+1}]$ and that $v(t)$ is an element of the ball $\overline{B}_\alpha(u(t), (2M_0)^{j+1-N})$ for $t \in [b_j, b_{j+1}]$, which belongs to $\overline{B}_\alpha(0, \overline{\rho})$ because of the basic bound $\overline{\rho} \geq 1 + \|u(t)\|_\alpha$ for $t \in [0, b_N]$. This claim then yields $t_+(v_0) > b$.

3) We prove the claim. First let $j = 0$. Since $\delta < 1$, the vector v_0 is contained in $\overline{B}_\alpha(0, \overline{\rho})$. From estimate (3.8) with $w = u$ we deduce

$$\|v(t) - u(t)\|_\alpha \leq 2M_0 \|v_0 - u_0\|_\alpha \leq 2M_0\delta = (2M_0)^{1-N}$$

for all $t \in [0, b_1]$, as asserted for $j = 0$.

Second, assume that the claim has been established for all $k \in \{0, \dots, j-1\}$ and some $j \in \{1, \dots, N-1\}$. It follows $\|v(b_j)\|_\alpha \leq \overline{\rho}$. Lemma 3.2 thus shows that v exists at least on $[0, b_{j+1}]$. Moreover, the inequality (3.8) can be applied to $v(t + b_j) = \varphi(t, v(b_j))$ and $u(t + b_j) = \varphi(t, u(b_j))$ for $t \in [0, \overline{b}]$. Using also the induction hypothesis, we infer the bound

$$\|v(t + b_j) - u(t + b_j)\|_\alpha \leq 2M_0 \|v(b_j) - u(b_j)\|_\alpha \leq (2M_0)^{j+1-N}$$

for $t \in [0, \overline{b}]$. So the claim is true.

4) It remains to prove the Lipschitz continuity asserted in c). Let $j \in \{0, \dots, N-1\}$ and $t \in [b_j, b_{j+1}]$. By the claim in 2), the vectors $v(b_j)$ and $w(b_j)$ belong to $\overline{B}_\alpha(0, \overline{\rho})$. As in step 3), inequality (3.8) implies

$$\begin{aligned}\|v(t+b_j) - w(t+b_j)\|_\alpha &= \|\varphi(t, v(b_j)) - \varphi(t, w(b_j))\|_\alpha \leq 2M_0 \|v(b_j) - w(b_j)\|_\alpha \\ &\leq \ldots \leq (2M_0)^{j+1} \|v_0 - w_0\|_\alpha \leq (2M_0)^N \|v_0 - w_0\|_\alpha. \qquad \square\end{aligned}$$

We stress that X_α is the adequate norm to decribe the behavior of the solutions to (3.1): They are continuous in the X_α–norm up to 0 and this norm gives the Lipschitz continuity and blow-up condition in assertions (b) and (c).

Observe that the assumptions still hold if we replace α by $\beta \in [\alpha, 1)$ and take $u_0 \in X_\beta$. Due to the uniqueness and the embeddings in Proposition 2.2, the solution u then belongs to $C([0, t_+(u_0)), X_\kappa)$ for all $\kappa \in [0, \beta]$.

The regularity of mild and classical solutions to (3.1) is studied in great detail in Chapter 7 of [**Lu1**], also for $u_0 \in X$ under additional restrictions on F. The above proofs completely break down if $\alpha = 1$; i.e., when the nonlinearity has the same order as the linear part. Under certain additional assumptions, one can also develop a theory on wellposedness and asymptotic behavior for such problems, which is similar to the semilinear case discussed here. This is done in Chapters 8 and 9 of [**Lu1**] based on the results on maximal regularity mentioned at the end of the previous chapter; see also the last chapter of this notes.

Since the data of equation (3.1) are not known exactly in applications, it is very important to know that the solution depends continuously on the system operators, where we restrict ourselves to F for simplicity. When discussing the positivity of reaction diffusion systems we will actually use a very special case of this continuous dependence (whose proof would not be much simpler). For this and other purposes, we need the *singular Gronwall inequality.* Let $0 \leq \varphi \in C(J')$, $\beta \in [0, 1)$ and $a, \kappa \geq 0$. Assume that

$$\varphi(t) \leq a + \kappa \int_0^t (t-s)^{-\beta}\, \varphi(s)\, \mathrm{d}s$$

holds for all $t \in J'$. Then there is a constant $c_0 > 0$ such that

$$\varphi(t) \le a + a\kappa c_0 t^{1-\beta}\, e^{c(\beta)\kappa^{1/(1-\beta)}t} \tag{3.9}$$

holds for all $t \in J$, where $c(\beta) := 2\Gamma(1-\beta)^{\frac{1}{1-\beta}}$, see Theorem II.3.3.1 in [**Am2**].

Proposition 3.5. *Let (3.2) be true, $v_0 \in X_\alpha$, $G : X_\alpha \to X$ be Lipschitz on bounded sets. Let u solve (3.1) and v solve (3.1) with nonlinearity G and initial value v_0. Fix $b \in (0, t_+(u_0, F))$. Then there are constants $\delta, \rho, c > 0$ (depending on b and u_0) with the following property: Let $\|u_0 - v_0\|_\alpha \le \rho$ and $\|F(u(t)) - G(u(t))\|_0 \le \delta$ for all $t \in [0, b]$. Then $t_+(v_0, G) > b$ and, for $t \in [0, b]$,*

$$\|u(t) - v(t)\|_\alpha \le c(\delta + \|u_0 - v_0\|_\alpha).$$

Proof. Fix $r > 0$ and $b \in (0, t_+(u_0, F))$, and let L be the Lipschitz constant of G on the bounded set $\bigcup\{\overline{B}_\alpha(u(t), r) \,|\, t \in [0, b]\}$. Take $\rho \in (0, r)$ and $v_0 \in \overline{B}_\alpha(u_0, \rho)$. The numbers

$$N_0 = \sup_{t\in[0,b]} \|T(t)\|_{\mathcal{B}(X_\alpha)} \quad \text{and} \quad N_1 = \sup_{t\in(0,b]} \|t^\alpha T(t)\|_{\mathcal{B}(X,X_\alpha)}$$

are finite by Theorem 2.14. Let b^* be the supremum of all times $\beta \in (0, b]$ such that $v(t)$ exists and $\|v(t) - u(t)\|_\alpha \le r$ for all $t \in [0, \beta]$. The continuity of $u - v$ yields $b^* > 0$ and $\|v(b^*) - u(b^*)\|_\alpha \le r$. Using the mild formulation of both evolution equations, we obtain

$$\begin{aligned}
u(t) - v(t) &= T(t)(u_0 - v_0) + \int_0^t T(t-s)(F(u(s)) - G(u(s)))\,\mathrm{d}s \\
&\quad + \int_0^t T(t-s)(G(u(s)) - G(v(s)))\,\mathrm{d}s, \\
\|u(t) - v(t)\|_\alpha &\le N_0\|u_0 - v_0\|_\alpha + N_1\delta \int_0^t (t-s)^{-\alpha}\,\mathrm{d}s \\
&\quad + N_1 L \int_0^t (t-s)^{-\alpha}\|u(s) - v(s)\|_\alpha\,\mathrm{d}s \\
&\le N_0\|u_0 - v_0\|_\alpha + \frac{\delta N_1 b^{1-\alpha}}{1-\alpha} + N_1 L \int_0^t (t-s)^{-\alpha}\|u(s) - v(s)\|_\alpha \mathrm{d}s
\end{aligned}$$

for all $t \in [0, b^*]$. The inequality (3.9) thus yields

$$\|u(t) - v(t)\|_\alpha \le c(\delta + \|u_0 - v_0\|_\alpha) \le c(\delta + \rho), \tag{3.10}$$

for all $t \in [0, b^*]$, where c depends on b, L, N_j and α, but not on t, δ or ρ. Fixing sufficiently small $\delta, \rho > 0$, we conclude that $\|u(t) - v(t)\|_\alpha \le r/2$ for all $t \in [0, b^*]$, implying that $b^* = b$ and that (3.10) holds for all $t \in [0, b]$. □

In the next example we give an introduction to the L^p–approach to *reaction-diffusion systems*, whereas in Section 7.3 of [**Lu1**] sup-norm setting is discussed.

Example 3.6. We first recall reaction equations without diffusion. As a simple example, we consider the chemical reaction $A + 2B \rightleftharpoons C$, where one mol of the substance A reacts with 2 mols of B to one mol of the product C, which in turn can decompose into one mol of A and two mols of B.

Let $a(t)$, $b(t)$ and $c(t)$ be the concentrations at time $t \geq 0$ of the species A, B and C, respectively. Roughly speaking, the two reactions take place with a 'probability' proportional to the products of $a(t)\,b(t)\,b(t)$ and $c(t)$ of the concentrations, where we denote the proportionality constants by k_+ and k_-, respectively. Each concentration then increases and decreases according to the two reactions, where the rate is given by the 'probability' times the number of mols needed of the respective substance. We arrive at the system

$$\begin{aligned}
&a'(t) = -k_+ a(t)b(t)^2 + k_- c(t), \qquad t \geq 0,\\
&b'(t) = -2k_+ a(t)b(t)^2 + 2k_- c(t), \qquad t \geq 0,\\
&c'(t) = k_+ a(t)b(t)^2 - k_- c(t), \qquad t \geq 0,\\
&a(0) = a_0, \quad b(0) = b_0, \quad c(0) = c_0,
\end{aligned}$$

with initial concentrations $a_0, b_0, c_0 \geq 0$. We write $f(a(t), b(t), c(t))$ for the vector of the reaction terms on the right-hand side.

This problem has a unique local non-negative solution due to the Picard–Lindelöf Theorem 4.6 and the positivity criterion Satz 4.9 of [**Ana4**]. Since $a' + c' = 0$ and $b' + 2c' = 0$, we have $a(t) + c(t) = a_0 + c_0$ and $b(t) + 2c(t) = b_0 + 2c_0$ as long as the solutions exist. Thanks to the positivity, the solutions thus stay bounded on their existence interval, so that they exist for all $t \geq 0$. These facts hold in much greater generality, see Section 8.7 of [**PW**].

In a reaction–diffusion system one takes into account that the concentrations of the species may differ at different points of the container G which is an open and bounded subset of $\mathbb{R}^m$ with $\partial G \in C^2$ and outer unit normal ν. For given ℓ species we thus consider concentration densities $u(t,x) = (u_1(t,x), \dots, u_\ell(t,x))$ at every time $t \geq 0$ and spatial point $x \in G$.

We assume that at each x a reaction-convection term $f(u(t,x), \nabla u(t,x))$ acts. Later we will focus on pure reaction terms $f(u(t,x))$ depending only on the concentrations $u(t,x)$ as in the ordinary differential equation above. If spatial gradients of $u(t,x)$ are involved, we also have (possibly nonlinear) convective effects. The function $f : \mathbb{C}^{\ell+m\ell} \to \mathbb{C}^\ell$ (or later $f : \mathbb{C}^\ell \to \mathbb{C}^\ell$) is given and assumed to be locally Lipschitz.

The species shall move in the container because of 'homogeneous' and 'isotropic' diffusion with constants $a_1, \dots, a_m > 0$, resulting in diffusion terms $a_j \Delta u_j(t,x)$. We assume that the species do not move through the boundary ∂G. It can be seen that this behaviour is decribed by the Neumann boundary condition $\partial_\nu u_j(t,x) = 0$ saying that in normal direction at the boundary the concentration does not change. Summing up, we arrive at the system

$$\begin{aligned}
&\partial_t u_j(t,x) = a_j \Delta u_j(t,x) + f_j(u(t,x), \nabla u(t,x)), \quad t > 0,\ x \in G,\ j \in \{1, \dots, \ell\},\\
&\partial_\nu u_j(t,x) = 0, \quad t > 0,\ x \in \partial G,\ j \in \{1, \dots, \ell\}, \qquad (3.11)\\
&u_j(0,x) = u_{j,0}(x), \quad x \in G,\ j \in \{1, \dots, \ell\},
\end{aligned}$$

for given initial distributions $u_{j,0} \geq 0$.

Recall from Example 5.2 of [**EE**] that the Neumann Laplacian Δ_N with domain $\mathrm{D}(\Delta_N) = \{v \in W^{2,p}(G) \,|\, \partial_\nu v = 0\}$ generates a contractive, positive, analytic C_0–semigroup $S(\cdot)$ on $L^p(G)$ for $p \in (1, \infty)$. We now set $E = L^p(G)^\ell$,

$0 \le u_0 = (u_{1,0}, \ldots, u_{\ell,0}) \in E$,

$$A = \begin{pmatrix} a_1\Delta_N & 0 & 0 & \cdots & 0 \\ 0 & a_2\Delta_N & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & a_\ell\Delta_N \end{pmatrix}, \qquad \mathrm{D}(A) = \mathrm{D}(\Delta_N)^\ell =: E_1, \tag{3.12}$$

and $[F(v)](x) = f(v(x), \nabla v(x))$ for $v \in W^{1,p}(G)^\ell$ and $x \in G$. Positivity in E means that each component of $u \in E$ is positive a.e.. It is easy to see that A generates the contractive, positive, analytic C_0–semigroup

$$T(t) = \begin{pmatrix} S(a_1 t) & 0 & 0 & \cdots & 0 \\ 0 & S(a_2 t) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & S(a_m t) \end{pmatrix}, \quad t \ge 0,$$

on E. We write $E_\alpha = D_A(\alpha, p)$ for $\alpha \in (0,1)$. Due to Theorem 2.14 the numbers $N_0 = \sup_{t\ge 0} \|T(t)\|_{\mathcal{B}(E_\alpha)}$ and $N_1 = \sup_{t>0} \|t^\alpha T(t)\|_{\mathcal{B}(E,E_\alpha)}$ are finite.

One could also treat more complicated diffusion phenomena. In the linear case, heterogeneous and anisotropic diffusion is described by the term $\mathrm{div}(a\nabla u_j)$ for a coefficient function a on G taking values in the symmetric positive definite matrices, which could even depend on time. If the diffusion coefficients $a = a(u)$ depend on the solution u itself, one has 'quasilinear diffusion' which can be treated by more sophisticated methods, see [**Lu1**] or the last chapter. Moreover, interactions between the species can lead to nondiagonal diffusion terms.

We look for a framework in which F becomes Lipschitz on bounded sets. We first let $v, w \in C^1(\overline{G})^\ell$ with C^1–norm less or equal r, where $C^1(\overline{G})$ contains the C^1–functions v on G such that v and ∇v have continuous extensions to ∂G. Denoting by $L_0(r)$ the Lipschitz constant of f on $\overline{B}_{|\cdot|_\infty<}(0,r)$, we estimate

$$\begin{aligned} |F(v)(x) - F(w)(x)| &\le L_0(r) \max\{|v(x) - w(x)|_\infty, |\nabla v(x) - \nabla w(x)|_\infty\} \\ &\le L_0(r)\|v - w\|_{C^1}. \end{aligned}$$

This means that $F : C^1(\overline{G})^\ell \to C(\overline{G})^\ell$, and thus $F : C^1(\overline{G})^\ell \to L^p(G)^\ell$, are Lipschitz on bounded sets. Take $p > m$ and $\alpha \in (\frac{1}{2} + \frac{m}{2p}, 1)$, so that $2\alpha - \frac{m}{p} > 1$. Example 2.12 and the fractional Sobolev embedding theorem (see Theorem 4.6.1 in [**Tr1**]) then imply that

$$E_\alpha \hookrightarrow W^{2\alpha,p}(G)^\ell \hookrightarrow C^1(\overline{G})^\ell. \tag{3.13}$$

As a result, $F : E_\alpha \to E$ is Lipschitz on bounded sets. If we have a pure reaction term $f(u)$, then the same arguments show that $F : E_\alpha \to E$ is Lipschitz on bounded sets already if $p > m/2$ and $\alpha \in (\frac{m}{2p}, 1)$.

We can thus write (3.11) as the semilinear parabolic problem (3.1) on E, and apply Theorem 3.4. For $u_0 \in E_\alpha$ it yields a unique solution u in $C(J'_+(u_0), E_\alpha) \cap C^1(J_+(u_0), E) \cap C(J_+(u_0), E_1)$ of (3.1), where $J_+(u_0) = (0, t_+(u_0))$. By the embedding (3.13), u belongs to $C(J'_+(u_0), C^1(\overline{G})^\ell)$. Such a solution satisfies the first line of (3.11) for a.e. $x \in G$ and the second line for all $x \in \partial G$. Observe that u is also the only solution of (3.11) in this sense. Hence, the regularity properties hold for all $p \in (m, \infty)$ and $\alpha \in (\frac{1}{2} + \frac{m}{2p}, 1)$. If $u_0 \in \mathrm{D}(A)$, we can include $t = 0$ here. ♢

In the following we restrict ourselves to pure reaction equations with $F(u)(x) = f(u(x))$ for a function $f : \mathbb{C}^\ell \to \mathbb{C}^\ell$ being locally Lipschitz. To study positivity, we further need *Hopf's lemma*. For functions $w \in C^2(B) \cap C^1(\overline{B})$, it is a special case of the lemma in Section 6.4.2 in [**Ev**]. Our result is shown in the same way using Proposition 3.1.10 of [**Lu1**]. Sobolev's embedding implies that $w \in C^1(\overline{G})$ under the assumptions of the lemma.

Lemma 3.7. *Let $B = B(y, \rho) \subset \mathbb{R}^m$ be an open ball and w belong to $W^{2,p}(B)$ for all $p \in (1, \infty)$ and satisfy $0 \le \Delta w \in C(\overline{B})$. Assume that there is a point $x_0 \in \partial B$ such that $w(x_0) > w(x)$ for all $x \in B$. Then $\partial_\nu w(x_0) > 0$ for the outer normal $\nu(x) = \rho^{-1}(x - y)$ of ∂B.*

We show a local wellposedness theorem for the reaction diffusion system (3.11) with $f(u)$. Besides basic properties already discussed in Example 3.6, it contains a much improved blow-up condition, a compactness result, and a positivity criterion which is analogous to the ODE case. These facts are the basis for later studies of the long-term behavior of (3.11). We stress that only non-negative solutions are relevant here.

Theorem 3.8. *Consider the situation of Example 3.6 with a locally Lipschitz map $f : \mathbb{C}^\ell \to \mathbb{C}^\ell$. Let $m < p < \infty$, $\alpha \in (\frac{m}{2p}, 1 - \frac{m}{2p})$, $F(v) = f \circ v$, A be given by (3.12), and $u_0 \in E_\alpha = D_A(\alpha, p)$. Then the following assertions hold.*

a) Problem (3.11) has a unique maximal solution u in $C([0, t_+(u_0)), E_\alpha) \cap C^1((0, t_+(u_0)), E) \cap C((0, t_+(u_0)), E_1)$. Assertion c) of Theorem 3.4 holds analogously. Moreover, the maps $\partial_t u$ and $\Delta_N u$ belong to $C((0, t_+(u_0)) \times \overline{G})^\ell$ and u to $C((0, t_+(u_0)), W^{2,q}(G))^\ell$ for all $q < \infty$. If $u_0 \in \mathrm{D}(A)$, then we can replace the interval $(0, t_+(u_0))$ by $[0, t_+(u_0))$.

b) If $t_+(u_0) < \infty$, then $\lim_{t \to t_+(u_0)^-} \|u(t)\|_\infty = \infty$.

c) If $t_+(u_0) = \infty$ and u is bounded on $\mathbb{R}_{\ge 0} \times \overline{G}$, the orbit $\{u(t) \,|\, t \ge 0\}$ is relatively compact in E_α.

d) Let $f(\mathbb{R}^\ell) \subseteq \mathbb{R}^\ell$ and $u_0 \ge 0$. Then u also takes real values. If f also satiesfies the positivity condition

$$f_k(r_1, \dots, r_{k-1}, 0, r_{k+1}, \dots, r_\ell) \ge 0 \quad \textit{for all} \;\; r_j \ge 0, \; j, k \in \{1, \cdots, \ell\}, \quad (3.14)$$

then $u(t) \ge 0$ for all $t \in [0, b(u_0))$.

Proof. a) The first part follows from Theorem 3.4 except for the additional regularity result, cf. Example 3.6. Let $0 < \varepsilon < b < t_+(u_0)$. As in Lemma 3.1, Theorems 2.14 and 2.15 show that $F(u) \in C^{1-\alpha}([\varepsilon, b], E)$, where one may take $\varepsilon = 0$ if $u_0 \in E_1$. Corollary 4.3.6 of [**Lu1**] then implies that $u' = \partial_t u : [\varepsilon, b] \to D_A(1 - \alpha, \infty)$ is bounded. Let $0 < \varepsilon \le s \le t \le b$ and $\beta \in (\frac{m}{2p}, 1 - \alpha)$. Corollary 1.24 in [**Lu2**] and Corollary 2.10 yield

$$\|u'(t) - u'(s)\|_\beta \lesssim \|u'(t) - u'(s)\|_{1-\alpha}^{\frac{\beta}{1-\alpha}} \, \|u'(t) - u'(s)\|_0^{1 - \frac{\beta}{1-\alpha}} \lesssim \|u'(t) - u'(s)\|_0^{1 - \frac{\beta}{1-\alpha}},$$

where the right-hand side tends to 0 as $t-s \to 0$. We deduce that $\partial_t u$ belongs to $C([\varepsilon, b]), E_\beta) \hookrightarrow C([\varepsilon, b], C(\overline{G})^\ell)$, using also the fractional Sobolev embedding Theorem 4.6.1 in [**Tr1**], Example 2.12 and $2\beta - \frac{m}{p} > 0$, cf. (3.13). Hence, $\partial_t u$ is contained in $C((0, t_+(u_0)) \times \overline{G})^\ell$ and the same holds for $\Delta_N u$ because of (3.11) and $F(u) \in C([0, t_0(u_0)) \times \overline{G})^\ell$.

In particular, u_j and $\Delta_N u_j$ are continuous with values in $L^q(G)$ for all $q < \infty$. Since $\mathrm{D}(\Delta_N) \subseteq W^{2,q}(G)$ by Example 5.2 in [**EE**], we obtain that u is an element of $C((0, t_+(u_0)), W^{2,q}(G))^\ell$.

b) Let $t_+ = t_+(u_0) < \infty$ and $|u(t,x)|_\infty \le R$ on $[0, t_+) \times \overline{G}$. It follows that

$$|f(u(t,x))| \le |f(u(t,x)) - f(0)| + |f(0)| \le L(R)|u(t,x)| + |f(0)| \le RL(R) + |f(0)|$$

for all $(t,x) \in [0, t_+(u_0)) \times \overline{G}$, and thus

$$\|F(u(t))\|_E \le \mathrm{vol}(G)^{\frac{1}{p}} (RL(R) + |f(0)|) =: C.$$

Employing Example 3.6, we can then control u via the mild formula (3.3) by

$$\begin{aligned}
\|u(t)\|_\alpha &\le \|T(t)u_0\|_\alpha + \int_0^t \|T(t-s)\|_{\mathcal{B}(E,E_\alpha)} \|F(u(s))\|_0 \,\mathrm{d}s \\
&\le N_0 \|u_0\|_\alpha + CN_1 \int_0^t (t-s)^{-\alpha} \,\mathrm{d}s = N_0 \|u_0\|_\alpha + \frac{CN_1}{1-\alpha} t_+^{1-\alpha}
\end{aligned}$$

for all $t \in [0, t_+)$. This bound contradicts the blow–up condition in Theorem 3.4, and thus $t_+ = \infty$.

c) Let $t_+(u_0) = \infty$ and u be bounded by R on $\mathbb{R}_{\ge 0} \times \overline{G}$. Using that u solves (3.1) on $[t-1, \infty)$ with initial value $u(t-1)$, we conclude as in part b) that

$$\begin{aligned}
\|u(t)\|_\beta &\le \|T(1)u(t-1)\|_\beta + \int_{t-1}^t \|T(t-s)\|_{\mathcal{B}(E,E_\beta)} \|F(u(s))\|_0 \,\mathrm{d}s \\
&\le \tilde{N}_1 \|u(t-1)\|_0 + \frac{C\tilde{N}_1}{1-\beta} \le R\tilde{N}_1 \,\mathrm{vol}(G)^{\frac{1}{p}} + \frac{C\tilde{N}_1}{1-\beta}
\end{aligned}$$

for some constants $\tilde{N}_j \ge 1$ and for all $t \ge 1$ and $\beta \in (\alpha, 1)$. Proposition 2.13 now shows assertion c), since E_1 is compactly embedded into E and the set $\{u(t) \,|\, 0 \le t \le 1\}$ is compact in E_α by continuity.

d) Let f be real and $u_0 \ge 0$. When discussing (3.11), one can then replace throughout the scalar field $\mathbb{C}$ by $\mathbb{R}$, and thus obtain a real-valued solution. (We use $\mathbb{C}$ only to construct the analytic semigroup $S(\cdot)$ generated by Δ_N. But since it is positive, it leaves invariant real-valued functions. See also Remark 3.14.)

Let $\varepsilon > 0$ and $b \in (0, t_+(u_0))$. We set $u_{0,\varepsilon} = u_0 + \varepsilon\mathbb{1} > 0$ and $F_\varepsilon(v) = F(v) + \varepsilon\mathbb{1}$. Let u_ε solve (3.11) for $u_{0,\varepsilon}$ and F_ε. Proposition 3.5 yields a number $\varepsilon_0(b) > 0$ such that $u_\varepsilon(t)$ exists for all $t \in [0, b]$ and $\varepsilon \in (0, \varepsilon_0(b)]$ and such that $u_\varepsilon(t)$ tends to $u(t)$ in $E_\alpha \hookrightarrow C(\overline{G})^\ell$ as $\varepsilon \to 0$ uniformly for $t \in [0, b]$. It thus suffices to show that $u_\varepsilon(t) > 0$ for all $t \in [0, b]$ and $\varepsilon \in (0, \varepsilon_0(b)]$.

Suppose that this was not the case. Since $u_\varepsilon(0) > 0$, there would exist $t_0 > 0$, $x_0 \in \overline{G}$ and $k \in \{1, \ldots, m\}$ such that $v(t_0, x_0) = 0$, $u_\varepsilon(t,x) > 0$ and $u_\varepsilon(t_0, x) \ge 0$ for all $t \in (0, t_0)$ and $x \in \overline{G}$, where we put $v = (u_\varepsilon)_k$. Hence, $\partial_t v(t_0, x_0) \le 0$ and $v(t_0, x_0)$ is a minimum of the function $x \mapsto v(t_0, x)$ on $\overline{G}$. Moreover, the condition (3.14) shows that $f_k(u_\varepsilon(t_0,x_0)) \ge 0$. Thus the differential equation in (3.11) implies that

$$a_k \Delta v(t_0,x_0) = \partial_t v(t_0,x_0) - f_k(u_\varepsilon(t_0,x_0)) - \varepsilon \le -\varepsilon < 0.$$

Note that then $\Delta v(t_0,x) \le 0$ for $x \in \overline{G}$ which are close to x_0, as $\Delta v(t_0)$ is continuous on $\overline{G}$ by part a).

If $x_0 \in G$, Proposition 3.1.10 of [**Lu1**] says that $\Delta v(t_0,x_0) \ge 0$ which is impossible. We can thus assume that $x_0 \in \partial G$ and $v(t_0,x) > 0$ for all $x \in G$. Since $\partial G \in C^2$, we find a ball $B \subseteq G$ such that $x_0 \in \partial B$ and $\Delta v(t_0) \le 0$ on $\overline{B}$. We apply Hopf's Lemma 3.7 to $w = -v(t_0)$ and obtain that $\partial_\nu v(t_0,x_0) < 0$, contradicting the boundary condition in (3.11). Consequently, $u_\varepsilon(t) > 0$ for all $t \in [0,b]$ and $\varepsilon \in (0,\varepsilon_0(b)]$, as needed. (Note that we use the extra regularity established in step a) to apply Proposition 3.1.10 of [**Lu1**] and Lemma 3.7.) □

We next show the weak maximum parabolic principle in our regularity framework. This result and slight variants are used below in the examples to obtain sup-norm estimates on solutions.

Proposition 3.9. *Let $G \subseteq \mathbb{R}^m$ be bounded and open with $\partial G \in C^2$, $a > 0$, and let the real-valued function v belong to $C([0,T] \times \overline{G}) \cap C^1((0,T], C(\overline{G})) \cap C((0,T], W^{2,p}(G))$ for all $p \in (1,\infty)$ and satisfy $\Delta v \in C((0,T] \times \overline{G})$. Assume that $\partial_t v - a\Delta v \le 0$ on $(0,T] \times \overline{G}$ and $\partial_\nu v \le 0$ on $(0,T] \times \partial G$. We then have*

$$\max_{(t,x)\in[0,T]\times\overline{G}} v(t,x) \le \max_{x\in\overline{G}} v(0,x).$$

Proof. We first observe that the maxima in the assertion exist. Let $v_\varepsilon(t,x) = v(t,x) - \varepsilon t$ for $\varepsilon > 0$ and $(t,x) \in [0,T] \times \overline{G}$. The maps v_ε satisfy the same assumptions with $\partial_t v_\varepsilon - a\Delta v_\varepsilon \le -\varepsilon < 0$ on $(0,T] \times \overline{G}$.

Suppose there were $\varepsilon > 0$, $t_0 \in (0,T]$ and $x_0 \in \overline{G}$ such that $M := v_\varepsilon(t_0,x_0) > \max_{\overline{G}} v(0)$. We define

$$t_1 = \sup\{t \in (0,T] \,|\, \max_{x\in\overline{G}} v_\varepsilon(s,x) < M \text{ for all } s \in [0,t]\} \in (0,t_0].$$

Note that there is a point $x_1 \in \overline{G}$ with $v_\varepsilon(t_1,x_1) = M$. Hence, $\partial_t v_\varepsilon(t_1,x_1) \ge 0$ and $v_\varepsilon(t_1,x_1)$ is a maximum of $v_\varepsilon(t_1,\cdot)$ on $\overline{G}$. As in the proof of Theorem 3.8 d), these facts lead to a contradiction implying the assertion. □

In Example 3.6 we have indicated that many (ordinary) reaction equations are globally solvable under reasonable assumptions. If combined with diffusion, the situation is much more complicated as discussed in the survey article [**Pi**]. To give a flavor of this topic, we investigate a simple example.

Example 3.10. In the framework of Example 3.6, we let $p > m$, $p > 3/2$ if $m = 1$, $\ell = 2$ with the (different) species u and v, and $f(u,v) = (u^k v^l, -u^k v^l)$ for some $k,l \in \mathbb{N}_0$. We show global existence for all initial data $0 \le (u_0,v_0) \in E_\alpha$.

Theorem 3.8 yields a unique nonnegative maximal solution (u,v) on $(0,t_+)$ since (3.14) holds. We suppose that $t_+ < \infty$. We first proceed as in the ODE system in Example 3.6 and deduce that

$$\partial_t(u+v) = a_1\Delta_N(u+v) + \frac{a_2 - a_1}{a_1}\, a_1\Delta_N v + u^k v^l - u^k v^l,$$

$$u(t)+v(t)=S(a_1t)(u_0+v_0)+\frac{a_2-a_1}{a_1}\,a_1\Delta_N\int_0^t S(a_1(t-s))v(s)\,\mathrm{d}s$$

for $t\in[0,t_+)$. Observe that $a_1\neq a_2$ as we have two different species. Therefore the sum $u+v$ satisfies an equation with the inhomogeneity $g=(a_2-a_1)\Delta_N v$ which has probably not a fixed sign. In the second equation above we see that the first summand is bounded on $\mathbb{R}_+\times\overline{G}$ since the semigroup $S(\cdot)$ generated by Δ_N is bounded on $L^p(G)$, and thus on $D_{\Delta_N}(\alpha,p)\hookrightarrow C(\overline{G})$, see Example 3.6. The second term is defined since v takes values in an interpolation space of Δ_N (cf. Theorem 2.14). However, the norm of v in these spaces could blow up as $t\to t_+$. Which uniform bounds do we know for v?

Since $\partial_t v-a_2\Delta_N v=-u^kv^l\leq 0$ by (3.11), Proposition 3.9 and Theorem 3.8 show that $\|v\|_\infty\leq\|v_0\|_\infty$ and hence

$$\|v\|_{L^q([0,t_+),L^q(G))}\leq t_+^{\frac{1}{q}}\operatorname{vol}(G)^{\frac{1}{q}}\,\|v_0\|_\infty.$$

for all $q\in(1,\infty)$. Unfortunately, the theory presented so far does not allow to use this global bound due to the presence of $a_1\Delta_N$ in front of the integral.

Here the deeper theory of *maximal regularity* of type L^q helps. It says that for certain classes of Banach spaces X and generators B of analytic semigroups $R(\cdot)$ on X (including Δ_N on $L^r(G)$ for all $r\in(1,\infty)$) the function $w(t)=\int_0^t R(t-s)g(s)\,\mathrm{d}s$ takes values in $\mathrm{D}(B)$ for a.e. $t\geq 0$ and that

$$\|Bw\|_{L^q([0,b],X)}\lesssim_{b,q}\|g\|_{L^q([0,b],X)}$$

for every $g\in L^q([0,b],X)$. (See Theorem 6.3.2 in [**PS**] or Section 7 in [**KW**] for this result. We refer to these two works (and our last chapter) for a treatment of this theory.) We recall that $\|BR(\cdot)x\|\lesssim_{b,q}\|x\|_{1-1/q,q}$ by Proposition 2.8.

With $B=a_1\Delta_N$ and $X=L^q(G)$ these facts imply that $u+v$, and thus u due to positivity, are bounded in $L^q([0,t_+),L^q(G))$ for every $q\in(1,\infty)$. Taking $q=kp$ and using that v is uniformly bounded, we deduce that u^kv^l belongs to $L^p([0,t_+),L^p(G))$. We next employ the equation $\partial_t u=a_1\Delta_N u+u^kv^l$ and again maximal regularity to conclude that $\Delta_N u$ is an element of $L^p([0,t_+),L^p(G))$. The equation then implies that $\partial_t u\in L^p([0,t_+),L^p(G))$. Finally, we can use the interpolative embedding

$$\begin{aligned}&L^p([0,t_+),[\mathrm{D}(\Delta_N)])\cap W^{1,p}([0,t_+),L^p(G))\\&\hookrightarrow C_b([0,t_+),D_{\Delta_N}(1-\tfrac{1}{p},p))\hookrightarrow C_b([0,t_+),W^{2-\frac{2}{p},p}(G))\hookrightarrow C_b([0,t_+),C(\overline{G})),\end{aligned}$$

see Theorem III.4.10.2 [**Am2**], Example 2.12 and the fractional Sobolev embedding Theorem 4.6.1 in [**Tr1**] with $2-\frac{2}{p}>\frac{m}{p}$. (Here we use that $p>3/2$ if $m=1$.) Summing up, the solution (u,v) is bounded on $[0,t_+)\times\overline{G}$ so that Theorem 3.8 b) yields that $t_+=\infty$.

One can show global existence also for (3.11) in the case of reactions $A+B\rightleftharpoons C$ with a more refined version of the above L^p approach, see [**Pr**]. ◊

3.2. Convergence to equilibria

Quite often one is particularly interested in certain classes of special solutions u_*. Here we only look at the simplest case of equilibria; a different one would be time-periodic solutions. We first characterize the stationary solutions of (3.1).

Lemma 3.11. *Let (3.2) be true. A vector $u_* \in X$ is a time-independent solution (an* equilibrium*) of (3.1) if and only if $u_* \in \mathrm{D}(A)$ and $Au_* = -F(u_*)$.*

Proof. If $u_* \in \mathrm{D}(A)$ and $Au_* = -F(u_*)$, then the function $u(t) = u_*$ clearly solves (3.1) for all $t \geq 0$. Conversely, if (3.1) has a stationary solution $u(t) = u_*$ for $t \geq 0$, then the mild formula (3.3) yields

$$\frac{1}{t}\,(T(t)u_* - u_*) = -\frac{1}{t}\int_0^t T(t-s)F(u_*)\,\mathrm{d}s \;\longrightarrow\; -F(u_*)$$

as $t \to 0$, and hence $u_* \in \mathrm{D}(A)$ with $Au_* = -F(u_*)$. □

At least some equilibria of (3.11) are easy to obtain.

Remark 3.12. The reaction diffusion system (3.11) possesses the spatially constant equilibrium $u_* = r_*\mathbb{1}$ if there is a vector $r_* \in \mathbb{R}^\ell$ with $f(r_*) = 0$; i.e., (3.11) inherits the equilibria of the correspondig ordinary differential equation $y' = f(y)$. (This works since we have chosen Neumann boundary conditions.) The construction of other, spatially heterogeneous equilibria is part of the theory of semilinear elliptic equation, not treated here.

More generally, for every constant initial value $u_0 = r_0\mathbb{1}$ the system (3.11) has the solution $u(t) = r(t)\mathbb{1}$ where $r' = f(r)$ and $r(0) = r_0 \in \mathbb{R}^\ell$, so that the reaction ODE is contained in the reaction diffusion system (3.11). ◊

In the applications one cannot exactly prescribe an equilibrium u_* as an initial value. Thus it is important whether small deviations of the initial value lead to small deviations of the solution for all $t \geq 0$. This property is called stability. In this section we treat the slightly different, but closely related, topic of convergence to u_*. We consider two basic results, both due to Lyapunov in the ODE case: a local one using the spectrum of the linearization and a global one using Lyapunov functions, cf. [**Ana4**] for ordinary differential equation.

The first result is called *principle of linearized stability*, and it is based on the idea that near an equilibrium u_* the problem $u'(t) = Au(t) + F(u(t))$ is very close to the linearized equation $w'(t) = Aw(t) + F'(u_*)w(t)$. We show that the exponential stability of the latter problem implies the 'local exponential stability' of u_* for (3.1). Recall that the spectral bound $\mathrm{s}(B)$ of a closed operator is the supremum of the real parts of all $\lambda \in \sigma(A)$.

Theorem 3.13. *Let (3.2) be true. Assume that u_* is an equilibrium of (3.1) such that F is differentiable at u_* and $\mathrm{s}(A + F'(u_*)) < 0$. Take $\kappa \in (0, -\mathrm{s}(A + F'(u_*)))$. Then there are constants $c, \rho > 0$ such that for all $u_0 \in \overline{B}_\alpha(u_*, \rho)$ we have $t_+(u_0) = \infty$ and*

$$\|u(t) - u_*\|_\alpha \leq c\mathrm{e}^{-\kappa t}\,\|u_0 - u_*\|_\alpha$$

holds for all $t \geq 0$, where u solves (3.1).

Proof. Theorem 3.10 of [**EE**] and Remark 2.11 imply that $B := A + F'(u_*)$ with domain $\mathrm{D}(A)$ generates an analytic C_0–semigroup $S(\cdot)$ on X. Due to Corollary 4.16 in [**EE**] and the assumption, we have $\omega_0(B) = \mathrm{s}(B) < -\kappa < 0$ and hence there are numbers $M \geq 1$ and $\delta' \in (\kappa, -\mathrm{s}(B))$ such that $\|S(t)\| \leq M\mathrm{e}^{-\delta' t}$ for all $t \geq 0$. We write X_α^B for the interpolation spaces of B and take $\delta \in (\kappa, \delta')$. Proposition 2.4 then yields that $\|S(t)\|_{\mathcal{B}(X_\alpha^B)} \leq M\mathrm{e}^{-\delta' t} \leq M\mathrm{e}^{-\delta t}$ for all $t \geq 0$. From Theorem 2.14 we infer $\|S(t)\|_{\mathcal{B}(X, X_\alpha^B)} \leq ct^{-\alpha} \leq c\mathrm{e}^{\delta} t^{-\alpha}\mathrm{e}^{-\delta t}$ for all $t \in (0,1]$. For $t \geq 1$, we further deduce

$$\|S(t)\|_{\mathcal{B}(X,X_\alpha^B)} \leq \|S(1)\|_{\mathcal{B}(X,X_\alpha^B)} \|S(t-1)\| \leq c\mathrm{e}^{-\delta' t} \leq ct^{-\alpha}\mathrm{e}^{-\delta t}.$$

To transfer these estimates to X_α, we note that $I : [\mathrm{D}(A)] \to [\mathrm{D}(B)]$ is an isomorphism by Theorem 3.10 in [**EE**]. Interpolation (see Theorem 2.9) then shows that $I : X_\alpha \to X_\alpha^B$ is also isomorphic, resulting in

$$\|S(t)\|_{\mathcal{B}(X_\alpha)} \leq M_0\,\mathrm{e}^{-\delta t} \qquad \text{and} \qquad \|S(t)\|_{\mathcal{B}(X,X_\alpha)} \leq M_1 t^{-\alpha}\mathrm{e}^{-\delta t} \tag{3.15}$$

for all $t > 0$ and some constants $M_0, M_1 \geq 1$.

Since $Au_* = -F(u_*)$, the function $v = u - u_*$ with initial value $v(0) = u_0 - u_* =: v_0$ satisfies the equation

$$\begin{aligned} v'(t) &= u'(t) = Au(t) + F(u(t)) - Au_* - F(u_*) \\ &= Bv(t) + F(u_* + v(t)) - F(u_*) - F'(u_*)v(t) =: Bv(t) + G(v(t)) \end{aligned} \tag{3.16}$$

for all $t \in [0, t_+(u_0))$. We can fix a number $\varepsilon > 0$ with

$$c_0 M_1 \varepsilon t^{1-\alpha} \exp(c(\alpha) M_1^{\frac{1}{1-\alpha}}\, \varepsilon^{\frac{1}{1-\alpha}}\, t)\, \mathrm{e}^{(\kappa-\delta)t} \leq \frac{1}{2}$$

for all $t \geq 0$, where the constants c_0 and $c(\alpha)$ are taken from (3.9). Because F is differentiable at u_*, there is a radius $r > 0$ such that

$$\forall\, x \in \overline{B}_\alpha(r) : \qquad \|G(x)\|_\alpha \leq \varepsilon \|x\|_\alpha. \tag{3.17}$$

To use this estimate, we have to restrict ourselves to times $t \geq 0$ such that $\|v(t)\|_\alpha \leq r$. Set $\rho = (2M_0)^{-1} r \in (0, r)$ and take $u_0 \in \overline{B}_\alpha(u_*, \rho)$ so that $\|v_0\|_\alpha \leq \rho < r$. We introduce the number

$$\tau = \sup\{t \in (0, t_+(u_0)) \,|\, \|v(s)\|_\alpha \leq r \ \text{ for all } s \in [0,t]\} \ \in \ (0, t_+(u_0)].$$

Equation (3.16) and estimates (3.15) and (3.17) now yield

$$\begin{aligned} \|v(t)\|_\alpha &\leq \|S(t)v_0\|_\alpha + \int_0^t \|S(t-s)G(v(s))\|_\alpha \,\mathrm{d}s \\ &\leq M_0\mathrm{e}^{-\delta t}\|v_0\|_\alpha + \varepsilon M_1 \int_0^t (t-s)^{-\alpha}\mathrm{e}^{-\delta(t-s)}\|v(s)\|_\alpha \,\mathrm{d}s, \\ \mathrm{e}^{\delta t}\|v(t)\|_\alpha &\leq M_0\|v_0\|_\alpha + \varepsilon M_1 \int_0^t (t-s)^{-\alpha}\,\mathrm{e}^{\delta s}\|v(s)\|_\alpha \,\mathrm{d}s \end{aligned}$$

for all $t \in [0, \tau)$. The singular Gronwall inequality (3.9) then implies

$$\begin{aligned} \mathrm{e}^{\delta t}\|v(t)\|_\alpha &\leq M_0\|v_0\|_\alpha\big(1 + c_0 M_1 \varepsilon t^{1-\alpha} \exp(c(\alpha) M_1^{\frac{1}{1-\alpha}}\, \varepsilon^{\frac{1}{1-\alpha}}\, t)\big), \\ \|v(t)\|_\alpha &\leq M_0\mathrm{e}^{-\kappa t}\|v_0\|_\alpha\big(1 + c_0 M_1 \varepsilon t^{1-\alpha} \exp(c(\alpha) M_1^{\frac{1}{1-\alpha}}\, \varepsilon^{\frac{1}{1-\alpha}}\, t)\mathrm{e}^{(\kappa-\delta)t}\big) \end{aligned}$$

$$\leq \tfrac{3}{2} M_0 \mathrm{e}^{-\kappa t} \|v_0\|_\alpha \leq \tfrac{3}{4} r,$$

for all $t \in [0, \tau)$, due to our choice of ε and ρ. If $\tau < t_+(u_0)$, we would obtain $\|v(\tau)\|_\alpha \leq 3r/4 < r$ by continuity, contradicting the definition of τ. Hence, $\tau = t_+(u_0)$ so that $\|v(t)\|_\alpha$ is bounded on $[0, t_+(u_0))$. Theorem 3.4 thus yields $t_+(u_0) = \infty$. The claim with $c := \frac{3M_0}{2}$ now follows from the above estimate. □

We note that refinements of Theorem 3.13 give a description of the neighborhood of an equilibrium depending on the spectrum of $A + F'(u_*)$, see e.g. Chapter 9 of [**Lu1**].

In Theorem 3.13 we have employed $\mathbb{C}$–linear derivatives. As we have seen in Corollary 1.18, there are many important nonlinearities which are only real differentiable. On the other hand, we have used complex scalars for spectral theory or construct and study analytic semigroups. In the next remark we indicate how to pass from real to complex scalars in our setting.

Remark 3.14. Let X be a real Banach space. We define its complexification

$$X_\mathbb{C} = X \otimes \mathrm{i}X = \{z = x + \mathrm{i}y \,|\, x, y \in X\},$$

and write $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$. It is straightforward to check that $X_\mathbb{C}$ is a complex vector space for the scalar multiplication

$$(\alpha + \mathrm{i}\beta)(x + \mathrm{i}y) := (\alpha x - \beta y) + \mathrm{i}(\beta x + \alpha y)$$

for $\alpha, \beta \in \mathbb{R}$ and $x, y \in X$. Moreover, $X_\mathbb{C}$ is a Banach space when endowed with

$$\|z\|_{X_\mathbb{C}} = \sup\nolimits_{\theta \in \mathbb{R}} \|\operatorname{Re}(\mathrm{e}^{\mathrm{i}\theta} z)\|_X.$$

Note that $\|z\|_{X_\mathbb{C}}$ is equivalent to $\|\operatorname{Re} z\|_X + \|\operatorname{Im} z\|_X$. Typical examples are the real-valued function spaces $L^p(\mu)$ or $C(K)$ whose complexifications are the corresponding $\mathbb{C}$–valued spaces. (Compare Appendix B.4 of [**HNVW**].)

For a real-linear operator B on X, one sets $\mathrm{D}(B_\mathbb{C}) = \mathrm{D}(B) \otimes \mathrm{i}\mathrm{D}(B)$ and $B_\mathbb{C} z = Bx + \mathrm{i}By$. Routine calculations show that $B_\mathbb{C}$ is $\mathbb{C}$-linear and closed if B is closed. In addition, if B is bounded, then $B_\mathbb{C}$ has the same norm. One can define C_0-semigroups $T(\cdot)$ and their generator A on X as before and analytic ones as in Theorem 2.25 d) of [**EE**].

We assume that A and F satisfy (3.2) on a real Banach space X, that $u_* \in \mathrm{D}(A)$ is a stationary solution for (3.1) with A and F, and that F is real differentiable at u_*. One then verifies that $A_\mathbb{C}$ generates the analytic C_0–semigroup $T_\mathbb{C}(\cdot)$ and that $D_{A_\mathbb{C}}(\alpha, p)$ and $D_{A_\mathbb{C}}(\alpha)$ are isomorphic to the complexifications of $D_A(\alpha, p)$ and $D_A(\alpha)$, respectively. Setting $F_\mathbb{C}(z) = F(\operatorname{Re} z) + \mathrm{i}F(\operatorname{Im} z)$, we obtain that $A_\mathbb{C}$ and $F_\mathbb{C}$ fulfill (3.2) on $X_\mathbb{C}$. Moreover, $u_* + \mathrm{i}u_*$ is an equilibrium for $A_\mathbb{C}$ and $F_\mathbb{C}$, and $F_\mathbb{C}$ possesses the $\mathbb{C}$-derivative $F'(u_*)_\mathbb{C}$ at $u_* + \mathrm{i}u_*$.

We also assume that $\mathrm{s}(A_\mathbb{C} + F'_\mathbb{C}(u_*)) < 0$. We can now apply Theorem 3.13 to the problem (3.1). Let $u_0 \in X_\alpha$ be real. The imaginary part $v = \operatorname{Im} u$ of the solution to (3.1) in $X_\mathbb{C}$ then satisfies $v' = Av + F(v)$ with $v(0) = 0$. As Proposition 2.6 of [**EE**] and Theorem 2.14 also work in the real case, Gronwall's inequality (3.9) yields $v = 0$. Cosequently, the solution u to (3.1) is real and satisfies the assertion of Theorem 3.13 with the same exponent. ◊

As a simple application of the principle of lienarized stability, we discuss spatially constant equilibria of reaction diffusion systems with two species. The treatment of general equilibria requires deeper investigations of spectral properties of elliptic systems with space-depending (heterogeneous) coefficients.

Example 3.15. We continue to work in the framework of Example 3.6 with $\ell = 2$, $m = 3$, $p > 2$, and $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$. We assume that $f(r_*, s_*) = 0$ for some $(r_*, s_*) \in \mathbb{R}^2_{\geq 0}$, and consider the equilibrium $(u_*, v_*) = (r_*, s_*)\mathbb{1}$. We write

$$f'(r_*, s_*) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} =: C.$$

To check that F is real differentiable, let $(u, v), (\hat{u}, \hat{v}) \in C(\overline{G})^2$ be real and $x \in \overline{G}$. We compute

$$\begin{aligned} D(u,v)(x) &:= \big(F(u + \hat{u}, v + \hat{v}) - F(\hat{u}, \hat{v}) - f'(\hat{u}, \hat{v})(u, v)\big)(x) \\ &= \int_0^1 \big[f'(\hat{u}(x)+\tau u(x), \hat{v}(x)+\tau v(x)) - f'(\hat{u}(x), \hat{v}(x))\big](u(x), v(x))\, \mathrm{d}\tau, \end{aligned}$$

$$\|D(u,v)\|_\infty \leq \|(u,v)\|_\infty \max_{|r|,|s| \leq \|(u,v)\|_\infty} \max_{x \in \overline{G}} |f'(\hat{u}(x)+r, \hat{v}(x)+s) - f'(\hat{u}(x),\hat{v}(x))|.$$

Since f', $\hat{u}$ and $\hat{v}$ are uniformly continuous on $\overline{G}$, the map $F : C(\overline{G})^2 \to C(\overline{G})^2$ has the real derivative given by

$$[F'(\hat{u}, \hat{v})(u, v)](x) = f'(\hat{u}(x), \hat{v}(x))(u(x), v(x)).$$

One can show in a similar way that $F' : C(\overline{G})^2 \to \mathcal{B}(C(\overline{G})^2)$ is continuous. Because $E_\alpha \hookrightarrow C(\overline{G})^2$ by (3.13) and $C(\overline{G})^2 \hookrightarrow E$, we conclude that F also belongs to $C^1(E_\alpha, E)$ with the same formula for $F'(\hat{u}, \hat{v})$, and hence

$$B(u, v) := [A + F'((u_*, v_*))](u, v) = \begin{pmatrix} a_1\Delta_N + c_{11}I & c_{12}I \\ c_{21}I & a_2\Delta_N + c_{22}I \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

If $\mathrm{s}(B) < 0$, then Remark 3.14 shows that all real solutions starting close to (u_*, v_*) converge to (u_*, v_*) in E_α exponentially as $t \to \infty$.

We want to obtain some information about $\mathrm{s}(B)$. We first note that $\mathrm{D}(B) = \{(u, v) \in W^{2,2}(G)^2 \,|\, \partial_\nu u = \partial_\nu v = 0 \text{ on } \partial G\}$ is compactly embedded into $E = L(G)^2$ by the Rellich–Kondrachov Theorem 3.29 in [**ST**]. Thus, B has pure point spectrum by Remark 2.14 and Theorem 2.16 of [**ST**]. To use the powerful L^2–theory, we consider the restriction B_2 of B to $L^2(G)^2$ with the analogous domain. As for B, one obtains $\sigma(B_2) = \sigma_p(B_2)$. We check that $\sigma_p(B_2) = \sigma(B)$.

From $p > 2$ it follows $\mathrm{D}(B) \subseteq \mathrm{D}(B_2)$ and hence each eigenvector of B is one of B_2 for the same eigenvalue, i.e.; $\sigma(B) \subseteq \sigma_p(B_2)$. Conversely, let $w = (u_1, u_2) \in \mathrm{D}(B_2)$ satisfy $\lambda w = B_2 w$. By Sobolev's embedding Theorem 3.26 of [**ST**], the eigenfunction w belongs to $C(\overline{G})^2 \hookrightarrow L^p(G)^2$ because of $2 - 3/2 > 0$. We further have $\partial_\nu u_j = 0$ on ∂G and

$$\Delta u_j - u_j = a_j^{-1}(\lambda u_j - c_{j1}u_1 - c_{j2}u_2) - u_j =: \varphi_j \ \in\ L^p(G)$$

for $j \in \{1, 2\}$. Since $\Delta_N - I$ is bijective on $L^p(G)$ by Example 5.2 of [**EE**], there is a function $v_j \in W^{2,p}(G) \subseteq W^{2,2}(G)$ such that $\partial_\nu v_j = 0$ and $\Delta v_j - v_j = \varphi_j$.

The injectivity of $\Delta_N - I$ on $L^2(G)$ implies that $u_j = v_j$ for $j \in \{1,2\}$, and so w is an eigenfunction of B for the eigenvalue λ. We have shown $\sigma(B) = \sigma_p(B_2)$.[3]

Arguing as in Example 1.52 of [**EE**] for the Dirichlet Laplacian, we see that Δ_N is selfadjoint on $L^2(G)$. The spectral theorem in the compact case (see Theorem 4.15 in [**ST**]) thus yields an orthonormal basis of eigenfunctions e_n for eigenvalues μ_n of Δ_N. For $(u,v) \in \mathrm{D}(B_2)$, we can now use the orthonormal series $u = \sum_{n\ge 0} \alpha_n \mathrm{e}_n$ and $v = \sum_{n\ge 0} \beta_n \mathrm{e}_n$ with the coefficients $\alpha_n = (u|\mathrm{e}_n)$ and $\beta_n = (v|\mathrm{e}_n)$. It further holds $\Delta_N u = \sum_{n\ge 0} \alpha_n \mu_n \mathrm{e}_n$ and $\Delta_N v = \sum_{n\ge 0} \beta_n \mu_n \mathrm{e}_n$. Hence, the equation $B_2(u,v) = \lambda(u,v)$ for some $\lambda \in \mathbb{C}$ is equivalent to

$$\sum_{n=0}^{\infty} \left[\begin{pmatrix} a_1 \mu_n \alpha_n \mathrm{e}_n \\ a_2 \mu_n \beta_n \mathrm{e}_n \end{pmatrix} + \begin{pmatrix} c_{11} \alpha_n \mathrm{e}_n + c_{12} \beta_n \mathrm{e}_n \\ c_{21} \alpha_n \mathrm{e}_n + c_{22} \beta_n \mathrm{e}_n \end{pmatrix} \right] = \sum_{n=0}^{\infty} \begin{pmatrix} \lambda \alpha_n \mathrm{e}_n \\ \lambda \beta_n \mathrm{e}_n \end{pmatrix}.$$

Since the functions e_n are orthonormal, this identity holds if and only if

$$M_n \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} := \begin{pmatrix} a_1 \mu_n + c_{11} & c_{12} \\ c_{21} & a_2 \mu_n + c_{22} \end{pmatrix} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} = \lambda \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} \qquad \text{for every } n \in \mathbb{N}_0.$$

This means that an eigenvalue λ of some M_k with eigenvector (α_k, β_k) is an eigenvalue of B_2 with eigenfunction $(\alpha_k \mathrm{e}_k, \beta_k \mathrm{e}_k)$. Conversely, every eigenfunction $w = (u,v)$ for an eigenvalue of B_2 gives an eigenvalue of M_n for those n such that $(\alpha_n, \beta_n) \neq 0$. The spectrum of B is thus given by

$$\sigma(B) = \sigma_p(B_2) = \bigcup\nolimits_{n\in\mathbb{N}_0} \sigma(M_n).$$

As $\sigma(B) \cap \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\,\lambda \ge -1\}$ is bounded and hence compact, we conclude

$$\mathrm{s}(B) < 0 \iff \forall\, n \in \mathbb{N}_0 :\ \mathrm{s}(M_n) < 0 \iff \forall\, n \in \mathbb{N}_0 :\ \mathrm{tr}\, M_n < 0,\ \det M_n > 0.$$

With these equivalences the stability problem is reduced to a countable family of inequalities involving the coefficients a_j and c_{jk} and the Neumann eigenvalues μ_n. We discuss this result a bit. As $\mu_0 = 0$, we have $M_0 = C = f'(r_*, s_*)$ so that $\mathrm{s}(C) < 0$ is a necessary condition for the local exponential stability of the reaction diffusion equation; i.e., (r_*, s_*) must be locally exponential stable for the ODE describing the pure reaction. If this is the case, we have $\mathrm{tr}\, C = c_{11} + c_{22} < 0$ and hence $\mathrm{tr}\, M_n = (a_1 + a_2)\mu_n + c_{11} + c_{12} < 0$ since $\mu_n \le 0$ for all $n \in \mathbb{N}_0$. Moreover, μ_n tends to $-\infty$ as $n \to \infty$ (see Theorem 4.15 in [**ST**]) so that $\det M_n > 0$ for all large n. However, for small $n \ge 1$ it can happen that

$$0 > \det M_n = a_1 a_2 \mu_n^2 + \mu_n (a_1 c_{22} + a_2 c_{11}) + \det C$$

despite $\det C > 0$ and $a_1 a_2 \mu_n^2 > 0$, as $\mu_n < 0$. Form instance, this behaviour can occur if $\det C$ is close to 0, a_1 is small, and $c_{11} \gg a_1 |\mu_1|$. (Typically, diffusion coefficients are small.) In this case one has 'diffusion induced instability'. $\Diamond$

It would be nice to know that the solutions converge to u_* for a 'larger' set of initial values (e.g., all positive ones). To obtain results in this direction, we need two more tools: Lyapunov functions and omega limit sets.

[3] The spectrum of restrictions of generators to subspaces is systematically studied in Section IV.2.b of [**EN**].

Definition 3.16. *Let* (3.2) *be true and* $\emptyset \neq D \subseteq X_\alpha$*. A map* $\Phi \in C(D, \mathbb{R})$ *is called* Lyapunov function *for* (3.1) *on* D *if the function* $\varphi_u(t) = \Phi(u(t))$ *does not increase for every solution* u *of* (3.1) *as long as* $u(t) \in D$ *and* $t \in [0, t_+(u_0))$*. A Lyapunov function is* strict *if* φ_u *decreases except for equilibria* u*.*

As we will see below, the existence of a Lyapunov function has a significant impact on the qualitative behavior of the system (3.1). So it is no surprise that there is not general recipe to find them. As a possible starting point we note that Lyapunov functions are often related to physical quantities such as energy or entropy for systems arising in science.

A Lyapunov function Φ allows to detect invariant sets for (3.1) and global solutions, if Φ blows up at ∂D and for large $\|x\|_\alpha$.

Proposition 3.17. *Let* (3.2) *be true,* $D \subseteq X_\alpha$ *be open, and* $\Phi \in C(D, \mathbb{R})$ *be a Lyapunov function for* (3.1)*. Then the following assertions hold.*

a) Let $D \neq X$ *and* $\Phi(x) \to \infty$ *as* $x \to \partial D$ *in* X_α*. Then* D *is invariant; i.e., for every* $u_0 \in D$ *we have* $\varphi(t, u_0) \in D$ *for all* $t \in [0, t_+(u_0))$*.*[4]

b) Let D *is invariant and* $\Phi(x) \to \infty$ *as* $\|x\|_\alpha \to \infty$ *for* $x \in D$*, then* $t_+(u_0) = \infty$ *for all* $u_0 \in D$*.*

Proof. Let $u_0 \in D$. The states $u(t) := \varphi(t, u_0)$ then belong to the set $\{x \in D \,|\, \Phi(x) \leq \Phi(u_0)\}$ as long as they stay in D and $t \in [0, t_+(u_0))$.

a) Let $\Phi(x) \to \infty$ as $x \to \partial D$ in X_α. Suppose there was a time $t_1 \in (0, t_+(u_0))$ with $u(t_1) \notin D$. Then the number

$$\tau := \sup\{t \in [0, t_1) \,|\, \forall\, s \in [0, t] : \; u(s) \in D\}$$

is contained in $(0, t_1]$. Since D is open and $u \in C(J'_+(u_0), X_\alpha)$, we have $u(\tau) \notin D$. For $t \in [0, \tau)$ the vectors $u(t)$ are elements of D and converge to $u(\tau)$ in X_α as $t \to \tau$. So $\Phi(u(t))$ tends to infinity, contradicting the initial observation.

b) Similarly, the conditions in b) imply that $\|u(t)\|_\alpha$ is bounded on $[0, t_+(u_0))$ so that $t_+(u_0) = \infty$ due to Theorem 3.4. □

As in [**Ana4**], one can use Lyapunov functions to detect (asymptotic) stability. We omit such results and focus on global convergence properties. To this end, we establish the basic properties of omega limit sets; i.e., the sets of accumulation points of solutions as $t \to \infty$.

We need the concept of connectedness in a metric space M. A set $C \subseteq M$ is *disconnected* if it decomposes into $C = C_1 \cup C_2$ for relatively open, disjoint, and non-empty subsets C_j. A subset is called *connected* if it is not disconnected.

We recall some basic properties of connected sets, see Section III.4 of [**AE1**]. The space M is connected if and only if only $\emptyset$ and M are the only open and closed subsets of M. For open subsets of a normed vector space, connectedness and pathwise connectedness coincide. Let N be a metric space, $f : M \to N$ be continuous, and $C \subseteq M$ be connected. Then the range $f(C)$ is connected.

[4] Here and below we restrict ourselves to 'forward concepts' for times $t \geq 0$. There also exist 'backward' versions.

Proposition 3.18. *Let* (3.2) *be true. Let* $u = \varphi(\,\cdot\,, u_0)$ *be a solution of* (3.1) *on* $\mathbb{R}_+$ *such that the orbit* $\kappa(u_0) := \{u(t) \,|\, t \geq 0\}$ *is relatively compact in* X_α*. Then the following assertions hold.*

a) The omega limit set

$$\omega(u_0) := \{x \in X_\alpha \,|\, \exists\, n_j \to \infty : u(t_{n_j}) \to x \text{ in } X_\alpha \text{ as } j \to \infty\}$$

is non-empty, compact and connected in X_α*, and it is invariant for* (3.1)*.*

b) We have $\mathrm{d}_\alpha(u(t), \omega(u_0)) := \mathrm{d}_{X_\alpha}(u(t), \omega(u_0)) \to 0$ *as* $t \to \infty$*.*

c) Let $v_0 \in \omega(u_0)$*. Then* $t_+(v_0) = \infty$ *and there is a solution* v *of* (3.1) *on* $\mathbb{R}$ *with* $v(0) = v_0$ *and* $v(t) \in \omega(u_0)$ *for all* $t \in \mathbb{R}$*. In particular,* $v_0 \in \mathrm{D}(A)$*.*

Proof. 1) The set $\omega(u_0)$ is not empty, since $\overline{\kappa(u_0)}^{\alpha}$ is compact in X_α. Let $x_n \in \omega(u_0)$ tend to x in X_α as $n \to \infty$. For each $n \in \mathbb{N}$ we choose $t_n \geq n$ such that $\|u(t_n) - x_n\|_\alpha \leq 1/n$. Hence, $\|x - u(t_n)\|_\alpha \leq \|x - x_n\|_\alpha + 1/n$ tends to 0 as $n \to \infty$. The vector x thus belongs to $\omega(u_0)$, and $\omega(u_0)$ is closed in X_α. The compactness of $\omega(u_0) \subseteq \overline{\kappa(u_0)}^{\alpha}$ now follows from its closedness in X_α.

2) Suppose that $\mathrm{d}_\alpha(u(t_n), \omega(u_0)) \geq \eta > 0$ for a sequence $t_n \to \infty$. Then there is a subsequence such that $u(t_{n_j})$ converges to some $x \in \omega(U_0)$ in X_α, which is a contradiction.

3) To show the connectedness of $\omega(u_0)$, we assume that there were non-empty disjoint subsets ω_j of $\omega(u_0)$ such that $\omega_j = O_j \cap \omega(u_0)$ for open sets $O_j \subseteq X_\alpha$ for $j \in \{1, 2\}$ and $\omega_1 \cup \omega_2 = \omega(u_0)$. Writing $\omega_j = \omega(u_0) \cap (X_\alpha \setminus O_i)$ for $i \neq j$, we see that ω_j is closed, hence compact, in X_α. It follows that $\|x_1 - x_2\|_\alpha \geq \delta > 0$ for some $\delta > 0$ and all $x_j \in \omega_j$.

Take some $x \in \omega_1$ and $y \in \omega_2$. There are times $s_n, t_n \to \infty$ such that $s_n < t_n < s_{n+1}$, $\|u(s_n) - x\|_\alpha < \delta/3$ and $\|v(t_n) - y\|_\alpha < \delta/3$ for all sufficiently large n. By continuity, we can then find a time $r_n \in (s_n, t_n)$ such that $\mathrm{d}_\alpha(u(r_n), \omega_1) = \delta/3$ and thus $\mathrm{d}_\alpha(u(r_n), \omega_2) \geq 2\delta/3$. But for a subsequence the states $u(r_{n_j})$ converge to an element z of $\omega(u_0)$ in X_α as $j \to \infty$, which is impossible. Consequently, $\omega(u_0)$ is connected.

4) It remains to show assertion c), which also implies the last part of a). Let $v_0 \in \omega(u_0)$. Then there are $t_n \to \infty$ such that $u(t_n) \to v_0$ in X_α as $n \to \infty$. The continuous dependence on initial data and the uniqueness of (3.1) imply

$$v(t) = \varphi(t, v_0) = \lim_{n\to\infty} \varphi(t, \varphi(t_n, u_0)) = \lim_{n\to\infty} \varphi(t + t_n, u_0)$$

in X_α for $t \in [0, t_+(v_0))$, see Theorem 3.4; i.e., all states $v(t)$ belong to $\omega(u_0)$. As a compact set, $\omega(u_0)$ is bounded in X_α. Theorem 3.4 c) now yields $t_+(v_0) = \infty$.

Inductively, for all $j \in \{0, 1, \ldots, m\}$ and $m \in \mathbb{N}$ we obtain vectors $v_j \in \omega(u_0)$ and subsequences $(t_{\nu_m(k)})_k$ of $(t_{\nu_{m-1}(k)})_k$ such that $\varphi(t_{\nu_m(k)} - j, u_0)$ tends to v_j in X_α as $k \to \infty$. Set $v^m = \varphi(\,\cdot\,, v_m)$ on $\mathbb{R}_{\geq 0}$. For $t \geq -j \geq -m$ we compute

$$\begin{aligned} v^m(t+m) &= \lim_{k\to\infty} \varphi(t+m, \varphi(t_{\nu_m(k)} - m, u_0)) = \lim_{k\to\infty} \varphi(t+j, \varphi(t_{\nu_m(k)} - j, u_0)) \\ &= v^j(t+j). \end{aligned}$$

We can thus define a solution of (3.1) on $\mathbb{R}$ by setting $v(t) = v^m(t+m)$ for all $t \geq -m$ and $m \in \mathbb{N}$. In particular, $v_0 = v(0)$ is an element of $\mathrm{D}(A)$. Since

$v(-m) = v_m \in \omega(u_0)$ for all $m \in \mathbb{N}_0$, the invariance of $\omega(u_0)$ implies that $v(t)$ belongs to $\omega(u_0)$ for each $t \in \mathbb{R}$. □

These results (except for the very last assertion) hold in a much more general setting, see Chapter 4 of [**He**]. The proof of the above proposition indicates again that one should describe the behavior of (3.1) in the norm of X_α. The compactness assumption in the above proposition is crucial, of course, as seen by the solution e^t of the ODE $u'(t) = u(t)$.

The above result only says that the solution tends to its omega limit set. In our next convergence theorem we can describe $\omega(u_0)$ in better way and then deduce that the solution has a limit. It is based on the above proposition and the observation that $\omega(u_0)$ only contains equlibria if the problem possesses a strict Lyapunov function.

Theorem 3.19. *Let (3.2) be true, $D \subseteq X_\alpha$, and $\Phi \in C(D, \mathbb{R})$ be a strict Lyapunov function for (3.1). Assume that $u_0 \in D$ satisfies $t_+(u_0) = \infty$ and that $\overline{\kappa(u_0)}^\alpha$ is compact in X_α and contained in D.*

Then the omega limit set $\omega(u_0)$ belongs to $\mathcal{E}_D$ and $\mathrm{d}_\alpha(u(t), \mathcal{E}_D)$ tends to 0 as $t \to \infty$, where $\mathcal{E}_D = \{u_* \in D \cap \mathrm{D}(A) \mid Au_* = -F(u_*)\}$ is the set of equilibria in D. If $\mathcal{E}_D$ is discrete, then u even converges to a vector $u_* \in \mathcal{E}_D$.

Proof. Since $\varphi_u = \Phi \circ u$ decreases and Φ is bounded on the compact set $\overline{\kappa(u_0)}^\alpha$, the function $\varphi_u(t)$ converges to some $\ell \in \mathbb{R}$. Take any $x \in \omega(u_0)$ and $t_n \to \infty$ such that $u(t_n) \to x$ in X_α. The vector x then belongs to D and $\Phi(x) = \lim_{n\to\infty} \varphi_u(t_n) = \ell$ which means that Φ is constant on $\omega(u_0)$. The orbit $\varphi(\,\cdot\,, x)$ stays in $\omega(u_0) \subseteq D$ by Proposition 3.18 so that x must belong to $\mathcal{E}_D$ because Φ is strict. The assertions now follow from Proposition 3.18. (Note that a discrete, connected, non-empty set has to be a singleton.) □

There is a variant of this theorem without the strictness assumption, called *LaSalle's invariance principle*, see Theorem 4.3.4 of [**He**]. In some situations one can also show convergence to an equilibrium if $\mathcal{E}_D$ is not discrete, see Sections 8.8 and 10.3 in [**PW**] for such results in an ODE setting.

Despite its surprisingly elementary proof, the above theorem is very powerful. In our reaction diffusion system (3.11) Theorem 3.8 shows that uniform boundedness already implies compactness of the orbit. For (strict) Lyapunov functions there are at least some candidates as the one used in the next example treating a predator-prey model. (Compare Beispiel 6.11 of [**Ana4**].)

Example 3.20. Again we work in the framework of Example 3.6 with $\ell = 2$, now assuming that G is connected. We consider the 'reaction term'

$$f(u, v) = \begin{pmatrix} (1 - \lambda_1 u - v)u \\ (\mu - \lambda_2 v + u)v \end{pmatrix}$$

with $\lambda_1, \lambda_2 > 0$ and $\mu \in \mathbb{R}$, describing the (normalized) interaction between the prey species u and the predators v. Let $0 \le u_0, v_0 \in E_\alpha \hookrightarrow C(\overline{G})^2$. Since the positivity condition (3.14) holds, there is a unique non-negative solution $(u(t), v(t))$ of (3.1) with the above f on a maximal existence interval $[0, t_+)$.

1) We first show that $u \le \kappa := \max\{\|u_0\|_\infty, 1/\lambda_1\}$. Suppose that there were $(t_0, x_0) \in (0, t_+) \times \overline{G}$ and $\delta > 0$ such that $u(t_0, x_0) \ge \delta + \kappa$. Set

$$t_1 := \sup\{t \in (0, t_0] \,|\, \forall\, s \in [0, t],\ x \in \overline{G}:\ u(s, x) < \delta + \kappa\} \ \in\ (0, t_0].$$

There thus exists a point $x_1 \in \overline{G}$ with $u(t_1, x_1) = \delta + \kappa$. Since $u(t_1, x_1) > 1/\lambda_1$ and $v \ge 0$, it follows that

$$\partial_t u(t_1, x_1) - a_1 \Delta u(t_1, x_1) \le u(t_1, x_1) - \lambda_1 u(t_1, x_1)^2 < 0.$$

As in the proof of the parabolic maximum principle Proposition 3.9, we obtain a contradiction so that $u \le \kappa$.

2) Step 1) implies the inequality $(\mu - \lambda_2 v + u)v \le (\mu + \kappa)v - \lambda_2 v^2$. We can now proceed as in 1) and show that $v \le \max\{(\mu + \kappa)/\lambda_2, \|v_0\|_\infty\}$. Theorem 3.8 thus yields that $t_+ = \infty$ and that the orbit is relatively compact in E_α.

3) From now, let $u_0 \ne 0$ and $v_0 \ne 0$. Below we will need the positivity of the solution. To this aim, set $\omega = \lambda_1 \|u\|_\infty + \|v\|_\infty - 1$ for the given solution. By (3.1), the rescaled function $\varphi(t) = \mathrm{e}^{\omega t} u(t)$ then satisfies the inequality

$$\partial_t \varphi - a_1 \Delta \varphi = \varphi(\omega + 1 - \lambda_1 u - v) \ge 0$$

on $\mathbb{R}_{\ge 0} \times \overline{G}$. If $u(t_0, x_0) = 0$ for some $(t_0, x_0) \in \mathbb{R}_+ \times \overline{G}$, then $\varphi = 0$ on $[0, t_0] \times \overline{G}$ due to the strong parabolic maximum principle (see e.g. Theorems 3.5 and 3.6 in [**PWe**][5]), contradicting $u_0 \ne 0$. In the same way one treats v. We have shown that $u, v > 0$ on $\mathbb{R}_+ \times \overline{G}$.

4) We only consider strictly positive equilibria. ByBeispiel 6.11 of [**Ana4**], f has a strictly positive zero (r_*, s_*) if and only if $\lambda_2 > \mu > -1/\lambda_1$, and then

$$(r_*, s_*) = \frac{1}{1 + \lambda_1 \lambda_2} (\lambda_2 - \mu, 1 + \mu \lambda_1).$$

This condition means that the internal reproduction coefficient μ of the predators is neither too negative nor bigger than the internal 'damping' coefficient λ_2 reflecting the competion between the predators. We thus study the equilibrium $(u_*, v_*) = (r_*, s_*)\mathbb{1}$ of (3.11), assuming $\lambda_2 > \mu > -1/\lambda_1$.

5) We introduce the set $D = \{w \in E_\alpha \,|\, w > 0 \text{ on } \overline{G}\}$ and the functional

$$\Phi(w) = \int_G \Big((w_1 - r_* \ln w_1) + (w_2 - s_* \ln w_2) \Big) \mathrm{d}x = \int_G \Psi(w) \, \mathrm{d}x.$$

As in Example 3.15, one sees that $\Psi \in C^1(D, C(\overline{G}))$ with derivative $\Psi'(w)[\overline{w}] = (1 - r_*/w_1)\overline{w}_1 + (1 - s_*/w_2)\overline{w}_2$ for all $\overline{w} \in E_\alpha$. The integral is just a linear functional on $C(\overline{G})$ so that $\Phi \in C^1(D, \mathbb{R})$. Since $(u(t), v(t)) \in D$ for all $t > 0$, the chain rule and (3.11) yield

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \Phi(u, v) &= \int_G \Big((1 - \frac{r_*}{u})\, u' + (1 - \frac{s_*}{v})\, v' \Big) \mathrm{d}x \\ &= \int_G \Big((1 - \frac{r_*}{u})\, a_1 \Delta u + (1 - \frac{s_*}{v})\, a_2 \Delta v \Big) \mathrm{d}x \end{aligned}$$

[5] To apply these results, one needs the connectness of G. The proofs given in [**PWe**] can be extended to the present situation.

$$+\int_G \Big((u-u_*)(1-\lambda_1 u - v) + (v-v_*)(\mu - \lambda_2 v + u)\Big)\,\mathrm{d}x,$$

omitting the variables $(t,x) \in \mathbb{R}_+ \times G$. We denote the integrals in the last two lines by J_1 and J_2, respectively. An integration by parts further shows that

$$J_1 = -\int_G \Big(\frac{a_1 r_* \, |\nabla u|^2}{u^2} + \frac{a_2 s_* \, |\nabla v|^2}{v^2}\Big)\mathrm{d}x \le 0.$$

Using $1 = \lambda_1 r_* + s_*$ and $\mu = \lambda_2 s_* - r_*$, we compute

$$\begin{aligned} J_2 &= \int_G \Big((u-u_*)(\lambda_1 u_* + v_* - \lambda_1 u - v) + (v-v_*)(\lambda_2 v_* - u_* - \lambda_2 v + u)\Big)\mathrm{d}x \\ &= -\int_G \Big(\lambda_1 (u-u_*)^2 + \lambda_2 (v-v_*)^2\Big)\mathrm{d}x \le 0. \end{aligned}$$

Summing up, we arrive at

$$\Phi(u(t),v(t)) \le \Phi(u(s),v(s)) - \int_s^t \int_G \Big(\lambda_1 (u(\tau)-u_*)^2 + \lambda_2 (v(\tau)-v_*)^2\Big)\mathrm{d}x\,\mathrm{d}\tau.$$

for all $t \ge s > 0$. As a consequence, $\Phi(u,v)$ decays along each orbit with nonzero non-negative initial data. If it is constant on $[s,t]$, we infer $u = u_*$ and $v = v_*$ on $[s,t]$, and thus on $\mathbb{R}_{\ge 0}$. In particular, (u_*, v_*) is the only equilibrium in D and Φ is a strict Lyapunov function on D.

6) To apply Theorem 3.19, we have to check that $\omega((u_0,v_0))$ is contained in D. Let $\widehat{w}_0 = (\widehat{u}_0, \widehat{v}_0) \in \omega((u_0,v_0))$. We have $\widehat{w}_0 \ge 0$ as a uniform limit of non-negative functions. Observe that $\Psi(u,v) \ge c_*$ for anumber $c_* \in \mathbb{R}$ and all $u, v \ge 0$. A varaint of Fatou's lemma thus implies that

$$\Phi(\widehat{w}_0) = \liminf_{t\to\infty} \Phi(u(t),v(t)) \le \Phi(u_0,v_0) < \infty$$

so that $\widehat{u}_0(x) > 0$ and $\widehat{v}_0(x) > 0$ must hold for a.e. $x \in G$. Recall from Proposition 3.18, that $\widehat{w}_0 = \widehat{w}(0)$ for a solution $\widehat{w}$ of (3.1) on $\mathbb{R}$ belonging to $\omega((u_0,v_0))$. In particular, $\widehat{u}(-1)(x) > 0$ and $\widehat{v}(-1)(x) > 0$ for a.e. $x \in G$, and hence $\widehat{w}_0 > 0$ on $\overline{G}$ by step 3). This means that also the closure of $\kappa((u_0,v_0))$ in E_α belongs to D.

6) Theorem 3.19 now implies that the solution $(u(t),v(t))$ converges to (u_*,v_*) in E_α as $t \to \infty$ if $\lambda_2 > \mu > -1/\lambda_1$ and $0 \le (u_0,v_0) \in E_\alpha$ with $u_0 \ne 0$ and $v_0 \ne 0$. (Observe that the decay of the L^1 type quantity $\Phi(u(t),v(t))$ already implies the convergence of $(u(t),v(t))$ to the equilibirium in all norms $\|\cdot\|_\alpha$.) $\Diamond$

CHAPTER 4

The nonlinear Schrödinger equation

In this chapter we investigate the nonlinear Schrödinger equation

$$\begin{aligned} \mathrm{i}\partial_t u(t,x) &= -\Delta u(t,x) + \mu |u(t,x)|^{\alpha-1} u(t,x), \qquad t \in J,\ x \in \mathbb{R}^m, \\ u(0,x) &= u_0(x), \qquad x \in \mathbb{R}^m. \end{aligned} \tag{4.1}$$

Equivalently, one can write

$$\partial_t u(t,x) = \mathrm{i}\Delta u(t,x) - \mathrm{i}\mu |u(t,x)|^{\alpha-1} u(t,x).$$

Throughout it is assumed that

$$\mu \in \{-1,1\}, \quad 1 < \alpha < \frac{m+2}{(m-2)_+} =: \alpha_c, \quad J^\circ \neq \emptyset \text{ is an interval}, \ \ 0 \in J. \tag{4.2}$$

(We also allow for negative times.) Note that $\alpha_c = \infty$ for $m \in \{1,2\}$, $\alpha_c = 5$ for $m = 3$, $\alpha_c = 3$ for $m = 4$, and $\alpha_c \searrow 1$ as $m \to \infty$. The results below can be extended to more general nonlinearities, see [**Ca**], but the model equation (4.1) already gives a very good insight in the field. In some cases we also treat the *critical exponent* $\alpha = \frac{m+2}{m-2}$ for $m \geq 3$, which is much harder to study in more advanced topics. An extended survey is given in [**Ta**]. If $\mu = 1$ one has the (in some sense simpler) *defocusing case* and for $\mu = -1$ the *focusing case*.

The nonlinear Schrödinger equation (and its variants) appears in quantum field theory; e.g., in the study of so called Bose-Einstein condensates. It is also used to describe (approximately) the amplitudes of wave packages in nonlinear optics, see [**MN**]. Natural numbers α play a significant role when nonlinear material laws are given by power series. Due to symmetry constraints one then often considers odd α.

In this chapter we write $W^{k,q} = W^{k,p}(\mathbb{R}^m)$ for $k \in \mathbb{N}_0$ and $q \in [1,\infty]$, $H^k = W^{k,2}$, and similarly for other function spaces on $\mathbb{R}^m$, where $W^{0,q} = L^q$. We motsly drop the domain $\mathbb{R}^m$ in integrals over $\mathbb{R}^m$. The norm on $W^{k,q}$ is denoted by $\|v\|_{k,q}$ and $\|v\|_{0,q} = \|v\|_q$.

4.1. Preparations

We start with a few (more or less explicit) special solutions of the differential equation in (4.1), which illustrate some phenomena occuring in the nonlinear Schrödinger equation. In the exercises we discuss symmetries and scaling properties of (4.1) which allow to construct new solutions out of a given one.

Example 4.1. We want to to construct *plane waves*. Given $\xi \in \mathbb{R}^m \setminus \{0\}$, we look for a function $\phi : \mathbb{R} \to \mathbb{C}$ such that the map $w_\xi(t,x) := \phi(t)\mathrm{e}^{\mathrm{i}\xi\cdot x}$ solves (4.1). We compute $\partial_t w_\xi(t,x) = \phi'(t)\mathrm{e}^{\mathrm{i}\xi\cdot x}$, $\partial_k u(t,x) = \mathrm{i}\xi_k w_\xi(t,x)$, and

$\Delta w_\xi(t,x) = -|\xi|^2 u(t,x)$ for $(t,x) \in \mathbb{R}^{1+m}$. Since $|w_\xi| = |\phi|$, the map w_ξ satisfies (4.1) if and only if

$$\phi'(t) = -\mathrm{i}(|\xi|^2 + \mu|\phi(t)|^{\alpha-1})\phi(t).$$

This ordinary differential equation can be solved leading to the plane wave

$$w_\xi(t,x) = a\mathrm{e}^{\mathrm{i}\xi\cdot x}\mathrm{e}^{-\mathrm{i}|\xi|^2 t}\mathrm{e}^{-\mathrm{i}\mu|a|^{\alpha-1}t}, \qquad (t,x) \in \mathbb{R}^{1+m}.$$

Here $a := \phi(0)$, $|a| = |w_\xi|$ is the amplitude, ξ is the wave vector, and $\omega = |\xi|^2 + \mu|a|^{\alpha-1}$ is proportional to the (temporal) frequency. Observe that the summand $|\xi|^2$ in ω comes from $-\Delta$, whereas $\mu|a|^{\alpha-1}$ is the contribution of the nonlinear part which depends on $|a|$. For $\mu = 1$ these two terms add up and so the nonlinearity increases the frequency and thus the time oscillation, whereas for $\mu = -1$ the oscillations partly cancel. We further have

$$w_\xi(t,x) = a \iff \frac{1}{|\xi|}\xi \cdot x = \Big(|\xi| + \mu\frac{|a|^{\mu-1}}{|\xi|}\Big)t =: v(\xi)t.$$

This plane moves along its unit normal vector $\frac{1}{|\xi|}\xi$ with the *phase velocity* $v(\xi)$ which depends on the length of the wave vector.

This behavior is called *dispersion*. Dispersion causes plane waves with different wave vectors ξ_j (say, having the same direction $\frac{1}{|\xi_j|}\xi_j = \eta$) to spread out in space as time evolves. This effect will be stronger in the defocusing case $\mu = 1$, since again the nonlinear effect adds to the linear one. In the case $\mu = -1$ the waves exhibit less dispersion, they longer stay *focused*. This explains the terminology of the two cases. ◊

In focusing case one can construct standing waves, which is not possible in the defocusing case, see Theorem 7.3.1 in [**Ca**]. In the latter situation dispersion destroys such persistent patterns.

Example 4.2. We look for a *standing wave* for (4.1); i.e., a solution given by

$$u_\omega(t,x) = \mathrm{e}^{\mathrm{i}\omega t}\varphi_\omega(x), \qquad (t,x) \in \mathbb{R}^{1+m},$$

for a frequency $\omega \in \mathbb{R}$ and a wave profile $\varphi_\omega \in H^2 \setminus \{0\}$. Such a function u_ω solves (4.1) if and only if

$$\mathrm{i}\,\partial_t u_\omega(t,x) = -\omega\mathrm{e}^{\mathrm{i}\omega t}\varphi_\omega(x) = -\mathrm{e}^{\mathrm{i}\omega t}\Delta\varphi_\omega(x) + \mu|\varphi_\omega(x)|^{\alpha-1}\mathrm{e}^{\mathrm{i}\omega t}\varphi_\omega(x),$$

$$-\Delta\varphi_\omega + \omega\varphi_\omega = -\mu|\varphi_\omega|^{\alpha-1}\varphi_\omega \tag{4.3}$$

on $\mathbb{R}^m$. The resulting semilinear elliptic problem for φ_ω can be solved in H^2 in the *focusing case* $\mu = -1$ if $\omega > 0$ and $1 < \alpha < \alpha_c$. One can even prove that $\varphi_\omega > 0$ and that φ_ω and $\nabla\varphi_\omega$ decay exponentially. (See §8.1 of [**Ca**] and the references given there.)

For $m = 1$, $\omega = 1$ and $\mu = -1$ one finds an explicit solution, namely

$$\varphi_1(x) = \left(\frac{\sqrt{\beta+1}}{\cosh(\beta x)}\right)^{\frac{1}{\beta}}, \qquad x \in \mathbb{R},$$

where we set $\beta = \frac{\alpha-1}{2}$. For $\alpha = 3$, one has $\beta = 1$ and $\varphi_1 = \frac{\sqrt{2}}{\cosh}$. ◊

These standing waves lead to a *blow-up* solution in the focusing case $\mu = -1$ with $\alpha = 1 + \frac{4}{m}$. Blow up also occurs for $\alpha \in [1 + \frac{4}{m}, \alpha_c)$ as shown in Theorem 6.5.10 of [**Ca**]. (The basic idea of the proof is similar to that of Proposition 1.21.) On the other hand, Theorem 4.21 below yields global existence if $\mu = 1$ and $\alpha < \alpha_c$ and if $\mu = -1$ and $\alpha < 1 + \frac{4}{m}$. Hence, stronger dispersion or weaker nonlinear effects prevent blow up.

Example 4.3. Let $\mu = -1$, $\alpha = 1 + \frac{4}{m} < \alpha_c$, $\omega > 0$, and take $0 < \varphi_\omega \in H^2$ as in Example 4.2. For $t \in [0, 1)$ and $x \in \mathbb{R}^m$ we define

$$u(t, x) = (\mathrm{i}(t-1))^{-\frac{m}{2}} \, \mathrm{e}^{\mathrm{i}\frac{|x|^2}{4(t-1)}} \, \mathrm{e}^{-\mathrm{i}\frac{\omega}{(t-1)}} \varphi_\omega\big(\tfrac{1}{t-1}x\big).$$

One can directly (and tediously) check that u solves (4.1), cf. p. 115 in [**Ta**]. Moreover, the substition $y = \frac{1}{t-1}x$ yields

$$\begin{aligned}
\|u(t)\|_2^2 &= \int_{\mathbb{R}^m} |t-1|^{-m} \, |\varphi_\omega\big(\tfrac{1}{t-1}x\big)|^2 \, \mathrm{d}x = \|\varphi_\omega\|_2^2, \\
\||\nabla u(t)|\|_2^2 &= \int_{\mathbb{R}^m} |t-1|^{-m} \left| \tfrac{\mathrm{i}}{2(t-1)} \varphi_\omega\big(\tfrac{1}{t-1}x\big)x + \tfrac{1}{t-1} \nabla \varphi_\omega\big(\tfrac{1}{t-1}x\big) \right|^2 \mathrm{d}x \\
&\geq \int_{\mathbb{R}^m} |t-1|^{-m-2} \left| \nabla \varphi_\omega\big(\tfrac{1}{t-1}x\big) \right|^2 \mathrm{d}x = |t-1|^{-2} \, \||\nabla \varphi_\omega|\|_2^2.
\end{aligned}$$

As a result, this solution explodes in H^1 as $t \to 1^-$ though it stays bounded in L^2 and the initial value $u(0)$ belongs to H^2 by the properties of φ_ω mentioned in Example 4.2. ◊

As we will see below, solutions of (4.1) preserve the L^2-norm (interpreted as 'mass') and the 'energy'. These conservation laws are the basic tools to study the longterm behavior of solutions and in particular to understand the blow-up behavior. To define the energy, we need some preparations.

Let $\alpha \in (1, \alpha_c]$ if $m \neq 2$ and $\alpha \in (1, \alpha_c)$ if $m = 2$. Sobolev's embedding shows

$$\begin{aligned}
H^1 &\hookrightarrow L^{1+\alpha}, \quad \|v\|_{1+\alpha} \leq C_{\mathrm{So}} \|v\|_{1,2}, \\
H^2 &\hookrightarrow L^{2\alpha}, \qquad \|w\|_{2\alpha} \leq C_{\mathrm{So}} \|w\|_{2,2},
\end{aligned} \tag{4.4}$$

for all $v \in H^1$ and $w \in H^2$. See Theorem 3.17 in [**ST**] and observe that

$$\begin{aligned}
m \geq 3: &\quad 1 + \alpha_c = 1 + \frac{m+2}{m-2} = \frac{2m}{m-2} \quad \Longrightarrow \quad 1 - \frac{m}{2} = \frac{2-m}{2} \geq -\frac{m}{1+\alpha}, \\
m \geq 4: &\quad 2\alpha_c < \frac{2m}{m-4} \quad \Longrightarrow \quad 2 - \frac{m}{2} = \frac{4-m}{2} > -\frac{m}{2\alpha}.
\end{aligned}$$

For the above α, we can define the 'energy' $\mathcal{E} : H^1 \to \mathbb{R}$ by

$$\mathcal{E}(v) = \frac{1}{2} \int_{\mathbb{R}^m} |\nabla v|^2 \, \mathrm{d}x + \frac{\mu}{\alpha+1} \int_{\mathbb{R}^m} |v|^{\alpha+1} \, \mathrm{d}x \tag{4.5}$$

for $v \in H^1 \hookrightarrow L^{1+\alpha}$. We stress that $2\mathcal{E}(v) + \|v\|_2^2 \geq \|v\|_{1,2}^2$ in the defocusing case $\mu = 1$, but that the energy may become negative if $\mu = -1$. These properties lead to global existence in the first case and the occurence of blow up in the second one (if $\alpha \geq 1 + \frac{4}{m}$), as noted above.

The embedding (4.4), the chain rule and Corollary 1.18 yield $\mathcal{E} \in C^1(H^1, \mathbb{R})$ with derivative

$$\mathcal{E}'(v)w = \operatorname{Re} \int_{\mathbb{R}^m} \big(\nabla v \cdot \nabla \overline{w} + \mu |v|^{\alpha-1} v \overline{w}\big) \, \mathrm{d}x \qquad \text{for} \quad v, w \in H^1. \tag{4.6}$$

We next show that regular solutions preserve the L^2-norm and the energy.

REMARK 4.4. Let $u \in C(J, H^2) \cap C^1(J, L^2)$ solve (4.1) on J. Let $t \in J$.

a) From equation (4.1) and an integration by parts we infer

$$\begin{aligned} \partial_t \|u(t)\|_2^2 &= 2 \operatorname{Re} \int \partial_t u(t) \, \overline{u(t)} \, \mathrm{d}x = 2 \operatorname{Re} \mathrm{i} \int (\Delta u(t) - \mu |u(t)|^{\alpha-1} u(t)) \, \overline{u(t)} \, \mathrm{d}x \\ &= 2 \operatorname{Im} \int (|\nabla u(t)|^2 + \mu |u(t)|^{\alpha+1}) \, \mathrm{d}x = 0, \\ \|u(t)\|_2 &= \|u_0\|_2. \end{aligned}$$

b) We cannot directly treat $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(u(t))$ by the chain rule and (4.6) as u may not belong to $C^1(J,H^1)$. To regularize, we use the Yosida approximations $R_n = nR(n, \Delta)$ which tend to I strongly in L^2 and $H^2 \hookrightarrow L^{2\alpha}$ (and hence uniformly on compact sets) as $n \to \infty$, see (4.4), Lemma 1.23 of [**EE**] and Lemma 4.5. Since $R_n u \in C^1(J, H^2)$, formula (4.6) and an integration by parts imply

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(R_n u(t)) &= \operatorname{Re} \int \big(\nabla R_n u(t) \cdot \nabla \partial_t R_n \overline{u}(t) + \mu |R_n u(t)|^{\alpha-1} R_n u(t) \partial_t R_n \overline{u}(t)\big) \mathrm{d}x \\ &= \operatorname{Re} \int \big(-\Delta R_n u(t) + \mu |R_n u(t)|^{\alpha-1} R_n u(t)\big) R_n \partial_t \overline{u}(t) \, \mathrm{d}x. \end{aligned}$$

Using the above mentioned properties of R_n and Hölder's inequality, one can let $n \to \infty$ locally uniformly in t on the right-hand side. Hence, $\mathcal{E}(u)$ is continuously differentiable with

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(u(t)) &= \operatorname{Re} \int \big(-\Delta u(t) + \mu |u(t)|^{\alpha-1} u(t)\big) \partial_t \overline{u}(t) \, \mathrm{d}x \\ &= \operatorname{Re} \int \mathrm{i} \, \partial_t u(t) \partial_t \overline{u}(t) \, \mathrm{d}x = 0, \\ \mathcal{E}(u(t)) &= \mathcal{E}(u_0), \end{aligned}$$

where we also employed (4.1). ♢

The above results indicate that one can control the H^1-norm of solutions at least in the defocusing case. We also note that there are no conservation laws involving second space derivatives, in general. Aiming at global existence, we are thus looking for a local wellposedness theory for the nonlinear Schrödinger equation (4.1) in which we can derive a blow-up condition involving only the H^1-norm. For such a theory it is reasonable to consider solutions which are continous with values in H^1 (for which one must then also show the above conservation laws). Hence, one has to extend the Laplacian to H^1. We first develop the necessary functional analytic framework.

To that purpose, we define the negative Sobolev spaces on $\mathbb{R}^m$ by

$$W^{-k,r} = W^{-k,r}(\mathbb{R}^m) := W^{k,r'}(\mathbb{R}^m)^*, \qquad H^{-k} := (H^k)^*$$

for $k \in \mathbb{N}$ and $r \in (1,\infty]$. The norm of the Banach space $W^{-k,r}$ is given by $\|\varphi\|_{-k,r} = \sup_{\|v\|_{k,r'}=1} |\varphi(v)|$. If $r \in (1,\infty)$, then $W^{-k,r}$ is separable and reflexive with a dual space isomorphic to $W^{k,r'}$. (See Proposition 3.7 of [**ST**] and Proposition 5.28 of [**FA**].)

Let $\alpha \in (1,\alpha_c]$ if $m \ge 3$ and $\alpha \in (1,\alpha_c)$ if $m \in \{1,2\}$. We set $q = \alpha+1$. By (4.4) and Proposition 4.13 of [**FA**], the inclusion $I : H^1 \to L^q$ is continuous and injective with dense range. Proposition 5.46 of [**FA**] thus yields that

$$I^* : L^{q'} \to H^{-1} \quad \text{is continuous and injective with dense range.} \tag{4.7}$$

In this context we note that

$$q = \alpha+1, \qquad q' = \frac{q}{q-1} = \frac{q}{\alpha} = 1 + \frac{1}{\alpha} \ge 1 + \frac{1}{\alpha_c} \qquad (> \quad \text{if } m \le 2). \tag{4.8}$$

Here and below, for $r \in [1,\infty)$ we identify $L^{r'}$ with $(L^r)^*$ via

$$\forall\, f \in L^r,\ g \in L^{r'} : \quad \langle f, g\rangle_{L^r \times (L^r)^*} = \int fg\,\mathrm{d}x.$$

(We do *not* identify H^1 with $(H^1)^* = H^{-1}$.) In this way a function $g \in L^{r'}$ with $r' \in [1+\frac{1}{\alpha_c}, 2]$ ($r' \in (1+\frac{1}{\alpha_c}, 2]$ if $m \le 2$) induces a functional $\varphi_g = I^* g$ acting on H^1 via

$$\forall\, v \in H^1 : \quad \langle v, \varphi_g\rangle_{H^1 \times H^{-1}} = \langle Iv, g\rangle_{L^r \times (L^r)^*} = \int vg\,\mathrm{d}x.$$

We set $F(v) = -\mathrm{i}\mu|v|^{\alpha-1}v$ for $v \in H^1$. Lemma 1.17, Corollary 1.18, (4.4), (4.7) and (4.8) yield that

$$F \in C^1_{\mathbb{R}}(L^{\alpha+1}, L^{\frac{\alpha+1}{\alpha}}) \cap C^1_{\mathbb{R}}(H^1, H^{-1}), \quad \|F'(v)\|_{\mathcal{B}(L^{\alpha+1}, L^{\frac{\alpha+1}{\alpha}})} \le \alpha \|v\|_{\alpha+1}^{\alpha-1}, \tag{4.9}$$

$$F \in C^1_{\mathbb{R}}(H^2, L^2), \quad \|F'(w)\|_{\mathcal{B}(H^2,L^2)} \le \alpha C_{\mathrm{So}}^{\alpha-1} \|w\|_{2,2}^{\alpha-1}, \tag{4.10}$$

for all $v \in L^{\alpha+1}$ and $w \in H^2$. In particular, the derivatives of F are bounded on bounded sets of H^1, respectively H^2.

We now turn our attention to the Laplacian. We first extend the partial derivative $\partial_j : H^1 \to L^2$ to a bounded map $\partial_j : L^2 \to H^{-1}$ via

$$\forall\, v \in H^1 : \quad \langle v, \partial_j u\rangle_{H^1 \times H^{-1}} = -\int \partial_j v\, u\,\mathrm{d}x$$

for $j \in \{1,\dots,m\}$ and $u \in L^2$. In this way we obtain bounded extensions $\partial_{jk}, \Delta : H^1 \to H^{-1}$. As in Example 1.52 of [**EE**] one shows the invertibility of $I - \Delta : H^1 \to H^{-1}$ using the quadratic form

$$a(u,v) = \int (u\overline{v} + \nabla u \cdot \nabla \overline{v})\,\mathrm{d}x \qquad \text{for} \quad u, v \in H^1.$$

For $u \in H^1$ we then compute

$$\begin{aligned} \|(I-\Delta)u\|_{-1,2} &= \sup_{\|v\|_{1,2}=1} \left|\langle \overline{v}, u - \Delta u\rangle_{H^1}\right| = \sup_{\|v\|_{1,2}=1} \left|\int (u\overline{v} + \nabla u \cdot \nabla \overline{v})\,\mathrm{d}x\right| \\ &= \sup_{\|v\|_{1,2}=1} \left|(u|v)_{H^1}\right| = \|u\|_{1,2}. \end{aligned} \tag{4.11}$$

As a result, $I-\Delta : H^1 \to H^{-1}$ is an isometric isomorphism and H^{-1} is a Hilbert space with the scalar product $(\varphi|\psi)_{H^{-1}} = \big((I-\Delta)^{-1}\varphi\big|(I-\Delta)^{-1}\psi\big)_{H^1}$.

By Example 1.46 of [**EE**] the Laplace operator Δ with domain H^2 is selfadjoint in L^2, and hence $\mathrm{i}\Delta$ is skewadjoint in L^2. Stone's Theorem 1.45 in [**EE**] thus shows that $\mathrm{i}\Delta$ generates a unitary C_0-group $T(\cdot)$ on L^2, which is called the *free Schrödinger group*. By the next result, this group looks like the diffusion semigroup with 'imaginary time' $\mathrm{i}t$. The resulting representation formula implies that $T(t)v \in C^\infty(\mathbb{R}^m)$ if $v \in L^2$ has compact support. However, there is no smoothing effect in the full space L^2 since $T(t)$ is bijective. We further extend $T(\cdot)$ to H^{-1} and show further regularity properties. Extensions and restrictions of $T(\cdot)$ and Δ are denoted by the same symbols.

LEMMA 4.5. *a) For $k \in \mathbb{Z}$ with $k \geq -1$, the operator Δ with $\mathrm{D}(\Delta) = H^{k+2}$ is selfadjoint and dissipative in H^k. The unitary group generated by $\mathrm{i}\Delta$ on H^k is an extension, respectively restriction, of $T(\cdot)$ on L^2. Moreover, $\partial_j T(t)u = T(t)\partial_j u$ in H^{-1} for all $u \in L^2$ and $j \in \{1,\dots,m\}$.*

b) For $v \in L^1 \cap L^2$, $t \in \mathbb{R}\setminus\{0\}$, and $x \in \mathbb{R}^m$ we have

$$T(t)v(x) = \frac{1}{(4\pi \mathrm{i}t)^{m/2}} \int_{\mathbb{R}^m} \mathrm{e}^{\mathrm{i}\frac{|x-y|^2}{4t}}\, v(y)\,\mathrm{d}y. \tag{4.12}$$

PROOF. 1) Let $k \in \mathbb{N}_0$ and $\mathcal{F} : L^2 \to L^2$; $\mathcal{F}u = \hat{u}$, be the (unitary) Fourier transform. Formula (1.24) in [**EE**] yields the characterization $H^k = \{u \in L^2 \mid |\xi|_2^k\, \hat{u} \in L^2\}$. As in Example 1.46 of [**EE**] one can then check that Δ with $\mathrm{D}(\Delta) = H^{k+2}$ is selfadjoint and dissipative in H^k, so that $\mathrm{i}\Delta$ generates a unitary C_0-group on H^k by Theorem 1.45 in [**EE**]. The uniqueness of the Cauchy problem implies that the groups on H^k extend each other.

2) For $\lambda > 0$ the operators $(I-\Delta)^{-1}$ and $R(\lambda, \mathrm{i}\Delta)$ commute on H^k. The resolvent approximation in Corollary 3.24 of [**EE**] then shows that also $(I-\Delta)^{-1}$ and $T(t)$ commute on H^k for $t \in \mathbb{R}$. Using the isomorphism $I-\Delta : H^1 \to H^{-1}$, see (4.11), we can thus extend $T(\cdot)$ to a unitary C_0-group on H^{-1} which is generated by $\mathrm{i}\Delta$ with domain H^1. Theorem 1.45 of [**EE**] now yields that Δ is selfadjoint in H^{-1}. If we restrict and extend the contraction semigroup $\mathrm{e}^{t\Delta}$ in the same way, the Hille–Yosida Theorem 1.27 in [**EE**] implies the dissipativity of Δ on H^{-1}.

3) Using $\partial_j \Delta = \Delta \partial_j$ on H^3, we compute $\partial_j R(\lambda, \mathrm{i}\Delta) = R(\lambda, \mathrm{i}\Delta)\partial_j$ on H^1 for $j \in \{1,\dots,m\}$ and $\lambda > 0$. As in step 2) one concludes that $\partial_j T(t) = T(t)\partial_j$ on H^1 for $t \in \mathbb{R}$. The last part of assertion a) then follows by approximation.[1]

4) The right-hand side of (4.12) defines a bounded map from L^1 to L^∞ for $t \neq 0$. Moreover, C_c^∞ is dense in $L^1 \cap L^2$ with respect to the sum norm $\|\cdot\|_1 + \|\cdot\|_2$ by Proposition 4.13 of [**FA**]. It thus suffices to show (4.12) for $v \in C_c^\infty$.

By (1.23) and (1.24) in [**EE**], we have $\mathcal{F}(\Delta\varphi) = -|\xi|^2 \mathcal{F}\varphi$ for $\varphi \in H^2$. Let $t \in \mathbb{R}$ and $x \in \mathbb{R}^m$. For $v \in C_c^\infty$, the map $u = T(\cdot)v$ belongs to $C(\mathbb{R}, H^2) \cap C^1(\mathbb{R}, L^2)$ and satisfies $u'(t) = \mathrm{i}\Delta u(t)$. It is then easy to check that $\widehat{u} \in C^1(\mathbb{R}, L^2)$ and

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\, \widehat{u}(t) = \mathcal{F}(\mathrm{i}\Delta u(t)) = -\mathrm{i}|\xi|^2 \widehat{u}(t), \qquad \widehat{u}(0) = \widehat{v}.$$

[1] The following steps of the proof were omitted in the lectures.

Solving this ordinary differential equation for fixed $\xi \in \mathbb{R}^m$, we arrive at

$$\widehat{u}(t,\xi) = \mathrm{e}^{-\mathrm{i}t|\xi|^2}\widehat{v}(\xi) = \gamma_{\mathrm{i}t}(\xi)\widehat{v}(\xi),$$

where $\gamma_z(\xi) := \mathrm{e}^{-z|\xi|^2}$ for $\xi \in \mathbb{R}^m$ and $z \in \mathbb{C}$. Since $\gamma_{\mathrm{i}t}$ is bounded, we deduce $u(t) = \mathcal{F}^{-1}(\gamma_{\mathrm{i}t}\widehat{v})$.

5) As $\gamma_{\mathrm{i}t}$ is not the Fourier transform of an L^1-function, we cannot directly apply the convolution formulas in (1.22) of [**EE**]. Instead we employ the regularization $m_\varepsilon(t) = \gamma_{\mathrm{i}t+\varepsilon} \in L^1 \cap L^2$ for $\varepsilon > 0$. Using the inversion formula for $\mathcal{F}$, see p.36 of [**EE**], we first compute

$$[\mathcal{F}^{-1}m_\varepsilon(t)](x) = (2\pi)^{-\frac{m}{2}}\int \mathrm{e}^{\mathrm{i}x\cdot\xi}\mathrm{e}^{-\mathrm{i}t|\xi|^2}\mathrm{e}^{-\varepsilon|\xi|^2}\,\mathrm{d}\xi = \prod_{k=1}^{m}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}x_k\xi_k-(\mathrm{i}t+\varepsilon)\xi_k^2}\,\mathrm{d}\xi_k.$$

By means of complex contour integrals, we establish in step 6) the identity

$$\int_{\mathbb{R}}\mathrm{e}^{-(\mathrm{i}t+\varepsilon)s^2}\mathrm{e}^{\mathrm{i}x_k s}\,\mathrm{d}s = \sqrt{\frac{\pi}{\mathrm{i}t+\varepsilon}}\,\mathrm{e}^{\frac{-x_k^2}{4(\mathrm{i}t+\varepsilon)}} \tag{4.13}$$

for $t \neq 0$. Hence, $\mathcal{F}^{-1}m_\varepsilon$ belongs to L^1.

We can thus apply (1.22) of [**EE**] to $m_\varepsilon\widehat{v}$. Since $|m_\varepsilon| \le \mathbb{1}$ and $m_\varepsilon(t)$ converges pointwise to $\gamma_{\mathrm{i}t}$, Lebesgue's theorem and the continuity of $\mathcal{F}^{-1}$ now yield

$$u(t) = \mathcal{F}^{-1}(\gamma_{\mathrm{i}t}\widehat{v}) = \lim_{\varepsilon\to 0}\mathcal{F}^{-1}(m_\varepsilon(t)\widehat{v}) = \lim_{\varepsilon\to 0}(2\pi)^{-\frac{m}{2}}(\mathcal{F}^{-1}m_\varepsilon(t)) * v,$$
$$u(t,x) = \frac{1}{(4\pi)^{\frac{m}{2}}}\lim_{\varepsilon\to 0}\int\frac{1}{(\mathrm{i}t+\varepsilon)^{\frac{m}{2}}}\mathrm{e}^{-\frac{|x-y|^2}{4(\mathrm{i}t+\varepsilon)}}v(y)\,\mathrm{d}y.$$

For fixed $t \neq 0$ and $x \in \mathbb{R}^m$, Lebesgue's theorem allows to let $\varepsilon \to 0$ in the integral since $v \in C_c^\infty$, and hence (4.12) holds.

6) It remains to check (4.13), where $x := x_k \in \mathbb{R}$, $t \neq 0$ and $\varepsilon > 0$ are fixed. We set $\zeta = \frac{\mathrm{i}x}{2(\mathrm{i}t+\varepsilon)}$ and compute

$$\int_{\mathbb{R}}\mathrm{e}^{-(\mathrm{i}t+\varepsilon)s^2}\mathrm{e}^{\mathrm{i}xs}\,\mathrm{d}s = \mathrm{e}^{-\frac{x^2}{4(\mathrm{i}t+\varepsilon)}}\int_{\mathbb{R}}\mathrm{e}^{-(\mathrm{i}t+\varepsilon)(s-\zeta)^2}\,\mathrm{d}s = \mathrm{e}^{-\frac{x^2}{4(\mathrm{i}t+\varepsilon)}}\int_{\mathbb{R}-\zeta}\mathrm{e}^{-(\mathrm{i}t+\varepsilon)z^2}\,\mathrm{d}z.$$

Writing $f(z) = \mathrm{e}^{-(\mathrm{i}t+\varepsilon)z^2}$ and $I = \int_{\mathbb{R}-\zeta} f\,\mathrm{d}z$, we have to prove $I = \big(\frac{\pi}{\mathrm{i}t+\varepsilon}\big)^{1/2}$.

To this purpose, we consider the counterclockwise oriented contour $\Gamma_n = \Gamma_n^{\mathrm{b}} \cup \Gamma_n^{\mathrm{r}} \cup \Gamma_n^{\mathrm{t}} \cup \Gamma_n^{\mathrm{l}}$, where

$$\Gamma_n^{\mathrm{b}} = [-n,n], \qquad \Gamma_n^{\mathrm{r}} = \{z = n + \tau\zeta \mid -1 \le \tau \le 0\},$$
$$\Gamma_n^{\mathrm{t}} = \{z = \tau n - \zeta \mid -1 \le \tau \le 1\}, \qquad \Gamma_n^{\mathrm{l}} = \{z = -n + \tau\zeta \mid -1 \le \tau \le 0\}$$

and $n \in \mathbb{N}$. Cauchy's theorem shows $\int_{\Gamma_n} f\,\mathrm{d}z = 0$. There is a constant $c = c(\varepsilon,t,x) > 0$ such that

$$\sup_{z\in\Gamma_n^{\mathrm{r}}\cup\Gamma_n^{\mathrm{l}}}|\mathrm{e}^{-(\mathrm{i}t+\varepsilon)z^2}| \le \mathrm{e}^{-\varepsilon n^2}\mathrm{e}^{cn},$$

and hence $\int_{\Gamma_n^j} f\,\mathrm{d}z \to 0$ as $n \to \infty$ for $j \in \{\mathrm{l},\mathrm{r}\}$. By a similar estimate one sees that $\int_{\Gamma_n^{\mathrm{t}}} f\,\mathrm{d}z$ tends to I. Letting $n \to \infty$, we then deduce that $I = 2\int_0^\infty f(s)\,\mathrm{d}s$.

Let $t > 0$ and set $\beta = \frac{1}{2}\arg(\mathrm{i}t+\varepsilon) \in (0, \frac{\pi}{4})$. Since $|\sqrt{\mathrm{i}t+\varepsilon}|\,\mathrm{e}^{\mathrm{i}\beta} = \sqrt{\mathrm{i}t+\varepsilon}$, the substitution $z = \sqrt{\mathrm{i}t+\varepsilon}\,s$ yields

$$I = \frac{2}{\sqrt{\mathrm{i}t+\varepsilon}} \int_{\mathrm{e}^{\mathrm{i}\beta}\mathbb{R}_{\geq 0}} \mathrm{e}^{-z^2}\,\mathrm{d}z.$$

To evaluate this integral, we use the contour

$$\Gamma_n = \mathrm{e}^{\mathrm{i}\beta}[0,n] \cup \{n\mathrm{e}^{\mathrm{i}\sigma} \,|\, 0 \leq \sigma \leq \beta\} \cup [0,n]$$

with positive orientation. Since $|\mathrm{e}^{-n^2\mathrm{e}^{2\mathrm{i}\sigma}}| \leq \mathrm{e}^{-n^2\cos 2\beta}$ for $\sigma \in [0,\beta]$, Cauchy's theorem now implies

$$I = \frac{2}{\sqrt{\mathrm{i}t+\varepsilon}} \int_0^\infty \mathrm{e}^{-s^2}\,\mathrm{d}s = \frac{\sqrt{\pi}}{\sqrt{\mathrm{i}t+\varepsilon}}\,,$$

as asserted. The case $t < 0$ is treated in the same way. □

This representation formula allows to describe the *dispersive* behavior of $T(t)$ in quantitative way. The next corollary says that $T(t)$ flattens initial data in $L^1 \cap L^2$ which become bounded immediately and then tend to 0 in all L^p–norms for $p > 2$ as $t \to \infty$. Since the L^2–norm is preserved, local concentrations of $T(t)v$ must be pushed towards infinity in $\mathbb{R}^m$.

Corollary 4.6. *Let $q \in [2,\infty]$. Then $T(t)$ extends from $L^1 \cap L^2$ to an operator in $\mathcal{B}(L^{q'}, L^q)$ for all $t \in \mathbb{R}\setminus\{0\}$, with norm less or equal $(4\pi|t|)^{m(\frac{1}{q}-\frac{1}{2})}$.*

Proof. By (4.12), $T(t)$ maps $(L^1 \cap L^2, \|\cdot\|_1)$ into L^∞ with norm less or equal $(4\pi|t|)^{-m/2}$. Moreover, it has norm 1 as an operator on L^2. Let $q \in [2,\infty]$. The Riesz–Thorin interpolation theorem then shows that we can extend $T(t)$ to an operator from $L^{q'}$ to L^q with norm less or equal $(4\pi|t|)^{-m/2(1-2/q)} = (4\pi|t|)^{m/q-m/2}$. See Theorem 2.26 in [**FA**] with $\theta := 2/q \in (0,1)$ and

$$\frac{1}{q'} = \frac{1-\theta}{1} + \frac{\theta}{2}, \qquad \frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2}.$$ □

4.2. Strichartz estimates

We first introduce our solution concepts, recalling that $F(v) = -\mathrm{i}\mu|v|^{\alpha-1}v$ induces a map $F \in C^1(H^1, H^{-1}) \cap C^1(H^2, L^2)$ by (4.9) and (4.10).

Definition 4.7. *Let $k \in \{1,2\}$ and (4.2) be true, where we also allow that $\alpha = \alpha_c$ if $m \geq 3$. A function $u \in C(J, H^k) \cap C^1(J, H^{k-2})$ satisfying (4.1) in H^{k-2} is called H^k–*solution *(on J).*

Since F is only defined on subspaces of H^{-1} or L^2, we cannot use the results of the first chapter to solve the nonlinear Schrödinger equation (4.1). We still want to follow the approach based on mild solutions and Duhamel's formula

$$u(t) = T(t)u_0 - \mathrm{i}\mu \int_0^t T(t-s)(|u(s)|^{\alpha-1}u(s))\,\mathrm{d}s, \qquad t \in J. \tag{4.14}$$

In Chapter 3, this was possible since an analytic semigroup maps into interpolation spaces of its generator on which F was defined. As the free Schrödinger group is bijective, this does not work here. On the other hand, in (4.14) we do not need regularization of $T(t-s)$ to treat the integral, it only has to improve integrability to counteract the power nonlinearity.

Corollary 4.6 already says that $T(t)$ improves integrability, though the corresponding norms blow up as $t \to 0$. Using also L^p-spaces in time, one can describe this dispersive behavior in a more convenient way, and it is possible to deal also with inhomogeneities as needed in (4.14). The resulting *Strichartz estimates* in Theorem 4.10 are crucial for the following sections.

Before proving the Strichartz estimates, we collect some tools needed to state and prove them. We refer to pp.92–93 of [**EE**] for a few remarks about Banach-space valued integration and Bochner-Lebesgue spaces. The basic results (up to Fubini's theorem) are analogous to the scalar-valued case. Moreover, for an interval $J \subset \mathbb{R}$, $k \in \mathbb{N}_0$ and $1 \le p, q < \infty$ (and $k \in \mathbb{Z}$ if $1 < q < \infty$) we have

$$L^p(J, W^{k,q})^* = L^{p'}(J, W^{-k,q'}) \quad \text{via } \langle f, g\rangle_{L^pW^{k,q}} = \int_J \langle f(t), g(t)\rangle_{W^{k,q}} \,\mathrm{d}t \quad (4.15)$$

for $f \in L^p(J, W^{k,q})$ and $g \in L^{p'}(J, W^{-k,q'})$, see Corollary 1.3.22 in [**HNVW**]. The space $L^p(J, W^{k,q})$ is thus reflexive for $p, q \in (1, \infty)$. One can show that $L^p(J, W^{k,q})$ is separable if $p, q \in [1, \infty)$, using the density of simple functions $u : J \to W^{k,q}$. For $f \in L^1_{\mathrm{loc}}(J, H^{-1})$, we define the one-sided convolution

$$(T *_+ F(u))(t) = \int_0^t T(t-s) f(s) \,\mathrm{d}s, \qquad t \in J,$$

by means of Lemma 4.5. We need a density result for Bochner spaces.[2]

Lemma 4.8. *Let $J \subseteq \mathbb{R}$ be open, $k \in \mathbb{N}_0$, and $1 \le p, q < \infty$. Then all spaces $C_c^\infty(J, W^{l,r})$ with $l \in \mathbb{N}_0$ and $r \in [1, \infty]$ are dense in $L^p(J, W^{k,q})$.*

Proof. Fix $f \in L^p(J, W^{k,q})$, $l \in \mathbb{N}_0$ and $r \in [1, \infty]$. Take $\varepsilon > 0$.

1) The standard mollifiers $G_{1/n}$ with $n \in \mathbb{N}$ are uniformly bounded in $W^{k,q}$ and tend strongly to I as $n \to \infty$. The same is true for the cut-off map $v \mapsto \phi_k v$ where $\phi_k(x) = \phi(|x|/k)$ for $k \in \mathbb{N}$ and $\phi \in C^\infty(\mathbb{R})$ with $0 \le \phi \le 1$, $\phi = 1$ on $[-1, 1]$ and $\operatorname{supp}\phi \subseteq (-2, 2)$. Using dominated convergence, we can thus fix indices $n, k \in \mathbb{N}$ and a function $g = G_{1/n}(\phi_k f)$ in $L^p(J, W^{l,r})$ such that $\|f - g\|_{L^p(J,W^{k,q})} \le \varepsilon$. (Compare Theorem 4.21 in [**FA**].)

2) Let $J_n \subseteq \overline{J_n} \subseteq J_{n+1} \subseteq J$ be open bounded intervals whose union is J. Pick maps $\psi_n \in C_c(J)$ with $0 \le \psi_n \le 1$, $\psi_n = 1$ on J_n, and $\operatorname{supp}\psi_n \subseteq J_{n+1}$. Lebesgue's theorem gives an index $N \in \mathbb{N}$ with $\|g - \psi_N g\|_{L^p(J,W^{l,r})} \le \varepsilon$. Using that $\psi_N g$ has compact support in J and mollifiers on $\mathbb{R}$, we then find a function h in $C_c^\infty(J, W^{l,r})$ satisfying $\|g - h\|_{L^p(J,W^{l,r})} \le 2\varepsilon$. (The usual properties of mollifiers also work in the Banach space-valued case.) □

We state the Hardy–Littlewood–Sobolev inequality, see Theorem 4.3 in [**LL**].

Lemma 4.9. *Let $\beta, \gamma \in (1, \infty)$ and $0 < \lambda < n$ satisfy $\frac{1}{\beta} + \frac{\lambda}{n} + \frac{1}{\gamma} = 2$, and $f \in L^\gamma(\mathbb{R}^n)$. We then have*

$$\Big(\int_{\mathbb{R}^n} \Big[\int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^\lambda} \,\mathrm{d}y\Big]^{\beta'} \mathrm{d}x\Big)^{\frac{1}{\beta'}} \le C\|f\|_\gamma.$$

[2] The next proof was omitted in the lectures.

This result resembles Young's convolution estimate from Theorem 2.14 in [**FA**]. To see this relation, let $\varphi_\lambda(x) = |x|^{-\lambda}$ for $x \in \mathbb{R}^n \setminus \{0\}$ and $\varphi_\lambda(0) = 0$, where $\lambda \in (0, n)$. If we had $\varphi_\lambda \in L^r(\mathbb{R}^n)$ with $\frac{1}{r} = 1 + \frac{1}{\beta'} - \frac{1}{\gamma}$ and $r, \beta, \gamma \in [1, \infty]$, then Young's inequality would give

$$\|\varphi_\lambda * g\|_{\beta'} \le \|\varphi_\lambda\|_r \, \|g\|_\gamma \qquad \text{for all } g \in L^\gamma(\mathbb{R}^n).$$

In view of Lemma 4.9, we need $\lambda r = n$. So φ_λ would belong to $L^r(\mathbb{R}^n)$ if $\int_{\mathbb{R}^n} |x|^{-n} \, \mathrm{d}x < \infty$ which is not quite true. Still, the Hardy–Littlewood–Sobolev inequality gives this estimate (for non-negative g) with $\|\varphi_\lambda\|_r$ replaced by C.

Strichartz estimates involve *(Schrödinger)-admissible* exponents (p, q); i.e.,

$$2 \le p, q \le \infty, \quad \frac{2}{p} + \frac{m}{q} = \frac{m}{2} \qquad \text{and} \quad (p, q) \ne (2, \infty) \text{ if } m = 2. \tag{4.16}$$

Observe that $(\infty, 2)$ is admissible (this will be the trivial case) and $(2, 2m/(m-2))$ for $m \ge 3$ is admissible (the endpoint case). The inverses $(\frac{1}{p}, \frac{1}{q})$ of admissible exponents (p, q) belong to the line from $(0, \frac{1}{2})$ to $(\frac{1}{2}, \frac{(m-2)_+}{2m})$, excluding the latter point if $m = 2$. So we have the embedding $L^{q'} \hookrightarrow H^{-1}$ for admissible (p, q) due to (4.7) and (4.8). This means that the convolution $T *_+ f$ is well-defined in H^{-1} for $f \in L^{p'}(J, L^{q'})$ as $T(\cdot)$ is a C_0-group on H^{-1}.

Let us explain admissibility by a scaling argument. Let $E = L^p(\mathbb{R}, L^q)$ for admissible (p, q). Assume that the map $\varphi \mapsto u = T(\cdot)\varphi$ is bounded from L^2 to E. For $\lambda > 0$ we set $u_\lambda(t, x) = u(\lambda^2 t, \lambda x)$ and $\varphi_\lambda(x) = \varphi(\lambda x)$ for $t \in \mathbb{R}$ and $x \in \mathbb{R}^m$. Then u_λ also solves $\partial_t u_\lambda = \mathrm{i}\Delta u_\lambda$ and thus $u_\lambda = T(\cdot)\varphi_\lambda$. Let $\varphi \in L^2 \setminus \{0\}$. Substituting $y = \lambda x$ and $t = \lambda^2 s$, we compute

$$\|\varphi_\lambda\|_2 = \Big(\int |\varphi(\lambda x)|^2 \, \mathrm{d}x \Big)^{\frac{1}{2}} = \lambda^{-\frac{m}{2}} \|\varphi\|_2,$$

$$\|u_\lambda\|_E^p = \Big(\int_{\mathbb{R}} \Big(\int |u(\lambda^2 s, \lambda x)|^q \, \mathrm{d}x \Big)^{\frac{p}{q}} \mathrm{d}s \Big)^{\frac{1}{p}} = \lambda^{-\frac{m}{q}} \lambda^{-\frac{2}{p}} \|u\|_E.$$

As a result, the claimed boundedness implies that $\lambda^{-\frac{m}{q}} \lambda^{-\frac{2}{p}} \|u\|_E \le C\lambda^{-\frac{m}{2}} \|\varphi\|_2$ for all $\lambda > 0$, and hence (p, q) must satisfy the equality in (4.16).

We now come to the main theorem of this section. The *homogeneous* Strichartz' estimate a) and the *inhomogeneous* estimate b) are needed to bound the two summands in the mild formula (4.14).

THEOREM 4.10. *Let (q, p) and $(\overline{q}, \overline{p})$ be admissible as in (4.16), $k \in \mathbb{N}_0$, $\varphi \in H^k$, $J \subseteq \mathbb{R}$ be an interval containing 0, and $f \in L^{\overline{p}'}(J, W^{k,\overline{q}'}) = \overline{E}_k'(J)$. Then $T *_+ f(t)$ exists in $W^{k,q}$ for a.e. $t \in J$, the maps $T(\cdot)\varphi$ and $T *_+ f$ belong to $L^p(J, W^{k,q}) = E_k(J)$, and there is a constant $C_{\mathrm{St}} \ge 1$ (independent of φ, f and J) such that*

a) $\|T(\cdot)\varphi\|_{E_k(J)} \le C_{\mathrm{St}} \|\varphi\|_{k,2}$,

b) $\|T *_+ f\|_{E_k(J)} \le C_{\mathrm{St}} \|f\|_{\overline{E}_k'(J)}$.

If $p = \infty$ and $q = 2$, we can replace L^∞ by C_b.

Compared to the L^2–setting, one gains space integrability from $q = 2$ to $q > 2$ and one looses time integrability from $p = \infty$ to $p < \infty$ (but gains some decay as $t \to \infty$). Moreover, in b) the exponents on the right-hand side are smaller than 2, whereas they are larger than 2 on the left-hand side. We point out that $(\overline{p},\overline{q})$ can be chosen independently of (p,q) in assertion b).

Part a) is wrong for non-admissible (p,q) as seen above and in §2.4 of [**Ca**], whereas part b) is true for some non-admissible exponents, cf. §2.4 of [**Ca**] and the exercises. The theorem and variants for the wave equation were proved by several authors in the case $p > 2$ and $\overline{p} > 2$ starting with Strichartz in the seventies, see §2.3 of [**Ca**]. The much more difficult endpoint case $p = 2$ and $\overline{p} = 2$ was established by Keel and Tao in [**KT**].

We will prove Theorem 4.10 only for $p, \overline{p} > 2$ and either for $(p,q) = (\overline{p},\overline{q})$ or for $(p,q) = (\infty, 2)$ and any admissible $(\overline{p},\overline{q})$, since we mostly work with these cases later on. In this situation the results follow from Corollary 4.6 and Lemma 4.9 in a nice way without deeper difficulties.

Exponents $(p,q) \neq (\overline{p},\overline{q})$ can be used to treat (4.1) with more general non-linearities, for instance. This case can be handled by another tool, the Christ-Kiselev lemma, see Section 2.3 in [**Ta**]. See also Theorem 2.3.3 in [**Ca**]. The endpoint case $(2, \frac{2m}{m-2})$ for $m \geq 3$ is needed to study (4.1) in the critical case $\alpha = \alpha_c$, see Theorem 4.17.

Proof of Theorem 4.10. As noted above we restrict ourselves to the cases $p, \overline{p} > 2$ and either $(p,q) = (\overline{p},\overline{q})$ or $(p,q) = (\infty, 2)$. Lemma 4.5 says that $\partial_j T(t) = T(t)\partial_j$ in H^{-1}, and ∂_j is closed in H^{-1} on its maximal domain (whichb can be used to take it out of integrals). Hence it is enough to show the result for $k = 0$. We will prove the assertions for $J = \mathbb{R}$. The case of general J can be reduced to $J = \mathbb{R}$ by extending f by 0 to $\mathbb{R}$ and by restricting $T(\cdot)\varphi$ and $T *_+ f$ from $\mathbb{R}$ to J. We write $E = E_0(\mathbb{R})$, $\overline{E}' = \overline{E}_0'(\mathbb{R})$, and note that $E^* = L^{p'}(\mathbb{R}, L^{q'})$ by (4.15).

1) Let $(p,q) = (\overline{p},\overline{q})$ be admissible, $2 < q < \infty$, $\varphi \in L^2$, and $f \in E^*$. We first prove estimate b). Corollary 4.6 and Lemma 4.9 (with $\lambda = \frac{m}{2} - \frac{m}{q}$, $n = 1$, $\beta = \gamma = p'$) imply the crucial estimate

$$I_1 := \Big[\int_{\mathbb{R}} \Big[\int_0^t \|T(t-s)f(s)\|_q \,\mathrm{d}s\Big]^p \mathrm{d}t\Big]^{\frac{1}{p}} \leq \Big[\int_{\mathbb{R}} \Big[\int_{\mathbb{R}} \frac{\|f(s)\|_{q'}}{(4\pi|t-s|)^{\frac{m}{2}-\frac{m}{q}}} \,\mathrm{d}s\Big]^p \mathrm{d}t\Big]^{\frac{1}{p}}$$
$$\leq C_0 \Big[\int_{\mathbb{R}} \|f(s)\|_{q'}^{p'} \,\mathrm{d}s\Big]^{\frac{1}{p'}} \tag{4.17}$$

where C_0 only depends on m, p and q. The conditions of Lemma 4.9 hold since (p,q) is admissible and $2 < q < \infty$. (The measurability of the integrand of I_1 is verified below.)

From this estimate assertion b) will follow by means of Fubini's theorem, but the details concerning integrability are a bit tricky. To this aim, take $l \in \mathbb{N}$ with $l \geq \frac{m}{2} - \frac{m}{q}$ so that $H^l \hookrightarrow L^q$ by Sobolev's embedding Theorem 3.17 in [**ST**]. Lemma 4.8 yields functions $g_n \in C_c(\mathbb{R}, H^l \cap L^{q'})$ that converge to f in E^* as $n \to \infty$. (Recall that $H^l \cap L^{q'}$ is a Banach space when endowed with the norm given by $\|v\|_{l,2} + \|v\|_{q'}$.)

The map $S_n : \mathbb{R}^2 \to L^q$; $(t,s) \mapsto T(t-s)g_n(s)$, is continuous for each $n \in \mathbb{N}$, since it is continuous in H^l by Lemma 4.5. There is a subsequence such that the functions $g_{n_j}(s)$ converge in $L^{q'}$ to $f(s)$ as $j \to \infty$ for a.e. $s \in \mathbb{R}$. Corollary 4.6

says that $T(t-s)$ maps $L^{q'}$ continuously into L^q for $t \neq s$. Therefore $(t,s) \mapsto T(t-s)g(s)$ is strongly measurable L^q outside a set of measure. Hence, the integral I_1 is defined. Similarly, one sees that $T(\cdot)\varphi : \mathbb{R} \to L^q$ is strongly measurable if $\varphi \in L^2 \cap L^{q'}$.

It now follows from Fubini's theorem and (4.17) that the integral $(T *_+ f)(t)$ exists in L^q for a.e. $t \in \mathbb{R}$ and that $T *_+ f : \mathbb{R} \to L^q$ is strongly measurable. Since $\|T *_+ f\|_E \le I_1$, assertion b) is shown in our case.

In the same way one derives $\|T * f\|_E \le C_0 \|f\|_{E^*}$ for the usual convolution.

2) We show part a) by a duality argument in the framework of step 1). Let $g \in C_c(\mathbb{R}, L^2 \cap L^{q'})$. Step 1) implies

$$I_2 := \int_{\mathbb{R}} \int \Big(\int_{\mathbb{R}} T(t-s) g(s)\, \mathrm{d}s \Big) \overline{g}(t)\, \mathrm{d}x\, \mathrm{d}t = \langle T * g, \overline{g} \rangle_E,$$
$$|I_2| \le \|T * g\|_E \, \|g\|_{E^*} \le C_0 \|g\|_{E^*}^2.$$

From Fubini's theorem we further deduce

$$\begin{aligned} I_2 &= \int_{\mathbb{R}} \int_{\mathbb{R}} (T(t-s)g(s) | g(t))_{L^2}\, \mathrm{d}s\, \mathrm{d}t = \int_{\mathbb{R}} \int_{\mathbb{R}} (T(-s)g(s) | T(-t)g(t))_{L^2}\, \mathrm{d}s\, \mathrm{d}t \\ &= \Big(\int_{\mathbb{R}} T(-s) g(s)\, \mathrm{d}s \Big| \int_{\mathbb{R}} T(-s) g(s)\, \mathrm{d}s \Big)_{L^2} = \Big\| \int_{\mathbb{R}} T(-s) g(s)\, \mathrm{d}s \Big\|_2^2, \end{aligned}$$

where all integrals are $\mathbb{C}$– or L^2–valued Riemann integrals. We have thus shown

$$\Big\| \int_{\mathbb{R}} T(-t) g(t)\, \mathrm{d}t \Big\|_2 \le \sqrt{C_0}\, \|g\|_{E^*}. \tag{4.18}$$

Let $\varphi \in C_c$. Observe that the scalar function $\langle T(\cdot)\varphi, \overline{g} \rangle_{L^q} = (T(\cdot)\varphi | g)_{L^2}$ is measurable. Estimate (4.18) leads to

$$\begin{aligned} \Big| \int_{\mathbb{R}} \langle T(t)\varphi, \overline{g(t)} \rangle_{L^p}\, \mathrm{d}t \Big| &= \Big| \int_{\mathbb{R}} (T(t)\varphi | g(t))_{L^2}\, \mathrm{d}t \Big| = \Big| \int_{\mathbb{R}} (\varphi | T(-t) g(t))_{L^2}\, \mathrm{d}t \Big| \\ &= \Big| \Big(\varphi \Big| \int_{\mathbb{R}} T(-t) g(t)\, \mathrm{d}t \Big)_{L^2} \Big| \le \sqrt{C_0}\, \|\varphi\|_2 \, \|g\|_{E^*}. \end{aligned}$$

Since $C_c(\mathbb{R}, L^2 \cap L^{q'})$ is dense in $E^* = L^{p'}(\mathbb{R}, L^{q'})$ by Lemma 4.8, the orbit $T(\cdot)\varphi$ belongs to $E = L^p(\mathbb{R}, L^q)$ and

$$\|T(\cdot)\varphi\|_E = \sup_{\|g\|_{E^*} \le 1} |\langle T(\cdot)\varphi, \overline{g} \rangle_E| \le \sqrt{C_0}\, \|\varphi\|_2.$$

By approximation, we derive assertion a) for $\varphi \in L^2$ and $q \in (2, \infty)$.

3) Let $(p,q) = (\infty, 2)$ and $(\overline{p}, \overline{q})$ be admissible with $\overline{p} > 2$. Then part a) is true with C_b instead of L^∞ since $T(\cdot)$ is a unitary C_0-group on L^2. To prove b), we set $f_t = \mathbb{1}_{[0,t]} f$ for $t > 0$ and $f_t = \mathbb{1}_{[t,0]} f$ for $t < 0$. We write (p,q) instead of $(\overline{p}, \overline{q})$. First, let $f \in C_c(\mathbb{R}, L^2 \cap L^{p'})$. Using also (4.18), we obtain $T *_+ f \in C_b(\mathbb{R}, L^2)$ and

$$\begin{aligned} \|T *_+ f\|_{C_b(\mathbb{R}, L^2)} &= \sup_{t \in \mathbb{R}} \Big\| \int_0^t T(t) T(-s) f_t(s)\, \mathrm{d}s \Big\|_2 = \sup_{t \in \mathbb{R}} \Big\| \int_{\mathbb{R}} T(-s) f_t(s)\, \mathrm{d}s \Big\|_2 \\ &\le \sup_{t \in \mathbb{R}} \sqrt{C_0}\, \|f_t\|_{E^*} = \sqrt{C_0}\, \|f\|_{E^*}\,. \end{aligned}$$

We approximate the given inhomogeneity f in E^* by $f_n \in C_c(\mathbb{R}, L^2 \cap L^{q'})$, The above estimate then shows that $(T *_+ f_n)_n$ converges to a function u in $C_b(\mathbb{R}, L^2)$. On the other hand, by step 1) for a subsequence the functions $(T *_+ f_{n_j})(t)$ tend to $(T *_+ f)(t)$ in L^q for a.e. $t \in \mathbb{R}$. Hence, $T *_+ f = u$ belongs to $C_b(\mathbb{R}, L^2)$ and assertion b) is true in the present case. □

4.3. Local wellposedness

In this section we establish the local wellposedness theory of the semilinear problem (4.1). The strategy of the proofs goes back to Kato. It is similar to the approach in Section 3.1 in the parabolic case. However, the smoothing effect of analytic semigroups is now replaced by Strichartz estimates, and many of the arguments are more sophisticated. We further show a part of the local wellposedness theory in the critical case $\alpha = 5$ if $m = 3$ in Theorem 4.17. Here one obtains a much less convenient blow-up condition, and one needs the endpoint case of Strichartz estimates. We start with some preparations.

We again reformulate (4.1) as fixed point problem for the operator

$$\Phi u(t) = (\Phi_{u_0}(u))(t) := T(t)u_0 + \int_0^t T(t-s)F(u(s))\,\mathrm{d}s, \qquad t \in J, \qquad (4.19)$$

for a given intial value $u_0 \in H^1$ and $F(v) = -\mathrm{i}\mu|v|^{\alpha-1}v$. Still assuming (4.2), we recall from (4.8) and (4.16) our assumptions and definitions:

$$q = \alpha+1 \in (2, \alpha_c), \quad \alpha_c = \frac{m+2}{(m-2)_+}, \quad q' = \frac{\alpha+1}{\alpha} = \frac{q}{\alpha}, \quad \frac{2}{p} = \frac{m}{2} - \frac{m}{q}. \qquad (4.20)$$

Moreover, F belongs to $C^1(H^1, H^{-1}) \cap C^1(L^q, L^{q'})$ by (4.9). The Strichartz estimates from Theorem 4.10 will compensate the loss of integrability caused by F in (4.19). For $k \in \mathbb{N}_0$ we introduce the spaces

$$E_k(J) = L^p(J, W^{k,q}), \quad E_k'(J) = L^{p'}(J, W^{k,q'}), \quad G_k(J) = L^\infty(J, H^k),$$
$$\mathcal{Z}_k(J) = E_k(J) \cap G_k(J) \quad \text{endowed with} \quad |||v|||_{k,J} = \max\{\|v\|_{E_k(J)},\ \|v\|_{G_k(J)}\}.$$

(If $k = 0$, we often omit the subscripts. For $J = [-b, b]$ we replace J by b.) We first collect the mapping properties of F needed below.

Lemma 4.11. *Let (4.20) be true and J be an interval of length a. Take $u, v \in G_1(J) \hookrightarrow E_0(J)$, $w \in \mathcal{F}_1(J)$, $\varphi, \psi \in L^q$, and $\chi \in W^{1,q}$. Set $r = \operatorname{ess\,sup}\{\|u(t)\|_{1,2}, \|v(t)\|_{1,2}, \|w(t)\|_{1,2} \,|\, t \in J\}$. Then we have $F(\varphi) \in L^{q'}$, $F(\chi) \in W^{1,q'}$, $F(u) \in E_0'(J)$, $F(w) \in E_1'(J)$, and the inequalities*

a) $\|F(\varphi) - F(\psi)\|_{q'} \le C_F\big(\|\varphi\|_q^{\alpha-1} + \|\psi\|_q^{\alpha-1}\big)\|\varphi - \psi\|_q$,

b) $\|F(u) - F(v)\|_{E_0'(J)} \le C_F r^{\alpha-1} a^{\frac{1}{p'}-\frac{1}{p}}\|u - v\|_{E_0(J)}$,

c) $\||\nabla F(\chi)|\|_{q'} \le C_F\|\chi\|_q^{\alpha-1}\||\nabla\chi|\|_q$,

d) $\||\nabla F(w)|\|_{E_0'(J)} \le C_F r^{\alpha-1} a^{\frac{1}{p'}-\frac{1}{p}}\||\nabla w|\|_{E_0(J)}$,

e) $\|F(w)\|_{E_1'(J)} \le C_F r^{\alpha-1} a^{\frac{1}{p'}-\frac{1}{p}}\|w\|_{E_1(J)}$

for a constant C_F only depending on α and m.

PROOF. From (4.9) and (4.20) we deduce

$$\begin{aligned}\|F(\varphi)-F(\psi)\|_{q'} &= \Big\|\int_0^1 F'(\varphi+\tau(\psi-\varphi))(\psi-\varphi)\,\mathrm{d}\tau\Big\|_{q'}\\ &\le \int_0^1 \|F'(\varphi+\tau(\psi-\varphi))\|_{\mathcal{B}(L^q,L^{q'})}\,\|\psi-\varphi\|_q\,\mathrm{d}\tau\\ &\le \alpha \sup_{\tau\in[0,1]} \|(1-\tau)\varphi+\tau\psi\|_q^{\alpha-1}\,\|\varphi-\psi\|_q\\ &\le c_\alpha\big(\|\varphi\|_q^{\alpha-1}+\|\psi\|_q^{\alpha-1}\big)\|\varphi-\psi\|_q.\end{aligned}$$

In this estimate we insert $u(t)$ and $v(t)$, and take the p-norm in time. Using Sobolev's embedding (4.4) and $\|u(t)\|_{1,2}, \|v(t)\|_{1,2}\le r$, we arrive at

$$\begin{aligned}\|F(u)-F(v)\|_{L^p(J,L^{q'})} &\le c_\alpha\Big(\int_J \big(\|u(t)\|_q^{\alpha-1}+\|v(t)\|_q^{\alpha-1}\big)^p\|u(t)-v(t)\|_q^p\,\mathrm{d}t\Big)^{\frac1p}\\ &\le 2c_\alpha C_{\mathrm{So}}^{\alpha-1} r^{\alpha-1}\|u-v\|_{E(J)}.\end{aligned}$$

With $\frac{1}{p'}=\frac1p+\frac{1}{p'}-\frac1p$ and $p>2>p'$, Hölder's inequality then yields

$$\|F(u)-F(v)\|_{E'(J)} \le a^{\frac{1}{p'}-\frac1p}\,\|F(u)-F(v)\|_{L^p(J,L^{q'})} \le c r^{\alpha-1} a^{\frac{1}{p'}-\frac1p}\|u-v\|_{E(J)}.$$

For c), we let $\phi(z)=-\mathrm{i}\mu|z|^{\alpha-1}z$ and $j\in\{1,\dots,m\}$. There are functions $\chi_n\in C_c^\infty$ converging to χ in $W^{1,q}$ as $n\to\infty$ by Theorem 3.15 in [**ST**]. Hölder's inequality with exponents $\frac{1}{q'}=\frac1q+\frac{\alpha-1}{q}$ yields

$$\|\partial_j F(\chi_n)\|_{q'} = \|\phi'(\chi_n)\partial_j\chi_n\|_{q'} \le \alpha\||\chi_n|^{\alpha-1}\partial_j\chi_n\|_{q'} \le \alpha\|\chi_n\|_q^{\alpha-1}\|\partial_j\chi_n\|_q.$$

Lemma 1.17 and Hölder's inequality show that the maps $\partial_j F(\chi_n)=\phi'(\chi_n)\partial_j\chi_n$ tend to $\phi'(\chi)\partial_j\chi$ and $F(\chi_n)$ to $F(\chi)$ in $L^{q'}$ as $n\to\infty$. From Lemma 3.5 in [**ST**] we then deduce that $F(\chi)\in W^{1,q'}$, and so inequality c) is true.

The estimates d) and e) follow as above (using also b) with $v=0$ for e)). □

Observe that the above estimates are uniform on a ball in H^1, which is often used below. Lemma 4.11 and the Strichartz estimates from Theorem 4.10 show that the operator Φ from (4.19) maps $\mathcal{F}_1(b)$ into itself and that it is Lipschitz on bounded sets of $\mathcal{F}_1(b)$, but only with respect to the metric of $\mathcal{F}_0(b)$. Fortunately, these properties still allow one to apply Banach's fixed point theorem, as seen in the next lemma. It relies on the Banach-Alaoglu theorem.

LEMMA 4.12. *Let (4.20) be true, $r>0$, and $J\subseteq\mathbb{R}$ be an interval. Then the ball $\Sigma(r,J)=\{v\in\mathcal{F}_1(J)\,|\,|||v|||_{1,J}\le r\}$ is a complete metric space when endowed with the metric induced by $|||\cdot|||_{0,J}$.*

PROOF. Let $(u_n)_n$ be a Cauchy sequence in $\Sigma(r,J)$ for $|||\cdot|||_{0,J}$. Since $\mathcal{F}_0(J)$ is a Banach space, $(u_n)_n$ converges in $\mathcal{F}_0(J)$ to a function $u\in\mathcal{F}_0(J)$ as $n\to\infty$. We have to show that $u\in\Sigma(r,J)$. As indicated after (4.15), the space $G_1(J)$ is the dual of $L^1(J,H^{-1})$ and $E_1(J)$ is reflexive with dual $L^{p'}(J,W^{-1,q'})$. Moreover, $L^1(J,H^{-1})$ is separable. The Banach-Alaoglu theorem thus provides a subsequence $(u_{n_j})_j$ which converges weakly in $E_1(J)$ to some v with $\|v\|_{E_1(J)}\le r$ and weakly* in $G_1(J)$ to some w with $\|w\|_{G_1(J)}\le r$

as $j \to \infty$. Since $E_0(J)^* \hookrightarrow E_1(J)^*$ and $L^1(J, L^2) \hookrightarrow L^1(J, H^{-1})$, the functions u_{n_j} also tend weakly in $E_0(J)$ to v and weak* in $G_0(J)$ to w. On the other hand, $(u_{n_j})_j$ has the limit u in both $E_0(J)$ and $G_0(J)$ so that $u = v$ and $u = w$ by the uniqueness of weak and weak* limits; i.e., u is an element $\Sigma(r, J)$. □

We further note that one can concatenate H^k–solutions.

Lemma 4.13. *Let (4.20) be true, $k \in \{1, 2\}$, and u and v be H^k–solutions of (4.1) on $[a, b]$ and $[b, c]$, respectively. Assume that $u(b) = v(b)$. Then the function w given by $w(t) = u(t)$ for $t \in [a, b]$ and $w(t) = v(t)$ for $t \in (b, c]$ is an H^k–solution of (4.1) with $w(a) = u(a)$.*

Proof. It is clear that $w \in C([a, c], H^k)$. Because of

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{-} w(b) = u'(b) = \mathrm{i}\Delta u(b) + F(u(b)) = \mathrm{i}\Delta v(b) + F(v(b)) = v'(b) = \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{+} w(b),$$

w also belongs to $w \in C^1([a, c], H^{k-2})$ and solves (4.1) on $[a, c]$. □

As a first part of the local wellposedness result, we establish the uniqueness of H^1–solutions which easily follows from Strichartz estimates and Lemma 4.11 if $\alpha < \alpha_c$.

Lemma 4.14. *Let (4.20) be true and $u_0 \in H^1$. Let u and v be H^1–solutions of (4.1) on intervals J_u and J_v containing 0, respectively. Then $u = v$ on $J_u \cap J_v$.*

Proof. If the assumption was not true, there would exist $\tau \in J_u \cap J_v$ such that $u = v$ on $[-\tau, \tau]$ and $u(t_n) \neq v(t_n)$ for certain $t_n \in J_u \cap J_v$ with, say, $t_n \to \tau^+$ as $n \to \infty$. (The case that $t_n \to (-\tau)^-$ is treated similarly.) Take $\delta_0 > 0$ such that $J_0 := [\tau, \tau + \delta_0] \subseteq J_u \cap J_v$. Let $r = \max\{\|u(t)\|_{1,2}, \|v(t)\|_{1,2} \,|\, t \in J_0\}$. Since u and v are H^1–solutions of (4.1), the maps $F(u)$ and $F(v)$ belong to $C(J_0, H^{-1})$ by (4.9). Proposition 2.6 of [**EE**] and Lemma 4.5 thus imply the mild formulas

$$u(t + \tau) = T(t)u(\tau) + \int_0^t T(t - s)F(u(s + \tau))\,\mathrm{d}s,$$
$$v(t + \tau) = T(t)u(\tau) + \int_0^t T(t - s)F(v(s + \tau))\,\mathrm{d}s$$

for all $t \in [0, \delta_0]$. Take any interval $J = [\tau, \tau + \delta] \subseteq J_0$. After a time shift, the Strichartz inequality from Theorem 4.10 and Lemma 4.11 then yield

$$|||u - v|||_{\mathcal{E}_0(J)} \leq C_{\mathrm{St}} C_F r^{\alpha - 1} \delta^{\frac{1}{q'} - \frac{1}{q}} \, |||u - v|||_{\mathcal{E}_0(J)}\,.$$

Choosing a sufficiently small $\delta \in (0, \delta_0]$, we deduce that $u = v$ on $[\tau, \tau + \delta]$ which contradicts $t_n \to \tau^+$. □

We can now establish the basic existence result for (4.1) employing the same tools as in the previous lemma and the fixed point space of Lemma 4.12.

Lemma 4.15. *Let (4.20) be true and $\rho > 0$. Then there is a number $b_0(\rho) > 0$ (see (4.22) below) such that for each initial value $u_0 \in \overline{B}_{H^1}(0, \rho)$ there is a unique H^1-solution $u \in \mathcal{F}_1(b_0(\rho))$ of (4.1) on the time interval $[-b_0(\rho), b_0(\rho)] =: J_0$. Moreover, $|||u|||_{1,J_0} \leq r := 1 + C_{\mathrm{St}}\rho$, where $C_{\mathrm{St}} \geq 1$ is taken from Theorem 4.10. We further have $u = \Phi_{u_0} u$ on J_0, cf. (4.19).*

Proof. Let $\rho > 0$ and $u_0 \in H^1$ with $\|u_0\|_{1,2} \le \rho$. Take $b > 0$ to be specified below. Fix $r = 1 + C_{\mathrm{St}}\rho$. Lemma 4.12 provides the complete metric space $\Sigma(r,b)$ with the metric $|||u - v|||_{0,b}$. Let $\Phi u = \Phi_{u_0} u$ be defined by (4.19) for $u \in \Sigma(r,b)$. Combining the Strichartz inequalities from Theorem 4.10 and Lemma 4.11, we estimate

$$|||\Phi u|||_{1,b} \le C_{\mathrm{St}}(\|u_0\|_{1,2} + \|F(u)\|_{E_1'(b)}) \le C_{\mathrm{St}}\rho + C_{\mathrm{St}} C_F r^\alpha (2b)^{\frac{1}{p'} - \frac{1}{p}}, \quad (4.21)$$

$$|||\Phi u - \Phi v|||_{0,b} \le C_{\mathrm{St}} \|F(u) - F(v)\|_{E_0'(b)} \le C_{\mathrm{St}} C_F r^{\alpha-1} (2b)^{\frac{1}{p'} - \frac{1}{p}} \|u - v\|_{E_0(b)}$$

for $u, v \in \Sigma(r,b)$, using that $p > 2 > p'$ by (4.20) and $\alpha < \alpha_c$. We now set

$$b_0(\rho) := \tfrac{1}{2} \min\Big\{ \big(C_{\mathrm{St}} C_F r^\alpha\big)^{\frac{p'p}{p'-p}}, \big(2C_{\mathrm{St}} C_F r^{\alpha-1}\big)^{\frac{p'p}{p'-p}} \Big\} > 0. \quad (4.22)$$

Let $b = b_0(\rho)$ and $J_0 = [-b_0(\rho), b_0(\rho)]$. It follows that $\Phi u \in \Sigma(r, J_0)$ and $|||\Phi u - \Phi v|||_{0,J_0} \le \frac{1}{2} |||u - v|||_{0,J_0}$. The contraction mapping principle then yields a unique fixed point $u = \Phi u$ in $\Sigma(r, J_0)$. (One can replace here $b_0(\rho)$ by any number $b \in (0, b_0(\rho)]$.)

Theorem 4.10 further shows that u belongs to $C(J_0, H^1)$, and hence $f := F(u)$ to $C(J_0, H^{-1})$ by (4.9). Since $u \in C(J_0, H^1)$ is a mild solution of $u' = \mathrm{i}\Delta u + f$ in H^{-1} with $u(0) = u_0$, Lemma 2.8 of [**EE**] and Lemma 4.5 imply that u is an H^1-solution of (4.1) on J_0. Uniqueness follows from Lemma 4.14. □

Before coming to the local wellposedness theorem, we define the *maximal existence times*

$$t_+(u_0) = \sup\big\{ b > 0 \,\big|\, \exists\, H^1\text{-solution } u_b \in C([0,b], H^1) \text{ of } (4.1) \big\},$$
$$t_-(u_0) = \inf\big\{ b < 0 \,\big|\, \exists\, H^1\text{-solution } u_b \in C([b,0], H^1) \text{ of } (4.1) \big\}.$$

Lemma 4.15 implies that $-t_-(u_0)$ and $t_+(u_0)$ belong to $[b_0(\|u_0\|_{1,2}), \infty]$. Using also Lemma 4.13 we can restart the system with initial value $u(t_\pm(u_0))$ and thus obtain $-t_-(u_0), t_+(u_0) > b_0(\|u_0\|_{1,2})$. As in Remark 1.10, Lemma 4.14 allows us to define H^1-solutions of (4.1) on $(t^-(u_0), 0]$ and $[0, t^+(u_0))$, and thus on $J(u_0) := (t^-(u_0), t^+(u_0))$ by Lemma 4.13. The H^1-solution u of (4.1) on $J(u_0)$ is called *maximal*. (In view of Theorem 4.16 c), we have to take an open time interval here.) It is unique by Lemma 4.14.

Our local wellposedness theorem follows the pattern of Theorem 1.11 though the underlying function spaces are adapted to the Strichartz estimates. Moreover, we only show the continuity of the solution map $u_0 \mapsto \varphi(\cdot, u_0)$ and not its Lipschitz continuity as in Theorems 1.11 and 3.4. One obtains the Lipschitz continuity under stronger conditions, see the exercises, or in weaker norms, cf. (4.24). We stress that we obtain a blow-up condition in H^1 which is the space of the initial values (as in Theorem 1.11 and 3.4).

Theorem 4.16. *Let (4.20) be true, $\rho > 0$, $u_0 \in H^1$ with $\|u_0\|_{1,2} \le \rho$, and $b_0(\rho)$ be defined by (4.22). Then the following assertions hold.*

a) There are numbers $0 < b_0(\rho) < \pm t_\pm(u_0) \le \infty$ and a unique maximal H^1-solution $u = \varphi(\cdot, u_0)$ of (4.1) on $J(u_0) = (t_-(u_0), t_+(u_0))$.

b) Let $[a,b] \subset J(u_0)$. Then $u \in L^p([a,b], W^{1,q})$.

c) If $\pm t_\pm(u_0) < \infty$, then $\limsup_{t \to t_\pm(u_0)} \|u(t)\|_{1,2} = \infty$.

d) Let $J \subseteq J(u_0)$ be a compact interval with $0 \in J$. Then there is a radius $\delta = \delta(J, u_0) > 0$ such that for $v_0 \in \overline{B}_{H^1}(u_0, \delta)$ we have $J \subseteq J(v_0)$ and the map

$$\overline{B}_{H^1}(u_0, \delta) \to C(J, H^1) \cap L^p(J, W^{1,q}), \quad v_0 \mapsto \varphi(\,\cdot\,, v_0),$$

is continuous.

Proof. 1) Part a) was proved before the theorem. Let $\tau \in J(u_0)$. Lemma 4.15 yields a time $\beta(\tau) > 0$ and an H^1-solution v of (4.1) with $v(0) = u(\tau)$ belonging to $\mathcal{F}_1(\beta(\tau))$. By the uniqueness Lemma 4.14, v is a restriction of $u(\tau + \cdot\,)$ and thus $u \in L^p([\tau - \beta(\tau), \tau + \beta(\tau)], W^{1,q})$. A compactness argument then yields assertion b).

Suppose that $t_+(u_0) < \infty$ and there were times $t_n \to t_+(u_0)^-$ with $\sup_n \|u(t_n)\|_{1,2} =: C < \infty$. Take an index with $t_N + b_0(C) > t_+(u_0)$. Using Lemmas 4.15 and 4.13 we can extend the given H^1-solution to $[0, t_N + b_0(C)]$ by considering (4.1) with initial value $u(t_N)$. This fact contradicts the definition of $t_+(u_0)$. One treats $t_-(u_0)$ in the same way. Hence, claim c) holds.

2) Fix $J = [T_0, T_1] \subseteq J(u_0)$ with $0 \in J$. We show that every sequence $(\psi_n)_n$ with limit u_0 in H^1 as $n \to \infty$ has a subsequence $(\psi_{n_j})_j$ such that $J \subseteq J(\psi_{n_j})$ for all j and the solutions $u_{n_j} = \varphi(\,\cdot\,, \psi_{n_j})$ tend to u in $\mathcal{F}_1(J)$ as $j \to \infty$. This fact implies assertion d) by a straightforward contradiction argument.

So let $(\psi_n)_n$ converge to u_0 in H^1 and set $u_n = u(\,\cdot\,, \psi_n)$. The uniform bound $\overline{\rho} := 1 + \max_{t \in J} \|u(t)\|_{1,2} < \infty$ will be crucial for our reasoning. There is an index $n_0 \in \mathbb{N}$ such that $\|\psi_n\|_{1,2} \le \overline{\rho}$ for all $n \ge n_0$. Lemma 4.15 then implies that $-t^-(\psi_n), t^+(\psi_n) > b_0(\overline{\rho}) =: b_0$ for all $n \ge n_0$. Let $n \ge n_0$. By Lemma 4.14, the restrictions of u and u_n to $J_0 := [-b_0, b_0]$ coincide with the solutions from Lemma 4.15 for the initial value u_0 and ψ_n, respectively. We thus have the equations $u = \Phi_{u_0}(u)$ and $u_n = \Phi_{\psi_n}(u_n)$ on J_0, and the core bound

$$|||u|||_{1,b_0},\ |||u_n|||_{1,b_0} \le \overline{r} := 1 + C_{\mathrm{St}}\overline{\rho}. \tag{4.23}$$

Due to (4.21) and the choice of b_0 in (4.22), the operator Φ_{u_0} is Lipschitz on $\Sigma(\overline{r}, b_0) = \overline{B}_{\mathcal{F}_1(b_0)}(0, \overline{r})$ for the metric induced by $|||\cdot|||_{0,b_0}$ with a constant bounded by $\frac{1}{2}$. Combining this fact with Strichartz estimates of Theorem 4.10, we derive the basic 0-order Lipschitz inequality

$$\begin{aligned}
|||u - u_n|||_{0,b_0} &\le |||\Phi_{u_0}(u) - \Phi_{u_0}(u_n)\,|||_{0,b_0} + |||\Phi_{u_0}(u_n) - \Phi_{\psi_n}(u_n)|||_{0,b_0} \\
&\le \tfrac{1}{2}|||u - u_n\,|||_{0,b_0} + |||T(\cdot)(u_0 - \psi_n)|||_{0,b_0} \\
&\le \tfrac{1}{2}|||u - u_n\,|||_{0,b_0} + C_{\mathrm{St}}\|u_0 - \psi_n\|_2, \\
|||u - u_n|||_{0,b_0} &\le 2C_{\mathrm{St}}\|u_0 - \psi_n\|_2
\end{aligned} \tag{4.24}$$

for all $n \ge n_0$. Unfortunately, this argument does not give the desired continuity of $v_0 \mapsto u(\,\cdot\,, v_0)$ from H^1 to $\mathcal{F}_1(b_0)$, so that we need a more sophisticated analysis than in Theorems 1.11 and 3.4.

3) The inequality (4.24) shows that, after passing to a subsequence $(u_k)_k$, the functions $u_k(t)$ tend to $u(t)$ in L^q as $k \to \infty$, for a.e. $t \in J$. Recall that

$$u_k - u = T(\cdot)(\psi_k - u_0) + T *_+ (F(u_k) - F(u)) \qquad \text{on } J_0.$$

Let $b \in (0, b_0]$ and $k \geq n_0$. Theorem 4.10 and Lemma 4.11 yield

$$|||u_k - u|||_{1,b} \leq C_{\mathrm{St}}\big(\|\psi_k - u_0\|_{1,2} + \|F(u_k) - F(u)\|_{E_1'(b)}\big). \tag{4.25}$$

Let $\phi(z) = -\mathrm{i}\mu|z|^{\alpha-1}z$ for $z \in \mathbb{R}^2$ and $j \in \{1, \dots, m\}$. As in the proof of Lemma 4.11, we compute

$$\begin{aligned}
&\|\partial_j F(u_k(t)) - \partial_j F(u(t))\|_{q'} \\
&\qquad \leq \|\phi'(u_k(t))[\partial_j u_k(t) - \partial_j u(t)]\|_{q'} + \|[\phi'(u_k(t)) - \phi'(u(t))]\partial_j u(t)\|_{q'} \\
&\qquad \leq \alpha \|u_k(t)\|_q^{\alpha-1} \|\partial_j u_k(t) - \partial_j u(t)\|_q + \|[\phi'(u_k(t)) - \phi'(u(t))]\partial_j u(t)\|_{q'} \\
&\qquad \leq \alpha C_{\mathrm{So}}^{\alpha-1} \overline{r}^{\alpha-1} \|u_k(t) - u(t)\|_{1,q} + \|[\phi'(u_k(t)) - \phi'(u(t))]\partial_j u(t)\|_{q'},
\end{aligned}$$

for $t \in J_0$ by the Sobolev embedding (4.4) and estimate (4.23). Taking the norm in $L^{p'}([-b, b])$, Lemma 4.11 b) and Hölder's inequality then lead to

$$\begin{aligned}
\|F(u_k) - F(u)\|_{E_1'(b)} \leq c_1 \overline{r}^{\alpha-1} b^{\frac{1}{p'} - \frac{1}{p}} \|u_k - u\|_{E_1(b)} \\
+ c_2 \|(\phi'(u_k) - \phi'(u))|\nabla u|\|_{E_0'(b)}.
\end{aligned}$$

Here and below, c_j are constants only depending on m and α. We fix

$$b = b(\overline{\rho}) = \min\big\{b_0, \big(2c_1 C_{\mathrm{St}} \overline{r}^{\alpha-1}\big)^{\frac{p'p}{p'-p}}\big\} \tag{4.26}$$

and insert the above inequality into (4.25), arriving at

$$\begin{aligned}
|||u_k - u|||_{1,b} &\leq C_{\mathrm{St}} \|\psi_k - u_0\|_{1,2} + \tfrac{1}{2}\|u_k - u\|_{E_1(b)} \\
&\qquad + c_2 C_{\mathrm{St}} \|(\phi'(u_k) - \phi'(u))|\nabla u|\|_{E_0'(b)}, \\
|||u_k - u|||_{1,b} &\leq 2C_{\mathrm{St}} \|\psi_k - u_0\|_{1,2} + 2c_2 C_{\mathrm{St}} \|(\phi'(u_k) - \phi'(u))|\nabla u|\|_{E_0'(b)}. \qquad (4.27)
\end{aligned}$$

It remains to control

$$\|(\phi'(u_k) - \phi'(u))|\nabla u|\|_{E_0'(b)}^{p'} = \int_{-b}^{b} \|(\phi'(u_k(t)) - \phi'(u(t)))|\nabla u(t)|\|_{q'}^{p'} \,\mathrm{d}t.$$

Lemma 1.17 implies that the functions $\phi'(u_k(t))|\nabla u(t)|$ tend to $\phi'(u(t))|\nabla u(t)|$ in $L^{q'}$ as $k \to \infty$ for a.e. $t \in [-b, b]$, since $u_k(t) \to u(t)$ in L^q. Combining Lemma 1.17 with (4.20), (4.4) and (4.23), we further estimate

$$\begin{aligned}
\|(\phi'(u_k(t)) - \phi'(u(t)))|\nabla u(t)|\|_{q'} &\leq c_3(\|u_k(t)\|_q^{\alpha-1} + \|u(t)\|_q^{\alpha-1}) \, \||\nabla u(t)|\|_q \\
&\leq 2c_3 C_{\mathrm{So}}^{\alpha-1} \overline{r}^{\alpha-1} \|u(t)\|_{1,q}.
\end{aligned}$$

Since $p' < p$, the function u belongs to $L^{p'}([-b, b], W^{1,q})$. Due to dominated convergence, the second term on the right-hand side in (4.27) thus tends to 0 as $k \to \infty$. As a result, $u_k \to u$ in $\mathcal{F}_1(b)$, and we can fix an index k_1 such that $\|u_k(\pm b)\|_{1,2} \leq \overline{\rho}$ for all $k \geq k_1$. (Here we use that $u, u_k \in C([-b, b], H^1)$.)

4) Let $T_0 < -b$ or $T_1 > b$. As $\|u_k(\pm b)\|_{1,2} \leq \overline{\rho}$ for $k \geq k_1$, we can repeat the above argument with initial time $-b$ or b, passing to further subsequences. This can be done with the step size b from (4.26) which only depends on $\overline{\rho}$, m and α. In finitely many steps, we thus construct a subsequence $(\psi_{n_j})_j$ with $J \subseteq J(\psi_{n_j})$ for all $j \in \mathbb{N}$ and $u_{n_j} \to u$ in $\mathcal{F}_1(J)$ as $j \to \infty$. Assertion d) then follows. □

The above arguments fail in the *critical case* $\alpha = \alpha_c$ and $m \geq 3$, where $p = p' = 2$ and the factor $b^{\frac{1}{p'} - \frac{1}{p}} = 1$ does not vanish as $b \to 0^+$. To treat this case, one uses the structure of the power nonlinearity in a more clever way. By means of Hölder and Sobelev inqualities one can bound a part of $F(v)$ in the $L^s((-b,b) \times \mathbb{R}^m)$ for a suitable $s > 1$. This space-time norm can be made small without restricting the size of v in $G_1(b)$, which would lead to small initial data u_0. This approach requires a more sophisticated fixed point space, and leads to a blow-up condition involving the the norm of $T(\cdot)u_0$ in $L^s((0,b) \times \mathbb{R}^m)$, which is much harder to control than the H^1-norm of $u(t)$.

We prove local existence, uniqueness and a blow-up criterion for $m = 3$ and $\alpha = 5$; one could treat $m \in \{4,5\}$ similarly. We do not treat the continuous dependence on data. See [**TV**] and Theorem 4.5.1 in [**Ca**] for the general case and results on continuous dependence.

We note that $L^s(J \times \mathbb{R}^m)$ is isometrically isomorphic to $L^s(J, L^s(\mathbb{R}^m))$ via $(\Psi f)(t) = f(t,\cdot)$ for $s \in [1,\infty)$ and intervals J, cf. Theorem X.6.22 in [**AE3**].

THEOREM 4.17. *Let $\mu \in \{-1,1\}$, $m = 3$, $\alpha = 5 = \alpha_c$, and $u_0 \in H^1$. Then there is a unique maximal H^1-solution u of (4.1) on $J(u_0) = (t_-(u_0), t_+(u_0))$. If $t_+(u_0) < \infty$, then $\|T(\cdot)u_0\|_{L^{10}([0,t_+(u_0))\times\mathbb{R}^3)} = \infty$, and analogously for $t_-(u_0)$.*

PROOF. 1) We first discuss the basic spaces and estimates needed in the proof. As before we use the Strichartz pairs $(\infty, 2)$ and $(2,6)$, where $6 = \alpha_c + 1 = q$. In addition we involve the pair $(10, r)$ with $r = \frac{30}{13} > 2$ noting that $\frac{2}{10} + \frac{39}{30} = \frac{3}{2}$. Since $1 - \frac{3}{r} = -\frac{3}{10}$ we have the Sobolev embedding

$$W^{1,r} \hookrightarrow L^{10}, \qquad \text{and hence} \;\; L^{10}(J, W^{1,r}) \hookrightarrow L^{10}(J \times \mathbb{R}^3) \tag{4.28}$$

with a constant C'_{So} independent of the interval J. We can thus control the norm of $L^{10}(J, L^{10})$ by the (Strichartz) norm of $L^{10}(J, W^{1,r})$. This fact shall be used to bound the nonlinearity in the (dual Strichartz) norm of $L^2(J, W^{k,\frac{6}{5}}) = E'_k(J)$, where $\frac{6}{5} = 6'$. To this aim, let $v \in L^{10}(J, L^{10})$ and $w \in L^{10}(J, L^r)$. Using Hölder's inequality with $\frac{5}{6} = \frac{2}{5} + \frac{1}{r}$ and with $\frac{1}{2} = \frac{2}{5} + \frac{1}{10}$, we estimate

$$\begin{aligned} \||v|^4 w\|_{E'_0(J)} &\leq \bigg(\int_J \Big[\Big(\int_{\mathbb{R}^3} |v(t)|^{\frac{4\cdot 5}{2}} \,\mathrm{d}x \Big)^{\frac{2\cdot 2}{5\cdot 2}} \Big(\int_{\mathbb{R}^3} |w(t)|^r \,\mathrm{d}x \Big)^{\frac{1}{r}} \Big]^2 \,\mathrm{d}t \bigg)^{\frac{1}{2}} \\ &= \Big(\int_J \big(\|v(t)\|_{10}^4 \, \|w(t)\|_r \big)^2 \,\mathrm{d}t \Big)^{\frac{1}{2}} \\ &\leq \Big[\int_J \|v(t)\|_{10}^{10} \,\mathrm{d}t \Big]^{\frac{2}{5}} \Big[\int_J \|w(t)\|_r^{10} \,\mathrm{d}t \Big]^{\frac{1}{10}} = \|v\|^4_{L^{10}_J L^{10}} \, \|w\|_{L^{10}_J L^r}, \end{aligned} \tag{4.29}$$

where we write $L^{10}_J L^r$ instead $L^{10}(J, L^r)$ etc. We now define the spaces

$$\mathcal{V}_k(J) = L^\infty(J, H^k) \cap L^2(J, W^{k,6}) \cap L^{10}(J, W^{k,r})$$

for $k \in \{0,1\}$, and put $\mathcal{V}_k(J) = \mathcal{V}_k(b)$ if $J = (0,b)$. They are Banach spaces with the norm $[\![v]\!]_{k,b}$ where we let $[\![v]\!]_{0,b}$ be the sum of the norms in $L^\infty(J, L^2)$, $L^2(J, L^6)$ and $L^{10}(J, L^r)$, and we set $[\![v]\!]_{1,b} = [\![v]\!]_{0,b} + [\![|\nabla v|_1]\!]_{0,b}$. (Accordingly, we use the equivalent norm $\|\varphi\|_{W^{1,r}} = \|\varphi\|_r + \||\nabla\varphi|_1\|_r$ on $W^{1,r}$.)

Let $u_0 \in H^1$ and $\rho \geq \|u_0\|_{1,2}$. Fix $R := 2C_{\mathrm{St}}\rho + 1$ with C_{St} from Theorem 4.10. For a time $b > 0$ and a radius $\varepsilon > 0$ we introduce the fixed point space

$$\Sigma(b, R, \varepsilon) = \{v \in \mathcal{V}_1(b) \mid [\![v]\!]_{1,b} \leq R,\ \|v\|_{L^{10}(J,L^{10})} \leq \varepsilon\}.$$

As in Lemma 4.12 one checks that $\Sigma(b, R, \varepsilon)$ is complete for the metric $[\![v - w]\!]_{0,b}$. (The inequality in L^{10} holds in the limit due to Fatou's lemma since convergence in $\mathcal{V}_0(b)$ implies pointwise convergence a.e. of a subsequence.) For $v \in \mathcal{V}_1(b)$ we further define fixed point operator

$$\Phi v(t) = T(t)u_0 - \mathrm{i}\mu \int_0^t T(t-s)(|v(s)|^4 v(s))\,\mathrm{d}s, \qquad t \in [0, b].$$

2) We show that Φ is a strict contraction on $\Sigma(b, R, \varepsilon)$ for small $b, \varepsilon > 0$. The Strichartz estimate in Theorem 4.10 a) and inequality (4.28) imply that $T(\cdot)u_0$ belongs to $L^{10}((0,b), W^{1,r}) \hookrightarrow L^{10}((0,b), L^{10}) =: L_b^{10}L^{10}$. We can thus fix a time $b = b(u_0, \varepsilon) > 0$ such that

$$\|T(\cdot)u_0\|_{L_b^{10}L^{10}} \leq \varepsilon/2. \tag{4.30}$$

Let $v \in \mathcal{V}_1(b)$. From Theorem 4.10 and (4.29) we deduce the inequalities

$$\begin{aligned}
[\![\Phi v]\!]_{0,b} &\leq C_{\mathrm{St}}(\|u_0\|_2 + \||v|^4 v\|_{E_0'(b)}) \leq C_{\mathrm{St}}\rho + C_{\mathrm{St}}\varepsilon^4 R,\\
[\![|\nabla \Phi v|]\!]_{0,b} &\leq C_{\mathrm{St}}(\|u_0\|_{1,2} + \||\nabla(|v|^4 v)|\|_{E_0'(b)}) \leq C_{\mathrm{St}}\rho + 5C_{\mathrm{St}}\||v|^4\,|\nabla v|\|_{E_0'(b)}\\
&\leq C_{\mathrm{St}}\rho + 5C_{\mathrm{St}}\varepsilon^4 R.
\end{aligned} \tag{4.31}$$

For $0 < \varepsilon \leq \varepsilon_0 := (6C_{\mathrm{St}}R)^{-1/4}$, the definition of R thus leads to

$$[\![\Phi v]\!]_{1,b} \leq 2C_{\mathrm{St}}\rho + 6C_{\mathrm{St}}\varepsilon^4 R \leq R.$$

Using also (4.30) and (4.28), we further estimate

$$\begin{aligned}
\|\Phi v\|_{L_b^{10}L^{10}} &\leq \tfrac{\varepsilon}{2} + C_{\mathrm{So}}'\,\|T * (|v|^4 v)\|_{L_b^{10}W^{1,r}}\\
&\leq \tfrac{\varepsilon}{2} + C_{\mathrm{So}}'C_{\mathrm{St}}\big(\||v|^4 v\|_{E_0(b)'} + 5\,\||v|^4|\nabla v|\|_{E_0(b)'}\big)\\
&\leq \tfrac{\varepsilon}{2} + C_{\mathrm{So}}'C_{\mathrm{St}}(\varepsilon^3 R + 5\varepsilon^3 R)\varepsilon \leq \varepsilon
\end{aligned} \tag{4.32}$$

provided that $0 < \varepsilon \leq \varepsilon_1 := \min\{\varepsilon_0, (12RC_{\mathrm{So}}'C_{\mathrm{St}})^{-1/3}\}$. For such ε and fixed $R = R(\rho)$ and $b = b_0(u_0, \varepsilon) > 0$, the operator Φ thus maps $\Sigma(b, R, \varepsilon)$ into itself.

Let also $w \in \Sigma(b, R, \varepsilon)$. Using Young's inequality, we can write

$$\begin{aligned}
|v|^4 v - |w|^4 w &= (v-w)|v|^4 + w(\overline{v} - \overline{w})|v|^2 v + |w|^2(v-w)|v|^2\\
&\quad + |w|^2 w(\overline{v} - \overline{w})v + |w|^4(v-w),\\
\big|\,|v|^4 v - |w|^4 w\,\big| &\leq \tfrac{5}{2}(|v|^4 + |w|^4)|v-w|.
\end{aligned}$$

The Strichartz estimate in Theorem 4.10 b) and (4.29) now yield

$$\begin{aligned}
[\![\Phi v - \Phi w]\!]_{0,b} &\leq C_{\mathrm{St}}\,\||v|^4 v - |w|^4 w\|_{E_0(b)'} \leq \tfrac{5}{2}C_{\mathrm{St}}\,\|(|v|^4 + |w|^4)|v-w|\|_{E_0(b)'}\\
&\leq 5C_{\mathrm{St}}\varepsilon^4 [\![v - w]\!]_{0,b} \leq \tfrac{1}{2}[\![v - w]\!]_{0,b}
\end{aligned}$$

for $0 < \varepsilon \leq \varepsilon_2 := \min\{\varepsilon_1, (10C_{\mathrm{St}})^{-1/4}\}$. Hence, there is a (unique) fixed point u of Φ in $\Sigma(b, R, \varepsilon)$ with $\varepsilon = \varepsilon_2$. By Theorem 4.10, the function u belongs to

$C([0,b],H^1)$. As in Lemma 4.15 we conclude that u is an H^1-solution of (4.1) on $[0,b]$ due to Lemma 2.8 of [**EE**] and Lemma 4.5.

3) Let u and v be H^1-solutions of (4.1) on J_u and J_v, respectively, such that $J := J_u \cap J_v$ contains more points than 0. If $0 \neq \max J$, we set

$$\tau = \sup\{t \in J \cap \mathbb{R}_{\geq 0} \,|\, \forall\, s \in [0,t] :\, u(s) = v(s)\}.$$

We suppose $\tau < \sup J$. The continuity of u and v then yields $u(\tau) = v(\tau) =: \varphi \in H^1$. Moreover there are times $J \ni t_n \to \tau^+$ with $u(t_n) \neq v(t_n)$ as $n \to \infty$.

Fix a time $\delta' > 0$ with $\tau + \delta' \in J$ and $\delta' \leq b(\varphi, \varepsilon)$, where $\varepsilon = \varepsilon_2$ and $b(\varphi,\varepsilon) =: b$ are chosen as in step 2) for $\rho = \|\varphi\|_{1,2}$. Let $R = 2C_{\mathrm{St}}\|\varphi\|_{1,2} + 1$. This step yields a solution $\tilde z \in \Sigma(\delta', R, \varepsilon)$ of (4.1) with $\tilde z(0) = \varphi$. Set $z = \tilde z(\cdot + \tau)$. It suffices to show that $u = z$ and $v = z$ on $[\tau, \tau+\delta]$ for some $\delta \in (0,\delta']$ to obtain a contradiction. We can thus assume that $v = z$ on $[\tau, \tau+\delta']$; i.e., $v_\tau := v(\cdot - \tau)$ is contained in $\Sigma(\delta', R, \varepsilon)$.

We set $w = u(\cdot - \tau) - v(\cdot - \tau)$ on $J' = [0, \delta']$. This function satisfies

$$\partial_t w = \mathrm{i}\Delta w + F(w + v_\tau) - F(v_\tau) =: \mathrm{i}\Delta w + f, \quad t \in J', \qquad w(0) = 0.$$

Moreover, w belongs to $C(J', H^1)$ and thus to $C(J', L^6)$. These properties are not enough to use directly the estimates of step 2). But we can combine them with the better behavior of $v_\tau \mathcal{V}_1(\delta')$. In particular, we need smallness in the argument. Here we have for any given $\eta > 0$ (to be fixed below) a time $\delta = \delta(\eta) \in (0, \delta']$ such that $\|w(t)\|_{1,2} \leq \eta$ for all $t \in [0,\delta]$ and $\|v_\tau\|_{L^{10}_\delta L^{10}} \leq \eta$.

Writing $|w + v_\tau|^4 = (w+v_\tau)^2(\overline{w} + \overline{v_\tau})^2$, we can expand $f = f_1 + \cdots + f_5$, where $f_5 = -\mathrm{i}\mu|w|^4 w$ and the other summands f_j are linear combinations of product containing j factors from $\{w, \overline{w}\}$ and $5-j$ factors from $\{v_\tau, \overline{v_\tau}\}$. The Strichartz estimate in Theorem 4.10 b) thus implies the bound

$$\|w\|_{L^2([0,\delta],L^6)} \leq C_{\mathrm{St}} \sum_{j=1}^{5} \|f_j\|_{L^{p_j'}([0,\delta],L^{q_j'})} \leq c_0 \sum_{j=1}^{5} \big\| |w|^j\, |v_\tau|^{5-j} \big\|_{L^{p_j'}([0,\delta],L^{q_j'})}$$

for a constant $c_0 > 0$ and the Strichartz pairs

$$(p_1,q_1) = (10, r) = \big(10, \tfrac{30}{13}\big), \quad (p_2,q_2) = \big(5, \tfrac{30}{11}\big), \quad (p_3,q_3) = \big(\tfrac{10}{3}, \tfrac{10}{3}\big),$$
$$(p_4,q_4) = \big(\tfrac{5}{2}, \tfrac{30}{7}\big), \quad (p_5,q_5) = (2,6).$$

Similar as in (4.29), on each product $|w|^j\, |v_\tau|^{5-j}$ we apply first Hölder inequality in space and then in time with the exponents

$$\begin{aligned}
j = 1:& \quad \frac{1}{q_1'} = \frac{17}{30} = \frac{1}{6} + \frac{4}{10}, \quad \frac{1}{p_1'} = \frac{9}{10} = \frac{1}{2} + \frac{4}{10},\\
j = 2:& \quad \frac{1}{q_2'} = \frac{19}{30} = \frac{1}{6} + \frac{1}{6} + \frac{3}{10}, \quad \frac{1}{p_2'} = \frac{4}{5} = \frac{1}{2} + \frac{1}{\infty} + \frac{3}{10},\\
j = 3:& \quad \frac{1}{q_3'} = \frac{7}{10} = \frac{1}{6} + \frac{2}{6} + \frac{2}{10}, \quad \frac{1}{p_3'} = \frac{7}{10} = \frac{1}{2} + \frac{1}{\infty} + \frac{2}{10},\\
j = 4:& \quad \frac{1}{q_4'} = \frac{23}{30} = \frac{1}{6} + \frac{3}{6} + \frac{1}{10}, \quad \frac{1}{p_4'} = \frac{3}{5} = \frac{1}{2} + \frac{1}{\infty} + \frac{1}{10},\\
j = 5:& \quad \frac{1}{q_5'} = \frac{5}{6} = \frac{1}{6} + \frac{4}{6}, \quad \frac{1}{p_5'} = \frac{1}{2} = \frac{1}{2} + \frac{1}{\infty}.
\end{aligned}$$

We thus obtain

$$\begin{aligned}\|w\|_{L^2_\delta L^6} \le c_0 \big(&\|w\|_{L^2_\delta L^6}\,\|v_\tau\|^4_{L^{10}_\delta L^{10}} + \|w\|_{L^2_\delta L^6}\,\|w\|_{L^\infty_\delta L^6}\,\|v_\tau\|^3_{L^{10}_\delta L^{10}}\\ &+ \|w\|_{L^2_\delta L^6}\,\|w\|^2_{L^\infty_\delta L^6}\,\|v_\tau\|^2_{L^{10}_\delta L^{10}} + \|w\|_{L^2_\delta L^6}\,\|w\|^3_{L^\infty_\delta L^6}\,\|v_\tau\|_{L^{10}_\delta L^{10}}\\ &+ \|w\|_{L^2_\delta L^6}\,\|w\|^4_{L^\infty_\delta L^6}\big)\\ \le{}& 5c_0\eta^4\,\|w\|_{L^2_\delta L^6}\,.\end{aligned}$$

We recall that w is an element of $C(J', L^6)$ and thus the norm $\|w\|_{L^2_\delta L^6}$ is finite. Fixing $\eta = (10c_0)^{-1/4}$ and consequently $\delta = \delta(\eta)$, we conclude that $w(t) = 0$ for every $t \in [0, \delta]$ and hence $u(t) = v(t)$ for $t \in [\tau, \tau + \delta]$. This is the desired contradiction. Negative times are treated analogously.[3]

4) We define as usual $t_+(u_0) > 0$ as the supremum of those $t > 0$ for which we have an H^1-solution $u^{(t)}$ of (4.1) on $[0, t]$. Step 3) then allows us to construct a unique maximal H^1-solution u on $[0, t_+(u_0))$. We next suppose that $t_+(u_0) =: t_+$ and $\|u\|_{L^{10}_{t_+} L^{10}} =: M$ are finite. Take $\kappa > 0$ to be fixed below. Since M is finite and $[0, t_+]$ is compact, there are times $t_0 = 0 < t_1 < \cdots < t_N = t_+$ such that $\|u\|_{L^{10}(J_k, L^{10})} \le \kappa$ for all intervals $J_k = [t_k, t_{k+1}]$.

Using this bound, as in (4.31) we find a constant $C > 0$ such that

$$[\![u]\!]_{1,J_k} \le C(\|u(t_k)\|_{1,2} + \kappa^4\,\|u\|_{E_1(b)}) \le C(\|u(t_k)\|_{1,2} + \kappa^4\,[\![u]\!]_{1,J_k})$$

for all $k \in \{0, 1, \ldots, N-1\}$. With $\kappa = (2C)^{-1/4}$ it follows

$$[\![u]\!]_{1,J_k} \le 2C\|u(t_k)\|_{1,2} \le 2C[\![u]\!]_{1,J_{k-1}} \le \cdots \le (2C)^{N-1}[\![u]\!]_{1,J_0} \le (2C)^N\|u_0\|_{1,2}$$

for all $k \le N-1$. As result, $\|u(t)\|_{1,2}$ is bounded by $\overline{\rho} := (2C)^N\rho$ for all $t \in [0, t_+)$ and $[\![u]\!]_{1,t_+}$ by $N\overline{\rho}$. Set $\overline{R} = 2C_{\mathrm{St}}\overline{\rho} + 1$. We define the number $\overline{\varepsilon} > 0$ corresponding to $\overline{R}$ as ε_2 in step 2).

For $\tau \in [0, t_+)$ and $J_\tau := (\tau, t_+)$, Theorem 4.10 b) and estimates (4.28) and (4.29) yield

$$\begin{aligned}\|u - T(\cdot - \tau)u(\tau)\|_{L^{10}_{J_\tau} L^{10}} &= \|T *_+ F(u)\|_{L^{10}_{J_\tau} L^{10}} \le C'_{\mathrm{So}}\|T *_+ F(u)\|_{L^{10}_{J_\tau} W^{1,r}}\\ &\le C'_{\mathrm{So}}C_{\mathrm{St}}\big(\||u|^5\|_{E'_0(J_\tau)} + 5\,\||u|^4|\nabla u|\|_{E'_0(J_\tau)}\big)\\ &\le 6C'_{\mathrm{So}}C_{\mathrm{St}}\|u\|^4_{L^{10}_{J_\tau} L^{10}}\,\|u\|_{L^{10}_{J_\tau} W^{1,r}}\\ &\le 6C'_{\mathrm{So}}C_{\mathrm{St}}N\overline{\rho}\,\|u\|^4_{L^{10}_{J_\tau} L^{10}} \;\longrightarrow\; 0\end{aligned}$$

as $\tau \to t_+$, since $\|u\|_{L^{10}_{J_\tau} L^{10}}$ tends to 0. We can now fix an initial time $\tau \in [0, t_+)$ such that $\|T(\cdot - \tau)u(\tau)\|_{L^{10}_{J_\tau} L^{10}} \le \overline{\varepsilon}/4$. Using that $T(\cdot - \tau)u(\tau)$ belongs to $L^{10}(\mathbb{R}, W^{1,r}) \hookrightarrow L^{10}(\mathbb{R}, L^{10})$, we then choose a time step $\beta > t_+ - \tau$ with

$$\|T(\cdot - \tau)u(\tau)\|_{L^{10}((\tau,\tau+\beta), L^{10})} \le \overline{\varepsilon}/2.$$

Step 2) now allows us to solve (4.1) on $[\tau, \tau + \beta]$ with intial value $u(\tau)$. Therefore we obtain a solution of (4.1) on $[0, \tau + \beta]$, contradicting the definition of

[3] Some parts of this step were only sketched in the lectures.

$t_+(u_0) = t_+$, so that the blow-up criterion is proved. Negative times are treated analogously. □

Conlcuding this section, we briefly explain why one calls the case $\alpha = \alpha_c$ *(energy)-critical*, where $m \geq 3$. Let u be an H^1-solution of (4.1). As seen in the exercises, the rescaled function $u_\lambda(t,x) := \lambda^{2/(\alpha-1)}u(\lambda^2 t, \lambda x)$ also solves (4.1) with initial value $\lambda^{2/(\alpha-1)}u_0(\lambda\,\cdot)$. Observe that

$$\|\nabla u_\lambda(t)\|_2 = \lambda^{\frac{2}{\alpha-1}}\lambda^{1-\frac{m}{2}}\,\|\nabla u(\lambda^2 t)\|_2 \quad \text{and} \quad \alpha \gtreqless \alpha_c \iff \frac{2}{\alpha-1} + 1 - \frac{m}{2} \lesseqgtr 0.$$

In the *energy-supercritical* case $\alpha > \alpha_c$ the possibly 'bad' behavior of a solution u at a time $t_0 \approx 1$ with large $\nabla u(0)$ is transfered to u_λ at small times $t = \lambda^{-2}t_0 \to 0$ as $\lambda \to \infty$, and one even has small $\nabla u_\lambda(0)$. This makes it hard to prove well-posedness.

In the energy-critical case $\alpha = \alpha_c$ we obtain $\|\nabla u_\lambda(t)\|_2 = \|\nabla u(\lambda^2 t)\|_2$ but $\|u_\lambda(t)\|_2 = \lambda^{-1}\|u(\lambda^2 t)\|_2$, so that the above effect is weaker, and there is hope for some wellposedness. In the *energy-subcritical* case $\alpha < \alpha_c$, the behavior is reversed: Good properties for small $\nabla u(0)$ should lead to good properties for large data at small times.

One can discuss in a similar way mass-criticality dropping the derivatives above. This leads to the alternative

$$\alpha \gtreqless \alpha_c^0 \iff \frac{2}{\alpha-1} - \frac{m}{2} \lesseqgtr 0 \qquad \text{with} \quad \alpha_c^0 := 1 + \frac{m}{4}.$$

(See pp.118–120 of [**Ta**] for more details.)

4.4. Asymptotic behavior

In this section we show global existence of solutions for $1 < \alpha < \alpha_c = \frac{m+2}{(m-2)_+}$ in the defocusing case, and in the focusing case if either α or u_0 are small. Moreover, we state a result on asymptotic stability in the defocusing case. A main ingredient of the proofs is the conservation of the L^2-norm and of the energy $\mathcal{E}$ for solutions to (4.1).

So far this property has been only shown for H^2-solutions in Remark 4.4. For such solutions one can expect a blow-up condition in H^2 which does not fit to the energy bound. We thus have to extend the conservation properties to H^1-solutions. This seems to be feasible since $\mathcal{E} \in C^1(H^1, \mathbb{R})$, H^2 is dense in H^1, and the solutions depend continuously on data in H^1 by Theorem 4.16. However, if we approximate a given initial value u_0 in H^1 by functions $\psi_n \in H^2$, we do not yet know whether the solutions $u_n = \varphi(\,\cdot\,, \psi_n)$ belong to $C(J, H^2)$ for an n-independent interval. (Theorem 4.16 only shows that $u_n \in C(J, H^1)$, for any compact $J \subseteq J(u_0)$ and large n.)

Fortunately, one can show that for $v_0 \in H^2$ the corresponding maximal H^1-solution v of (4.1) actually belongs to $C(J(v_0), H^2) \cap C^1(J(v_0), L^2)$, where $J(v_0)$ is the maximal existence interval as an H^1-solution from Theorem 4.16.

To this aim, we modify the proof of Lemma 4.15 involving extra time regularity. In this argument we use the *vector-valued Sobolev space* $W^{1,r}(J, X)$

for $r \in [1,\infty]$, an open interval $J \subset \mathbb{R}$ and a Banach space X.[4] A function u belongs to $W^{1,r}(J,X)$ if $u \in L^r(J,X)$ and there is a map $v \in L^r(J,X)$ with

$$u(t) = u(a) + \int_a^t v(s)\,\mathrm{d}s \tag{4.33}$$

for a.e. $t, a \in J$. One can then fix a number $a \in J$ (and thus a representative of u) such that (4.33) is valid for a.e. $t \in J$. Hence, u is continuous on J and has a continuous extension to $\overline{J}$ by dominated convergence, and so (4.33) holds for all $t, a \in \overline{J}$. We set $u' = v$.

We discuss a few properties of these spaces needed later on. The space $W^{1,r}(J,X)$ is a Banach space when equipped with the norm given by

$$\|u\|_{1,r} = \begin{cases} \big(\|u\|_r^r + \|u'\|_r^r\big)^{\frac{1}{r}}, & \text{if } 1 \le r < \infty, \\ \max\{\|u\|_\infty, \|u'\|_\infty\}, & \text{if } r = \infty, \end{cases}$$

where $\|\cdot\|_r$ is the norm on $L^r(J,X)$. Moreover $W^{1,r}(J,X)$ is isometrically isomorphic to a closed subspace of $L^r(J,X)^2$ via the map $u \mapsto (u,u')$. The remarks after (4.15) and standard results from functional analysis then yield parts a) and b) of the next result.

Remark 4.18. a) Let $1 \le r < \infty$ and X is separable. Then $W^{1,r}(J,X)$ is separable.

b) Let $1 < r < \infty$ and X is reflexive. Then $W^{1,r}(J,X)$ is reflexive.

c) Let X be reflexive. Then $W^{1,\infty}(J,X)$ is isometrically isomorphic to the space of bounded Lipschitz functions $u : \overline{J} \to X$. (See §1.2 in [**ABHN**].)

d) Let $a < b < c$ and $r \in [1,\infty)$. Let $u \in W^{1,r}((a,b),X)$ and $v \in W^{1.r}((b,c),X)$ satisfy $u(b) = v(b)$. Define $w(t) = u(t)$ for $t \in (a,b)$, $w(b) = u(b)$ and $w(t) = v(t)$ for $t \in (b,c)$. Set $g(t) = u'(t)$ for $t \in (a,b)$ and $g(t) = v'(t)$ for $t \in (b,c)$. It is then straightforward to check that $g \in L^r((a,c),X)$ is the derivative of w. The concatenation w thus belongs to $W^{1,r}((a,c),X)$. $\Diamond$

We further need a simple density and embedding result for $W^{1,r}(J,X)$.

Lemma 4.19. *Let $J \subseteq \mathbb{R}$ be an open and bounded interval. Then the following assertions hold.*

a) If $1 \le r < \infty$ then $C^1(\overline{J},X)$ is dense in $W^{1,r}(J,X)$.

b) If $1 \le r \le \infty$, then $W^{1,r}(J,X) \hookrightarrow C(\overline{J},X)$.

Proof. Let $u \in W^{1,r}(J,X)$. As noted after (4.33), we can fix a representative in $C(\overline{J},X)$. Part b) is shown as in Remark 1.42 of [**EE**].

Let $1 \le r < \infty$ and $\overline{J} = [a,b]$. We can approximate u' in $L^r(J,X)$ by $v_n \in C_c(J,X)$, $n \in \mathbb{N}$, as in the proof of Lemma 4.8. Setting

$$u_n(t) = u(a) + \int_a^t v_n(s)\,\mathrm{d}s, \qquad t \in \overline{J},\ n \in \mathbb{N},$$

[4] Most of material on $W^{1,r}(J,X)$ was omitted in the lectures.

we obtain functions $u_n \in C^1(\overline{J}, X)$ with $u_n' = v_n \to u'$ in $L^r(J, X)$. Hölder's inequality further yields

$$\|u_n(t) - u(t)\|_X^r \le \Big(\int_a^t \|v_n(s) - u'(s)\|_X \,\mathrm{d}s\Big)^r \le |J|^{\frac{r}{r'}} \int_J \|v_n(s) - u'(s)\|_X^r \,\mathrm{d}s,$$
$$\|u_n - u\|_r \le |J|\, \|v_n - u'\|_r \longrightarrow 0$$

as $n \to \infty$, where $|J|$ is the length of J. Assertion a) has been shown. □

We now come to the announced regularity result.[5]

PROPOSITION 4.20. *Let (4.20) be true and $u_0 \in H^2$. Then the maximal solution u obtained in Theorem 4.16 is an H^2-solution on $J(u_0)$. Moreover, u belongs to $W^{1,p}((a,b), L^q)$ for all intervals $[a,b] \subseteq J(u_0)$.*

PROOF. Let $u_0 \in H^2(\mathbb{R}^d)$ and let u be the maximal H^1-solution of (4.1) obtained in Theorem 4.16. Take any compact interval $J_0 \subseteq J(u_0)$ containing 0. We have to show that u is an H^2–solution on J_0 with $u \in W^{1,p}(J_0^\circ, L^q)$. By a refinement of the fixed point argument in the proof of Lemma 4.15, we first prove this claim on an interval $J_1 = [-b_1, b_1]$. It turns out that this time $b_1 > 0$ only depends on α, m, and the size

$$\overline{\rho} := \max_{t \in J_0} \|u(t)\|_{1,2}$$

of u in H^1. We can thus repeat the argument for the initial values $u(\pm b_1)$ with the same time step b_1 and deduce the assertion in finitely many iterations. Throughout we use the setting and the notation of the proof of Lemma 4.15 and Theorem 4.16, e.g., the spaces $\mathcal{F}_k(b)$ and their norm $|||v|||_{k,b}$.

1) We fix $\overline{r} := 1 + C_{\mathrm{St}}\overline{\rho}$. Lemma 4.15 and its proof say that the H^1-solution u is a fixed point of the operator Φ (see (4.19)) in the set $\Sigma(\overline{r}, b) = \overline{B}_{\mathcal{F}_1(b)}(0, \overline{r})$, where $0 < b \le b_0(\overline{\rho})$, see (4.22). Set $J = (-b, b)$. We will show that u is also a fixed point in a subset of more regular functions, using that $u_0 \in H^2$. To this aim, for $R \ge C_{\mathrm{St}}\|\Delta u_0\|_2 =: R_0$ (to be fixed below) we define the spaces

$$\mathcal{Z}(b) = \mathcal{F}_1(b) \cap W^{1,p}(J, L^q) \cap W^{1,\infty}(J, L^2),$$
$$\Theta = \Theta(R, b) = \{v \in \mathcal{Z}(b) \,|\, v(0) = u_0,\ |||v|||_{1,b} \le \overline{r},\ |||v'|||_{0,b} \le R\}.$$

The set Θ is non-empty since the Strichartz estimate in Theorem 4.10 a) yields $|||T(\cdot)u_0|||_{1,b} \le C_{\mathrm{St}}\overline{\rho} \le \overline{r}$ and $|||\frac{\mathrm{d}}{\mathrm{d}t}T(\cdot)u_0|||_{0,b} = |||T(\cdot)\Delta u_0|||_{0,b} \le C_{\mathrm{St}}\|\Delta u_0\|_2 \le R$. We endow $\Theta(R, b)$ with the metric given by $|||v - w|||_{0,b}$.

We recall from Remark 4.18 that $W^{1,\infty}(J, L^2)$ is isometrically isomorphic to the space of Lipschitz functions $f : \overline{J} \to L^2$. We first claim that $\Theta(R, b)$ is complete. In fact, take a Cauchy sequence $(v_n)_n$ in $\Theta(R, b)$. Lemma 4.12 yields that $(v_n)_n$ tends in $\mathcal{F}_0(b)$ to a function $v \in \mathcal{F}_1(b)$ with $|||v|||_{1,b} \le \overline{r}$ as $n \to \infty$. Since the maps $v_n : J \to L^2$ converge in $L^\infty(J, L^2)$ to v and have the uniform Lipschitz bound R, we conclude that $v_n \to v$ in $C(\overline{J}, L^2)$ as $n \to \infty$, $v(0) = u_0$, and $v : \overline{J} \to L^2$ is Lipschitz with bound R. By Remark 4.18, the limit v belongs to $W^{1,\infty}(J, L^2)$ with $\|v'\|_{L^\infty(J,L^2)} \le R$. Further, after passing to a subsequence, $(v_{n_j})_j$ has a weak limit w in $W^{1,p}(J, L^q)$ with $\|w'\|_{E_0(J)} \le R$. Since

[5] Parts of the proof were omitted in the lectures.

$E_0(J)^* = L^p(J, L^q)^* \hookrightarrow W^{1,p}(J, L^q)^*$ and $\frac{\mathrm{d}}{\mathrm{d}t} : W^{1,p}(J, L^q) \to E_0(J)$ is linear and bounded, v_{n_j} and v'_{n_j} converge weakly in $E_0(J)$ to w and w', respectively. It follows $v = w$, $v \in \mathcal{Z}(b)$, and $|||v'|||_{0,b} \le R$; i.e., $\Theta(R, b)$ is complete.

2) Let $t \in J$ and $v, w \in \Theta(R, b)$ for $R \ge R_0$ and $b \in (0, b_0(\overline{\rho})]$. We rewrite Φ as

$$\Phi(v)(t) = T(t)u_0 + \int_0^t T(t-s)F(v(s))\,\mathrm{d}s = T(t)u_0 + \int_0^t T(s)F(v(t-s))\,\mathrm{d}s.$$

We want to choose R and b so that $\Phi : \Theta(R, b) \to \Theta(R, b)$ becomes a strict contraction. In Lemma 4.15, (4.21) and (4.22) we have already shown that $\Phi(v)$ belongs to $\mathcal{F}_1(b) \cap C(\overline{J}, H^1)$ and satisfies

$$|||\Phi(v) - \Phi(w)|||_{0,b} \le \tfrac{1}{2} |||v - w\,|||_{0,b} \qquad \text{and} \qquad |||\Phi(v)|||_{1,b} \le \overline{r}. \tag{4.34}$$

3) To treat $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(v)$ in the next step, we first differentiate the convolution term with respect to t. This is done via an approximation argument. Corollary 1.18 and (4.20) show that $F : L^q \to L^{q'}$ is real continuously differentiable with derivative given by $F'(\varphi)\psi = \phi'(\varphi)\psi$ for $\varphi, \psi \in L^q$ and $\phi(z) = -\mathrm{i}\mu|z|^{\alpha-1}z$ for $z \in \mathbb{R}^2$. Moreover,

$$\|F'(\varphi)\psi\|_{q'} \le c_1 \|\varphi\|_q^{\alpha-1} \|\psi\|_q. \tag{4.35}$$

Here and below c_j is a constant only depending on α and m.

Lemma 4.19 allows us to approximate v in $W^{1,p}(J, L^q)$ by $w_n \in C^1(\overline{J}, L^q)$. Passing to a subsequence if necessary, the maps $w'_n(t)$ converge in L^q as $n \to \infty$ and $\|w'_n(t)\|_p \le h(t)$ for all $n \in \mathbb{N}$, a.e. $t \in J$, and a function $h \in L^p(J) \hookrightarrow L^{p'}(J)$, where we note that $p' < 2 < p$. Lemma 4.19 b) and its proof yield that $w_n(0) = v(0) = u_0$ and the sequence $(w_n)_n$ converges to v in $C(\overline{J}, L^q)$. It is thus bounded by a constant $\overline{c}$ in this space. By the properties of F, the functions $F'(w_n(t))w'_n(t)$ tend to $F'(v(t))v'(t)$ in $L^{q'}$ as $n \to \infty$ and satisfy

$$\sup_{n\in\mathbb{N}} \|F'(w_n(t))w'_n(t)\|_{q'} \le \sup_{s\in\overline{J}, n\in\mathbb{N}} c_1 \|w_n(s)\|_q^{\alpha-1} \|w'_n(t)\|_q \le c_1 \overline{c}^{\,\alpha-1} h(t).$$

From dominated convergence we deduce that the maps $F'(w_n)w'_n$ have the limit to $F'(v)v'$ in $L^{p'}(J, L^{q'})$ as $n \to \infty$.

Because $L^{q'} \hookrightarrow H^{-1}$ by (4.7), the function $F(w_n) : \overline{J} \to H^{-1}$ is continuously differentiable and so the derivative

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t T(s)F(w_n(t-s))\,\mathrm{d}s &= T(t)F(w_n(0)) + \int_0^t T(s)F'(w_n(t-s))w'_n(t-s)\,\mathrm{d}s \\ &= T(t)F(u_0) + \int_0^t T(t-s)F'(w_n(s))w'_n(s)\,\mathrm{d}s\end{aligned}$$

exists in H^{-1}. (In this calculation we identify $\mathbb{C}$ with $\mathbb{R}^2$.) Observe that $F(u_0)$ belongs to L^2 since $u_0 \in H^2$ and $F \in C^1(H^2, L^2)$, cf. (4.10). The Strichartz estimate in Theorem 4.10 b) thus implies that the right-hand side of the above identity is continuous in L^2 and converges to

$$T(t)F(u_0) + \int_0^t T(t-s)F'(v(s))v'(s)\,\mathrm{d}s$$

in L^2 uniformly in t as $n \to \infty$. Similarly, the integral on the left-hand side tends to $T *_+ F(v)(t)$ in L^2 uniformly in t. We can thus differentiate $T *_+ F(v)$ in L^2 and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t T(s)F(v(t-s))\,\mathrm{d}s = T(t)F(u_0) + \int_0^t T(t-s)F'(v(s))v'(s)\,\mathrm{d}s,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(v)(t) = T(t)(\mathrm{i}\Delta u_0 + F(u_0)) + \int_0^t T(t-s)F'(v(s))v'(s)\,\mathrm{d}s. \tag{4.36}$$

4) In this step we prove that $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(v)$ is an element of $\mathcal{F}_0(b)$ with $|||\frac{\mathrm{d}}{\mathrm{d}t}\Phi(v)|||_{0,b} \le R$. It is crucial that R will enter only linearly. Using inequality (4.35), Sobolev's embedding (4.4) and $\|v(s)\|_{1,2} \le \overline{r}$, we derive

$$\|F'(v(s))v'(s)\|_{q'} \le c_1 C_{\mathrm{So}}^{\alpha-1}\overline{r}^{\alpha-1}\|v'(s)\|_q$$

for all $s \in J$. The inhomogeneous Strichartz estimate and Hölder's inequality now allow us to bound the $\mathcal{F}_0(b)$–norm of the integral term in (4.36) by

$$\begin{aligned} C_{\mathrm{St}}\|F'(v)v'\|_{E_0'(b)} &\le C_{\mathrm{St}}c_1C_{\mathrm{So}}^{\alpha-1}\overline{r}^{\alpha-1}\|v'\|_{L^{p'}(J,L^q)} \le c_2\overline{r}^{\alpha-1}b^{\frac{1}{p'}-\frac{1}{p}}\|v'\|_{E_0(b)} \\ &\le c_2\overline{r}^{\alpha-1}b^{\frac{1}{p'}-\frac{1}{p}}R, \end{aligned} \tag{4.37}$$

using $v \in \Theta(R,b)$. By Sobolev's embedding (4.4), we have $H^2 \hookrightarrow L^{2\alpha}$ and hence $\|F(u_0)\|_2 = \|u_0\|_{2\alpha}^{\alpha} \le c_3\,\|u_0\|_{2,2}^{\alpha}$. Equation (4.36), the Strichartz estimates and estimate (4.37) thus yield that

$$|||\tfrac{\mathrm{d}}{\mathrm{d}t}\Phi(v)|||_{0,b} \le C_{\mathrm{St}}(\|\Delta u_0\|_2 + c_3\|u_0\|_{2,2}^{\alpha}) + c_2\overline{r}^{\alpha-1}b^{\frac{1}{p'}-\frac{1}{p}}R \tag{4.38}$$

and that $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(v)$ belongs to $C(\overline{J}, L^2)$. We fix

$$R_1 = 2C_{\mathrm{St}}(\|\Delta u_0\|_2 + c_3\|u_0\|_{2,2}^{\alpha}) \ge R_0 \quad \text{and} \quad b_1 = \min\big\{b_0(\overline{\rho}), (2c_2\overline{r}^{\alpha-1})^{\frac{p'-p}{p'p}}\big\}.$$

Since $\overline{r} = 1 + C_{\mathrm{St}}\overline{\rho}$, the number b_1 only depends on $\overline{\rho}$, α and m. The inequalities (4.34) and (4.38) show that $\Phi : \Theta(R_1, b_1) \to \Theta(R_1, b_1)$ is a strict contraction.

We thus obtain a fixed point $v_* = \Phi(v_*)$ in $\Theta(R_1, b_1)$ contained in $C(J_1, H^1) \cap C^1(J_1, L^2) \cap W^{1,q}(J_1^\circ, L^q)$, where $J_1 = [-b_1, b_1]$. The function v_* is an H^1-solution of (4.1) on J_1 by Lemma 2.8 of [**EE**]. Hence, $u = v_*$ by the uniqueness of (4.1).

5) We still have to show that $u \in C(J_1, H^2)$. To prove this fact, we use a 'boot-strapping' argument based on the invertibility of $I - \Delta : W^{2,r} \to L^r$ for all $r \in (1, \infty)$, see Example 2.29 in [**EE**]. If $\varphi \in C_c^\infty$, then there is a unique solution $v \in \bigcap_r W^{2,r}$ of $v - \Delta v = \varphi$, see (1.26) in [**EE**]. By density, we see that the inverses $(I - \Delta)^{-1}$ coincide on $L^r \cap L^s$ for $r, s \in (1, \infty)$.

As a starting point, we rewrite (4.1) as

$$u - \Delta u = u + \mathrm{i}u' - \mathrm{i}F(u) = f + g,$$

where $f := u + \mathrm{i}u'$ belongs to $C(J_1, L^2)$ and $g := -\mathrm{i}F(u)$ to $C(J_1, L^{q/\alpha})$ since $u \in C^1(J_1, L^2(\mathbb{R}^d)) \cap C(J_1, L^q)$. We have $\frac{q}{\alpha} = q' \in (1,2)$ by (4.20). It follows that $(I - \Delta)^{-1}f \in C(J_1, H^2)$ and $(I - \Delta)^{-1}g \in C(J_1, W^{2,q/\alpha})$. Because of $2 - \frac{m}{2} > 1 - \frac{m\alpha}{q}$, Sobolev's embedding Theorem 3.17 of [**ST**] yields

$$u = (I - \Delta)^{-1}(f + g) \in C(J_1, L^{r_1})$$

for $r_1 = \frac{m}{m\alpha - 2q} q =: \gamma q > q$ if $m\alpha > 2q$ and for any $r_1 \in (2, \infty)$ otherwise (e.g. if $m \in \{1, 2\}$). Observe that $\gamma > 1$ if $m\alpha > 2q$ since $q = \alpha + 1$ and $\alpha < \frac{m+2}{m-2}$.

This extra integrability of u implies that g is an element of $C(J_1, L^{r_1/\alpha})$, by (4.9). If $r_1 \geq 2\alpha$, the function g belongs to $C(J_1, L^2)$ since then $L^{q/\alpha} \cap L^{r_1/\alpha} \hookrightarrow L^2$ by Hölder's inequality. As a result, $u = (I - \Delta)^{-1}(f + g)$ is contained in $C(J_1, H^2)$ in this case.

If $r_1 < 2\alpha$, as above we infer that $u \in C(J_1, L^{r_2})$ for $r_2 = \frac{mr_1}{m\alpha - 2r_1} \geq \gamma r_1 = \gamma^2 q$ if $m\alpha > 2r_1$ and for any $r_2 \in (2, \infty)$ if $m\alpha \leq 2r_1$. Since $\gamma > 1$, in finitely many steps we arrive at $r_N \geq \gamma^N q \geq 2\alpha$, and hence $u \in C(J_1, H^2)$.

6) We can now finish the proof. If $J_0 \subseteq J_1$ we are done. If not, assume that $\max J_0 > b_1$. Since $u(b_1) \in H^2$, we can repeat steps 2) – 5) with initial value $u(b_1)$ and the same time step b_1. We then obtain an H^2-solution u_1 of (4.1) on $[b_1, 2b_1]$ with $u_1(b_1) = u(b_1)$. Lemma 4.13 allows us to glue together these functions to an H^2-solution v on $[-b_1, 2b_1]$ with $v \in W^{1,p}((-b_1, 2b_1), L^q)$. The uniqueness of H^1–solutions yields that $v = u$ on $[-b_1, 2b_1]$. The assertion now follows in finitely many iterations. □

Based on the above result we can show global existence in the defocusing and in the focusing case if $\alpha < 1 + \frac{4}{m}$ or u_0 is small in H^1. We stress that there is blow-up if $\alpha \in [1 + \frac{4}{m}, \alpha_c)$ and $\mu = -1$ by Theorem 6.5.10 in [**Ca**].

THEOREM 4.21. *Let (4.20) be true, $u_0 \in H^1$, and u be the maximal H^1-solution of (4.1) on $J(u_0)$. Then the following assertions are true.*

a) We have $\|u(t)\|_2 = \|u_0\|_2$ and $\mathcal{E}(u(t)) = \mathcal{E}(u_0)$ for all $t \in J(u_0)$.

b) Let $\mu = 1$. Then $J(u_0) = \mathbb{R}$ for all $u_0 \in H^1$.

c) Let $\mu = -1$ and $\alpha < 1 + \frac{4}{m}$. Then $J(u_0) = \mathbb{R}$ for all $u_0 \in H^1$.

d) Let $\mu \in \{-1, 1\}$ and $\alpha \in (1, \alpha_c)$. Then there is a constant $\varepsilon_0 > 0$ with the following property: For all $\varepsilon \in (0, \varepsilon_0]$ there is a radius $\rho = \rho(\varepsilon) > 0$ such that for all $u_0 \in \overline{B}_{H^1}(0, \rho)$ we have $J(u_0) = \mathbb{R}$ and $\|u(t)\|_{1,2} \leq \varepsilon$ for all $t \in \mathbb{R}$.

In all three cases the nonlinearity is relatively tame so that it does not destroy the global existence that we have in the linear case: In b) we have the right sign $\mu = 1$. In c) the nonlinearity does not grow too much. In d) the solution u is small initially which leads to an even smaller nonlinearity $|u|^{\alpha-1}u$. Moreover, the trivial fixed point $u_* = 0$ is stable in all cases. As we will see in the proof, in these situations the mass and energy of a solution control its H^1-norm, so that the blow-up condition implies global existence.

These results and their proofs are typical for evolution equations possessing

- a 'full' local wellposedness theory as in Theorem 4.16,
- a conserved quantity which dominates the norm of the blow-up condition (directly or under a smallness condition) at least for a class of 'good solutions',
- and a regularity result which shows that these good solutions exist as long as those from the local wellposedness theorem.

PROOF OF THEOREM 4.21. a) The first assertion is true for H^2-solutions by Remark 4.4. There are functions $\psi_n \in H^2$ tending to u_0 in H^1 as $n \to \infty$. Set

$u_n = \varphi(\,\cdot\,, \psi_n)$. Proposition 4.20 says that u_n is an H^2-solution on the maximal interval $J(\psi_n)$ from Theorem 4.16. Take any compact interval $J \subseteq J(u_0)$. Due to Theorem 4.16, there is an index $N_J \in \mathbb{N}$ such that $J \subseteq J(\psi_n)$ for $n \geq N_J$ and $u_n(t)$ converges to $u(t)$ in H^1 as $n \to \infty$, uniformly for $t \in J$. Assertion a) then follows by approximation since $\mathcal{E} \in C^1(H^1, \mathbb{R})$, cf. (4.6).

b) Let $\mu = 1$. In this case, from step a) and (4.5) we derive

$$\|u(t)\|_{1,2}^2 \leq 2\mathcal{E}(u(t)) + \|u(t)\|_2^2 = \mathcal{E}(u_0) + \|u_0\|_2^2$$

for all $t \in J(u_0)$. The blow–up criterion in Theorem 4.16 now yields $J(u_0) = \mathbb{R}$.

c) Let $1 < \alpha < 1 + \frac{4}{m}$ and $\mu = -1$. We consider $m \geq 3$, the proof for $m \in \{1, 2\}$ is similar. Observe that

$$\frac{1}{\alpha+1} = \frac{1-\theta}{2} + \theta\,\frac{m-2}{2m} \qquad \text{for} \quad \theta = \frac{m}{2} - \frac{m}{\alpha+1} \in (0,1).$$

The interpolation and Sobolev inequalities (see (3.16) in [**ST**]) thus imply the estimate[6]

$$\|v\|_{\alpha+1}^{\alpha+1} \leq \Big(\|v\|_2^{1-\theta}\,\|v\|_{\frac{2m}{m-2}}^{\theta}\Big)^{\alpha+1} \leq c\,\|v\|_2^{\alpha+1-m(\alpha-1)/2}\,\||\nabla v|\|_2^{m(\alpha-1)/2}$$

for all $v \in H^1$. We have $\beta := \frac{4}{m(\alpha-1)} > 1$ by our assumption. Young's inequality with β and β' leads to

$$\frac{1}{\alpha+1}\,\|v\|_{\alpha+1}^{\alpha+1} \leq \frac{1}{4}\,\||\nabla v|\|_2^2 + c\,\|v\|_2^{\beta'(\alpha+1-m(\alpha-1)/2)}$$

for a constant only depending on α and m. Denoting the last summand by $k(\|v\|_2)$, we infer from step a) that

$$\begin{aligned}\mathcal{E}(u_0) = \mathcal{E}(u(t)) &= \frac{1}{2}\,\||\nabla u(t)|\|_2^2 - \frac{1}{\alpha+1}\,\|u(t)\|_{\alpha+1}^{\alpha+1} \\ &\geq \frac{1}{4}\,\||\nabla u(t)|\|_2^2 - k(\|u(t)\|_2) = \frac{1}{4}\,\||\nabla u(t)|\|_2^2 - k(\|u_0\|_2)\end{aligned}$$

for all $t \in J(u_0)$. Hence, $\|u(t)\|_{1,2}^2 \leq 4\mathcal{E}(u_0) + 4k(\|u_0\|_2) + \|u_0\|_2^2$ for all $t \in J(u_0)$, and again it follows that $J(u_0) = \mathbb{R}$.

d) Definition (4.5), part a), and Sobolev's embedding (4.4) yield

$$\begin{aligned}\frac{1}{2}\,\|u(t)\|_{1,2}^2 &= \frac{1}{2}\|u(t)\|_2^2 + \mathcal{E}(u(t)) - \frac{\mu}{\alpha+1}\|u(t)\|_{\alpha+1}^{\alpha+1} \\ &\leq \frac{1}{2}\|u_0\|_2^2 + \mathcal{E}(u_0) + c_0\|u(t)\|_{1,2}^{\alpha-1}\,\|u(t)\|_{1,2}^2 \end{aligned} \tag{4.39}$$

for all $t \in J(u_0)$ and with $c_0 = C_{\mathrm{So}}^{\alpha+1}/(\alpha+1)$. We set $\varepsilon_0 = (4c_0)^{1/(1-\alpha)}$ and take any $0 < \rho < \varepsilon \leq \varepsilon_0$. Let $\|u_0\|_{1,2} \leq \rho$. We now define

$$\tau = \sup\{t \in (0, t_+(u_0)) \mid \|u(s)\|_{1,2} \leq \varepsilon \text{ for all } s \in [0,t]\}$$

and observe that $\tau \in (0, t_+(u_0)]$. Estimates (4.39) and (4.4) then lead to

$$\frac{1}{2}\,\|u(t)\|_{1,2}^2 \leq \frac{1}{2}\|u_0\|_2^2 + \mathcal{E}(u_0) + \frac{1}{4}\|u(t)\|_{1,2}^2,$$

[6] Such estimates are called Gagliardo–Nirenberg inequalities.

$$\|u(t)\|_{1,2}^2 \le 2\|u_0\|_{1,2}^2 + \frac{4}{\alpha+1}\|u_0\|_{\alpha+1}^{\alpha+1} \le c_1(\rho^2 + \rho^{\alpha+1}) \tag{4.40}$$

for all $t \in [0,\tau)$ and a constant $c_1 > 0$ depending only on α and m. We finally choose $\rho = \rho(\varepsilon) \in (0,\varepsilon)$ such that $c_1(\rho^2 + \rho^{\alpha+1}) \le \varepsilon/2$. Then (4.40) would be impossible if $\tau < t_+(u_0)$. We thus obtain $\tau = t_+(u_0)$ and so Theorem 4.16 d) gives $t_+(u_0) = \infty$ and the asserted bound for $t \ge 0$. Similarly one treats negative times, possibly decreasing ρ. □

Global existence holds in the defocusing case also if $\alpha = \alpha_c$ and $m \ge 3$. This deep result is far beyond the scope of these lectures, see Chapter 5 of [**Ta**] for an extended survey.

The next result is a direct consequence of the (hard to prove) 'pseudo–confomal' conservation law for (4.1), Theorem 7.2.1 in [**Ca**]. It says that solutions decay to 0 as $|t| \to \infty$ in the defocusing case.

Proposition 4.22. *Let $1 + 4/m \le \alpha < \alpha_c$, $\mu = 1$ and $u_0 \in H^1$ with $|x|u_0 \in L^2$,and u be the corresponding H^1-solution of (4.1) on $\mathbb{R}$. Then there is a constant $c > 0$ such that*

$$\|u(t)\|_{\alpha+1} \le c\,|t|^{-\frac{2}{\alpha+1}}\,\||x|u_0\|_2^{\frac{2}{\alpha+1}} \qquad \text{for } \; t \in \mathbb{R}.$$

Proof. Let $t \in J(u_0) = \mathbb{R}$. Theorem 7.2.1 in [**Ca**] yields

$$\|(x+2\mathrm{i}t\nabla)u(t)\|_2^2 + \frac{8t^2}{\alpha+1}\,\|u(t)\|_{\alpha+1}^{\alpha+1} = \|xu_0\|_2^2 + 4\frac{m+4-\alpha m}{\alpha+1}\int_0^t s\,\|u(s)\|_{\alpha+1}^{\alpha+1}\,\mathrm{d}s.$$

Since the last summand is non-positive, the assertion follows. □

A more precise version of this result is given in Theorem 7.3.1 of [**Ca**]. It also covers the full range $\alpha \in (1,\alpha_c)$ and allows for $r \in (2, 1+\alpha_c]$ ($r \in (2, 1+\alpha_c)$ if $m = 2$). In many cases one obtains the decay $c|t|^{\frac{m}{r}-\frac{m}{2}}$ as in Corollary 4.6.

In the setting of the proposition it can be shown that there are maps $u_\pm \in H^1$ with $|x|u_\pm \in L^2$ such $T(-t)u(t) \to u_\pm$ as $t \to \pm\infty$ in the norm $\|v\|_{1,2} + \|xv\|_2$. See Theorem 7.4.1 of [**Ca**]. There are variants of these results in H^1 without weights, see Sections 7.7 and 7.8 in [**Ca**].

In the focusing case, the qualitative behavior is completey different. In Example 4.2 we have seen that (4.1) admits standing waves if $\mu = -1$ and $\alpha \in (1,\alpha_c)$, see Section 8.1 of [**Ca**]. Because of symmetries, these standing waves form a finite dimensional manifold in H^1. It is unstable in H^1 (due to blow up) if $\alpha \ge 1 + \frac{4}{m}$, and stable if $\alpha < 1 + \frac{4}{m}$ by Sections 8.2 and 8.3 of [**Ca**].

CHAPTER 5

Quasilinear parabolic problems

So far we have investigated semilinear problems in which a generator was perturbed by a nonlinear term of 'lower order'. In a quasilinear problem the linear operator itself depends nonlinearly on the solution though this dependence is again of 'lower order'.

In this short chapter we prove the local wellposedness of a basic class of such systems in the parabolic case. We study the equation

$$u'(t) = A(u(t))u(t) + F(u(t)), \quad t \in J, \qquad u(0) = u_0, \tag{5.1}$$

requiring that $A(v)$ with *fixed* domain X_1 is a sectorial operator in X of angle $\varphi > \pi/2$ depending on vectors $v \in X_\gamma := (X, X_1)_{1-\frac{1}{p},p}$ for a fixed $p \in (1,\infty)$, cf. Section 2.1. We assume that the mappings $A : X_\gamma \to \mathcal{B}(X_1, X)$ and $F : X_\gamma \to X$ are Lipschitz on balls, that $u_0 \in X_\gamma$, and that $J \subseteq \mathbb{R}$ is a non-empty open interval with $0 = \inf J$. We first present a prototypical example.

Example 5.1. Let $G \subseteq \mathbb{R}^m$ be an open and bounded set with a C^2 boundary and let $a \in C^2(\mathbb{R}, \mathbb{R}^{m\times m})$ satisfy $a = a^\top \geq \eta I$ for a number $\eta > 0$. Fixing an exponent $p \in (m+2, \infty)$, we set $X = L^p(G)$ and $X_1 = W^{2,p}(G) \cap W^{1,p}(G)$. As explained in Example 2.12, one has the embedding $X_\gamma \hookrightarrow W^{2-\frac{2}{p},p}(G)$. The fractional Sobolev embedding theorem then implies that $X_\gamma \hookrightarrow C^1(\overline{G})$ since $2 - \frac{2}{p} - \frac{m}{p} > 0$, cf. Theorem 4.6.1 in [**Tr1**]. We define the operator $A(v)$ by $A(v)u = \operatorname{div}(a(v)\nabla u)$ for $u \in X_1$ and $v \in X_\gamma$, and let $F(v) = f(v, \nabla v)$ be a reaction-convection term as in Example 3.6.

In this case the quasilinar problem (5.1) becomes the reaction-diffusion equation with state-depending anisotropic diffusion coefficients

$$\begin{aligned}
\partial_t u(t,x) &= \sum_{j,k=1}^m \partial_j\big(a_{jk}(u(t,x)\partial_k u(t,x)\big) + f(u(t,x), \nabla u(t,x)), \quad t > 0,\ x \in G, \\
u(t,x) &= 0, \quad t > 0,\ x \in \partial G, \\
u(0,x) &= u_0(x), \quad x \in G.
\end{aligned} \tag{5.2}$$

We come back to this equation in Example 5.8. One can also treat analogous systems. Neumann-type boundary conditions are not covered by our setting since then the domain of $A(v)$ contains the condition $\operatorname{tr}(\nu(a(v)\nabla u) = 0$ and thus depends on v.[1] See also the comments after Theorem 5.7. $\Diamond$

[1] The domain becomes v-independent, however, if one replaces X_1 by $W^{1,q}(G)$ and passes to a weak formulation.

We want to solve (5.1) by a fixed point procedure again. For a solution $u \in C(\overline{J}, X_1)$ we write

$$u'(t) = A(u_0)u(t) + (A(u(t)) - A(u_0))u(t) + F(u(t)). \tag{5.3}$$

One now replaces u on the right-hand side by a given function v in our fixed point space, say $v \in C(\overline{J}, X_1)$. Using the analytic C_0-semigroup $T(\cdot)$ generated by $A(u_0)$, Duhamel's formula then yields

$$u(t) = T(t)u_0 + \int_0^t T(t-s)\big(((A(v(s)) - A(u_0))v(s) + F(v(s))\big)\,\mathrm{d}s =: \Phi(v).$$

In a fixed point procedure, the image $\Phi(v)$ must also belong to $C(\overline{J}, X_1)$. However, the inhomogeneity $f = (A(v) - A(u_0))v + F(v)$ is just an element of $C(\overline{J}, X)$ so that

$$T *_+ f(t) = \int_0^t T(t-s)f(s)\,\mathrm{d}s$$

is not contained in X_1, in general, since $T(t-s)$ has norm $c/(t-s)$ in $\mathcal{B}(X, X_1)$. See Example 4.1.7 in [**Lu1**].

To overcome this problem, one can pass to classes of more regular functions, e.g., v being Hölder continuous in time or taking values in interpolation spaces smaller than X_1. This is done in [**Am1**], [**Lu1**] or [**Ya**] in various ways.

Here we follow a different route by reducing the regularity level a bit. For this we have to introduce a new concept and discuss its basic properties. Let A be a closed, densely defined operator in X, where $X_1 = [\mathrm{D}(A)]$. We say that A has *maximal regularity of type* L^p on J if for all $f \in L^p(J, X) =: E_0(J)$ there exists a unique function $u \in W^{1,p}(J, X) \cap L^p(J, X_1) =: E_1(J)$ solving[2]

$$u'(t) = Au(t) + f(t), \quad \text{a.e. } t \in J, \qquad u(0) = 0. \tag{5.4}$$

We then write $A \in \mathrm{MR}_p(J)$.

Let $S : E_0(J) \to E_1(J);\ f \mapsto u$, be the solution operator of (5.4). Let $f_n \to 0$ in $E_0(J)$ and $u_n = Sf_n \to u$ in $E_1(J)$ as $n \to \infty$. Then u solves (5.4) with $f = 0$ and it thus must be equal to 0 by uniqueness. The closed graph theorem now shows that $\|u\|_{E_1(J)} \le C_p \|f\|_{E_0(J)}$ for a constant $C_p > 0$ and all $f \in E_0(J)$.

We next collect other basic facts about these spaces and this concept; many of them are employed in the proof of the local wellposedness result below.

Remark 5.2. We use the notation and definions introduced above.

a) We have the embeddings $E_1(J) \hookrightarrow C(\overline{J}, X_\gamma)$ and $E_1(\mathbb{R}_+) \hookrightarrow C_0(\mathbb{R}_{\ge 0}, X_\gamma)$. See Theorem III.4.10.2 in [**Am2**] and statement (3.81) in [**PS**]. The initial condition of (5.4) is understood in this sense. Moreover, the constant $c(J)$ of the embeddings can be bounded uniformly for intervals J whose length b is larger than a fixed number $b_0 > 0$. As $b \to 0$ the constant $c(b)$ will blow up e.g. for functions $t \mapsto v(t) = v_0$ with $v_0 \in X_1 \setminus \{0\}$, which is a severe obstacle in a fixed point argument. On a subspace of $E_1(J)$ this does not happen:

Let $b > 0$ and $v \in E_1(b)$ with $v(0) = 0$. We reflect v at b and extend the resulting function by 0 for $t > 2b$. This yields an extension $\tilde{v} \in E_1(\mathbb{R}_+)$ with

[2] We write $E_k(b)$ if $J = (0, b)$.

norm less or equal $2\|v\|_{E_1(b)}$. We then obtain the uniform bound

$$\|v\|_{C([0,b],X_\gamma)} = \|\tilde{v}\|_{C_0(\mathbb{R}_{\geq 0},X_\gamma)} \leq c(\mathbb{R}_+)\|\tilde{v}\|_{E_1(\mathbb{R}_+)} \leq 2c(\mathbb{R}_+)\|v\|_{E_1(b)}.$$

b) Let $A \in \mathrm{MR}_p(J)$ for an interval J and some $p \in (1,\infty)$. Then there is a number $\omega \geq 0$ such that A is sectorial of angle $\varphi > \pi/2$. In the case $J = \mathbb{R}_+$ one can choose $\omega = 0$, and A is invertible. See Proposition 3.5.2 in [**PS**]. The proof implies that ω and the sectoriality constants (K,ϕ) of A are bounded by constants only depending on J, p, and C_p above. In turn, Lemma 2.22 of [**EE**] gives an exponential bound of the semigroup in terms of ω, K and ϕ.

The solution of (5.4) is thus given by $u = T *_+ f$ for the analytic C_0-semigroup $T(\cdot)$ generated by A due to an L^p-variant of Proposition 2.6 of [**EE**]. If one includes an initial value $u_0 \neq 0$ in (5.4), we have the solution $u = T(\cdot)u_0 + T *_+ f$. In the next item we treat the orbit $T(\cdot)u_0$.

c) Let A generate an analytic C_0-semigroup $T(\cdot)$ and $p \in (1,\infty)$. By Theorem 2.14, $T(\cdot)$ induces an analytic C_0-semigroup on X_γ by restriction. Proposition 2.8 says that a vector u_0 belongs to X_γ if and only if the orbit $T(\cdot)u_0$ is an element of $E_1(1)$. In this case one has $\|T(\cdot)u_0\|_{E_1(1)} \leq c\|u_0\|_{X_\gamma}$ for some constant $c > 0$. It follows that $\|T(\cdot)u_0\|_{E_1(b)} \leq c\|u_0\|_{X_\gamma}$ for $b \in (0,1)$. We further estimate

$$\|T(\cdot)u_0\|_{E_1(1,2)} = \|T(\cdot)T(1)u_0\|_{E_1(1)} \leq c\|T(1)\|_{\mathcal{B}(X_\gamma)}\,\|u_0\|_{X_\gamma}.$$

Iteratively we obtain that $\|T(\cdot)\|_{E_1(b)} \leq c(b_0)\|u_0\|_{X_\gamma}$ for $b \in (0,b_0]$ and any given $b_0 > 0$. If $\omega_0(A) < 0$, one finds a uniform constant for all $b > 0$ by this argument.

d) Let $A \in \mathrm{MR}_p(J)$. Let $J_0 \subseteq J = (0,b)$ be an open subinterval with $\inf J_0 = 0$. We extend $f \in E_0(J_0)$ by 0 to a function $\tilde{f} \in E_0(J)$. Then one has $T *_+ f = T *_+ \tilde{f}$ on J_0 and hence

$$\|u\|_{E_1(J_0)} \leq \|u\|_{E_1(J)} \leq c(J)\|\tilde{f}\|_{E_0(J)} = c(J)\|f\|_{E_0(J_0)}$$

for the solution u of (5.4). Let $f \in E_0(2b)$. For $t \in (b,2b)$ we compute

$$\begin{aligned}\int_0^t T(t-s)f(s)\,\mathrm{d}s &= \int_b^t T(t-s)f(s)\,\mathrm{d}s + T(t-b)\int_0^b T(b-s)f(s)\,\mathrm{d}s\\ &= \int_0^{t-b} T(t-b-r)f(r+b)\,\mathrm{d}r + T(t-b)\int_0^b T(b-s)f(s)\,\mathrm{d}s.\end{aligned}$$

The first term can be estimated in $E_1((b,2b))$ by the maximal regularity on $(0,b)$. Part a) implies that the norm in X_γ of the last integral is bounded by $\|f\|_{E_0(b)}$. Hence, the second summand is controlled in $E_1(b,2b)$ using part c). Summing up, we have maximal regularity on $(0,2b)$. This procedure can be iterated, so that $A \in \mathrm{MR}_p(b')$ for all $b' > 0$. This argument also yields that A has maximal regularity on $\mathbb{R}_+$ if $T(\cdot)$ is exponentially stable, in addition.

e) It is easy to check that A has maximal regularity on a bounded interval J if and only $A - \omega I \in \mathrm{MR}_p(J)$, since the later operator generates $\mathrm{e}^{-\omega t}T(t)$. For $\omega > \omega_0(A)$, the operator $A - \omega I$ then has maximal regularity on $\mathbb{R}_+$ by part d).

f) If an operator has maximal regularity of type L^p for some $p \in (1,\infty)$, it has this property for all $p \in (1,\infty)$. See Theorem 4.2 in [**Do**]. (This result requires some harmonic analysis.)

g) There exist quite explicit examples of sectorial operators of angle $\varphi > \pi/2$ on spaces $L^1(\mu)$ or $C(K)$ without maximal regularity of type L^p. On L^q spaces with $q \neq 2$ such examples also exist, but they are not very explicit. See Paragraphs 1.14 and 1.15 of [**KW**]. ◊

We describe two classes of operators having maximal regularity of type L^p. The first one says that this property comes for free on Hilbert spaces, and the second one suffices for our Example 5.1.

Example 5.3. Let X be a Hilbert space and A be sectorial of angle $\varphi > \pi/2$ on X. Then A has maximal regularity of type L^p.

Proof. By Remark 5.2 it is enough to consider the case $p = 2$, and one can assume that $\omega_0(A) < 0$ and $J = \mathbb{R}_+$. We extend $f \in E_0(\mathbb{R}_+)$ and $T(\cdot)$ by 0 to $t < 0$ so that $T *_+ f(t) = T * f(t)$ for $t \geq 0$.

First take $f \in L^p(\mathbb{R}_+, X_1)$. Since X is a Hilbert space, Plancherel's theorem is valid for the Fourier transform $\mathcal{F}$ on $L^2(\mathbb{R}, X)$, see Theorem 1.8.2 in [**ABHN**]. On further has the usual convolution theorem for $\mathcal{F}$ by Theorem 1.8.1. Using also the resolvent formula (4.3) from [**EE**], we compute

$$\begin{aligned}
\|T *_+ f\|_{L^2(\mathbb{R}_+,X_1)} &\leq \|T * f\|_{L^2(\mathbb{R},X)} + \|AT * f\|_{L^2(\mathbb{R},X)} \\
&= \|\mathcal{F}(T * f)\|_{L^2(\mathbb{R},X)} + \|\mathcal{F}(T * Af)\|_{L^2(\mathbb{R},X)} \\
&= \sqrt{2\pi}\big(\|\widehat{T}\widehat{f}\|_{L^2(\mathbb{R},X)} + \|\widehat{T}A\widehat{f}\|_{L^2(\mathbb{R},X)}\big) \\
&= \|R(\mathrm{i}\,\cdot, A)\widehat{f}\|_{L^2(\mathbb{R},X)} + \|AR(\mathrm{i}\,\cdot, A)\widehat{f}\|_{L^2(\mathbb{R},X)} \\
&\leq c\,\|\widehat{f}\|_{L^2(\mathbb{R},X)} = c\,\|f\|_{L^2(\mathbb{R}_+,X)}.
\end{aligned}$$

Note that the operators $R(\mathrm{i}\tau, A)$ and $AR(\mathrm{i}\tau, A) = \mathrm{i}\tau R(\mathrm{i}\tau, A) - I$ are uniformly bounded for $\tau \in \mathbb{R}$ as A generates an exponentially stable analytic C_0-semigroup. By approximation the result then follows. □

Example 5.4. Let A generate an analytic C_0-semigroup $T(\cdot)$ on $L^2(\mu)$ for a measure space $(S, \mathcal{A}, \mu)$. Assume that there is a constant $\omega \geq 0$ such that $\|\mathrm{e}^{-\omega t}T(t)f\|_q \leq \|f\|_q$ for all $f \in L^2(\mu) \cap L^q(\mu)$, $q \in [1,\infty]$, and $t \geq 0$. Then A has maximal regularity of type L^p, see Theorem 4.4 in [**Do**].

Moreover, one has maximal regularity for each generator A of a positive and contractive analytic C_0-semigroup on $L^q(\mu)$ for some $q \in (1,\infty)$ as indicated in Example 1.13 and Note 1.13 in [**KW**]. These results rely on deeper tools from operator theory and harmonic analysis. ◊

In view of the above example, it should be noted that semigroups generated by elliptic systems often fail to be contractive. Fortunately there is a quite convenient characterization of maximal regularity in 'good' Banach spaces due to Lutz Weis (2001). We present it now without giving many details.

We first describe the relevant class of Banach spaces. Let $f \in C_c^1(\mathbb{R}, X)$. By p. 374 of [**HNVW**] the limit

$$Hf(t) = \lim_{\varepsilon \to 0^+, r \to \infty} \int_{\varepsilon < |s| < r} \frac{f(s)}{\pi(t-s)} \, \mathrm{d}s$$

exist for all $t \in \mathbb{R}$. One calls Hf the *Hilbert transform* of f. We call X a *UMD space*[3] if H has a continuous extension to $L^p(\mathbb{R}, X)$ for some (and then all) $p \in (1, \infty)$, cf. Theorem 5.1.1 in [**HNVW**]. The spaces needed in our examples belong to this class.

Example 5.5. The spaces $X = L^q(\mu)$ for a measure space $(S, \mathcal{A}, \mu)$ and $q \in (1, \infty)$ are of class UMD by Proposition 4.2.15 of [**HNVW**]. Proposition 4.2.17 of this monograph further shows that Cartesian products, closed subspaces, quotient spaces, duals, isomorphic images, and real (and complex) interpolation spaces with exponent $r \in (1, \infty)$ of UMD spaces have the same property. Hence, (closed subspaces of) Sobolev-Slobodetski $W^{\alpha,q}(G)$ and Besov spaces $B^{\alpha}_{q,r}(G)$ are UMD if $\alpha \geq 0$ and $1 < q, r < \infty$, cf. Example 2.3 and 2.5. $\Diamond$

We also need a stronger sectoriality concept. Let $\varepsilon_n : \Omega \to \{-1, 1\}$ be measurable functions on a probability space which are (stochastically) independent and have expectation 0 for $n \in \mathbb{N}$. An example are the Rademacher functions $r_n(t) = \operatorname{sign} \sin(2^n \pi t)$ on $\Omega = (0, 1)$ with the Lebesgue measure. A set $\mathcal{T} \subseteq \mathcal{B}(X)$ is called *$\mathcal{R}$-bounded* if there is a constant $C > 0$ such that

$$\forall\, N \in \mathbb{N},\ x_n \in X,\ T_n \in \mathcal{T}: \quad \Big\| \sum_{n=1}^{N} \varepsilon_n T_n x_n \Big\|_{L^2(\Omega;X)} \leq C \Big\| \sum_{n=1}^{N} \varepsilon_n x_n \Big\|_{L^2(\Omega;X)}.$$

See Paragraph 1.9 and Remark 2.6 of [**KW**]. Roughly speaking, this means that we can estimate the operators T_n in sums with random signs if we take averages. Uniform boundedness follows from $\mathcal{R}$-boundedness (take $N = 1$), and these notions are equivalent in a Hilbert space X, see Paragraph 1.9 of [**KW**].

A closed and densely defined operator A is called *$\mathcal{R}$-sectorial* of angle $\phi \in (0, \pi]$ if the set $\mathcal{T} = \{AR(\lambda, A) \,|\, \lambda \in \Sigma_\phi\}$ is $\mathcal{R}$-bounded.

We state Weis' result, see Theorem 1.11 of [**KW**] or Theorem 4.4.4 of [**PS**].

Theorem 5.6. *Let X be an UMD space. A closed and densely defined operator A on X has maximal regularity of type L^p if and only if it is $\mathcal{R}$-sectorial of angle $\phi > \pi/2$.*

In Hilbert spaces X one already knows this characterization in view of the above remarks and Example 5.3. Remark 5.2 g) shows that one needs a stronger condition than sectoriality even on L^q with $q \in (1, \infty) \setminus \{2\}$. In Paragraphs 1.15 of [**KW**] it is also explained that the UMD property is crucial here.

Since $\mathcal{R}$-boundedness is a complicated property, one wonders whether $\mathcal{R}$-sectoriality can be checked in examples. Fortunately, now a powerful theory is available to deal with this concept. First, there are more accessible sufficient conditions for it (like the properties mentioned in Example 5.4). Moreover, one

[3] UMD stands for 'uniform martingal difference' which refers to the standard definition of UMD spaces in the literature, see Definition 4.2.1 in [**HNVW**].

has developed a machinery that allows to show $\mathcal{R}$-sectoriality based on standard techniques for the proof of sectoriality of elliptic operators with boundary conditions on $L^q(G)$. The monographs [**KW**] and [**PS**] give an introduction to this theory and its applications.

We now come to a local wellposedness result for the quasilinear parabolic problem (5.1). In contrast to earlier sections we allow for nonlinearities defined only on an open subset $V_\gamma \subseteq X_\gamma$. By a *solution* of (5.1) on J we mean a function $u \in E_1(J) \hookrightarrow C(\overline{J}, X_\gamma)$ with values in V_γ and $u(0) = u_0$ which solves the differential equation (5.1) pointwise a.e. in X. Then all terms in (5.1) belong to $L^p(J, X)$. In quasilinear problems local boundedness is not enough to ensure global existence even if $X_\gamma = V_\gamma$. Below we require uniform continuity in X_γ, compare Theorem 4.17 for a different condition. The subscript γ refers to the norm of X_γ.

Theorem 5.7. *Let X_1 be densely embdded into X, $X_\gamma = (X, X_1)_{1-\frac{1}{p},p}$ for some $p \in (1,\infty)$, $V_\gamma \subseteq X_\gamma$ be open and nonempty. Assume that the maps $A : V_\gamma \to \mathcal{B}(X_1, X)$ and $F : V_\gamma \to X$ are Lipschitz on bounded subsets of V_γ and that $A(v)$ has maximal regularity of type L^q for each $v \in V_\gamma$. Let $\overline{u}_0 \in V_\gamma$. Then the following assertions are true.*

a) There is a radius $\rho = \rho(\overline{u}_0) > 0$ and a time $b = b(\overline{u}_0) > 0$ such that for all $u_0 \in \overline{B}_\gamma(\overline{u}_0, \rho) \subseteq V_\gamma$ we have a unique solution $u = \varphi(\,\cdot\,, u_0) \in E_1(b)$ of (5.1). These solutions satisfy

$$\|\varphi(\,\cdot\,, u_0) - \varphi(\,\cdot\,, v_0)\|_{E_1(b)} \le c\, \|u_0 - v_0\|_\gamma \tag{5.5}$$

for a constant $c = c(\overline{u}_0) > 0$ and all $u_0, v_0 \in \overline{B}_\gamma(\overline{u}_0, \rho)$.

b) We can extend u from a) to a unique solution $u = \varphi(\,\cdot\,, u_0) \in E_1(t_+(u_0))$ of (5.1) on the maximal existence interval $J(u_0) = [0, t_+(u_0))$. Here $t_+(u_0) < \infty$ implies that $u : J(u_0) \to X_\gamma$ is not uniformly continuous or that the distance $\mathrm{dist}_\gamma(u(t), \partial V_\gamma)$ tends to 0 as $t \to t_+(u_0)^-$.

c) Assume that the constants of maximal regularity of $A(v)$ are uniformly bounded for $v \in V_\gamma$. Let $\overline{b} \in J(\overline{u}_0)$. Then there is a radius $\delta = \delta(\overline{u}_0, \overline{b}) > 0$ such that for all $u_0, v_0 \in \overline{B}_\gamma(\overline{u}_0, \delta) \subseteq V_\gamma$ we have $t_+(u_0), t_+(v_0) > \overline{b}$ and the estimate (5.5) is true for $t \in [0, \overline{b}]$.[4]

In contrast to Theorem 3.4 the above result does not provide a pointwise regularization of the solution. This can be achieved by an extension of our approach using weights in time, see Theorem 5.1.1 of [**PS**]. In this way one can also obtain compactness in X_γ of bounded orbits if X_1 is compactly embedded into X as shown in Theorem 5.7.1 of [**PS**] combined with Proposition 2.13.

In Theorem 5.7 all operators $A(v)$ have the same domain. One can show variants of it without this assumption using 'maximal regularity of type C^α,' roughly speaking. We refer to [**Am1**], [**Lu1**] or [**Ya**] for such results. In the context of quasilinear parabolic partial differential equations results of this type have been achieved since the sixties, also employing maximal regularity proved directly for a specific class of PDEs. The methods discussed here can also

[4]This part was omitted in the lectures.

be extended to such PDE with nonlinear boundary conditions or with moving interfaces as presented in [**PS**]. Actually, such applications were a main motivation for this line of research.

Proof of Theorem 5.7. 1) We first collect auxiliary facts and prove the basic estimates. Let $b \in (0,1]$ and denote by $T(\cdot)$ the semigroup generated by $A(\overline{u}_0)$. Remark 5.2 yields the following inequalities with constants ≥ 1 not depending on b, where M_γ, C_{MR} and C^0_{MR} are functions of $\overline{u}_0$. We write $L^\infty_b X_\gamma$ for $L^\infty((0,b),X_\gamma)$ etc.

$$\forall\, v \in E_1(b) \text{ with } v(0)=0: \quad v \in C([0,b],X_\gamma), \quad \|v\|_{L^\infty_b X_\gamma} \leq C_\gamma \|v\|_{E_1(b)}, \tag{5.6}$$

$$\forall\, v_0 \in X_\gamma: \quad T(\cdot)v_0 \in C([0,b],X_\gamma)\cap E_1(b), \quad \|T(\cdot)v_0\|_{L^\infty_b X_\gamma} \leq M_\gamma \|v_0\|_\gamma, \tag{5.7}$$

$$\|T(\cdot)v_0\|_{E_1(b)} \leq C^0_{\mathrm{MR}} \|v_0\|_\gamma, \tag{5.8}$$

$$\forall\, f \in E_0(b): \quad T *_+ f \in E_1(b), \quad \|T *_+ f\|_{E_1(b)} \leq C_{\mathrm{MR}}\, \|f\|_{E_0(b)}. \tag{5.9}$$

Fix a radius $\rho_0 > 0$ with $B_0 := \overline{B}_\gamma(\overline{u}_0,\rho_0) \subseteq V_\gamma$. Take $\rho \in (0,\rho_0]$ and $u_0 \in B_\gamma(\overline{u}_0,\rho)$. We set $u_* = T(\cdot)u_0$ and $\overline{u}_* = T(\cdot)\overline{u}_0$. To obtain smallness below, we need the limit

$$\kappa_0(b) := \max_{t\in[0,b]} \|\overline{u}_*(t) - \overline{u}_0\|_\gamma \;\longrightarrow\; 0, \qquad b \to 0.$$

The estimate (5.6) is only uniform as $b \to 0$ for functions vanishing at 0. For this reason, we incorporate the initial condition $v(0) = v_0$ in our fixed point space and substract u_* from v. The other constants are tied to $\overline{u}_0$ and we will thus linearize the equation at $\overline{u}_0$ (and not at the initial value u_0 as in (5.3)). So the difference $v - \overline{u}_0$ appears naturally. Let $v \in E_1(b)$ with $v(0) = u_0$ and $\|v - \overline{u}_*\|_{E_1(b)} \leq r$ for some $r > 0$. Using (5.6), (5.7) and (5.8), we estimate

$$\begin{aligned}
\|v-\overline{u}_0\|_{L^\infty_b X_\gamma} &\leq \|v-u_*\|_{L^\infty_b X_\gamma} + \|T(\cdot)(u_0-\overline{u}_0)\|_{L^\infty_b X_\gamma} + \|\overline{u}_*-\overline{u}_0\|_{L^\infty_b X_\gamma}\\
&\leq C_\gamma \|v-u_*\|_{E_1(b)} + M_\gamma\|u_0-\overline{u}_0\|_\gamma + \kappa_0(b)\\
&\leq C_\gamma \|v-\overline{u}_*\|_{E_1(b)} + C_\gamma\|T(\cdot)(u_0-\overline{u}_0)\|_{E_1(b)} + M_\gamma\rho + \kappa_0(b)\\
&\leq C_\gamma r + C_\gamma C^0_{\mathrm{MR}}\rho + M_\gamma\rho + \kappa_0(b)\\
&\leq C_\gamma r + \frac{r}{3} + \frac{r}{3} \leq \rho_0,
\end{aligned} \tag{5.10}$$

where we take $r \in (0,r_0]$, $\rho \in (0,\rho_1]$ and $b \in (0,b_0]$ with $b_0 \leq 1$

$$r_0 = \frac{\rho_0}{3C_\gamma} \leq \rho_0, \qquad \rho_1 = \frac{r}{3(C_\gamma C^0_{\mathrm{MR}} + M_\gamma)} \leq \rho_0, \qquad \kappa_0(b_0) \leq \frac{r}{3}. \tag{5.11}$$

Observe that the numbers r_0, $\rho_1 = \rho_1(r)$ and $b_0 = b_0(r)$ only depend on $\overline{u}_0$, ρ_0, and r. As in the above computation, we infer

$$\|u_* - \overline{u}_*\|_{E_1(b)} \leq C^0_{\mathrm{MR}}\|u_0-\overline{u}_0\|_\gamma \leq C^0_{\mathrm{MR}}\rho \leq r/3. \tag{5.12}$$

2) For ρ, b and r as above, we take $u_0 \in \overline{B}_\gamma(\overline{u}_0,\rho)$ and define

$$\Sigma(r,b) := \{v \in E_1(b)\,|\, v(0) = u_0,\; \|v-\overline{u}_*\|_{E_1(b)} \leq r\}.$$

This set contains u_* due to (5.12). It is complete for the metric $\|v-w\|_{E_1(b)}$ because of (5.6). Let $v,w \in \Sigma(r,b)$. Inequality (5.10) shows that $v(t) \subseteq B_0 \subseteq$

V_γ for all $t \in [0,b]$. Let L_0 be the maximum of the Lipschitz constants of A and F on B_0. We set

$$G(v) = (A(v) - A(\overline{u}_0))v + F(v) \quad E_0(b).$$

The assumption and Remark 5.2 provide a solution $u =: \Phi(v) = \Phi_{u_0}(v) \in E_1(b)$ of the linear problem

$$u'(t) = Au(t) + G(v(t)), \quad t \in (0,b), \qquad u(0) = u_0. \tag{5.13}$$

It is given by $u = T(\cdot)u_0 + T *_+ G(v)$.

We set $\kappa_1(b) = \|\overline{u}_*\|_{E_1(b)}$ and note that this number tends to 0 as $b \to 0$. The inequalities (5.8), (5.9), (5.10), and (5.12) imply

$$\begin{aligned}
&\|\Phi(v) - \overline{u}_*\|_{E_1(b)} \le \|u_* - \overline{u}_*\|_{E_1(b)} + \|T *_+ G(v)\|_{E_1(b)} \\
&\quad \le C^0_{\mathrm{MR}}\|u_0 - \overline{u}_0\|_\gamma + C_{\mathrm{MR}}\|(A(v) - A(\overline{u}_0))v\|_{E_0(b)} + C_{\mathrm{MR}}\|F(v) \mp F(\overline{u}_0)\|_{E_0(b)} \\
&\quad \le C^0_{\mathrm{MR}}\rho + C_{\mathrm{MR}}L_0\|v - \overline{u}_0\|_{L^\infty_b X_\gamma}\,\|v - \overline{u}_* + \overline{u}_*\|_{E_1(b)} \\
&\qquad\quad + C_{\mathrm{MR}}b^{\frac{1}{p}}(L_0\|v - \overline{u}_0\|_{L^\infty_b X_\gamma} + \|F(\overline{u}_0)\|_\gamma) \\
&\quad \le \tfrac{r}{3} + +C_{\mathrm{MR}}L_0 r(C_\gamma + \tfrac{2}{3})(r + \kappa_1(b)) + C_{\mathrm{MR}}b^{\frac{1}{p}}(L_0\rho_0 + \|F(\overline{u}_0)\|_\gamma) \\
&\quad \le \tfrac{r}{3} + \tfrac{r}{3} + \tfrac{r}{3} = r.
\end{aligned} \tag{5.14}$$

Here we fix $r = r_1$ and $\rho = \rho_1(r_1)$ and take $b \in (0, b_2]$ with

$$\begin{aligned}
&r_1 = \min\{r_0, (6\hat{c})^{-1}\}, \qquad \hat{c} = C_{\mathrm{MR}}L_0(C_\gamma + \tfrac{2}{3}), \qquad \kappa_1(b_1) \le r, \\
&b_2 = \min\big\{b_0, b_1, r^p\big(3C_{\mathrm{MR}}(L_0\rho_0 + \|F(\overline{u}_0)\|_\gamma)\big)^{-p}\big\}.
\end{aligned} \tag{5.15}$$

Similarly, using estimates (5.9), (5.10), (5.6), (5.14), (5.15) and $(v-w)(0) = 0$, we compute

$$\begin{aligned}
&\|\Phi(v) - \Phi(w)\|_{E_1(b)} \le C_{\mathrm{MR}}\|G(v) - G(w)\|_{E_0(b)} \\
&\quad \le C_{\mathrm{MR}}\big(\|(A(v) - A(\overline{u}_0))(v - w)\|_{E_0(b)} + \|(A(v) - A(w))w\|_{E_0(b)} \\
&\qquad\quad + \|F(v) - F(w)\|_{E_0(b)}\big) \\
&\quad \le C_{\mathrm{MR}}L_0\big(\|v - \overline{u}_0\|_{L^\infty_b X_\gamma}\,\|v - w\|_{L^p_b X_1} + \|v - w\|_{L^\infty_b X_\gamma}\,\|w - \overline{u}_* + \overline{u}_*\|_{L^p_b X_1} \\
&\qquad\quad + \|v - w\|_{L^p_b X_\gamma}\big) \\
&\quad \le \big(\hat{c}r + C_{\mathrm{MR}}L_0C_\gamma(r + \kappa_1(b)) + C_{\mathrm{MR}}L_0 b^{\frac{1}{p}}c_\gamma\big)\|v - w\|_{E_1(b)} \\
&\quad \le (\tfrac{1}{6} + \tfrac{1}{3} + \tfrac{1}{6})\,\|v - w\|_{E_1(b)} = \tfrac{2}{3}\|v - w\|_{E_1(b)}\,,
\end{aligned} \tag{5.16}$$

Here c_γ is the norm of the embedding $X_\gamma \hookrightarrow X_1$ from Proposition 2.2, and we have taken

$$0 < b \le b_3 := \min\{b_2, (3C_{\mathrm{MR}}L_0c_\gamma)^{-p}\}. \tag{5.17}$$

As a consequence, $\Phi = \Phi_{u_0} : \Sigma(r_1, b) \to \Sigma(r_1, b)$ is a strict contraction for each initial value $u_0 \in \overline{B}_\gamma(\overline{u}_0, \rho_1(r_1))$. The fixed point $u \in \Sigma(r_1, b)$ solves (5.1) on $[0,b]$ uniquely in $\Sigma(r_1, b)$.

3) Let $u \in E_1(J_u)$ and $v \in E_1(J_v)$ solve (5.1) on open intervals J_u and J_v, respectively. We suppose that $u \ne v$ on $J = J_u \cap J_v$. Then there are times t_n in J with limit $\tau \in J \cup \{0\}$ such that $u = v$ on $[0,\tau]$, $t_n > \tau$, and $u(t_n) \ne v(t_n)$ for all $n \in \mathbb{N}$. We replace in steps 1) and 2) the vector $\overline{u}_0 \in V_\gamma$ by

$\overline{u}_0' = u(\tau) = v(\tau) \in V_\gamma$, where we fix a radius $\rho_0' > 0$ such that $\overline{B}_\gamma(\overline{u}_0', \rho_0') \subseteq V_\gamma$. This leads to numbers r_1' and b_3' as in (5.11), (5.15), and (5.17). We thus have a unique fixed point $w_b \in \Sigma(r_1', b)$ of (5.1) with initial value $\overline{u}_0'$ on $(0, b)$ for each $b \in (0, b_3']$.

On the other hand, there is a number $\beta \in (0, b_3']$ such that $\tau + \beta \in J$, $\|u(\cdot - \tau) - T(\cdot)\overline{u}_0'\|_{E_1(\beta)} \le r_1'$, and $\|v(\cdot - \tau) - T(\cdot)\overline{u}_0'\|_{E_1(\beta)} \le r_1'$. This means that $u(\cdot - \tau)$ and $v(\cdot - \tau)$ both belong to $\Sigma(r_1', \beta)$ and thus have to coincide. This contradiction shows that $u = v$ on $J_u \cap J_v$.

4) Let u and v solve (5.1) on $[0, b_3] = [0, b]$ for initial data $u_0, v_0 \in \overline{B}(\overline{u}_0, \rho_1(r_1))$ as found in step 2). We thus have $u = \Phi_{u_0}(u)$ and $v = \Phi_{v_0}(v)$. Observing that $\Phi_{u_0}(u) - \Phi_{v_0}(u) = T(\cdot)(u_0 - v_0)$, we derive from (5.8) and (5.16) the bound

$$\begin{aligned} \|u - v\|_{E_1(b)} &\le \|T(\cdot)(u_0 - v_0)\|_{E_1(b)} + \|\Phi_{v_0}(u) - \Phi_{v_0}(v)\|_{E_1(b)} \\ &\le C_{\mathrm{MR}}^0 \, \|u_0 - v_0\|_\gamma + \tfrac{2}{3}\|u - v\|_{E_1(b)}, \\ \|u - v\|_{E_1(b)} &\le 3C_{\mathrm{MR}}^0 \, \|u_0 - v_0\|_\gamma \, . \end{aligned} \tag{5.18}$$

We thus have shown assertion a).

5) Let $u_0 \in V_\gamma$. Based on step 3), we obtain as usual a unique solution $\varphi(\cdot\,, u_0)$ of (5.1) on $J(u_0) = [0, t_+(u_0))$ with

$$t_+(u_0) = \sup\{b > 0 \,|\, \exists\, u_b \in E_1(b) \text{ solving (5.1) on } (0, b)\}.$$

Note that $t_+(u_0) \ge b_3(u_0)$ for the number b_3 from (5.17) if one replaces $\overline{u}_0$ by u_0 as in step 3). One can also restart the problem at $b_3(u_0)$ and obtains a solution on a larger interval so that $t_+(u_0) > b_3(u_0)$.

Suppose that $t_+(u_0) < \infty$ and that $u : J(u_0) \to X_\gamma$ is unibformly continuous and $\mathrm{dist}_\gamma(u(t), \partial V_\gamma) \ge \delta > 0$ for all $t \in [0, t_+(u_0))$. Then the limit $u(t) \to u_1$ as $t \to T_+(u_0)$ exists in X_γ and u_1 still belongs to V_γ. One can thus extend the solution as above and obtains a contradiction, so that b) holds. as $b \to 0$

6) Take $\overline{b} \in (0, t_+(\overline{u}_0))$ and fix $b' \in (\overline{b}, t_+(\overline{u}_0))$. We write $\overline{u}(t) = \varphi(t, \overline{u}_0)$ and note that the orbit $\Gamma = \{\overline{u}(t) \,|\, t \in [0, b']\}$ is compact in V_γ. It thus has a positive distance to ∂V_γ, and we can redefine $\rho_0 > 0$ from step 1) such that $\mathrm{dist}_\gamma(\Gamma, \partial V_\gamma) > \rho_0$. In parts 1) and 2) we replace $\overline{u}_0$ by $v \in \Gamma$ in the definition of all constants and of κ_j keeping the notation. By the assumption in c), the constants C_{MR}^0 and C_{MR} are then uniformly bounded $v \in \Gamma$. The same is true for M_γ by Remark 5.2 and Proposition 2.4. Moreover, the functions $\kappa_j(b)$ tend to 0 as $b \to 0$ uniformly for $v \in \Gamma$ by the compactness of Γ and (5.8). As a result, the numbers $r = r_1$, $\rho = \rho_1(r_1) \le \rho_0$, and $b = b_3$ in (5.15) and (5.17) can be chosen uniformly for the vectors $\overline{u}(t)$ with $t \in [0, b']$ instead of $\overline{u}_0$.

Let $t_k = kb$ and $J_k = [t_{k-1}, t_k]$ for $k \in \mathbb{N}$ and N be the first integer with $t_N > b$. If $Nb > b'$, we redefine $T_N = b'$. We set $C = 3C_\gamma C_{\mathrm{MR}}^0 \ge 1$ and $\delta = C^{-N}\rho$. Let $u_0 \in \overline{B}_\gamma(\overline{u}_0, \delta)$. Estimates (5.18) and (5.6) imply that

$$\|u(b) - \overline{u}(b)\|_\gamma \le C_\gamma \|u - \overline{u}\|_{E_1(b)} \le 3C_\gamma C_{\mathrm{MR}}^0 \, \|u_0 - \overline{u}_0\|_\gamma \le C\delta \le \rho.$$

We can thus extend u to $[0, t_2]$ by step 1) and deduce

$$\|u(t_2) - \overline{u}(t_2)\|_\gamma \le C\|u(t_1) - \overline{u}(t_1)\|_\gamma \le C^2\delta \le \rho.$$

Iteratively we see that u exists on $[0, b_N]$; i.e., $t_+(u_0) > \overline{b}$.

Let also $v_0 \in \overline{B}_\gamma(\overline{u}_0, \delta)$. We can now replace in the above argument $\overline{u}$ by v, obtaining

$$\|u - v\|_{E_1(J_k)} \le 3C^0_{\mathrm{MR}} \|u(t_{k-1}) - v(t_{k-1})\|_\gamma \le 3C^0_{\mathrm{MR}} C^{N-1} \|u_0 - v_0\|_\gamma$$

for all $k \in \{1, \dots, N\}$. So assertion c) follows by the triangle inequality. □

One can apply Theorem 5.7 to the quasilinear reaction-diffusion problem (5.1) in Example 5.1 as we briefly sketch.

Example 5.8. In the setting of Example 5.1, let $V_\gamma = X_\gamma$ and write $\partial_j(a_{jk}(u)\partial_k u) = a_{jk}(u)\partial_{jk}u + \partial_k a_{jk}(u)\partial_k u$. (One would employ sets of the form $V_\gamma = \{v \in X_\gamma \,|\, a(v) > \alpha I > 0\}$ if one only assumes that $a = a^\top > 0$.) Using $X_\gamma \hookrightarrow C^1(\overline{G})$, one can the check that $A : V_\gamma \to \mathcal{B}(X_1, X)$ and $F : V_\gamma \to X$ are Lipschitz on bounded subsets of X_γ similar as in Example 3.6. The maximal regularity of $A(v)$ can de defuced from Example 5.3 above and Theorem 7.3.6 of [**Pa**], where positivity is shown as in Example 5.2 of [**EE**].

As in Theorem 3.8 global existence here follows again from uniform boundedness if $F(v) = f(v)$. See Theorem 5.2 of [**Am1**], where even certain elliptic systems were studied in a somewhat different setting. ◊

The asymptotic behavior of problems such as (5.1) was investigated e.g. in [**Lu1**], [**Pr**] or [**PS**].

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