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Strang splitting for a semilinear Schrödinger equation with damping and forcing

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Abstract We propose and analyze a Strang splitting method for a cubic semilinear Schrödinger equation with forcing and damping terms. The nonlinear part can be solved analytically, whereas the linear part – space derivatives, damping and forcing – is approximated by the exponential trapezoidal rule. The necessary operator exponentials and ϕ-functions can be computed efficiently by fast Fourier transforms if space is discretized by spectral collocation. We show well-posedness of the problem and $H^4(\mathbb{T})$ regularity of the solution for initial data in $H^4(\mathbb{T})$ and sufficiently smooth forcing. Under these regularity assumptions, we prove a first-order error bound in $H^1(\mathbb{T})$ and a second-order error bound in $L^2(\mathbb{T})$ on bounded time-intervals.

Keywords Nonlinear Schrödinger equation · Strang splitting · error analysis · stability · well-posedness · regularity

Mathematics Subject Classification (2000) 65M12 · 65M15 · 65M70 · 35Q55 · 35B65 · 65J15

1 Introduction

Nonlinear Schrödinger equations (NLS) occur in many different forms and describe a multitude of different phenomena, such as Bose-Einstein condensates, small-amplitude surface water waves, Langmuir waves in hot plasmas, or signal processing through optical fibers, to name but a few. The intriguing properties – for example conservation of norm, energy, and momentum, near-conservation of actions over long times, existence of solitary waves, or possible blow-up – have inspired and challenged mathematicians for a long time. Surveys about these topics can be found, e.g., in the monographs [3, 20].

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In most applications, the solution of the NLS has to be approximated by a numerical scheme. For problems on the d-dimensional torus $\mathbb{T}^d$, splitting methods with spectral collocation in space are particularly popular. These integrators are based on the observation that the linear and the nonlinear part of the NLS can be solved at low computational costs in the absence of the other part. The splitting approach can also be applied to the Gross-Pitaevskii equation, a NLS on $\mathbb{R}^d$ with constraining polynomial potential, by using the basis of Hermite functions for the space discretization. The accuracy of such integrators has been analyzed in [1, 5, 6, 8, 13, 15, 16, 18, 21]. The long-time behavior of numerical solutions, in particular the (near-)conservation of invariants over long times and the stability of plane waves, has been investigated in [6, 7, 9].

In this article we consider a cubic, focusing NLS which, in contrast to the "classical" NLS, contains a damping and a forcing term. This special form has first been considered in [17] and is called *Lugiato-Lefever equation* in physics and electronic engineering. The Lugiato-Lefever equation has been proposed as a model for the formation of Kerr-frequency combs in microresonators coupled to optical waveguides and driven by an external pump tuned to a resonance wavelength [4, 11]. The frequency combs generated by such a device can be used as optical sources for high-speed data transmission. In the mathematical model, the forcing term represents the external pump, whereas the radiation into the waveguide is modeled by the damping term. Clearly, these terms destroy the Hamiltonian structure of the NLS, and in contrast to the "classical" NLS the energy, momentum and norm of the solution do not remain constant in time.

However, the solution can still be approximated with a splitting approach, because the linear inhomogeneous part (including the space derivatives and the forcing/damping terms) can be efficiently propagated by an exponential integrator [12]. In this article we analyze the accuracy of the semi-discretization in time with such a splitting method. Our main result states that if the initial data and the forcing are sufficiently regular, then the method converges on bounded time-intervals with the classical order 2 in $L^2(\mathbb{T})$, and with order 1 in $H^1(\mathbb{T})$; see Theorem 3.1 below. The proof is rather long and consists of several steps which are formulated as self-contained results. Our masterplan mimics the line of arguments in [16]: The classical concept "consistency plus stability yields convergence" must be suitably adapted, because the stability result (Theorem 3.4 below) assumes the numerical solution to be bounded in $H^1(\mathbb{T})$. This is verified by proving an error bound for the local error in $H^1(\mathbb{T})$, in addition to the local error bound in $L^2(\mathbb{T})$ required for consistency. The proofs in [5, 6, 8, 13, 15, 16, 18, 21] are based on the calculus of Lie derivatives and commutator bounds. In our situation, however, the iterated commutators between the linear and nonlinear part are not the only source of error, because the forcing term is coupled to the space derivatives and to the nonlinear part in a rather complicated way. For this reason, we decided to avoid the notationally rather involved Lie derivatives.

In the next section, we prove global well-posedness of the Lugiato-Lefever equation and $H^4(\mathbb{T})$ regularity of the solution for initial data in $H^4(\mathbb{T})$ and a forcing function in $C^{2-j}\big([0,T];H^j(\mathbb{T})\big)$ for $j=0,1,2$. This result is the cornerstone of the error analysis of the numerical method. It is derived by means of a modified energy functional and regularity properties of several differentiated versions of the equation. The splitting method for the Lugiato-Lefever equation is introduced in section 3, and we formulate the error bounds for the global error (Theorem 3.1) along with the results required for its proof (bounds of the local error in $L^2(\mathbb{T})$ and $H^1(\mathbb{T})$ and stability of the scheme). All following sections are devoted to the proofs of these assertions. In section 4, we prove stability of the numerical scheme, and we compile a number of auxiliary results which are often used throughout the paper. The bounds of the local errors are shown in sections 5 and 6, respectively. The proof of the main result follows in section 7.

2 The Lugiato-Lefever equation

The cubic semilinear Schrödinger equation

$$\partial_t u(t,x) = -u(t,x) + \mathrm{i}\partial_x^2 u(t,x) + \mathrm{i}|u(t,x)|^2 u(t,x) + g(t,x)\,, \qquad t>0 \tag{2.1a}$$

$$u(0,x) = u_0(x) \tag{2.1b}$$

on the one-dimensional torus $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$ is known as the Lugiato-Lefever equation in physics and electronic engineering. The terms $-u(t,x)$ and $g(t,x)$ model damping and external forcing, respectively, and do not appear in the "classical" NLS. Solely the one-dimensional torus is considered, because this is the relevant setting for modeling frequency comb generation; cf. [4,11].

The evolution equation (2.1) is considered on $L^2(\mathbb{T})$, i.e. on the Hilbert space of square integrable functions with the inner product

$$\langle v,w\rangle = \int_{\mathbb{T}} v(x)\overline{w(x)}\,\mathrm{d}x\,, \qquad v,w\in L^2(\mathbb{T})$$

and induced norm $\|v\|_{L^2} = \sqrt{\langle v,v\rangle}$. The Sobolev space of all functions $v:\mathbb{T}\to\mathbb{C}$ with partial derivatives up to order $k\in\mathbb{N}_0$ in $L^2(\mathbb{T})$ is denoted by $H^k(\mathbb{T})$. For every k, $H^k(\mathbb{T})$ is a Hilbert space with norm

$$\|v\|_{H^k}^2 = \sum_{j=0}^{k} \|\partial_x^j v\|_{L^2}^2\,.$$

In particular, we identify $H^0(\mathbb{T}) = L^2(\mathbb{T})$. We further define the Sobolev spaces of negative order by duality, i.e., we set

$$H^{-k}(\mathbb{T}) := H^k(\mathbb{T})^* \qquad \text{with norm} \qquad \|\varphi\|_{H^{-k}} = \sup\{|\phi(v)| \,\big|\, \|v\|_{H^k}\le 1\}$$

for $k\in\mathbb{N}$. We assume the regularity

$$u_0\in H^4(\mathbb{T}) \quad \text{and} \quad g\in C^{2-j}\big([0,T];H^j(\mathbb{T})\big), \quad j=0,1,2, \tag{2.2}$$

for the initial data u_0 and the forcing g, respectively, where $T > 0$ is fixed. It will be shown in Theorem 2.1 that these assumptions guarantee the global existence and uniqueness of a sufficiently smooth solution.

Henceforth, we will usually omit the space variable and write $u(t)$ instead of $u(t,x)$, and so on. Throughout the paper, $C > 0$ and $C(\cdot) > 0$ denote universal constants, possibly taking different values at various appearances. The notation $C(\cdot)$ means that the constant depends only on the values specified in the brackets.

2.1 Analytic setting

For $v \in H^1(\mathbb{T})$ and $w \in L^2(\mathbb{T})$, the Sobolev embedding $H^{j+1}(\mathbb{T}) \hookrightarrow C^j(\mathbb{T})$ for $j \in \mathbb{N}_0$ implies the inequality

$$\|vw\|_{L^2} \leq \|v\|_{L_\infty} \|w\|_{L^2} \leq C\|v\|_{H^1}\|w\|_{L^2} \,. \tag{2.3}$$

For every $k \in \mathbb{N}_0$ we define

$$k^* = \max\{1, k\}. \tag{2.4}$$

Then, for $v \in H^{k^*}(\mathbb{T})$ and $w \in H^k(\mathbb{T})$ the bound

$$\|vw\|_{H^k} \leq C\|v\|_{H^{k^*}}\|w\|_{H^k} \,, \quad k \geq 0 \tag{2.5}$$

follows from (2.3). For every $k \in \mathbb{Z}$, the operator

$$A := \mathrm{i}\partial_x^2 - I \qquad \text{with domain} \qquad H^{k+2}(\mathbb{T})$$

generates a strongly continuous group e^{tA} in $H^k(\mathbb{T})$. The level k of regularity is not expressed in our notation since the respective operators are restrictions of each other.

2.2 Existence and uniqueness of solutions

We need a unique (global) solution u of (2.1) on $[0,T]$ in H^4, under assumption (2.2). We refer to e.g. [3] for local wellposedness in such regularity classes and for related results on global wellposedness without forcing. In the next theorem we construct the desired solution, and we also establish additional results on the longtime behavior (on the time interval $\mathbb{R}_+$).

Theorem 2.1 *Let assumption* (2.2) *hold. Then,* (2.1) *has a unique solution*

$$u \in C\big(\mathbb{R}_+; H^4(\mathbb{T})\big) \cap C^1\big(\mathbb{R}_+; H^2(\mathbb{T})\big) \cap C^2\big(\mathbb{R}_+; L^2(\mathbb{T})\big). \tag{2.6}$$

Under additional assumptions on g, the following bounds hold.

(a) If $g \in L^\infty(\mathbb{R}_+, L^2(\mathbb{T}))$, then $\|u(t)\|_{L^2} \leq C$ for $t \geq 0$.
(b) If $g, \partial_t g \in L^\infty(\mathbb{R}_+; L^2(\mathbb{T}))$, then $\|u(t)\|_{H^1} \leq C\sqrt{1+t}$ for $t \geq 0$.

(c) If $g \in L^2(\mathbb{R}_+; L^2(\mathbb{T}))$, then $u \in L^2(\mathbb{R}_+; L^2(\mathbb{T}))$.
(d) If $g \in H^1(\mathbb{R}_+; L^2(\mathbb{T}))$, then $\|u(t)\|_{H^1} \le C$ for $t \ge 0$.

Here C only depends on $\|u_0\|_{L^2}$ in (a) and on $\|u_0\|_{H^1}$ in (b) and (d), as well as on the respective norms of g.

Before we start with the proof, we make a few preparations. First, we set

$$F(t,v) = \mathrm{i}\,|v|^2 v + g(t)$$

for $t \in \mathbb{R}$ and $v \in H^1(\mathbb{T})$ and define for $a \in (0,\infty]$ and $k \in \mathbb{Z}$ the space

$$E_k := C\big([0,a); H^k(\mathbb{T})\big) \cap C^1\big([0,a); H^{k-2}(\mathbb{T})\big).$$

Second, we state and prove the following lemma which leads to the existence of solutions to (2.1) with values in $H^1(\mathbb{T})$.

Lemma 2.1 *a) Let $g \in C^1(\mathbb{R}_+; H^1(\mathbb{T}))$. Then the map $F : \mathbb{R}_+ \times H^1(\mathbb{T}) \to H^1(\mathbb{T})$ is continuously differentiable. Its derivative with respect to $v \in H^1$ is given by*

$$\partial_v F(t,v)w = \mathrm{i}\,|v|^2 w + 2\mathrm{i}\ \mathrm{Re}(\overline{v}w)v$$

for $t \in \mathbb{R}$ and $v, w \in H^1(\mathbb{T})$. Moreover, F is Lipschitz on bounded subsets.
b) Let $u \in E_1$ and $f(t) = |u(t)|^2 u(t)$ for $t \in [0,a)$. Then $f \in E_1$ and

$$f' = |u|^2 \partial_t u + 2\,\mathrm{Re}(\overline{u}\partial_t u)u =: h.$$

Proof We omit the well known and straightforward proof of part a). For a given $v \in H^1(\mathbb{T})$ the map $\phi \mapsto v\phi$ is bounded in $H^{-1}(\mathbb{T})$ by duality and (2.5). In view of (2.5), we obtain $f \in C([0,a); H^1(\mathbb{T}))$ and $h \in C([0,a); H^{-1}(\mathbb{T}))$. Observe that

$$\begin{aligned} f(t+s) - f(t) = (u(t+s) - u(t))|u(t+s)|^2 + (u(t+s) - u(t))\overline{u}(t+s)u(t) \\ + (\overline{u}(t+s) - \overline{u}(t))u(t)^2. \end{aligned}$$

Using $u \in E_1$ and (2.5), one can differentiate f in $H^{-1}(\mathbb{T})$ with $f' = h$. □

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1.

Step 1. We first look for a local solution of (2.1) in E_1, starting with the integral equation

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} F(s, u(s))\,\mathrm{d}s. \tag{2.7}$$

Theorem 6.1.4 in [19] and Lemma 2.1 yield a unique maximal solution u in $C([0,a); H^1(\mathbb{T}))$ of (2.7) for some $a > 0$. Moreover, if a is finite, then $\|u(t)\|_{H^1}$ becomes unbounded as $t \to a$. Differentiating (2.7) in $H^{-1}(\mathbb{T})$, one obtains a solution $u \in E_1$ of (2.1). It is unique since every solution of (2.1) in E_1 satisfies (2.7). In the following steps we improve the regularity of u iteratively.

Step 2. Lemma 2.1 allows us to differentiate the right hand side of (2.1) in $H^{-3}(\mathbb{T})$ with respect to t, and so there exists $\partial_t^2 u$ in $C([0,a);H^{-3}(\mathbb{T}))$. We set $v = \partial_t u \in E_{-1}$ and $v_0 := \mathrm{i}\partial_x^2 u_0 - u_0 + \mathrm{i}\,|u_0|^2 u_0 + g(0)$. Note that $v_0 \in H^2(\mathbb{T})$ by assumption (2.2). These functions then satisfy

$$\begin{aligned} \partial_t v &= \mathrm{i}\partial_x^2 v - v + \mathrm{i}\,|u|^2 v + 2\mathrm{i}\,\mathrm{Re}(\overline{u}v)u + \partial_t g, \\ v(0) &= v_0. \end{aligned} \tag{2.8}$$

Since u is bounded on $[0, a-\delta] \times \mathbb{T}$ for each $\delta \in (0,a)$, there is a unique function w in $C([0,a);L^2(\mathbb{T}))$ solving

$$w(t) = e^{tA} v_0 + \int_0^t e^{(t-s)A}\Big(\mathrm{i}\,|u(s)|^2 w(s) + 2\mathrm{i}\,\mathrm{Re}(\overline{u}(s)w(s))u(s) + \partial_t g(s)\Big)\,\mathrm{d}s, \tag{2.9}$$

see Theorem 6.1.2 in [19]. Because $v \in E_{-1}$ solves (2.8), it also satisfies (2.9). As noted above, the multiplication by u or $\overline{u}$ is a bounded operator on $H^{-1}(\mathbb{T})$. If we now subtract the two integral equations for v and w, we derive

$$\|v(t) - w(t)\|_{H^{-1}} \le C(\delta) \int_0^t \|v(s) - w(s)\|_{H^{-1}}\,\mathrm{d}s \quad \text{for} \quad 0 \le t \le a - \delta.$$

Gronwall's inequality implies that $v = w \in C([0,a);L^2(\mathbb{T}))$; i.e., u is contained in $C^1([0,a);L^2(\mathbb{T}))$. Equation (2.1) then yields $u \in C([0,a);H^2(\mathbb{T}))$ since $(\mathrm{i}\partial_x^2 - I)^{-1}$ maps $L^2(\mathbb{T})$ continuously into $H^2(\mathbb{T})$. Moreover, u belongs to $C^2([0,a);H^{-2}(\mathbb{T}))$ by (2.8).

Step 3. We next differentiate (2.8) in $H^{-3}(\mathbb{T})$ with respect to x. For instance, we have $\partial_x(|u|^2 v) = |u|^2 \partial_x v + 2\,\mathrm{Re}(\overline{u}\partial_x u)v$ since

$$\begin{aligned} \langle \partial_x(|u|^2 v), \varphi\rangle &= -\langle v, |u|^2 \partial_x \varphi\rangle = -\langle v, \partial_x(|u|^2\varphi)\rangle + \langle v, 2\,\mathrm{Re}(\overline{u}\partial_x u)\varphi\rangle \\ &= \langle |u|^2 \partial_x v, \varphi\rangle + \langle 2\,\mathrm{Re}(\overline{u}\partial_x u)v, \varphi\rangle \end{aligned}$$

for $u, \varphi \in H^1(\mathbb{T})$, $v \in L^2(\mathbb{T})$, and the duality between $H^{-1}(\mathbb{T})$ and $H^1(\mathbb{T})$. The function $w := \partial_x v = \partial_t \partial_x u$ then fulfills

$$\begin{aligned} \partial_t w &= \mathrm{i}\partial_x^2 w - w + \mathrm{i}|u|^2 w + 2\mathrm{i}\,\mathrm{Re}(\overline{u}w)u + \partial_t\partial_x g + h, \\ w(0) &= \partial_x v_0 \in H^1(\mathbb{T}), \end{aligned} \tag{2.10}$$

where $h := 2\mathrm{i}(\mathrm{Re}(u\partial_x\overline{u})v + \mathrm{Re}(v\partial_x\overline{u})u + \mathrm{Re}(v\overline{u})\partial_x u)$ and $\partial_t\partial_x g$ belong to $C([0,a);L^2(\mathbb{T}))$ by step 2) and assumption (2.2). As in step 2), we see that the integrated version of (2.10) has a solution $\varphi \in C([0,a);L^2(\mathbb{T}))$ and that $w = \varphi$. Hence, $u \in C^1([0,a);H^1(\mathbb{T}))$.

Step 4. In (2.8) we have a non-autonomous perturbation given by

$$P(t)\psi = \mathrm{i}\,|u(t)|^2\psi + 2\mathrm{i}\,\mathrm{Re}(\overline{u}(t)\psi)u(t), \qquad \psi \in L^2(\mathbb{T}).$$

Thanks to step 3), the linear maps $P(t)$ are strongly continuously differentiable with respect to $t \in [0,a)$ as bounded operators on $L^2(\mathbb{T})$. Also, v_0 is contained in $H^2(\mathbb{T})$ and $\partial_t g$ in $C^1(\mathbb{R}_+, L^2(\mathbb{T}))$ by assumption (2.2). So (2.8) possesses a unique solution φ in $C^1([0,a);L^2(\mathbb{T}))\cap C([0,a);H^2(\mathbb{T}))$ by Theorem 6.3 in [19]. The uniqueness of the corresponding integral equation (2.9) then shows that $\varphi = v = \partial_t u$, and so u belongs to $C^2([0,a);L^2(\mathbb{T}))\cap C^1([0,a);H^2(\mathbb{T}))$.

Step 5. We finally differentiate (2.1) twice with respect to x and obtain

$$\partial_t\partial_x^2 u = \mathrm{i}\partial_x^4 u - \partial_x^2 u + P(t)\partial_x^2 u + 4\mathrm{i}\,\mathrm{Re}(\overline{u}\partial_x u)\partial_x u + 2\mathrm{i}\,|\partial_x u|^2 u + \partial_x^2 g$$

at first in $H^{-2}(\mathbb{T})$. This equation, assumption (2.2) and the previous steps imply that $(\mathrm{i}\partial_x^2 - I)\partial_x^2 u$ is contained in $C([0,a);L^2(\mathbb{T}))$; i.e., $u\in C([0,a);H^4(\mathbb{T}))$.

Step 6. It remains to show global existence, where we employ the regularity of u established above. Using (2.1) and integrating by parts, we first compute

$$\begin{aligned}
\partial_t\|u(t)\|_{L^2}^2 &= 2\,\mathrm{Re}\int_{\mathbb{T}}\overline{u}(t)\partial_t u(t)\,\mathrm{d}x\\
&= 2\,\mathrm{Re}\int_{\mathbb{T}}\overline{u}(t)\Big(\mathrm{i}\partial_x^2 u(t) - u(t) + \mathrm{i}|u(t)|^2u(t) + g(t)\Big)\,\mathrm{d}x\\
&= -2\,\|u(t)\|_{L^2}^2 + 2\,\mathrm{Re}\int_{\mathbb{T}}\overline{u}(t)g(t)\,\mathrm{d}x\\
&\le -2\,\|u(t)\|_{L^2}^2 + 2\,\|u(t)\|_{L^2}\,\|g(t)\|_{L^2}\\
&\le -\|u(t)\|_{L^2}^2 + \|g(t)\|_{L^2}^2 \qquad (2.11)
\end{aligned}$$

for $t\in[0,a)$. Hence, $\partial_t(e^t\|u(t)\|_{L^2}^2)\le e^t\|g(t)\|_{L^2}^2$ and integration yields

$$\begin{aligned}
\|u(t)\|_{L^2}^2 &\le e^{-t}\,\|u_0\|_{L^2}^2 + \int_0^t e^{s-t}\,\|g(s)\|_{L^2}^2\,\mathrm{d}s\\
&\le \|u_0\|_{L^2}^2 + \max_{0\le s\le b}\|g(s)\|_{L^2}^2 =: C_0(b) \qquad (2.12)
\end{aligned}$$

for $0\le t\le b<a$. We further need a modified energy of (2.1) given by

$$\mathcal{E}(t,v) = \tfrac{1}{2}\,\|\partial_x v\|_{L^2}^2 - \tfrac{1}{4}\,\|v\|_{L^4}^4 + \mathrm{Re}\int_{\mathbb{T}}\mathrm{i}g(t)\overline{v}\,\mathrm{d}x \qquad (2.13)$$

for $v\in H^1(\mathbb{T})$ and $t\ge 0$. Proceeding as above, we obtain

$$\begin{aligned}
\partial_t\mathcal{E}(t,u(t)) &= \mathrm{Re}\int_{\mathbb{T}}\big[\partial_x u(t)\partial_{tx}\overline{u}(t) - (|u(t)|^2u(t) - \mathrm{i}g(t))\partial_t\overline{u}(t) + \mathrm{i}\overline{u}(t)\partial_t g(t)\big]\mathrm{d}x\\
&= \mathrm{Re}\int_{\mathbb{T}}\big[(-\partial_x^2 u(t) - |u(t)|^2u(t) + \mathrm{i}g(t))\partial_t\overline{u}(t) + \mathrm{i}\overline{u}(t)\partial_t g(t)\big]\mathrm{d}x\\
&= \mathrm{Re}\int_{\mathbb{T}}\mathrm{i}\big[(\partial_t u(t) + u(t))\partial_t\overline{u}(t) + \overline{u}(t)\partial_t g(t)\big]\mathrm{d}x\\
&= \mathrm{Re}\int_{\mathbb{T}}\mathrm{i}\Big[u(t)(\overline{g}(t) - \mathrm{i}\partial_x^2\overline{u}(t) - \overline{u}(t) - \mathrm{i}|u(t)|^2\overline{u}(t)) + \overline{u}(t)\partial_t g(t)\Big]\mathrm{d}x\\
&= \mathrm{Re}\int_{\mathbb{T}}\big[\mathrm{i}\,(u(t)\overline{g}(t) + \overline{u}(t)\partial_t g(t)) + |u(t)|^4 - |\partial_x u(t)|^2\big]\,\mathrm{d}x \qquad (2.14)
\end{aligned}$$

for $0 \le t \le b < a$. On the other hand, Sobelev's embedding theorem and complex interpolation (see Sect. 7.4.2 and 7.4.5 in [22]) yield

$$\begin{aligned} \|v\|_{L^4} &\le C\,\|v\|_{H^{1/4}} \le C\,\|v\|_{L^2}^{\frac{3}{4}}\,\|v\|_{H^1}^{\frac{1}{4}}, \\ \|v\|_{L^4}^4 &\le \|\partial_x v\|_{L^2}^2 + \|v\|_{L^2}^2 + C\,\|v\|_{L^2}^6. \end{aligned} \tag{2.15}$$

We thus deduce

$$\begin{aligned} \partial_t \mathcal{E}(t,u(t)) &\le \operatorname{Re}\int_{\mathbb{T}} \mathrm{i}(u(t)\overline{g}(t) + \overline{u}(t)\partial_t g(t))\,\mathrm{d}x + \|u(t)\|_{L^2}^2 + C\,\|u(t)\|_{L^2}^6 \\ &\le 2\,\|u(t)\|_{L^2}^2 + \tfrac{1}{2}\,\|g(t)\|_{L^2}^2 + \tfrac{1}{2}\,\|\partial_t g(t)\|_{L^2}^2 + C\,\|u(t)\|_{L^2}^6 \\ &\le 2C_0(b) + CC_0(b)^3 + C_1(b) \end{aligned} \tag{2.16}$$

with $C_1(b) := \frac{1}{2}\max_{0\le t\le b}(\|g(t)\|_{L^2}^2 + \|\partial_t g(t)\|_{L^2}^2)$, where we also used (2.12). The estimate (2.15) further leads to the lower bound

$$\mathcal{E}(t,u(t)) \ge \tfrac{1}{4}\,\|\partial_x u(t)\|_{L^2}^2 - \tfrac{3}{4}C_0(b) - CC_0(b)^3 - \tfrac{1}{2}\,\|g(t)\|_{L^2}^2. \tag{2.17}$$

Combining (2.12), (2.16) and (2.17), we arrive at

$$\|u(t)\|_{H^1}^2 \le 4\mathcal{E}(0,u_0) + C(1+b)(C_0(b) + C_0(b)^3 + C_1(b)) \tag{2.18}$$

for $0 \le t \le b < a$. If a was finite, we could take here $b = a$ and obtain a contradiction to the blow-up condition stated in Step 1). Hence, $a = \infty$.

Step 7. If g is bounded in $L^2(\mathbb{T})$, then (2.12) shows that $u(t)$ is bounded in $L^2(\mathbb{T})$ for $t \ge 0$. In particular, we can replace $C_0(b)$ by C in this case.

If also $\partial_t g$ is bounded in $L^2(\mathbb{T})$, then (2.18) implies that $u(t)$ grows at most as $\sqrt{1+t}$ in $H^1(\mathbb{T})$.

Next, if g is contained in $L^2(\mathbb{R}_+; L^2(\mathbb{T}))$, then we infer from the line before (2.12) and Young's convolution inequality that $u \in L^2(\mathbb{R}_+; L^2(\mathbb{T}))$.

If even $g \in H^1(\mathbb{R}_+; L^2(\mathbb{T}))$, then g is also bounded in $L^2(\mathbb{T})$. Thus u belongs $L^6(\mathbb{R}_+; L^2(\mathbb{T}))$. Integrating in t, we now deduce from the line before (2.16) that $\mathcal{E}(t,u(t))$ is uniformly bounded. Hence, the boundedness of $u(t)$ in $H^1(\mathbb{T})$ follows from (2.17). □

3 Strang splitting for the Lugiato-Lefever equation

In order to formulate a numerical method for (2.1), it is convenient to define the nonlinear mapping

$$B : L^2(\mathbb{T}) \longrightarrow L_1(\mathbb{T}), \qquad B(w) = \mathrm{i}|w|^2.$$

If $w \in H^1(\mathbb{T})$, then $B(w) \in L_\infty(\mathbb{T})$ due to the Sobolev embedding $H^1(\mathbb{T}) \hookrightarrow L_\infty(\mathbb{T})$. For a *fixed* $w \in H^1(\mathbb{T})$, the function $x \mapsto B(w)(x) = \mathrm{i}|w(x)|^2$ will be identified with the multiplication operator

$$B(w) : L^2(\mathbb{T}) \to L^2(\mathbb{T}), \quad B(w)v = \mathrm{i}|w|^2 v$$

which generates a unitary group $(e^{tB(w)})_{t\in\mathbb{R}}$ on $L^2(\mathbb{T})$. For $v \in H^{k^*}(\mathbb{T})$ and $w \in H^k(\mathbb{T})$ it follows from (2.5) that

$$\|B(v)w\|_{H^k} \leq C\|v\|_{H^{k^*}}^2 \|w\|_{H^k}\,, \qquad k \geq 0\,. \tag{3.1}$$

As in 2.1 we let $A := \mathrm{i}\partial_x^2 - I$. Then, the Lugiato-Lefever equation (2.1) reads

$$\partial_t u = Au + B(u)u + g, \tag{3.2a}$$
$$u(0) = u_0. \tag{3.2b}$$

The solution is supposed to be approximated on the time-interval $[0, T]$ for $T > 0$.

3.1 ϕ-functions

In the construction and analysis of the splitting method for (2.1) we use the operator-valued functions $\phi_j(tA)$ defined by

$$\phi_j(tA)v = \int_0^1 \frac{\theta^{j-1}}{(j-1)!} e^{(1-\theta)tA} v \,\mathrm{d}\theta, \quad j \in \mathbb{N}, \qquad \phi_0(tA) = e^{tA}, \tag{3.3}$$

cf. [12]. For every $j, k \in \mathbb{N}_0$ and $t \geq 0$, the operator $\phi_j(tA) : H^k(\mathbb{T}) \to H^k(\mathbb{T})$ is bounded, and

$$\|\phi_j(tA)v\|_{H^k(\mathbb{T})} \leq \frac{1}{j!}\|v\|_{H^k(\mathbb{T})} \qquad \text{for all } v \in H^k(\mathbb{T}).$$

For every $v \in L^2(\mathbb{T})$ and $t > 0$, the recurrence relation

$$\phi_{j+1}(tA)v = (tA)^{-1}\left(\phi_j(tA)v - \frac{1}{j!}v\right), \qquad j \in \mathbb{N}_0 \tag{3.4}$$

follows from (3.3) via integration by parts. This recursion yields the Taylor expansions

$$e^{tA}v = \phi_0(tA)v = \sum_{k=0}^{m-1} \frac{t^k}{k!}A^k v + (tA)^m \phi_m(tA)v \tag{3.5}$$

for $m \in \mathbb{N}$ and $v \in D(A^m)$. Similar to (3.3) we define for $j \in \mathbb{N}$, $w \in H^1(\mathbb{T})$ and $v \in L^2(\mathbb{T})$

$$\phi_j(tB(w))v = \int_0^1 \frac{\theta^{j-1}}{(j-1)!} e^{(1-\theta)tB(w)} v \,\mathrm{d}\theta, \qquad \phi_0(tB(w))v = e^{tB(w)}v. \tag{3.6}$$

Equation (3.5) still holds if A is replaced by $B(w)$.

3.2 Time-integration scheme

Strang splitting methods for (3.2) are based on the observation that solving each of the two sub-problems

$$\partial_t v(t) = Av(t) + g(t), \tag{3.7}$$
$$\partial_t w(t) = B(w(t))w(t), \tag{3.8}$$

is much easier than solving (3.2a). Let $t_n = n\tau$ with step-size $\tau > 0$. Applying the variation-of-constants formula to (3.7) yields

$$v(t_{n+1}) = e^{\tau A} v(t_n) + \int_0^\tau e^{(\tau - s)A} g(t_n + s)\, \mathrm{d}s.$$

After $s \mapsto g(t_n + s)$ has been approximated by the linear interpolation

$$s \mapsto g(t_n) + s\frac{g(t_{n+1}) - g(t_n)}{\tau},$$

the integral can be computed analytically via integration by parts, and we obtain the exponential trapezoidal rule

$$v_{n+1} = e^{\tau A} v_n + \tau\Big(\phi_1(\tau A) g(t_n) + \phi_2(\tau A)\big(g(t_{n+1}) - g(t_n)\big)\Big) \tag{3.9}$$

which yields approximations $v_n \approx v(t_n)$ to the solution of (3.7); cf. [12].

The sub-problem (3.8) can even be solved exactly: Since

$$\partial_t \left(|w(t)|^2\right) = 2\mathrm{Re}\left(\overline{w(t)}\partial_t w(t)\right) = 2\mathrm{Re}\left(\mathrm{i}|w(t)|^4\right) = 0\,,$$

it follows that $|w(t)| = |w(0)|$ is time invariant, and hence the solution of (3.8) is given explicitly by

$$w(t) = e^{tB(w(0))} w(0)\,.$$

This is a well-known fact; see [6], e.g. Approximations $u_n \approx u(t_n)$ to the solution of the *full* problem (3.2) can now be computed recursively with the Strang splitting

$$u_n^+ = e^{\tau B(u_n)/2} u_n, \tag{3.10a}$$
$$u_n^* = e^{\tau A} u_n^+ + \tau\Big(\phi_1(\tau A) g(t_n) + \phi_2(\tau A)\big(g(t_{n+1}) - g(t_n)\big)\Big), \tag{3.10b}$$
$$u_{n+1} = e^{\tau B(u_n^*)/2} u_n^*\,. \tag{3.10c}$$

Every time-step $u_n \mapsto u_{n+1}$ of the Strang splitting consists of three sub-steps. First, (3.8) is solved over the interval $[t_n, t_n + \frac{\tau}{2}]$ with initial data $w(t_n) = u_n$, which yields an update $u_n^+ = w(t_n + \frac{\tau}{2})$. Then, one step of the exponential trapezoidal rule (3.9) with step-size τ and $v_n = u_n^+$ is carried out, which turns u_n^+ into u_n^*. Finally, (3.8) is propagated over the interval $[t_n + \frac{\tau}{2}, t_{n+1}]$, which gives the new approximation $u_{n+1} \approx u(t_{n+1})$. Note that for $A = \mathrm{i}\Delta$ and

$g(t) \equiv 0$, (3.10) reduces to the method considered in [16] for solving the NLS in absence of damping and forcing.

For every $\theta \geq 0$, the result after $n \in \mathbb{N}_0$ steps of the Strang splitting (3.10) with step-size $\tau > 0$ starting at time θ with initial data u_θ will be denoted by

$$\Phi^n_{\tau,\theta}(u_\theta).$$

If $n = 1$, then we simply write $\Phi_{\tau,\theta}(u_\theta)$ instead of $\Phi^1_{\tau,\theta}(u_\theta)$. For any $\tau > 0$, $n \in \mathbb{N}$, u_θ, u_0, the relations

$$\Phi^0_{\tau,\theta}(u_\theta) = u_\theta, \qquad \Phi^n_{\tau,0}(u_0) = \Phi_{\tau,t_{n-1}}\left(\Phi^{n-1}_{\tau,0}(u_0)\right) = \Phi^{n-1}_{\tau,t_1}\left(\Phi_{\tau,0}(u_0)\right)$$

follow directly from the definition. In addition to the numerical flow $\Phi^n_{\tau,\theta}(u_\theta)$, we also consider the exact flow given by the exact solution of (3.2a)

$$t \mapsto \Psi_{t,\theta}(u_\theta)$$

with initial data $u(\theta) = u_\theta$ at time θ.

For the discretization of space the spectral collocation method can be used, i.e. the solution $u(t) = u(t,x)$ is approximated by a trigonometric polynomial which satisfies (2.1a) in $m \in \mathbb{N}$ equidistant collocation points $x_k = 2\pi k/m$; see [6] for details. If v is such a trigonometric polynomial, then $e^{\tau A}v$ can be easily computed by means of the fast Fourier transform. Terms like $e^{\tau B(v)/2}v$ are approximated with a trigonometric polynomial which interpolates the values $e^{\mathrm{i}\tau|v(x_k)|^2/2}v(x_k)$ in the collocation points. Hence, all terms in (3.10) can be evaluated quickly at low computational costs. In this paper, however, only the semidiscretization in time with the Strang splitting (3.10) and without any approximation in space will be analyzed.

3.3 Error analysis: main results

Our goal is to prove that the Strang splitting converges with order 1 in $H^1(\mathbb{T})$ and with order 2 in $L^2(\mathbb{T})$ on bounded time-intervals. In order to state our results, we define the abbreviations

$$m^k_u := \sup_{t\in[0,T]} \|u(t)\|_{H^k}, \qquad m^k_g := \sup_{t\in[0,T]} \|g(t)\|_{H^k},$$
$$m^k_{g'} := \sup_{t\in[0,T]} \|\partial_t g(t)\|_{H^k}, \qquad m^k_{g''} := \sup_{t\in[0,T]} \|\partial^2_t g(t)\|_{H^k}.$$

Observe that for $k \leq 4$ the number m^k_u is finite by Theorem 2.1 and assumption (2.2). An inspection of the proof of Theorem 2.1 shows that m^4_u only depends on the norms of u_0 and g in the spaces involved in (2.2). For a solution $u(t) \in H^4(\mathbb{T})$ of (3.2) we immediately obtain the estimates

$$\sup_{t\in[0,T]} \|\partial_t u(t)\|_{H^k} \leq C\left(m^{k+2}_u, m^k_g\right), \qquad \text{for } 0 \leq k \leq 2, \tag{3.11}$$

$$\sup_{t\in[0,T]} \|\partial^2_t u(t)\|_{L^2} \leq C\left(m^4_u, m^2_g, m^0_{g'}\right). \tag{3.12}$$

The following theorem is the main result of this paper.

Theorem 3.1 *Let $u(t) = \Psi_{t,0}(u_0)$ be the exact solution of* (3.2) *and assume that the initial data u_0 and the forcing g have the regularity* (2.2)*. Then, the global error of the splitting method* (3.10) *is bounded by*

$$\|\Phi^n_{\tau,0}(u_0) - u(t_n)\|_{H^1} \le \tau C\left(T, m^3_u, m^1_g, m^1_{g'}\right), \tag{3.13}$$
$$\|\Phi^n_{\tau,0}(u_0) - u(t_n)\|_{L^2} \le \tau^2 C\left(T, m^4_u, m^2_g, m^1_{g'}, m^0_{g''}\right) \tag{3.14}$$

for all $n \in \mathbb{N}$ with $t_n = n\tau \le T$ and sufficiently small $\tau > 0$.

Theorem 3.1 is shown in Section 7. In (7.6) and (7.8) we give upper bounds for the step-size τ, but we remark that this step-size restriction is typically too pessimistic in practice. The outline of the proof is taken from [16]. The first two ingredients are bounds for the local error of (3.10) in $H^1(\mathbb{T})$ and in $L^2(\mathbb{T})$, respectively. They are established in Section 5 and 6.

Theorem 3.2 (Local error in $H^1(\mathbb{T})$) *Let $n \in \mathbb{N}$ with $t_{n+1} = t_n + \tau \le T$. If $u(t_n) \in H^3(\mathbb{T})$ and if $g \in C^{2-j}\big([0,T]; H^j(\mathbb{T})\big)$ for $j = 0, 1, 2$, then the error after one step of the splitting method* (3.10) *is bounded by*

$$\big\|\Phi_{\tau,t_n}\big(u(t_n)\big) - \Psi_{\tau,t_n}\big(u(t_n)\big)\big\|_{H^1} \le \tau^2 C(m^3_u, m^1_g, m^1_{g'}).$$

Theorem 3.3 (Local error in $L^2(\mathbb{T})$) *Let $n \in \mathbb{N}$ with $t_{n+1} = t_n + \tau \le T$. Under assumption* (2.2) *the error after one step of the splitting method* (3.10) *is bounded by*

$$\big\|\Phi_{\tau,t_n}\big(u(t_n)\big) - \Psi_{\tau,t_n}\big(u(t_n)\big)\big\|_{L^2} \le \tau^3 C(m^4_u, m^2_g, m^1_{g'}, m^0_{g''}).$$

The error bound for the global error of (3.10) is obtained by combining the bounds for the local error with the following stability result, proved in Section 4.

Theorem 3.4 (Stability) *Let $n \in \mathbb{N}$ with $t_{n+1} = t_n + \tau \le T$. For $v, w \in H^1(\mathbb{T})$ with $\|v\|_{H^1} \le M$ and $\|w\|_{H^1} \le M$, the splitting method* (3.10) *satisfies*

$$\|\Phi_{\tau,t_n}(v) - \Phi_{\tau,t_n}(w)\|_{H^k} \le e^{C\cdot(M_*^2 + M^2 - 1)}\|v - w\|_{H^k}, \qquad k = 0, 1 \tag{3.15}$$

with constant

$$M_* = e^{(CM^2-1)\tau}M + \tau C m^1_g. \tag{3.16}$$

4 Stability and auxiliary results

Now we state three lemmas which will be used frequently throughout the paper. The first lemma asserts a stability result for the mapping $v \mapsto e^{tB(v)}v$. As before in (2.4), we let $k^* = \max\{1, k\}$.

Lemma 4.1 *If $v, w \in H^{k^*}(\mathbb{T})$ with $\|v\|_{H^{k^*}} \le M$ and $\|w\|_{H^{k^*}} \le M$ for some $k \in \mathbb{N}_0$, then*

$$\|e^{tB(v)}v - e^{tB(w)}w\|_{H^k} \le e^{CM^2 t}\|v-w\|_{H^k}\,, \qquad t \ge 0\,, \tag{4.1a}$$

$$\|e^{tB(v)}v\|_{H^k} \le Me^{CM^2 t}\,, \qquad t \ge 0\,. \tag{4.1b}$$

Note that for the stability in $L^2(\mathbb{T})$ (i.e. $k=0$ and $k^*=1$) the functions v and w have to belong to $H^1(\mathbb{T})$.

Proof The proof uses ideas of [16]. Let $k \in \mathbb{N}_0$ and let $v, w \in H^{k^*}(\mathbb{T})$ with $\|v\|_{H^{k^*}} \le M$ and $\|w\|_{H^{k^*}} \le M$. Then, the functions $x(t) = e^{tB(v)}v$ and $y(t) = e^{tB(w)}w$ are the solutions of the initial value problems

$$\begin{aligned} x'(t) &= B(v)x(t)\,, \quad x(0) = v\,, \quad t \ge 0,\\ y'(t) &= B(w)y(t)\,, \quad y(0) = w\,, \quad t \ge 0, \end{aligned} \tag{4.2}$$

respectively, cf. Section 3.2. The inequality (2.5) implies that $B(v) \in H^k(\mathbb{T})$ and hence $e^{tB(v)} \in H^k(\mathbb{T})$, and applying (2.5) once again shows that $x(t) = e^{tB(v)}v \in H^k(\mathbb{T})$ for every $t \in [0,T]$. The same arguments yield that $y(t) \in H^k(\mathbb{T})$ for every $t \in [0,T]$.

First, we examine $\|x(t)\|_{H^k}$. From (3.1) we derive the estimate

$$\|B(v)x(t)\|_{H^k} \le CM^2\|x(t)\|_{H^k}\,, \quad t \ge 0, \tag{4.3}$$

and hence

$$\|x(t)\|_{H^k} = \|x(0)\|_{H^k} + \int_0^t \|B(v)x(s)\|_{H^k}\,\mathrm{d}s \le M + CM^2\int_0^t \|x(s)\|_{H^k}\,\mathrm{d}s\,.$$

Gronwall's lemma now yields

$$\|x(t)\|_{H^k} \le Me^{CM^2 t}, \tag{4.4}$$

which proves (4.1b). In order to show (4.1a), we consider the difference

$$\begin{aligned} B(v)x(t) - B(w)y(t) &= \mathrm{i}|v|^2x(t) - \mathrm{i}|w|^2y(t)\\ &= \mathrm{i}(v-w)\bar{v}x(t) + \mathrm{i}w(\bar{v}-\bar{w})x(t) + \mathrm{i}w\bar{w}\big(x(t)-y(t)\big)\,. \end{aligned}$$

Using also (2.5) and (4.4), we derive

$$\begin{aligned} \|B(v)x(t)-B(w)y(t)\|_{H^k} &\le C\big[2M\|x(t)\|_{H^{k^*}}\|v-w\|_{H^k} + M^2\|x(t)-y(t)\|_{H^k}\big]\\ &\le C\big[2M^2e^{CM^2t}\|v-w\|_{H^k} + M^2\|x(t)-y(t)\|_{H^k}\big]. \end{aligned}$$

The equations (4.2) thus imply

$$\begin{aligned} \|x(t)-y(t)\|_{H^k} &= \|v-w\|_{H^k} + \int_0^t \|B(v)x(s)-B(w)y(s)\|_{H^k}\,\mathrm{d}s\\ &\le \left(1 + 2CM^2\int_0^t e^{CM^2s}\,\mathrm{d}s\right)\|v-w\|_{H^k}\\ &\quad + CM^2\int_0^t \|x(s)-y(s)\|_{H^k}\,\mathrm{d}s \end{aligned}$$

for $t \geq 0$. Since $0 \leq (e^{CM^2t} - 1)^2$ yields $2e^{CM^2t} - 1 \leq e^{2CM^2t}$, it follows that

$$1 + 2CM^2 \int_0^t e^{CM^2s} \, \mathrm{d}s = 1 + 2(e^{CM^2t} - 1) \leq e^{2CM^2t}.$$

Applying Gronwall's lemma once again, we arrive at

$$\|e^{tB(v)}v - e^{tB(w)}w\|_{H^k} = \|x(t) - y(t)\|_{H^k} \leq e^{\widehat{C}M^2t}\|v - w\|_{H^k}$$

with $\widehat{C} = 3C$. □

The next lemma concerns technical estimates regarding the quantity u^* in the splitting method (3.10).

Lemma 4.2 *For a given v, $n \in \mathbb{N}$ and $\tau \in (0, T]$ we define*

$$v^*(\tau) = e^{\tau A}e^{\tau B(u_n)/2}v + \tau\Big(\phi_1(\tau A)g(t_n) + \phi_2(\tau A)\big(g(t_n + \tau) - g(t_n)\big)\Big). \tag{4.5}$$

(i) If $v, u_n \in H^1(\mathbb{T})$ with $\|v\|_{H^1}, \|u_n\|_{H^1} \leq M$, then

$$\|v^*(\tau)\|_{H^k} \leq e^{(CM^2-1)\tau}M + \tau Cm_g^k \qquad \textit{for } k = 0, 1\,.$$

(ii) If $k \in \{0, 1\}$ and $v, u_n \in H^{k+2}(\mathbb{T})$ with $\|v\|_{H^{k+2}}, \|u_n\|_{H^{k+2}} \leq M$, then

$$\|\partial_\tau v^*(\tau)\|_{H^k} \leq C(T, M, m_g^k, m_{g'}^k)\,.$$

(iii) If $v, u_n \in H^4(\mathbb{T})$ with $\|v\|_{H^4}, \|u_n\|_{H^4} \leq M$, then

$$\|\partial_\tau^2 v^*(\tau)\|_{L^2} \leq C(T, M, m_g^2, m_{g'}^0, m_{g''}^0).$$

Remark. If $v = u_n$, then $v^*(\tau) = u_n^*$ defined in (3.10b). In the error analysis below, however, Lemma 4.2 will sometimes also be applied with $v = u(t_n)$.

Proof The first assertion follows from Lemma 4.1 and the boundedness of the operators $\phi_j(\tau A)$. For the proof of (ii) and (iii), it is useful to represent the derivative $\partial_t\phi_j(\tau A)$ in terms of $\phi_{j-1}(\tau A)$ and $\phi_j(\tau A)$: If $v \in D(A)$, then (3.3) yields

$$\begin{aligned}
\partial_\tau\phi_j(\tau A)v &= \int_0^1 \frac{\theta^{j-1}}{(j-1)!}(1-\theta)Ae^{(1-\theta)\tau A}v \; \mathrm{d}\theta \\
&= \int_0^1 \frac{\theta^{j-1}}{(j-1)!}Ae^{(1-\theta)\tau A}v \; \mathrm{d}\theta - j\int_0^1 \frac{\theta^j}{j!}Ae^{(1-\theta)\tau A}v \; \mathrm{d}\theta \\
&= \big(\phi_j(\tau A) - j\phi_{j+1}(\tau A)\big)Av,
\end{aligned} \tag{4.6}$$

and with (3.4) we obtain

$$\begin{aligned}
\partial_\tau\phi_0(\tau A)v &= e^{\tau A}Av\,, && v \in D(A), \quad \tau \geq 0, \\
\partial_\tau\phi_j(\tau A)v &= \frac{1}{\tau}\big(\phi_{j-1}(\tau A) - j\phi_j(\tau A)\big)v\,, && v \in L^2(\mathbb{T}), \quad j > 0, \quad \tau > 0.
\end{aligned}$$

Now (ii) and (iii) can be shown with straightforward calculations using (3.1), Lemma 4.1, and the boundedness of $\phi_j(\tau A)$. □

After these preparations we are ready to prove stability of the Strang splitting scheme (3.10). In order to simplify notation (in particular in sections 5 and 6), we define

$$B_{1/2}(u) = \frac{\mathrm{i}}{2}|u|^2 = \frac{1}{2}B(u). \tag{4.7}$$

Proof of Theorem 3.4.
Let $v, w \in H^1(\mathbb{T})$ with $\|v\|_{H^1} \le M$ and $\|w\|_{H^1} \le M$. As in (3.10), we define

$$\begin{aligned} v^* &= e^{\tau A} e^{\tau B_{1/2}(v)} v + \tau\Big(\phi_1(\tau A) g(t_n) + \phi_2(\tau A)\big(g(t_n+\tau) - g(t_n)\big)\Big), \\ w^* &= e^{\tau A} e^{\tau B_{1/2}(w)} w + \tau\Big(\phi_1(\tau A) g(t_n) + \phi_2(\tau A)\big(g(t_n+\tau) - g(t_n)\big)\Big). \end{aligned}$$

According to Lemma 4.2, we have $\|v^*\|_{H^1} \le M_*$ and $\|w^*\|_{H^1} \le M_*$ with M_* defined in (3.16). Applying Lemma 4.1 twice results in the estimates

$$\begin{aligned} \|\Phi_{\tau,t_n}(v) - \Phi_{\tau,t_n}(w)\|_{H^k} &= \|e^{\tau B_{1/2}(v^*)} v^* - e^{\tau B_{1/2}(w^*)} w^*\|_{H^k} \\ &\le e^{CM_*^2 \tau} \|v^* - w^*\|_{H^k} \\ &= e^{CM_*^2 \tau} \|e^{\tau A} e^{\tau B_{1/2}(v)} v - e^{\tau A} e^{\tau B_{1/2}(w)} w\|_{H^k} \\ &\le e^{(CM_*^2 - 1)\tau} \|e^{\tau B_{1/2}(v)} v - e^{\tau B_{1/2}(w)} w\|_{H^k} \\ &\le e^{C\cdot(M_*^2 + M^2 - 1)\tau} \|v - w\|_{H^k} \end{aligned}$$

for $k \in \{0, 1\}$. $\square$

The last lemma in this subsection will be useful in the proofs of Theorems 3.2 and 3.3.

Lemma 4.3 *For a given $n \in \mathbb{N}$ and $\tau > 0$ let*

$$b(\tau) = B\big(u(t_n+\tau)\big) - B\big(v^*(\tau)\big),$$

where $v^\star(\tau)$ is defined by (4.5) *with $v = u(t_n)$. Under the assumption* (2.2), *we have*

$$b(\tau) = \int_0^\tau \partial_\tau b(s)\,\mathrm{d}s = \int_0^\tau \int_0^s \partial_\tau^2 b(r)\,\mathrm{d}r\,\mathrm{d}s.$$

Proof The fundamental theorem of calculus gives

$$b(\tau) = b(0) + \int_0^\tau \partial_\tau b(s)\,\mathrm{d}s = b(0) + \tau\partial_\tau b(0) + \int_0^\tau \int_0^s \partial_\tau^2 b(r)\,\mathrm{d}r\,\mathrm{d}s.$$

As $v^\star(0) = v = u(t_n)$ by assumption, it is clear that $b(0) = 0$. Hence, we only have to show that $\partial_\tau b(0) = 0$. By definition, we have

$$\partial_\tau b(\tau) = 2\mathrm{iRe}\left(\overline{u(t_n+\tau)}\partial_\tau u(t_n+\tau)\right) - 2\mathrm{iRe}\left(\overline{v^\star(\tau)}\partial_\tau v^\star(\tau)\right).$$

Since $u(t_n) = v = v^\star(0)$,

$$\begin{aligned} \partial_\tau u(t_n+\tau)\Big|_{\tau=0} &= Au(t_n) + B(u(t_n))u(t_n) + g(t_n), \\ \partial_\tau v^\star(\tau)\Big|_{\tau=0} &= Av + B_{1/2}(v)v + g(t_n), \end{aligned}$$

we deduce

$$\begin{aligned} \partial_\tau b(0) &= 2\mathrm{iRe}\left(\overline{u(t_n)}\partial_\tau u(t_n)\right) - 2\mathrm{iRe}\left(\overline{u(t_n)}\partial_\tau v^\star(0)\right) \\ &= 2\mathrm{iRe}\left(\overline{u(t_n)}B(u(t_n))u(t_n)\right) - 2\mathrm{iRe}\left(\overline{u(t_n)}B_{1/2}(u(t_n))u(t_n)\right) = 0 \end{aligned}$$

because $\overline{u(t_n)}B(u(t_n))u(t_n) = \mathrm{i}|u(t_n)|^4$ and $\overline{u(t_n)}B_{1/2}(u(t_n))u(t_n) = \frac{\mathrm{i}}{2}|u(t_n)|^4$ are both purely imaginary functions; cf. [2]. □

5 Local error in $H^1(\mathbb{T})$: Proof of Theorem 3.2

Without loss of generality we assume that $n = 0$, i.e. $u(t_n) = u(0) = u_0$ and $\Psi_{\tau,t_n}(u(t_n)) = \Psi_{\tau,0}(u_0) = u(\tau)$.

Step 1. The variation-of-constants formula yields the representation

$$u(\tau) = e^{\tau A}u_0 + \int_0^\tau e^{(\tau-s)A}\Big(B(u(s))u(s) + g(s)\Big)\,\mathrm{d}s$$

of the exact solution of (3.2). Substituting the formula a second time for $u(s)$, we obtain

$$u(\tau) = e^{\tau A}u_0 + I_1 + I_2 + R_1\,, \tag{5.1}$$

where we set

$$I_1 = \int_0^\tau e^{(\tau-s)A}B(u(s))e^{sA}u_0\ \mathrm{d}s, \tag{5.2}$$

$$I_2 = \int_0^\tau e^{(\tau-s)A}g(s)\ \mathrm{d}s, \tag{5.3}$$

$$R_1 = \int_0^\tau \int_0^s e^{(\tau-s)A}B(u(s))e^{(s-\sigma)A}\left[B(u(\sigma))u(\sigma) + g(\sigma)\right]\ \mathrm{d}\sigma\ \mathrm{d}s.$$

Using (3.1), it can be shown that

$$\|R_1\|_{H^1} \le \tau^2 C(m_u^1\,, m_g^1)\,. \tag{5.4}$$

The approximation $u_1 = \Phi_{\tau,0}(u_0)$ of the numerical method after one step reads

$$\begin{aligned} u_1 &= e^{\tau B_{1/2}(u_0^*)}e^{\tau A}e^{\tau B_{1/2}(u_0)}u_0 \\ &\quad + \tau e^{\tau B_{1/2}(u_0^*)}\Big(\phi_1(\tau A)g(0) + \phi_2(\tau A)\big(g(\tau)-g(0)\big)\Big) \end{aligned} \tag{5.5}$$

with $B_{1/2}(\cdot)$ defined in (4.7) and

$$u_0^* = e^{\tau A} e^{\tau B_{1/2}(u_0)} u_0 + \tau\Big(\phi_1(\tau A) g(0) + \phi_2(\tau A)\big(g(\tau) - g(0)\big)\Big).$$

We also consider u_0^* as a function of $\tau > 0$. Using further the expansion

$$e^{\tau B_{1/2}(u_0)} v = \sum_{k=0}^{m-1} \frac{\tau^k}{k!} B_{1/2}^k(u_0) v + \tau^m B_{1/2}^m(u_0) \phi_m(\tau B_{1/2}(u_0)) v$$

with $m \in \{1, 2\}$, see (3.5), we derive

$$u_1 = e^{\tau A} u_0 + T_1 + T_2 + R_2\,, \tag{5.6}$$

where we also employ (3.5) with $m = 1$ and define

$$T_1 = \tau\Big(B_{1/2}(u_0^*) e^{\tau A} + e^{\tau A} B_{1/2}(u_0)\Big) u_0, \tag{5.7}$$

$$T_2 = \tau\Big(\phi_1(\tau A) g(0) + \phi_2(\tau A)\big(g(\tau) - g(0)\big)\Big), \tag{5.8}$$

$$\begin{aligned} R_2 = \tau^2 \Big[& e^{\tau A} B_{1/2}^2(u_0) \phi_2\big(\tau B_{1/2}(u_0)\big) u_0 + B_{\frac12}(u_0^*) e^{\tau A} B_{\frac12}(u_0) \phi_1\big(\tau B_{1/2}(u_0)\big) u_0 \\ & + B_{1/2}^2(u_0^*) \phi_2\big(\tau B_{1/2}(u_0^*)\big) e^{\tau A} e^{\tau B_{1/2}(u_0)} u_0 \\ & + B_{1/2}(u_0^*) \phi_1\big(\tau B_{1/2}(u_0^*)\big) \Big(\phi_1(\tau A) g(0) + \phi_2(\tau A)\big(g(\tau) - g(0)\big)\Big) \Big]\,. \end{aligned}$$

Estimate (3.1) and Lemmas 4.1 and 4.2 imply

$$\|R_2\|_{H^1} \le \tau^2 C(m_u^1, m_g^1)\,. \tag{5.9}$$

Step 2. We compare the exact solution (5.1) with the numerical solution (5.6). Using also (5.4) and (5.9), we infer

$$\|u(\tau) - u_1\|_{H^1} \le \|I_1 - T_1\|_{H^1} + \|I_2 - T_2\|_{H^1} + \tau^2 C(m_u^1, m_g^1).$$

Our goal is now to bound the terms $\|I_1 - T_1\|_{H^1}$ and $\|I_2 - T_2\|_{H^1}$. With the abbreviations

$$h_1(s) = e^{(\tau - s)A} B(u(s)) e^{sA} u_0\,, \tag{5.10}$$

$$b(\tau) = B(u(\tau)) - B(u_0^*(\tau))\,, \tag{5.11}$$

the first term can be represented as $I_1 - T_1 = Q_1 + E_1$, where

$$Q_1 = \int_0^\tau h_1(s)\,\mathrm{d}s - \frac{\tau}{2}\big(h_1(0) + h_1(\tau)\big) \tag{5.12}$$

is the local quadrature error of the trapezoidal rule and

$$E_1 = \frac{\tau}{2} b(\tau) e^{\tau A} u_0 \tag{5.13}$$

is a remainder term. The order of the trapezoidal rule is two, and hence its local error scales like $\mathcal{O}\big(\tau^3\big)$ if the integrand is smooth enough. For the proof of Theorem 3.2, however, the bound

$$\|Q_1\|_{H^1} \le \tau^2 C \sup_{s\in[0,\tau]} \|\partial_s h_1(s)\|_{H^1} \tag{5.14}$$

is sufficient. Applying Lemma 4.3 with $n = 0$ and $t_n = 0$, the remainder term E_1 can be bounded by

$$\|E_1\|_{H^1} \le \tau^2 C m_u^1 \sup_{s\in[0,\tau]} \|\partial_s b(s)\|_{H^1} \,.$$

The difference

$$I_2 - T_2 = \int_0^\tau e^{(\tau-s)A} g(s) \;\mathrm{d}s - \tau\Big(\phi_1(\tau A) g(0) + \phi_2(\tau A)\big(g(\tau) - g(0)\big)\Big)$$

is the local error of the exponential trapezoidal rule so that

$$\|I_2 - T_2\|_{H^1} \le \tau^2 C m_{g'}^1 \,,$$

see Theorem 2.7 in [12].

Step 3. To complete the proof of Theorem 3.2, it remains to show that the terms

$$\sup_{s\in[0,\tau]} \|\partial_s h_1(s)\|_{H^1} \quad \text{and} \quad \sup_{s\in[0,\tau]} \|\partial_s b(s)\|_{H^1}$$

are bounded. The equations (3.1) and (3.11) yield

$$\sup_{s\in[0,\tau]} \|\partial_s h_1(s)\|_{H^1} \le C\left(m_u^3\,, m_g^1\right).$$

Finally, in view of (3.11) and Lemma 4.2, the term

$$\partial_s b(s) = 2\mathrm{i}\left(\mathrm{Re}(\overline{u(s)}\partial_s u(s)) - \mathrm{Re}(\overline{u_0^*(s)}\partial_s u_0^*(s))\right)$$

can be estimated

$$\sup_{s\in[0,\tau]} \|\partial_s b(s)\|_{H^1} \le C\left(\tau\,, m_u^3\,, m_g^1\,, m_{g'}^1\right)\,,$$

which completes the proof of Theorem 3.2. □

6 Local error in $L^2(\mathbb{T})$: Proof of Theorem 3.3

To prove the third-order local error bound in $L^2(\mathbb{T})$, we mimic the proof for the second-order bound of the local error in $H^1(\mathbb{T})$. However, we have to expand the analytical solution and the numerical scheme to a higher order. As before, we assume without any loss of generality that $n = 0$ and let $u(\tau) = \Psi_{\tau,0}(u_0)$.

Step 1. We expand the exact solution further by inserting the variation of constant formula for $u(\sigma)$ into (5.1). It follows that

$$u(\tau) = e^{\tau A}u_0 + I_1 + I_2 + I_3 + I_4 + \hat{R}_1,$$

where I_1 and I_2 have been defined in (5.2) and (5.3), respectively, and we introduce

$$\begin{aligned}
I_3 &= \int_0^\tau \int_0^s e^{(\tau-s)A} B(u(s)) e^{(s-\sigma)A} B(u(\sigma)) e^{\sigma A} u_0 \, \mathrm{d}\sigma \, \mathrm{d}s \\
I_4 &= \int_0^\tau \int_0^s e^{(\tau-s)A} B(u(s)) e^{(s-\sigma)A} g(\sigma) \, \mathrm{d}\sigma \, \mathrm{d}s \\
\hat{R}_1 &= \int_0^\tau \int_0^s \int_0^\sigma e^{(\tau-s)A} B(u(s)) e^{(s-\sigma)A} \\
&\qquad\qquad \times\ B(u(\sigma)) e^{(\sigma-\xi)A} \big[B(u(\xi))u(\xi) + g(\xi) \big] \, \mathrm{d}\xi \, \mathrm{d}\sigma \, \mathrm{d}s.
\end{aligned}$$

The estimate (3.1) yields

$$\|\hat{R}_1\|_{L^2} \le \tau^3 C(m_u^1, m_g^0)\,. \tag{6.1}$$

Substituting the expansion

$$e^{\tau B_{1/2}(\cdot)} = I + \tau B_{1/2}(\cdot) + \frac{\tau^2}{2} B_{1/2}^2(\cdot) + \tau^3 B_{1/2}^3(\cdot) \phi_3\big(\tau B_{1/2}(\cdot)\big)$$

into the splitting method (5.5), we derive

$$u_1 = e^{\tau A}u_0 + T_1 + T_2 + T_3 + T_4 + \hat{R}_2$$

with T_1, T_2 from (5.7), (5.8) and

$$\begin{aligned}
T_3 &= \frac{\tau^2}{2} \Big(B_{1/2}^2(u_0^*) e^{\tau A} + 2B_{1/2}(u_0^*) e^{\tau A} B_{1/2}(u_0) + e^{\tau A} B_{1/2}^2(u_0) \Big) u_0 \\
T_4 &= \tau^2 B_{1/2}(u_0^*) \Big(\phi_1(\tau A) g(0) + \phi_2(\tau A) \big(g(\tau) - g(0) \big) \Big) \\
\hat{R}_2 &= \tau^3 \Big[e^{\tau A} B_{1/2}^3(u_0) \phi_3\big(\tau B_{1/2}(u_0)\big) u_0 + B_{\frac{1}{2}}(u_0^*) e^{\tau A} B_{1/2}^2(u_0) \phi_2\big(\tau B_{\frac{1}{2}}(u_0)\big) u_0 \\
&\qquad + \frac{1}{2} B_{1/2}^2(u_0^*) e^{\tau A} B_{1/2}(u_0) \phi_1\big(\tau B_{1/2}(u_0)\big) u_0 \\
&\qquad + B_{1/2}^3(u_0^*) \phi_3\big(\tau B_{1/2}(u_0^*)\big) e^{\tau A} e^{\tau B_{1/2}(u_0)} u_0 \\
&\qquad + B_{1/2}^2(u_0^*) \phi_2\big(\tau B_{1/2}(u_0^*)\big) \Big(\phi_1(\tau A) g(0) + \phi_2(\tau A) \big(g(\tau) - g(0) \big) \Big) \Big].
\end{aligned}$$

Inequality (3.1) and Lemma 4.1 imply

$$\|\hat{R}_2\|_{L^2} \leq \tau^3 C(m_u^1, m_g^0)\,. \tag{6.2}$$

Step 2. Comparing the exact solution with the numerical solution and using (6.1) and (6.2), we estimate

$$\begin{aligned} \|u(\tau) - u_1\|_{L^2} \leq & \|I_1 - T_1\|_{L^2} + \|I_2 - T_2\|_{L^2} + \|I_3 - T_3\|_{L^2} \\ & + \|I_4 - T_4\|_{L^2} + \tau^3 C(m_u^1, m_g^0). \end{aligned}$$

As before, the terms of the numerical solution are splitted into a suitable quadrature formula and a remainder term. In addition to $h_1(s)$ defined in (5.10) and $b(\tau)$ defined in (5.11), we employ the abbreviations

$$\begin{aligned} h_2(s,\sigma) &= e^{(\tau-s)A} B(u(s)) e^{(s-\sigma)A} B(u(\sigma)) e^{\sigma A} u_0\,, \\ h_3(s) &= e^{(\tau-s)A} B(u(s)) \phi_1(sA) g(0). \end{aligned}$$

We still use the decomposition $I_1 - T_1 = Q_1 + E_1$ with the quadrature error Q_1 from (5.12) and the remainder E_1 from (5.13). Since now we aim at a local error in $L^2(\mathbb{T})$ of third order, we replace the error bound (5.14) by

$$\|Q_1\|_{L^2} \leq \tau^3 C \sup_{s\in[0,\tau]} \|\partial_s^2 h_1(s)\|_{L^2}\,,$$

see [14]. Lemma 4.3 implies

$$\|E_1\|_{L^2} \leq \tau^3 C m_u^1 \sup_{s\in[0,\tau]} \|\partial_s^2 b(s)\|_{L^2}\,.$$

The difference

$$I_2 - T_2 = \int_0^\tau e^{(\tau-s)A} g(s)\ \mathrm{d}s - \tau\Big(\phi_1(\tau A) g(0) + \phi_2(\tau A)\big(g(\tau) - g(0)\big)\Big)$$

is the local error of the exponential trapezoidal rule. We thus acquire

$$\|I_2 - T_2\|_{L^2} \leq \tau^3 C m_{g''}^0\,,$$

see [12]. For the third error term we use the partition $I_3 - T_3 = Q_3 + E_3$ with

$$\begin{aligned} Q_3 &= \int_0^\tau \int_0^s h_2(s,\sigma)\ \mathrm{d}\sigma\ \mathrm{d}s - \frac{\tau^2}{8}\big(h_2(0,0) + 2h_2(\tau,0) + h_2(\tau,\tau)\big), \\ E_3 &= \frac{\tau^2}{8} b(\tau)\Big(2e^{\tau A} B(u_0) + \big[B(u(\tau)) + B(u_0^*(\tau))\big] e^{\tau A}\Big) u_0, \end{aligned}$$

and $b(\tau)$ defined in (5.11). We identify Q_3 as the error of a cubature formula which integrates constant functions exactly. It follows

$$\|Q_3\|_{L^2} \leq C\tau^3 \left(\sup_{\triangle} \|\partial_s h_2(s,\sigma)\|_{L^2} + \sup_{\triangle} \|\partial_\sigma h_2(s,\sigma)\|_{L^2} \right)\,,$$

where $\triangle$ is the triangle $0 \le s \le \tau\,, 0 \le \sigma \le s$, see p. 362 in [14]. From Lemma 4.3 we infer

$$\|E_3\|_{L^2} \le \tau^3 C m_u^1 \sup_{s\in[0,\tau]} \|\partial_s b(s)\|_{L^2}\,.$$

The fourth term is decomposed into three parts

$$I_4 - T_4 = E_4^1 + E_4^2 + E_4^3$$

given by

$$\begin{aligned}
E_4^1 &= \int_0^\tau e^{(\tau-s)A} B(u(s)) F(s)\,\mathrm{d}s,\\
F(s) &= \int_0^s e^{(s-\sigma)A} g(\sigma)\ \mathrm{d}\sigma - s\phi_1(sA)g(0),\\
E_4^2 &= \int_0^\tau s h_3(s)\,\mathrm{d}s - \tau^2 B_{1/2}(u_0^*)\phi_1(\tau A)g(0),\\
E_4^3 &= -\tau^2 B_{1/2}(u_0^*)\phi_2(\tau A)\big(g(\tau)-g(0)\big).
\end{aligned}$$

Since $F(s)$ is the local error of the exponential Euler rule, we can estimate

$$\max_{s\in[0,\tau]} \|F(s)\|_{L^2} \le \tau^2 C m_{g'}^0\,,$$

see [12]. This inequality and (3.1) lead to

$$\|E_4^1\|_{L^2} \le \tau^3 C(m_u^1\,, m_{g'}^0)\,.$$

Integrating by parts, we calculate

$$\int_0^\tau s h_3(s)\,\mathrm{d}s = \frac{\tau^2}{2} h_3(\tau) - \frac{1}{2}\int_0^\tau s^2 \partial_s h_3(s)\,\mathrm{d}s,$$

so that

$$E_4^2 = \frac{\tau^2}{2} b(\tau)\phi_1(\tau A)g(0) - \frac{1}{2}\int_0^\tau s^2 \partial_s h_3(s)\ \mathrm{d}s.$$

Lemma 4.3 then yields

$$\|E_4^2\|_{L^2} \le C\tau^3 \Big(m_g^0 \|\partial_s b(s)\|_{L^2} + \sup_{s\in[0,\tau]} \|\partial_s h_3(s)\|_{L^2} \Big).$$

Exploiting the regularity of g, we obtain

$$\|g(\tau) - g(0)\|_{L^2} \le \tau m_{g'}^0\,.$$

Estimate (3.1), Lemma 4.2 and the boundedness of $\phi_j(\tau A)$ finally imply

$$\|E_4^3\|_{L^2} \le \tau^3 C(m_u^1\,, m_g^0\,, m_{g'}^0)\,.$$

Step 3. To complete the proof of Theorem 3.3, it remains to show that the terms

$$\sup_{s\in[0,\tau]} \|\partial_s^2 h_1(s)\|_{L^2}\,, \quad \sup_{\triangle}\|\partial_s h_2(s,\sigma)\|_{L^2}\,, \quad \sup_{\triangle}\|\partial_\sigma h_2(s,\sigma)\|_{L^2}\,,$$
$$\sup_{s\in[0,\tau]} \|\partial_s h_3(s)\|_{L^2} \quad \text{and} \quad \sup_{s\in[0,\tau]} \|\partial_s^2 b(s)\|_{L^2}$$

are bounded. Formulas (3.1), (4.2) and (4.6) yield

$$\begin{aligned}
\sup_{s\in[0,\tau]} \|\partial_s^2 h_1(s)\|_{L^2} &\le C(m_u^4\,, m_g^2\,, m_{g'}^0),\\
\sup_{\triangle}\|\partial_s h_2(s,\sigma)\|_{L^2} &\le C(m_u^2\,, m_g^0),\\
\sup_{\triangle}\|\partial_\sigma h_2(s,\sigma)\|_{L^2} &\le C(m_u^2\,, m_g^0),\\
\sup_{s\in[0,\tau]} \|\partial_s h_3(s)\|_{L^2} &\le C(m_u^2\,, m_g^2).
\end{aligned}$$

We then apply 3.11 and Lemma 4.2 to

$$\partial_s^2 b(s) = 2i\Big(|\partial_s u(s)|^2 + \mathrm{Re}(\overline{u(s)}\partial_s^2 u(s)) - |\partial_s u^*(s)|^2 - \mathrm{Re}(\overline{u^*(s)}\partial_s^2 u^*(s))\Big),$$

and conclude the last needed bound

$$\sup_{s\in[0,\tau]} \|\partial_s^2 b(s)\|_{L^2} \le C\left(\tau\,, m_u^4\,, m_g^2\,, m_{g'}^1\,, m_{g''}^0\right). \qquad \square$$

7 Proof of Theorem 3.1

In order to prove the global error estimates in Theorem 3.1, the local error bounds from Theorems 3.2 and 3.3 are combined with the stability result from Theorem 3.4 in the classical construction known as Lady Windermere's fan [10]. However, the stability result (3.15) can only be applied if the numerical solution $\Phi^n_{\tau,0}(u_0)$ stays bounded in $H^1(\mathbb{T})$ for all $n \in \mathbb{N}$ with $\tau n \le T$. This condition can be shown by the following induction argument. Let $u_0 \in H^3(\mathbb{T})$ and assume that there is a constant $\widehat{M} > m_u^1$ such that

$$\|\Phi^k_{\tau,t_\ell}\big(u(t_\ell)\big)\|_{H^1} \le \widehat{M} \quad \text{for all } \ell \in \mathbb{N}_0, \quad k = 0,\dots,n-1, \quad t_{\ell+k} \le T. \tag{7.1}$$

We will prove that

$$\|\Phi^n_{\tau,t_\ell}\big(u(t_\ell)\big)\|_{H^1} \le \widehat{M} \quad \text{for all } \ell \in \mathbb{N}_0, \quad t_{\ell+n} \le T \tag{7.2}$$

provided that the step-size τ is sufficiently small. Since the argument is the same for all ℓ, we assume that $\ell = 0$ with no loss of generality. Representing $\Phi^n_{\tau,0}(u_0)$ by the telescoping sum

$$\Phi^n_{\tau,0}(u_0) = u(t_n) + \sum_{j=0}^{n-1} \Phi^{n-j}_{\tau,t_j}\big(u(t_j)\big) - \Phi^{n-j-1}_{\tau,t_{j+1}}\big(u(t_{j+1})\big) \tag{7.3}$$

with $u(t_n) = \Psi_{t_n,0}(u_0)$ and $u(t_0) = u_0$ gives

$$\|\Phi_{\tau,0}^n(u_0)\|_{H^1} \le \|u(t_n)\|_{H^1} + \sum_{j=0}^{n-1} \|\Phi_{\tau,t_j}^{n-j}\big(u(t_j)\big) - \Phi_{\tau,t_{j+1}}^{n-j-1}\big(u(t_{j+1})\big)\|_{H^1} \tag{7.4}$$

According to (7.1), Theorem 3.4 can be applied and yields for $n-j-1 \ge 1$ that

$$\begin{aligned} &\Big\|\Phi_{\tau,t_j}^{n-j}\big(u(t_j)\big) - \Phi_{\tau,t_{j+1}}^{n-j-1}\big(u(t_{j+1})\big)\Big\|_{H^1} \\ &= \Big\|\Phi_{\tau,t_{n-1}}\Big(\Phi_{\tau,t_j}^{n-j-1}\big(u(t_j)\big)\Big) - \Phi_{\tau,t_{n-1}}\Big(\Phi_{\tau,t_{j+1}}^{n-j-2}\big(u(t_{j+1})\big)\Big)\Big\|_{H^1} \\ &\le e^{C\cdot(\widehat{M}_*^2+\widehat{M}^2-1)}\|\Phi_{\tau,t_j}^{n-j-1}\big(u(t_j)\big) - \Phi_{\tau,t_{j+1}}^{n-j-2}\big(u(t_{j+1})\big)\|_{H^1} \end{aligned} \tag{7.5}$$

with constant

$$\widehat{M}_* = e^{(C\widehat{M}^2-1)\tau}\widehat{M} + \tau C m_g^1,$$

cf. (3.16). If τ is sufficiently small, then $\widehat{M}_* \le C\widehat{M}$ so that $e^{C\cdot(\widehat{M}_*^2+\widehat{M}^2-1)} \le e^{C\widehat{M}^2}$. Applying (7.5) recursively, we then obtain

$$\begin{aligned} \|\Phi_{\tau,t_j}^{n-j}(u(t_j)) - \Phi_{\tau,t_{j+1}}^{n-j-1}(u(t_{j+1}))\|_{H^1} &\le e^{C\widehat{M}^2(n-j-1)\tau}\|\Phi_{\tau,t_j}(u(t_j)) - u(t_{j+1})\|_{H^1} \\ &\le e^{CT\widehat{M}^2} C_{loc}\tau^2 \end{aligned}$$

due to Theorem 3.2, with the constant C_{loc} from the local error bound. So (7.4) yields

$$\|\Phi_{\tau,0}^n(u_0)\|_{H^1} \le \|u(t_n)\|_{H^1} + ne^{CT\widehat{M}^2}C_{loc}\tau^2 \le m_u^1 + e^{CT\widehat{M}^2}C_{loc}T\tau.$$

If τ is so small that

$$\tau \le \frac{\widehat{M} - m_u^1}{C_{loc}T} e^{-CT\widehat{M}^2}, \tag{7.6}$$

then $\|\Phi_{\tau,0}^n(u_0)\|_{H^1} \le \widehat{M}$, as required.

It is now easy to show the bound for the global error in $L^2(\mathbb{T})$. The telescopic sum (7.3) yields

$$\|\Phi_{\tau,0}^n(u_0) - u(t_n)\|_{L^2} \le \sum_{j=0}^{n-1} \|\Phi_{\tau,t_j}^{n-j}\big(u(t_j)\big) - \Phi_{\tau,t_j}^{n-j-1}\big(u(t_{j+1})\big)\|_{L^2},$$

and with Theorem 3.4 and Theorem 3.3 we obtain similar as before

$$\|\Phi_{\tau,0}^n(u_0) - u(t_n)\|_{L^2} \le ne^{CT\widehat{M}^2}\tilde{C}_{loc}\tau^3 \le e^{CT\widehat{M}^2}\tilde{C}_{loc}T\tau^2 \tag{7.7}$$

with $\tilde{C}_{loc}$ denoting the constant from the local error bound in Theorem 3.3. The bound for the global error in $H^1(\mathbb{T})$ is obtained upon replacing $\|\cdot\|_{L^2}$ by $\|\cdot\|_{H^1}$, τ^p by τ^{p-1}, and $\tilde{C}_{loc}$ by C_{loc}. $\square$

Remark. According to (7.7) the global error is small if

$$\tau^2 \ll \frac{1}{\tilde{C}_{loc}T} e^{-CT\widehat{M}^2}. \tag{7.8}$$

Hence, even if one could avoid the step-size restriction (7.6) imposed by *stability*, there is still a similar step-size restriction imposed by *accuracy*. Of course, both (7.6) and (7.8) are usually too pessimistic in practice. These step-size restrictions are *not* a characteristic property of the equation (2.1) nor of the splitting method (3.10). For example, the error bound for the global error of Runge-Kutta methods for solving *ordinary* differential equations is similar to (7.7); cf. Theorem 3.6 in chapter II in [10].

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