Lecture Notes
Nonlinear Maxwell Equations

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These lecture notes are based on my course from winter semester 2019/20. A shorter version appears in the volume [17] within the series Oberwolfach Seminars. Typically, the proofs and calculations in the notes are a bit shorter than those given in the course. The drawings and many additional oral remarks from the lectures are omitted here. On the other hand, the notes contain some material omitted in the lectures. It is assumed that the reader has a solid background in functional analysis and Sobolev spaces. Occasionally I use notation and definitions of my lecture notes Functional Analysis and Spectral Theory without further notice.

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CHAPTER 1

Introduction and local wellposedness on $\mathbb{R}^3$

In this section we develop a local wellposedness theory for the quasilinear Maxwell equations on $\mathbb{R}^3$. Our approach is based on energy methods and a fixed-point argument, which make use of the linear system with time-depending coefficients. One has to work in Sobolev spaces $\mathcal{H}^s$ with $s > \frac{5}{2}$ in this context, where we take $s = 3$ for simplicity. Actually we treat general symmetric hyperbolic systems on $\mathbb{R}^3$. In the first subsection we introduce Maxwell equations and discuss some facts used throughout these notes. We then investigate the linear case, first in L^2 and then in $\mathcal{H}^3$, also establishing the finite speed of propagation. Our main tools are energy estimates, duality arguments for existence in L^2, approximation by mollifiers for regularity and uniqueness, and finally a transformation from L^2 to $\mathcal{H}^3$. The non-linear problem is solved by means of fixed-point arguments going back to Kato [**30**] at least, where the derivation of blow-up conditions in $W^{1,\infty}$ and the continuous dependence of data in $\mathcal{H}^3$ require significant additional efforts. Finally, for the isotropic Maxwell system, we show the preservation of energy and construct a blow-up example in $\mathcal{H}^1$.

The wellposedness results on $\mathbb{R}^3$ are due to Kato [**31**], but our proof differs from Kato's and instead uses (well known) energy methods from the theory of symmetric hyperbolic PDE, see [**5**], [**7**], [**11**], [**36**], for instance. The problem on domains is treated also via energy methods in Chapter 2, and so core features of these arguments can first be presented in a simpler situation on $\mathbb{R}^3$.

1.1. The Maxwell system

The Maxwell equations relate the *electric field* $E(t,x) \in \mathbb{R}^3$, the (electric) *displacement field* $D(t,x) \in \mathbb{R}^3$, the *magnetic field* $B(t,x) \in \mathbb{R}^3$ and the *magnetizing field* $H(t,x) \in \mathbb{R}^3$ via the Maxwell–Ampère and Maxwell–Faraday laws

$$\partial_t D = \operatorname{curl} H - J_e, \qquad \partial_t B = -\operatorname{curl} E, \qquad t \geq 0,\ x \in G, \tag{1.1}$$

where $G \subseteq \mathbb{R}^3$ is open and $J_e(t,x) \in \mathbb{R}^3$ is the *current density.* (See e.g. [**29**] for the background in physics.) If $G \neq \mathbb{R}^3$ we have to add boundary conditions to (1.1) as discussed in Chapter 2. We use the standard differential expressions

$$\operatorname{curl} u = \nabla \times u = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \operatorname{div} u = \nabla \cdot u = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3,$$

where the derivatives are interpreted in a weak sense if needed (see Section 1.2). Since $\operatorname{div}\operatorname{curl} = 0$, solutions to (1.1) fulfill Gauß' laws

$$\rho_e(t) := \operatorname{div} D(t) = \operatorname{div} D(0) - \int_0^t \operatorname{div} J_e(s)\,\mathrm{d}s, \qquad \operatorname{div} B(t) = \operatorname{div} B(0), \tag{1.2}$$

for $t \geq 0$. The *electric charge density* ρ_e is thus determined by the initial charge and the current density. As there are no magnetic charges in physics, one often requires $\operatorname{div} B(0) = 0$.

To complete the Maxwell system (1.1), we have to connect the fields via *material laws.* They involve the *polarization* $P = D - \varepsilon_0 E$ and the *magnetization* $M = B - \mu_0 H$ which describe the material response to the fields E and B. Below we set $\varepsilon_0 = \mu_0 = 1$ for simplicity (thus destroying physical units). In these notes we use instantaneous constitutive relations, namely

$$(D, B) = \theta(x, E, H) = \theta(x, u) \quad \text{for regular} \quad \theta : G \times \mathbb{R}^6 \to \mathbb{R}^6. \tag{1.3}$$

We choose $u = (E, H)$ as state because this fits best to energy estimates. Other choices are possible since transformations like $\theta(x, \cdot)$ are typically invertible. Our main hypothesis will be that $\partial_u \theta(x, u)$ is symmetric and $a_0 \geq \eta I$ for some number $\eta > 0$. Finally, the current is modelled as the sum

$$J_e = \sigma(x, E, H)E + J_0 \tag{1.4}$$

of a given external current density $J_0 : \mathbb{R}_{\geq 0} \times G \to \mathbb{R}^3$ and a current induced via Ohm's law for a (possibly state-depending) *conductivity* $\sigma : G \times \mathbb{R}^6 \to \mathbb{R}^{3\times 3}$.

Example 1.1. A basic example in nonlinear optics is the Kerr law

$$D = \chi_1(x)E + \chi_3(x)|E|^2 E, \qquad H = B,$$

for bounded functions $\chi_j : G \to \mathbb{R}$ with $\chi_1(x) \geq 2\eta > 0$ for all x, see [**2**], [**23**] and also Example 1.21. It is isotropic; i.e., $D(t, x)$ and $E(t, x)$ are parallel. The Kerr law satisfies our assumption $a_0 = a_0^\top \geq \eta I$ for small E (and for all E if $\chi_3 \geq 0$). The latter also holds for the more general laws $D = \chi_e(x)E + \beta_e(x, |E|^2)E$ and $H = \chi_m(x)B + \beta_m(x, |B|^2)B$ for 3×3 matrices $\chi_j = \chi_j^\top \geq 2\eta I$ and smooth scalar β_j with $\beta_j(0) = 0$. $\diamondsuit$

In physics material laws often also contain a time retardation, see [**2**], [**9**] or [**23**]. Here we stick to the instantaneous case which stays within the PDE framework. (But we expect that we can treat the Maxwell system with retardation by variants of our methods.)

It is often convenient to rewrite (1.1) with (1.3) and (1.4) as a quasilinear symmetric hyperbolic system. To this end, we first introduce the matrices

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{satisfying}$$

$$\operatorname{curl} = S_1\partial_1 + S_2\partial_2 + S_3\partial_3, \qquad a \times b = (a_1 S_1 + a_2 S_2 + a_3 S_3)b$$

for vectors $a, b \in \mathbb{R}^3$. We then define $\partial_0 = \partial_t$,

$$A_j^{\mathrm{co}} = \begin{pmatrix} 0 & -S_j \\ S_j & 0 \end{pmatrix}, \quad a_0(u) = \partial_u \theta(\cdot, u), \quad d = \begin{pmatrix} \sigma \\ 0 \end{pmatrix}, \quad f = \begin{pmatrix} -J_0 \\ 0 \end{pmatrix} \tag{1.5}$$

for $j \in \{1, 2, 3\}$. Note that the matrices A_j^{co} are symmetric.

Then the Maxwell system (1.1) with material laws (1.3) and (1.4) becomes

$$L(u)u := a_0(u)\partial_t u + \sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j u + d(u)u = f. \tag{1.6}$$

Our strategy to solve this problem goes (at least) back to Kato [30]. One freezes a function v from a suitable space $\mathcal{E}$ in the nonlinearities, setting $A_0 = a_0(v)$ and $D = d(v)$. One next solves the resulting non-autonomous linear problem $L(v)u = \sum_{j=0}^3 A_j\partial_j u + Du = f$ in the space $\mathcal{E}$. For small times $(0,T)$ one finds a fixed point of the map $v \mapsto u$ which then solves (1.6) and (1.1). The first linear step is more difficult; here it is crucial to control very well how the constants in the estimates depend on the coefficients. We carry out this program for $G = \mathbb{R}^3$ in the following sections.

1.2. The linear problem on $\mathbb{R}^3$ in L^2

Let $J = (0,T)$. We solve the linear problem in the space $C(\overline{J}, L^2(\mathbb{R}^3,\mathbb{R}^6)) = C(\overline{J}, L^2_x)$ for coefficients and data subject to the assumptions

$$\begin{aligned} &A_j = A_j^\top \in W^{1,\infty}_{t,x} = W^{1,\infty}(J\times\mathbb{R}^3,\mathbb{R}^{6\times 6}),\ j\in\{0,1,2,3\}, \quad A_0 = A_0^\top \ge \eta I > 0,\\ &D \in L^\infty_{t,x} = L^\infty(J\times\mathbb{R}^3,\mathbb{R}^{6\times 6}), \quad u_0 \in L^2_x, \quad f\in L^2_{t,x} = L^2(J\times\mathbb{R}^3,\mathbb{R}^6). \quad (1.7)\end{aligned}$$

We often omit range spaces as $\mathbb{R}^6$ in the notation. We use the subscript t to indicate a function space over $t\in J$ or other time intervals, and x for a space over $x\in\mathbb{R}^3$ (or over $x \in U \subseteq \mathbb{R}^m$). Compared to (1.6) we allow for D and f with non-zero 'magnetic' components, as needed in our analysis. We also deal with general symmetric (t,x)-depending coefficients A_1, A_2 and A_3, and thus with linear *symmetric hyperbolic systems*. Those occur in many applications, see [7], [30] or [36]; and our reasoning would not differ much if we restricted to $A_j = A_j^{\mathrm{co}}$. Moreover, when treating the Maxwell system on domains by localization arguments, one obtains x-depending coefficients. It is useful to see them first in an easier case.

Assuming (1.7), we look for a solution $u\in C(\overline{J}, L^2_x)$ of the system

$$Lu := \sum_{j=0}^{3} A_j\partial_j u + Du = f, \quad t\ge 0, \qquad u(0) = u_0, \qquad (1.8)$$

with $\partial_0 = \partial_t$. Here the derivatives are understood in a weak sense.

To explain this, we assume that the reader is familiar with Sobolev spaces $W^{k,p}(U) = W^{k,p}$ for an open subset U of $\mathbb{R}^m$, $k\in\mathbb{N}_0$, and $p\in[1,\infty]$. (See [1] or [8], for instance.) We mostly work with real scalars, endow $W^{k,p}$ with the (complete) norm $\|v\|_{k,p}^p = \sum_{0\le|\alpha|\le k}\|\partial^\alpha v\|_p^p$ (obvious modification for $p=\infty$), and write $\mathcal{H}^k := W^{k,2}$ (which is a Hilbert space), $L^p = W^{0,p}$ and $\|v\|_p := \|v\|_{0,p}$. By $W^{k,p}_0(U)$ we denote the closure of test functions $C_c^\infty(U)$ in $W^{k,p}(U)$. If ∂U is compact and C^k (or Lipschitz if $k=1$), say, then $W^{k,p}_0$ is the closed subspace in $W^{k,p}$ of functions whose (weak) derivatives of order up to $k-1$ have trace 0. One can check that $W^{k,p}_0(\mathbb{R}^m) = W^{k,p}(\mathbb{R}^m)$.

Let $\mathcal{H}^{-k}(U)$ be the dual space $\mathcal{H}^k_0(U)^*$, where we restrict ourselves to $p=2$ for simplicity. For $\varphi\in L^2(U)$, $j\in\{1,\dots,m\}$ and $v\in\mathcal{H}^1_0(U)$, we define the weak derivative $\partial_j\varphi\in\mathcal{H}^{-1}(U)$ by setting

$$(\partial_j\varphi)(v) = \langle v, \partial_j\varphi\rangle_{\mathcal{H}^1_0} := -\langle\partial_j v,\varphi\rangle_{L^2}.$$

(The brackets $\langle\cdot,\cdot\rangle_X$ designate the duality pairing between a Banach space X and its dual X^*.) Since $|\langle\partial_j v,\varphi\rangle| \le \|v\|_{1,2}\,\|\varphi\|_2$, the linear map $\partial_j : L^2(U) \to \mathcal{H}^{-1}(U)$ is bounded. Iteratively, one obtains bounded maps $\partial_j : \mathcal{H}^{-k}(U) \to \mathcal{H}^{-k-1}(U)$, and analogously $\partial^\alpha : \mathcal{H}^{-k}(U) \to \mathcal{H}^{-k-|\alpha|}(U)$ for multi-indices $\alpha \in \mathbb{N}_0^m$ and $k \in \mathbb{N}_0$. The definitions imply that these derivatives commute.

For $a \in W^{1,\infty}(U)$ and $\varphi \in \mathcal{H}^{-1}(U)$, we next define the map $a\varphi \in \mathcal{H}^{-1}(U)$ by

$$(a\varphi)(v) = \langle v, a\varphi\rangle_{\mathcal{H}_0^1} := \langle av,\varphi\rangle_{\mathcal{H}_0^1}, \qquad v \in \mathcal{H}_0^1(U).$$

Because of $\|av\|_{1,2} \lesssim \|a\|_{1,\infty}\,\|v\|_{1,2}$, we see as above that the multiplication operator $M_a : \varphi \mapsto a\varphi$ is bounded on $\mathcal{H}^{-1}(U)$. (Here and below $A \lesssim_\alpha B$ stands for $A \le cB$ for a generic constant $c = c(\alpha)$ which is non-decreasing in each component of $\alpha \in \mathbb{R}^n_{\ge 0}$.) These facts easily extend to $\mathbb{R}^l$–valued functions.

We infer that $Lu \in \mathcal{H}^{-1}_{t,x}$ if $u \in L^2_{t,x}$. If $Lu = f$ is contained in $L^2_{t,x}$, we obtain

$$\partial_t u = A_0^{-1} f - \sum_{j=1}^3 A_0^{-1} A_j \partial_j u - A_0^{-1} D u \;\in\; L^2_t \mathcal{H}^{-1}_x = L^2(J, \mathcal{H}^{-1}_x), \tag{1.9}$$

and so u belongs to $\mathcal{H}^1_t\mathcal{H}^{-1}_x \hookrightarrow C(\overline{J}, \mathcal{H}^{-1}_x)$. Accordingly, the initial condition in (1.8) is understood in $\mathcal{H}^{-1}_x$.

We will first show the basic *energy* (or *apriori*) *estimate*. Here we use the temporal weights $e_{-\gamma}(t) := \mathrm{e}^{-\gamma t}$ for $\gamma \ge 0$ and $t \in J$ (or $t \in \mathbb{R}$) and the weighted spaces $L^2_\gamma \mathcal{H}^k_x$ of functions with (finite) norm

$$\|v\|_{L^2_\gamma \mathcal{H}^k_x} := \|e_{-\gamma} v\|_{L^2_t \mathcal{H}^k_x}.$$

On J, we have the equivalence $\|v\|_{L^2_\gamma \mathcal{H}^k_x} \le \|v\|_{L^2 \mathcal{H}^k_x} \le \mathrm{e}^{\gamma T}\|v\|_{L^2_\gamma \mathcal{H}^k_x}$. Taking large γ in these norms, we can produce small constants in front of the contribution of f in the inequality below. This fact will be used to absorb error terms by the left-hand side, for instance. The estimate and the precise form of the constants is also crucial for the nonlinear problem. We write $\operatorname{div} A = \sum_{j=0}^3 \partial_j A_j$.

LEMMA 1.2. *Assume that* (1.7) *is true and* $u \in \mathcal{H}^1(J \times \mathbb{R}^3)$ *solves* (1.8). *Let* $C := \frac{1}{2}\operatorname{div} A - D$, $\gamma \ge \gamma_0'(L) := \max\{1, 4\eta^{-1}\|C\|_\infty\}$, *and* $t \in \overline{J}$. *We then obtain*

$$\tfrac{\gamma\eta}{4}\,\|u\|^2_{L^2_\gamma((0,t),L^2_x)} + \tfrac{\eta}{2}\mathrm{e}^{-2\gamma t}\,\|u(t)\|^2_{L^2_x} \le \tfrac{1}{2}\|A_0(0)\|_\infty\,\|u_0\|^2_{L^2_x} + \tfrac{1}{2\gamma\eta}\,\|f\|^2_{L^2_\gamma((0,t),L^2_x)}\,.$$

PROOF. Set $v = e_{-\gamma} u$ and $g = e_{-\gamma} f$. We have $\gamma A_0 v + Lv = g$. Using the symmetry of A_j, we derive

$$\begin{aligned}
\langle g, v\rangle &= \gamma\langle A_0 v, v\rangle + \sum_{j=0}^3 \langle A_j \partial_j v, v\rangle + \langle Dv, v\rangle \\
&= \gamma\langle A_0 v, v\rangle + \frac{1}{2}\sum_{j=0}^3 \Big(\int_0^t \int_{\mathbb{R}^3} \partial_j (A_j v\cdot v)\,\mathrm{d}x\,\mathrm{d}s - \langle \partial_j A_j v, v\rangle\Big) + \langle Dv, v\rangle,
\end{aligned}$$

where we drop the subscript $L^2((0,t), L^2_x)$ of the brackets and denote the scalar product in $\mathbb{R}^6$ by a dot. Integration yields

$$\gamma\langle A_0 v, v\rangle + \tfrac{1}{2}\langle A_0(t)v(t), v(t)\rangle_{L^2_x} = \tfrac{1}{2}\langle A_0(0)v(0), v(0)\rangle_{L^2_x} + \langle Cv, v\rangle + \langle g, v\rangle.$$

We now replace $v = e_{-\gamma}u$, $g = e_{-\gamma}f$ as well as $u(0) = u_0$, and use (1.7) and $\|C\|_\infty \le \gamma\eta/4$. It follows

$$\begin{aligned}\gamma\eta\|u\|_{L^2_\gamma L^2_x} + \tfrac{\eta}{2}\mathrm{e}^{-2\gamma t}\|u(t)\|^2_{L^2_x} \\ &\le \tfrac{1}{2}\|A_0(0)\|_\infty \|u_0\|^2_{L^2_x} + \|C\|_\infty \|u\|^2_{L^2_\gamma L^2_x} + \tfrac{\sqrt{\gamma\eta}}{\sqrt{\gamma\eta}}\|u\|_{L^2_\gamma L^2_x}\|f\|_{L^2_\gamma L^2_x} \\ &\le \tfrac{1}{2}\|A_0(0)\|_\infty \|u_0\|^2_{L^2_x} + \big(\tfrac{\gamma\eta}{4} + \tfrac{\gamma\eta}{2}\big)\|u\|^2_{L^2_\gamma L^2_x} + \tfrac{1}{2\gamma\eta}\|f\|^2_{L^2_\gamma L^2_x},\end{aligned}$$

which implies the assertion. □

Below we use the above estimate for

$$\gamma \ge \gamma_0(r,\eta) := \max\{1, 12r/\eta\} \ge \gamma_0'(L) \tag{1.10}$$

where $\|\partial_j A_j\|_\infty, \|D\|_\infty \le r$. For $\gamma = 0$ its proof yields the *energy equality*

$$\int_{\mathbb{R}^3} A_0(t)u(t)\cdot u(t)\,\mathrm{d}x = \int_{\mathbb{R}^3} A_0(0)u_0\cdot u_0\,\mathrm{d}x + 2\int_0^t\int_{\mathbb{R}^3}\big(C(s)u(s)+f(s)\big)\cdot u(s)\,\mathrm{d}x\,\mathrm{d}s. \tag{1.11}$$

In the term with $C = \frac{1}{2}\operatorname{div} A - D$ we have damping effects (if $D = D^\top \gneqq 0$) and extra errors terms coming from the t- or x-dependence of A_j.

Lemma 1.2 yields uniqueness of $\mathcal{H}^1$-solutions to (1.8). However, we need uniqueness (and the energy estimate) for solutions in $C(\overline{J}, L^2_x)$. This fundamental gap can be closed by a crucial regularization argument based on mollifiers. We recall the definition and basic properties of this core tools, see e.g. [8].

We set $g_\varepsilon(x) = \varepsilon^{-m}g(\varepsilon^{-1}x)$ for any function g on $\mathbb{R}^m$, $\varepsilon > 0$, and $x \in \mathbb{R}^m$. Take $0 \le \rho \in C_c^\infty(\mathbb{R}^m)$ with $\int \rho\,\mathrm{d}x = 1$, support $\operatorname{supp}\rho$ in the closed unit ball $\overline{B}(0,1)$, and $\rho(x) = \rho(-x)$ for $x \in \mathbb{R}^m$. Note that $\|\rho_\varepsilon\|_1 = 1$. For $\varepsilon > 0$ and $v \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$, we define the *mollifiers* R_ε by

$$R_\varepsilon v(x) = \rho_\varepsilon * v(x) = \int_{\mathbb{R}^m} \rho_\varepsilon(x-y)v(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^m.$$

One can check that $R_\varepsilon v \in C^\infty(\mathbb{R}^m)$, $\operatorname{supp} R_\varepsilon v \subseteq \operatorname{supp} v + \overline{B}(0,\varepsilon)$, and $\partial^\alpha R_\varepsilon v = R_\varepsilon \partial^\alpha v$ for $v \in W^{|\alpha|,p}(\mathbb{R}^m)$. Young's inequality for convolutions yields $\|R_\varepsilon v\|_{k,p} \le \|v\|_{k,p}$ for $p \in [1,\infty]$ and $k \in \mathbb{N}_0$. Using this estimate, one derives that $R_\varepsilon v \to v$ in $W^{k,p}(\mathbb{R}^m)$ for $v \in W^{k,p}(\mathbb{R}^m)$ as $\varepsilon \to 0$ if $p < \infty$, since this limit is true for test functions v. Differentiating $\rho_\varepsilon(x-y)$ in x, one also obtains the smoothing estimate $\|R_\varepsilon v\|_{k,p} \lesssim_{\varepsilon,k} \|v\|_p$.

Finally, for $\varphi \in \mathcal{H}^{-k}(\mathbb{R}^m)$, $v \in \mathcal{H}^k(\mathbb{R}^m)$ and $k \in \mathbb{N}$, we set

$$(R_\varepsilon\varphi)(v) = \langle v, R_\varepsilon\varphi\rangle_{\mathcal{H}^k} := \langle R_\varepsilon v, \varphi\rangle_{\mathcal{H}^k}.$$

This definition is consistent with the symmetry $R_\varepsilon^* = R_\varepsilon$ on $L^2(\mathbb{R}^m)$ which follows from the symmetry of ρ and Fubini's theorem. By means of its properties in $\mathcal{H}^k(\mathbb{R}^m)$, one can show that R_ε is contractive on $\mathcal{H}^{-l}(\mathbb{R}^m)$ and that it maps this space into $\mathcal{H}^k(\mathbb{R}^m)$ for all $l \in \mathbb{N}$. Moreover, it commutes with ∂^α.

Hence, the commutator $[R_\varepsilon, M_a] := R_\varepsilon M_a - M_a R_\varepsilon$ tends to 0 strongly in L^2_x if $a \in L^\infty_x$. It even gains a derivative if $a \in W^{1,\infty}_x$, which is crucial for our analysis.

Proposition 1.3. *Let* $a\in W^{1,\infty}(\mathbb{R}^m)$*,* $u\in L^2(\mathbb{R}^m)$*,* $j\in\{1,\dots,m\}$*, and* $\varepsilon>0$*. Set* $C_\varepsilon u := R_\varepsilon(a\partial_j u) - a\partial_j(R_\varepsilon u)$*. Then there is a constant* $c=c(\rho)$ *such that*

$$\|C_\varepsilon u\|_2 \le c\|a\|_{1,\infty}\,\|u\|_2 \qquad \text{and} \qquad C_\varepsilon u\to 0 \quad \text{in } L^2_x \quad \text{as } \ \varepsilon\to 0.$$

Proof. Let $v\in\mathcal{H}^1(\mathbb{R}^m)$. Using the above indicated facts, we compute

$$\langle v, C_\varepsilon u\rangle_{\mathcal{H}^1} = \langle aR_\varepsilon v, \partial_j u\rangle_{\mathcal{H}^1} - \langle av, R_\varepsilon\partial_j u\rangle_{\mathcal{H}^1} = \langle \partial_j(R_\varepsilon(av) - aR_\varepsilon v), u\rangle_{L^2}.$$

We set $C'_\varepsilon v = \partial_j(R_\varepsilon(av) - aR_\varepsilon v)$ and R^j_ε for the convolution with $(|\partial_j\rho|)_\varepsilon$. For a.e. $x\in\mathbb{R}^m$, differentiation and $|x-y|\le\varepsilon$ yield

$$C'_\varepsilon v(x) = \int_{B(x,\varepsilon)} \varepsilon^{-m}(\partial_j\rho)(\varepsilon^{-1}(x-y))\,\varepsilon^{-1}(a(y)-a(x))v(y)\,\mathrm{d}y - \partial_j a(x)R_\varepsilon v(x),$$

$$|C'_\varepsilon v(x)| \le \|a\|_{1,\infty}\,(|R^j_\varepsilon v(x)| + |R_\varepsilon v(x)|).$$

(Recall that $W^{1,\infty}(\mathbb{R}^m)$ is isomorphic to the space of bounded Lipschitz functions [8].) Young's inequality now implies the first assertion. The second one is true for u in the dense subspace $\mathcal{H}^1(\mathbb{R}^m)$ and thus on $L^2(\mathbb{R}^m)$ by the uniform estimate. □

With this tool we can extend Lemma 1.2 to all solutions of (1.8) in $C(\overline{J}, L^2_x)$.

Proposition 1.4. *Let* (1.7) *hold and* $u\in C(\overline{J}, L^2_x)$ *solve* (1.8)*. Then the statement of Lemma 1.2 and* (1.11) *are also valid for* u*. Hence,* (1.8) *has at most one solution in* $C(\overline{J}, L^2_x)$*.*

Proof. We note that $R_\varepsilon u$ belongs to $C(\overline{J}, \mathcal{H}^k_x)$ for all $\varepsilon>0$ and $k\in\mathbb{N}$. Moreover, $R_\varepsilon u$ tends to u in $C(\overline{J}, L^2_x)$ as $\varepsilon\to 0$ since $u(\overline{J})$ is compact and $R_\varepsilon\to I$ strongly in L^2_x. As $\|R_\varepsilon f(t)\|_2\le\|f(t)\|_2$, dominated convergence also yields $R_\varepsilon f\to f$ in $L^2_{t,x}$. Using $Lu=f$ and (1.9), we compute

$$\begin{aligned} LR_\varepsilon u &= R_\varepsilon f + [D,R_\varepsilon]u + \sum_{j=1}^{3}[A_j,R_\varepsilon]\partial_j u + [A_0,R_\varepsilon]\partial_t u \qquad (1.12)\\ &= R_\varepsilon f + [D,R_\varepsilon]u + [A_0,R_\varepsilon]A_0^{-1}(f-Du) + \sum_{j=1}^{3}\big([A_j,R_\varepsilon] - [A_0,R_\varepsilon]A_0^{-1}A_j\big)\partial_j u. \end{aligned}$$

Proposition 1.3 shows that the right-hand side belongs to $L^2_{t,x}$ with uniform bounds. Hence, $R_\varepsilon u$ is also contained $\mathcal{H}^1_t L^2_x$ by (1.9). Arguing as above, we further see that the commutator terms tend to 0 in $L^2_{t,x}$ and thus in $L^2_\gamma L^2_x$. Lemma 1.2 and (1.11) for $R_\varepsilon u$ now lead to the first assertion letting $\varepsilon\to 0$. The second one follows from linearity. □

Combining the energy estimate with a clever duality argument, one can also deduce the existence of a solution.

Theorem 1.5. *Let* (1.7) *be true. Then there is a unique map* u *in* $C(\overline{J}, L^2_x)$ *solving* (1.8)*. It satisfies the estimate in Lemma 1.2 and* (1.11)*.*

Proof. 1) We need the (formal) adjoint $L^\circ = -\sum_{j=0}^3 A_j\partial_j + D^\circ$ of L with $D^\circ = D^\top - \operatorname{div} A$. Let $V = \big\{v\in\mathcal{H}^1(J\times\mathbb{R}^3,\mathbb{R}^6)\,\big|\, v(T)=0\big\}$, $v\in V$, and

$L^\circ v = h$. We introduce $\tilde v(t) = v(T-t)$ and $f(t) = h(T-t)$ for $t \in \overline{J}$ and the operator $\tilde L$ with coefficients $\tilde A_0(t) = A_0(T-t)$, $\tilde A_j(t) = -A_j(T-t)$ for $j \in \{1,2,3\}$ and $\tilde D(t) = D^\circ(T-t)$. Note that $\tilde L \tilde v = f$ and $\tilde v(0) = 0$. Applied at time $T-t$ to $\tilde L$, $\tilde v$ and $\gamma = \gamma_0(r,\eta)$ from (1.10), Lemma 1.2 yields the estimate

$$\begin{aligned}
\|v(t)\|_2^2 = \|\tilde v(T-t)\|_2^2 &\le \frac{2\mathrm{e}^{2\gamma(T-t)}}{\eta\cdot 2\eta\gamma}\int_0^{T-t} \mathrm{e}^{-2\gamma\tau}\|h(T-\tau)\|_2^2\,\mathrm{d}\tau\\
&\le \frac{\mathrm{e}^{2\gamma T}}{\gamma\eta^2}\int_t^T \|h(s)\|_2^2\,\mathrm{d}s,\\
\|v\|_{L^2_{t,x}} \le \kappa\sqrt{T}\,\|L^\circ v\|_{L^2_{t,x}}, &\qquad \kappa := \tfrac{1}{\eta\sqrt{\gamma}}\mathrm{e}^{\gamma T}.
\end{aligned} \tag{1.13}$$

In particular, $L^\circ : V \to L^2(J\times\mathbb{R}^3)^6$ is injective. We can thus define the functional

$$\ell_0 : L^\circ V \to \mathbb{R}; \quad \ell_0(L^\circ v) = \langle v, f\rangle_{L^2_{t,x}} + \langle v(0), A_0(0)u_0\rangle_{L^2_x}.$$

The Cauchy–Schwarz inequality and estimate (1.13) imply

$$|\ell_0(L^\circ v)| \le \big(\|f\|_{L^2_{t,x}} + \|A_0(0)u_0\|_{L^2_x}\big)\,\kappa\big(\sqrt{T}+1\big)\|L^\circ v\|_{L^2_{t,x}}.$$

By the Hahn–Banach theorem, ℓ_0 has an extension ℓ in $(L^2_{t,x})^*$ which can be represented by a function $u \in L^2(J, L^2_x)$ via

$$\begin{aligned}
\langle v, f\rangle_{L^2_{t,x}} + \langle v(0), A_0(0)u_0\rangle_{L^2_x} &= \ell(L^\circ v) = \langle L^\circ v, u\rangle_{L^2_{t,x}}\\
&= \langle v, Du\rangle - \sum_{j=0}^{3}\int_0^T\!\!\int_{\mathbb{R}^3} \partial_j(A_j v)\cdot u\,\mathrm{d}x\,\mathrm{d}t \quad (\forall\, v\in V).
\end{aligned} \tag{1.14}$$

2) To evaluate (1.14), we first take $v \in \mathcal{H}^1_0(J\times\mathbb{R}^3)$. The definition of weak derivatives then leads to $\langle v, f\rangle_{L^2_{t,x}} = \langle v, Lu\rangle_{\mathcal{H}^1_0}$; i.e., $Lu = f$ in $\mathcal{H}^{-1}_{t,x}$. Hence, u belongs to $\mathcal{H}^1_t\mathcal{H}^{-1}_x$ because of (1.9) and $f \in L^2_{t,x}$. For $v \in V$, we can now integrate by parts the summand in (1.14) with $j = 0$ in $\mathcal{H}^{-1}_x$; the others are treated as before. As $v(T) = 0$, it follows

$$\langle v, f\rangle_{L^2_{t,x}} + \langle v(0), A_0(0)u_0\rangle_{L^2_x} = \langle v, Lu\rangle_{\mathcal{H}^1_0} + \langle A_0(0)v(0), u(0)\rangle_{L^2_x}.$$

Since $A_0(0)$ is symmetric and $Lu = f$, we have also shown that $u(0) = u_0$.

3) We next use (1.12) for $w_{n,m} = R_{1/n}u - R_{1/m}u$. As in the proof of Proposition 1.4, Proposition 1.3 implies that $w_{n,m}$ is contained in $\mathcal{H}^1_{t,x}$ and satisfes $Lw_{n,m} \to 0$ in $L^2_{t,x}$ and $w_{n,m}(0) \to 0$ in L^2_x as $n, m \to \infty$. So $(R_{1/n}u)$ is a Cauchy sequence in $C(\overline{J}, L^2_x)$ by Lemma 1.2, and it converges to u in $L^2_{t,x}$. Thus, u belongs to $C(\overline{J}, L^2_x)$. The other assertions were proven in Proposition 1.4. □

In the time-independent Maxwell case ($A_0 = A_0(x)$ and $A_j = A_j^{\mathrm{co}}$) one can show a similar result if A_0 is only bounded and positive definite (even with boundary conditions), see e.g. Theorem 5.2.5 in [4] or §7.8 in [**23**]. In the non-autonomous case there are blow-up solutions even for the wave equation on $G = \mathbb{R}$ with Hölder continuous and x-independent coefficients, as shown in [**12**].

As indicated in Section 1.1 and described in the next example, the above result can easily be applied to the linear Maxwell system

$$\partial_t(\varepsilon E) = \operatorname{curl} H - \sigma E - J_0, \qquad \partial_t(\mu H) = -\operatorname{curl} E, \qquad t \geq 0,\ x \in \mathbb{R}^3, \quad (1.15)$$

which is (1.1) on $G = \mathbb{R}^3$ with the material laws $D = \varepsilon(t,x)E$ and $B = \mu(t,x)H$. We write $\mathbb{R}^{n\times n}_\eta$ for the space of real $n \times n$ matrices $M = M^\top \geq \eta I$.

Example 1.6. Let $\varepsilon, \mu \in W^{1,\infty}(J \times \mathbb{R}^3, \mathbb{R}^{3\times 3}_\eta)$ for some $\eta > 0$, $\sigma \in L^\infty(J \times \mathbb{R}^3, \mathbb{R}^{3\times 3})$, $E_0, H_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ and $J_0 \in L^2(J \times \mathbb{R}^3, \mathbb{R}^3)$. As in (1.5), we set $A_0 = \operatorname{diag}(\varepsilon, \mu)$, $A_j = A_j^{\mathrm{co}}$ for $j = \{1,2,3\}$, $D = \operatorname{diag}(\sigma + \partial_t\varepsilon, \partial_t\mu)$, $f = (-J_0, 0)$, and $u_0 = (E_0, H_0)$. Theorem 1.5 then yields a unique solution $(E,H) \in C(\overline{J}, L^2_x)$ of (1.15) with $E(0) = E_0$ and $H(0) = H_0$. It satisfies the energy equality

$$\begin{aligned} \|\varepsilon(t)^{\frac12} E(t)\|_2^2 + \|\mu(t)^{\frac12} H(t)\|_2^2 &= \|\varepsilon(0)^{\frac12} E_0\|_2^2 + \|\mu(0)^{\frac12} H_0\|_2^2 \\ &\quad - \int_0^t \int_{\mathbb{R}^3} \big((2\sigma + \partial_t\varepsilon\, E + 2J_0)\cdot E + \partial_t\mu H \cdot H\big) \mathrm{d}x\, \mathrm{d}s. \quad \Diamond \end{aligned}$$

One of the key features of hyperbolic systems is the finite propagation speed of their solutions. To see a simple example first, we look at the standard wave equation $\partial_t^2 u = c^2 \partial_{xx} u$ on $\mathbb{R}$ for the wave speed $c > 0$ equipped with the initial conditions $u(0) = u_0$ and $\partial_t u(0) = v_0$. (One can put this second-order equation in the above first-order framework for the new state $(\partial_t u, \partial_x u)$.) The solution of this wave problem is given by d'Alembert's formula

$$u(t,x) = \tfrac12 (u_0(x+ct) + u_0(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(s)\, \mathrm{d}s, \qquad t \geq 0,\ x \in \mathbb{R}.$$

Hence, the solution at (x,t) only depends on the initial data on $[x-ct, x+ct]$; for instance, $u(t,x) = 0$ if u_0 and v_0 vanish on $[x-ct, x+ct]$. Conversely, the value of u_0 and v_0 at y influences u at most for (t,x) with $|x-y| \leq ct$; i.e., on a triangle with vertex $(y,0)$ and lateral sides of slope $\pm c$. In this sense, c is the speed of propagation.

We extend these observations to the system (1.8), assuming (1.7). In the statement we use the backward 'light' cone

$$\Gamma(x_0, R, K) = \big\{(t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \,\big|\, |x - x_0| < R - Kt\big\}.$$

It has the base $B(x_0, R)$ at $t = 0$ and the apex $(\frac{R}{K}, x_0)$. Set

$$k_0^2 = \|A_1\|_\infty^2 + \|A_2\|_\infty^2 + \|A_3\|_\infty^2$$

with the operator norm for $|\cdot|_2$ on $\mathbb{R}^{6\times 6}$. Note that $k_0 = \sqrt{3}$ in the Maxwell example.

Below we see (for $f = 0$) that u vanishes on $\Gamma(x_0, R, k_0/\eta)$ if $u_0 = 0$ on $B(x_0, R)$. Hence, if two initial functions u_0 and $\tilde u_0$ coincide on $B(x_0, R)$ then the corresponding solutions u and $\tilde u$ are equal on $\Gamma(x_0, R, k_0/\eta)$. In other words, the values of u_0 outside $B(x_0, R)$ influence $u(t)$ only off $\Gamma(x_0, R, k_0/\eta)$, that is, with maximal speed k_0/η. Our proof is based on energy estimates with an exponential weight, and the arguments are taken from §4.2.2 of [5].

Theorem 1.7. *Let (1.7) be true. Assume that $u_0 = 0$ on $B(x_0, R)$ and $f = 0$ on $\Gamma(x_0, R, k_0/\eta)$ for some $R > 0$ and $x_0 \in \mathbb{R}^3$. Then the solution $u \in C(\overline{J}, L^2_x)$ of (1.8) also vanishes on $\Gamma(x_0, R, k_0/\eta)$.*

Proof. 1) Let $\delta, R > 0$ and $x_0 \in \mathbb{R}^3$ be given. There is a function $\psi \in C^\infty(\mathbb{R}^3)$ with $|\nabla\psi| \le \eta/k_0$ (for the euclidean norm) and

$$-2\delta + \eta k_0^{-1}(R - |x - x_0|) \le \psi(x) \le -\delta + \eta k_0^{-1}(R - |x - x_0|), \qquad x \in \mathbb{R}^3. \quad (1.16)$$

We construct ψ as in Theorem 6.1 of [**51**]. Take $\chi(s) = -\frac{3}{2}\delta + \eta k_0^{-1}(R - |s|)$ for $s \in \mathbb{R}$. This function is Lipschitz with constant η/k_0. The same is true for the mollified map $\chi_\varepsilon = R_\varepsilon \chi$ as $\nabla\chi_\varepsilon = R_\varepsilon \nabla\chi$. Also, χ_ε tends uniformly to χ as $\varepsilon \to 0$ since

$$|\chi_\varepsilon(s) - \chi(s)| \le \int_{\mathbb{R}} \varepsilon^{-1}\rho(\varepsilon^{-1}\tau)\,|\chi(s-\tau) - \chi(s)|\,\mathrm{d}\tau \le \eta k_0^{-1}\varepsilon \int_{\mathbb{R}} \rho(\sigma)\,|\sigma|\,\mathrm{d}\sigma.$$

We fix a small $\varepsilon > 0$ such that χ_ε satisfies (1.16) with s instead of $|x - x_0|$ and $5/3$ instead of 2. Then $\psi(x) = \chi_\varepsilon((\delta_0^2 + |x - x_0|^2)^{1/2})$ does the job, where $\delta_0 = k_0\delta(3\eta)^{-1}$.

Set $\phi(t, x) = \psi(x) - t$ and $u_\tau = \mathrm{e}^{\tau\phi}u$ for $\tau > 0$. Inequality (1.16) yields $\psi(x) \le -\delta + t$ if $|x - x_0| \ge R - k_0 t/\eta$ (i.e., $(t, x) \notin \Gamma(x_0, R, k_0/\eta)$), so that $\mathrm{e}^{\tau\phi} \le \mathrm{e}^{-\tau\delta} \le 1$ off $\Gamma(x_0, R, k_0/\eta)$ and $\mathrm{e}^{\tau\phi}$ is bounded on $J \times \mathbb{R}^3$. We further have $\nabla \mathrm{e}^{\tau\phi} = \tau\nabla\psi \mathrm{e}^{\tau\phi}$ and $\partial_t \mathrm{e}^{\tau\phi} = -\tau \mathrm{e}^{\tau\phi}$. As a result, u_τ is an element of $C(\overline{J}, L^2_x)$ and the right-hand side of

$$Lu_\tau = \mathrm{e}^{\tau\phi} f - \tau\Big(A_0 - \sum_{j=1}^{3} A_j \partial_j \psi\Big)u_\tau$$

belongs to $L^2_{t,x}$. The matrix in parentheses is denoted by M.

2) For $\xi \in \mathbb{R}^6$ we have $M\xi \cdot \xi \ge (\eta - k_0|\nabla\psi|)|\xi|^2 \ge 0$. Set $C = \frac{1}{2}\operatorname{div} A - D$ and $\kappa = \|C\|_\infty$. By Theorem 1.5, the function u_τ satisfies the energy equality

$$\|A_0(t)^{\frac{1}{2}} u_\tau(t)\|^2_{L^2_x} = \|A_0(0)^{\frac{1}{2}} u_\tau(0)\|^2_{L^2_x} + 2\langle (C - \tau M)u_\tau + \mathrm{e}^{\tau\phi} f, u_\tau\rangle_{L^2_{t,x}}.$$

Using Cauchy–Schwarz, the above inequalities and Gronwall, we estimate

$$\eta\,\|u_\tau(t)\|^2_{L^2_x} \le \|A_0(0)\|_\infty\,\|\mathrm{e}^{\tau\phi}u_0\|^2_{L^2_x} + \|\mathrm{e}^{\tau\phi} f\|^2_{L^2_{t,x}} + (2\kappa + 1)\int_0^t \|u_\tau(s)\|^2_{L^2_x}\,\mathrm{d}s,$$
$$\|\mathrm{e}^{\tau\phi}u(t)\|^2_{L^2_x} \lesssim_T \|\mathrm{e}^{\tau\phi}u_0\|^2_{L^2_x} + \|\mathrm{e}^{\tau\phi} f\|^2_{L^2_{t,x}}.$$

The right-hand side tends to 0 as $\tau \to \infty$ since u_0 and f vanish on $\Gamma(x_0, R, k_0/\eta)$ and $\mathrm{e}^{\tau\phi} \to 0$ uniformly off $\Gamma(x_0, R, k_0/\eta)$. Hence, $u(t)$ has to be 0 on $\{\phi > \delta\} = \{\psi > t + \delta\}$. By (1.16), this set includes points (t, x) with $|x - x_0| < R - k_0\eta^{-1}(t + 3\delta)$. Since $\delta > 0$ is arbitrary here, u equals 0 on $\Gamma(x_0, R, k_0/\eta)$. □

1.3. The linear problem on $\mathbb{R}^3$ in $\mathcal{H}^3$

As noted in Section 1.1, to solve the nonlinear problem (1.6) we will set $A_0 = a_0(v)$ for functions v having the same regularity as the desired solution u. Since A_0 has to be Lipschitz in Theorem 1.5, the same must be true for v. Working in $\mathcal{H}^k_x$ spaces, we thus need solutions in $L^\infty_t \mathcal{H}^3_x \cap W^{1,\infty}_t \mathcal{H}^2_x$ at least. We want to reduce the problem in $\mathcal{H}^3_x$ to that in L^2_x by means of a transformation. (One could also perform the proof of Theorem 1.5 in $\mathcal{H}^3_x$ instead of L^2_x, see e.g. [**7**] or [**11**], which would require more work in our context.)

To this end, we define the square root $\Lambda = (I-\Delta)^{1/2} = \mathcal{F}^{-1}(1+|\xi|^2)^{1/2}\mathcal{F}$ of the shifted Laplacian on $L^2(\mathbb{R}^3)$, where $\mathcal{F}$ is the Fourier transform. Using standard properties of $\mathcal{F}$, one can check that Λ commutes with derivatives and that it can be extended, respectively restricted, to isomorphisms $\mathcal{H}^k_x \to \mathcal{H}^{k-1}_x$ for $k \in \mathbb{Z}$ with inverse given by $\Lambda^{-1} = (I-\Delta)^{-1/2} = \mathcal{F}^{-1}(1+|\xi|^2)^{-1/2}\mathcal{F}$. Observe that $\Lambda = (I-\Delta)\Lambda^{-1}$ and that Λ^{-1} is a convolution operator with positive kernel, see Proposition 6.1.5 in [**25**]. Hence, Λ leaves invariant real-valued functions.

Our analysis relies on a commutator estimate for Λ^3 and $M_a : \varphi \mapsto a\varphi$ which gains a derivative. In Lemma A2 in [**30**] it is shown that

$$\|[\Lambda^3, M_a]\|_{\mathcal{B}(\mathcal{H}^2(\mathbb{R}^3),L^2(\mathbb{R}^3))} \lesssim \|\nabla a\|_{\mathcal{H}^2(\mathbb{R}^3)}. \tag{1.17}$$

Here the space dimension 3 is crucial; on $\mathbb{R}^m$ one obtains e.g. an analogous bound for $[\Lambda^k, M_a] : \mathcal{H}^{k-1}_x \to L^2_x$ with $k > \frac{m}{2}+1$. (Noninteger k are also allowed here.)

Guided by (1.17) and (1.7), we introduce the space

$$\tilde{\mathcal{F}}^k(J) = \tilde{\mathcal{F}}^k(T) = \big\{A \in W^{1,\infty}(J\times\mathbb{R}^3, \mathbb{R}^{6\times 6}) \,\big|\, \nabla_{t,x}A \in L^\infty_t\mathcal{H}^{k-1}_x\big\}, \qquad k \in \mathbb{N},$$

for the coefficients, endowed with its natural norm. We will usually take $k=3$. We use the same notation for vector- or scalar-valued functions of the same regularity. The subscript sym will refer to symmetric matrices and η to those with $A = A^\top \geq \eta I$ with $\eta > 0$. We state the hypotheses of the present section:

$$\begin{aligned} &A_0 \in \tilde{\mathcal{F}}^3_\eta(J), \quad A_1, A_2, A_3 \in \tilde{\mathcal{F}}^3_{\mathrm{sym}}(J), \quad D \in \tilde{\mathcal{F}}^3(J), \\ &u_0 \in \mathcal{H}^3_x = \mathcal{H}^3(\mathbb{R}^3,\mathbb{R}^6), \quad f \in \mathcal{Z}^3(J) = \mathcal{Z}^3(T) := L^2(J,\mathcal{H}^3_x)\cap\mathcal{H}^1(J,\mathcal{H}^2_x). \end{aligned} \tag{1.18}$$

Set $\|f\|^2_{\mathcal{Z}^3_\gamma(J)} = \|e_{-\gamma}f\|^2_{L^2_t\mathcal{H}^3_x} + \|e_{-\gamma}\partial_t f\|^2_{L^2_t\mathcal{H}^2_x}$ for $\gamma \geq 0$. We also use the spaces

$$\hat{\mathcal{H}}^k_x = \big\{v \in L^\infty(\mathbb{R}^3) \,\big|\, \nabla_x v \in \mathcal{H}^{k-1}_x\big\}, \quad \tilde{\mathcal{G}}^k(J) = \tilde{\mathcal{G}}^k(T) = C(\overline{J},\mathcal{H}^k_x)\cap C^1(\overline{J},\mathcal{H}^{k-1}_x)$$

with their natural norms. (Such spaces will also be considered on other time intervals.) We state product and inversion rules which will be used throughout, cf. [**53**]. Here one can replace $\mathbb{R}^3$ by all Lipschitz domains. In the proof and also later on, we employ Sobolev embeddings such as $\mathcal{H}^2 \hookrightarrow L^p$ for $p \in [2,\infty]$ and $\mathcal{H}^1 \hookrightarrow L^q$ for $q \in [2,6]$ on (Lipschitz domains in) $\mathbb{R}^3$.

LEMMA 1.8. *Let $k \geq \max\{j,2\}$.*

a) For $v \in \mathcal{H}^k_x$ and $w \in \mathcal{H}^j_x$ we have the estimate

$$\|vw\|_{\mathcal{H}^j} \lesssim \|v\|_{\mathcal{H}^k}\|w\|_{\mathcal{H}^j}.$$

Here one can replace $\mathcal{H}^k_x$ by $\hat{\mathcal{H}}^k_x$, as well as $\mathcal{H}^j_x$ and $\mathcal{H}^k_x$ by $\tilde{\mathcal{G}}^j(J)$ and $\tilde{\mathcal{G}}^j(J)$ (or $\tilde{\mathcal{F}}^j(J)$), or by $\tilde{\mathcal{F}}^j(J)$ and $\tilde{\mathcal{F}}^k(J)$.

b) Also, if $A \in \hat{\mathcal{H}}^k_\eta$ for $k \in \mathbb{N}$, then A^{-1} belongs to $\hat{\mathcal{H}}^k_\eta$ with norm bounded by $c(\eta,k)(1+\|A\|_{\hat{\mathcal{H}}^k_x})^{k-1}\|A\|_{\hat{\mathcal{H}}^k_x}$.

PROOF. a) For the first claim, by the product rule (and interpolative inequalities) we have to control $\partial_x^\beta v \partial_x^{\alpha-\beta} w$ for multi-indices $0 \leq \beta \leq \alpha$ with

$|\alpha| = j$. Observe that $\partial_x^\beta v \in \mathcal{H}_x^{k-|\beta|}$ and $\partial_x^{\alpha-\beta} w \in \mathcal{H}_x^{|\beta|}$. This product can be estimated in L_x^2 as needed if $k-|\beta| \ge 2$ or $|\beta| \ge 2$ since then v and w are bounded, respectively. As $k \ge 2$, only the case $|\beta| = 1$ remains. Here $\partial_x^\beta v$ and $\partial_x^{\alpha-\beta} w$ belong to $\mathcal{H}_x^1 \hookrightarrow L_x^4$ and thus the product to L_x^2. The other variants are proved similarly.

b) Observe that $\nabla_x^3 A^{-1}$ is a linear conbination of terms like

$$A^{-1}\nabla_x^3 AA^{-1}, \quad A^{-1}\nabla_x^2 AA^{-1}\nabla_x AA^{-1}, \quad A^{-1}\nabla_x AA^{-1}\nabla_x AA^{-1}\nabla_x AA^{-1}.$$

These terms clearly satisfy the asserted estimate, and the lower-order ones are treated similarly. □

We look for a solution $u \in \tilde{\mathcal{G}}^3(J)$ of (1.8) assuming (1.18). The basic idea is to solve a modified problem for $w = \Lambda^3 u$ in $C(\overline{J}, L_x^2)$. Since the inequality (1.17) only improves space regularity, we first replace the equation $Lu = f$ by $\hat{L}u = \hat{f} := A_0^{-1} f$ where $\hat{L}$ has the coefficients $\hat{A}_0 = I$, $\hat{A}_j = A_0^{-1} A_j$ and $\hat{D} = A_0^{-1} D$. We then obtain

$$\hat{L}w = \Lambda^3 \hat{f} + \sum_{j=1}^3 [\hat{A}_j, \Lambda^3]\partial_j u + [\hat{D}, \Lambda^3]u,$$

$$Lw = A_0\Lambda^3 \hat{f} + \sum_{j=1}^3 A_0[\hat{A}_j, \Lambda^3]\partial_j u + A_0[\hat{D}, \Lambda^3]u =: g(f,u). \tag{1.19}$$

We now replace in g the unknown u by a given function $v \in C(\overline{J}, \mathcal{H}_x^3)$. Theorem 1.5 will give a solution $w \in C(\overline{J}, L_x^2)$ of $Lw = g(f,v)$ with $w(0) = \Lambda^3 u_0$. The energy estimate from Lemma 1.2 (with a large γ) then implies that $\Phi : v \mapsto \Lambda^{-3} w$ is a strict contraction on $L_\gamma^\infty \mathcal{H}_x^3$. This fact will lead to the desired regularity result. Let λ be the maximum of $\|\Lambda^k\|_{\mathcal{B}(\mathcal{H}^k, L^2)}$ and $\|\Lambda^{-k}\|_{\mathcal{B}(L^2,\mathcal{H}^k)}$ for $k \in \{2,3\}$. It will be important in the fixed-point argument for the nonlinear problem that the contant c_0 in (1.20) only depends on r_0 (and η), but not on r.

Theorem 1.9. *Let (1.18) be true. Then there is a unique u in $C(\overline{J}, \mathcal{H}_x^3) \cap C^1(\overline{J}, \mathcal{H}_x^2)$ solving (1.8). For $t \in \overline{J}$ and $\gamma \ge \gamma_1(r,\eta) := \max\{\gamma_0(r,\eta), \sqrt{c_1}\}$, see (1.10), we have*

$$\begin{aligned}\gamma\|u\|^2_{\mathcal{Z}^3_\gamma(0,t)} &+ \mathrm{e}^{-2\gamma t}(\|u(t)\|^2_{\mathcal{H}^3_x} + \|\partial_t u(t)\|^2_{\mathcal{H}^2_x}) \\ &\le c_0(\|u_0\|^2_{\mathcal{H}^3_x} + \|f(0)\|^2_{\mathcal{H}^2_x}) + \tfrac{c_1}{\gamma}\,\|f\|^2_{\mathcal{Z}^3_\gamma(0,t)}\,,\end{aligned} \tag{1.20}$$

where $\|A_j(0)\|_{\hat{\mathcal{H}}^2_x}, \|D(0)\|_{\hat{\mathcal{H}}^2_x} \le r_0$, $\|A_j\|_{\tilde{\mathcal{F}}^3(J)}, \|D\|_{\tilde{\mathcal{F}}^3(J)} \le r$ for $j \in \{0,1,2,3\}$, and $c_0 = c_0(r_0,\eta)$ and $c_1 = c_1(r,\eta)$ are constants described in the proof.

Proof. 1) Take $v \in C(\overline{J}, \mathcal{H}_x^3)$ and $\gamma \ge \gamma_0(r,\eta)$ from (1.10). Using Lemma 1.8 and (1.17), we see that the square of the norm in $L^2_\gamma L^2_x$ of $g(f,v)$ from (1.19) is bounded by $c_1'(\|f\|^2_{L^2_\gamma \mathcal{H}^3_x} + \|v\|^2_{L^2_\gamma \mathcal{H}^3_x})$ for a constant $c_1' = c_1'(r,\eta)$. Theorem 1.5 yields a solution $w \in C(\overline{J}, L_x^2)$ of $Lw = g(f,v)$ and $w(0) = \Lambda^3 u_0 =: w_0$ which satisfies

$$\tfrac{\gamma\eta}{4}\|w\|^2_{L^2_\gamma L^2_x} + \tfrac{\eta}{2}\|w\|^2_{L^\infty_\gamma L^2_x} \le c_0'\|u_0\|^2_{\mathcal{H}^3_x} + \tfrac{c_1'}{2\gamma\eta}\big(\|f\|^2_{L^2_\gamma \mathcal{H}^3_x} + \|v\|^2_{L^2_\gamma \mathcal{H}^3_x}\big) \tag{1.21}$$

with $c_0' = \frac{\lambda^2}{2}\|A_0(0)\|_\infty$. The map w also belongs to $C^1(\overline{J}, \mathcal{H}_x^{-1})$ because of (1.9) and $f \in \mathcal{Z}^3(J)$. Set $\Phi v = \Lambda^{-3} w \in \tilde{\mathcal{G}}^3(J)$. Let $\overline{w}$ satisfy $L\overline{w} = g(f, \overline{v})$ and $\overline{w}(0) = w_0$ for some $\overline{v} \in C(\overline{J}, \mathcal{H}_x^3)$. For $w - \overline{w}$ estimate (1.21) applies with $u_0 = 0$ and $f = 0$ so that

$$\|\Phi(v - \overline{v})\|_{L^\infty_\gamma \mathcal{H}^3_x} = \|\Lambda^{-3}(w - \overline{w})\|_{L^\infty_\gamma \mathcal{H}^3_x} \le \frac{\lambda\sqrt{c_1' T}}{\sqrt{\gamma\eta}} \|v - \overline{v}\|_{L^\infty_\gamma \mathcal{H}^3_x}.$$

Fixing a large $\gamma = \gamma(r, \eta, T)$, we obtain a fixed point u of Φ in $L^\infty_\gamma \mathcal{H}^3_x$. It actually belongs to $\tilde{\mathcal{G}}^3(J)$ and satisfies $u(0) = u_0$. Equation (1.19) implies that $Lu = f$. Uniqueness of solutions was already shown in Proposition 1.4.

2) It remains to establish (1.20). We first insert $u = v$ and $w = \Lambda^3 u$ in (1.21) and take $\gamma \ge \max\left\{\gamma_0(r, \eta), \frac{2\lambda\sqrt{c_1'}}{\eta}\right\}$. Absorbing $\|u\|^2_{L^2_\gamma \mathcal{H}^3_x}$ by the left-hand side, we infer

$$\frac{\gamma\eta}{8}\|u\|^2_{L^2_\gamma \mathcal{H}^3_x} + \frac{\eta}{2}\|u\|^2_{L^\infty_\gamma \mathcal{H}^3_x} \le c_0'\lambda^2\|u_0\|^2_{\mathcal{H}^3_x} + \frac{c_1'\lambda^2}{2\gamma\eta}\|f\|^2_{L^2_\gamma \mathcal{H}^3_x}. \tag{1.22}$$

If we estimated $\partial_t u$ in $\mathcal{H}^2_x$ by means of (1.9) and (1.22), we would obtain a constant depending on r in front of the norm of u_0. Instead we use that $\partial_t u \in C(\overline{J}, \mathcal{H}^2_x)$ satisfies

$$L\partial_t u = \partial_t f - \partial_t D u - \sum_{j=0}^3 \partial_t A_j \partial_j u =: h,$$

$$\partial_t u(0) = A_0(0)^{-1} f(0) - A_0(0)^{-1} D(0) u_0 - \sum_{j=1}^3 A_0(0)^{-1} A_j(0) \partial_j u_0 =: v_0.$$

Lemma 1.8 yields

$$\|h(t)\|_{\mathcal{H}^2_x} \le \|\partial_t f(t)\|_{\mathcal{H}^2_x} + \overline{c}(r)(\|u(t)\|_{\mathcal{H}^3_x} + \|\partial_t u(t)\|_{\mathcal{H}^2_x}),$$
$$\|v_0\|_{\mathcal{H}^2_x} \le c(r_0, \eta)(\|f(0)\|_{\mathcal{H}^2_x} + \|u_0\|_{\mathcal{H}^3_x}).$$

The commutator $[M_a, \Lambda^2] = [M_a, -\Delta] : \mathcal{H}^1_x \to L^2_x$ is bounded if $a \in W^{1,\infty}_x$ and $D^2 a \in \mathcal{H}^1_x \hookrightarrow L^3_x$. Starting from $L\partial_t u = h$, as in (1.19) and (1.21) we thus deduce

$$\begin{aligned}
&\tfrac{\gamma\eta}{4}\|\partial_t u\|^2_{L^2_\gamma \mathcal{H}^2_x} + \tfrac{\eta}{2}\|\partial_t u\|^2_{L^\infty_\gamma \mathcal{H}^2_x} \\
&\quad \le \hat{c}_0\lambda^2\big(\|u_0\|^2_{\mathcal{H}^3_x} + \|f(0)\|^2_{\mathcal{H}^2_x}\big) + \tfrac{\hat{c}_1\lambda^2}{2\gamma\eta}\big(\|\partial_t f\|^2_{L^2_\gamma \mathcal{H}^2_x} + \|u\|^2_{L^2_\gamma \mathcal{H}^3_x} + \|\partial_t u\|^2_{L^2_\gamma \mathcal{H}^2_x}\big)
\end{aligned}$$

for constants $\hat{c}_0 = \hat{c}_0(r_0, \eta)$ and $\hat{c}_1 = \hat{c}_1(r, \eta)$. Set $c_0 = 16\lambda^2\eta^{-1}(c_0' + \hat{c}_0)$ and $c_1 = \frac{8\lambda^2}{\eta^2}\max\{c_1', \hat{c}_1\}$. We add the above inequality to (1.22) and take $\gamma \ge \gamma_1(r, \eta) := \max\left\{\gamma_0(r, \eta), \sqrt{c_1}\right\}$. Estimate (1.20) follows after some calculations. □

In the above result we control more space than time derivatives. Under stronger assumptions on A_j, D and f, one can obtain analogous estimates on $\partial_t^2 u$ in $\mathcal{H}^1_x$ and $\partial_t^3 u$ in L^2_x by differentiating (1.8) in time, see (2.27). We discuss variants of the above theorem partly needed below.

PROPOSITION 1.10. *Let A_j and D be as in Theorem 1.9, as well as $u_0 \in \mathcal{H}^2_x$ and $f \in L^2(J, \mathcal{H}^2_x)$. Then there is a unique solution $u \in C(\overline{J}, \mathcal{H}^2_x) \cap C^1(\overline{J}, \mathcal{H}^1_x)$ of (1.8). For $t \in \overline{J}$ and $\gamma \geq \tilde{\gamma}_1(r, \eta) := \max\{\gamma_0(r, \eta), \sqrt{\tilde{c}_1}\}$, we have*

$$\gamma \|u\|^2_{L^2_\gamma((0,t),\mathcal{H}^2_x)} + +\mathrm{e}^{-2\gamma t}\|u(t)\|^2_{\mathcal{H}^2_x} \leq \tilde{c}_0 \|u_0\|^2_{\mathcal{H}^2_x} + \tfrac{\tilde{c}_1}{\gamma} \|f\|^2_{L^2_\gamma((0,t),\mathcal{H}^2_x)}$$

for constants $\tilde{c}_0 = \tilde{c}_0(r_0, \eta)$ and $\tilde{c}_1 = \tilde{c}_1(r, \eta)$. If $\partial_t f \in L^2(J, \mathcal{H}^1_x)$ we also obtain

$$\gamma \|\partial_t u\|^2_{L^2_\gamma((0,t),\mathcal{H}^1_x)} + \mathrm{e}^{-2\gamma t}\|\partial_t u(t)\|^2_{\mathcal{H}^1_x} \leq \tilde{c}_0 (\|u_0\|^2_{\mathcal{H}^2_x} + \|f(0)\|^2_{\mathcal{H}^1_x}) + \tfrac{\tilde{c}_1}{\gamma} \|f\|^2_{\mathcal{Z}^2_\gamma(0,t)}$$

where $\mathcal{Z}^k(J) := L^2(J, \mathcal{H}^k_x) \cap \mathcal{H}^1(J, \mathcal{H}^{k-1}_x)$ for $k \in \mathbb{N}$.

The result is shown as Theorem 1.9, replacing Λ^3 by Λ^2 in its proof up to (1.22) and Λ^2 by Λ afterwards. For the second part one also uses that the commutator $[M_a, \Lambda]$ is bounded on L^2_x by Proposition 4.1.A in [**54**] if $a \in W^{1,\infty}_x$.

REMARK 1.11. In Theorem 1.9 we have focused on the space $\mathcal{H}^3_x$ needed for the quasilinear problem. Actually, one obtains a unique solution $u \in \tilde{\mathcal{G}}^k(J)$ of (1.8) satisfying the analogue of (1.20) if $u_0 \in \mathcal{H}^k_x$, $f \in \mathcal{Z}^k(J)$, $A_j, D \in \tilde{\mathcal{F}}^k(J)$, $A_j = A_j^\top$, $A_0 \geq \eta I$, and $k \in \mathbb{N} \setminus \{2\}$. For $k = 2$ one needs another assumption stated below. This can be shown as for $k = 3$, one only has to take care of estimates for products, inverse matrices and commutators.

Indeed, for $k > 3$ one can use the product and inversion results mentioned above and the higher-order version of (1.17) in [**30**]. For $k = 1$ (thus for coefficients in $W^{1,\infty}_{t,x}$) the needed product and inversion bounds are easy to check, and we have just seen that $[M_a, \Lambda]$ is bounded on L^2_x if $a \in W^{1,\infty}_x$. For $k = 2$ the second-order derivatives of A_j also have to belong to $L^\infty_t L^3_x$. Then the commutator $[M_a, \Lambda^2] = [M_a, -\Delta] : \mathcal{H}^1_x \to L^2_x$ is bounded, and the extra condition is preserved by products and inverses.

Moreover, there is no problem to change the range space $\mathbb{R}^6$ to $\mathbb{R}^n$. Also other spatial domains $\mathbb{R}^m$ can be treated analogously, though one has to modify the assumptions on the coefficients in this case. Finally, invoking a bit more harmonic analysis one can also work in fractional Sobolev spaces $\mathcal{H}^s_x$ instead of $\mathcal{H}^k_x$, see [**31**]. $\Diamond$

REMARK 1.12. In (1.18) we have required that the derivatives of the coefficients belong to $\mathcal{H}^2_x$. So local singularities are allowed to some extent, but one enforces a certain decay at infinity which is an unnecessary restriction. Actually, Theorem 1.9 remains valid if we replace the space $\tilde{\mathcal{F}}^3(J)$ by $\tilde{\mathcal{F}}^3_\infty(J) = \tilde{\mathcal{F}}^3(J) + W^{3,\infty}_{t,x}$, and $\hat{\mathcal{H}}^2_x$ by $\hat{\mathcal{H}}^2_\infty = \hat{\mathcal{H}}^2_x + W^{2,\infty}_x$. (They have the norm of sums $X + Y$, namely $\|z\|_{X+Y} = \inf_{z=x+y} \|x\|_X + \|y\|_Y$.) To show this fact, we note that $[M_A, \Lambda^2] : \mathcal{H}^2_x \to \mathcal{H}^1_x$ is bounded uniformly in t if $A \in \hat{\mathcal{F}}^3(J) + W^{3,\infty}_{t,x}$, and so the same is true for

$$[M_A, \Lambda^3] = [M_A, \Lambda]\Lambda^2 + \Lambda[M_A, \Lambda^2] : \mathcal{H}^2_x \to L^2_x.$$

(Recall the boundedness of $[M_a, \Lambda]$ on L^2_x.) One can further show the appropriate bounds for products and inversions involving $\tilde{\mathcal{F}}^3(J) + W^{3,\infty}_{t,x}$ and $\hat{\mathcal{H}}^2_x + W^{2,\infty}_x$, as well as $\tilde{\mathcal{G}}^3(J)$. The analogue of Theorem 1.9 can now be proven as before. $\Diamond$

As a preparation for Theorem 1.19 on the wellposedness of the nonlinear problem we show an approximation result for the coefficients.

LEMMA 1.13. *Let $u_0 \in L^2_x$, $f \in L^2_{t,x}$, $n \in \mathbb{N} \cup \{\infty\}$, $j \in \{0,1,2,3\}$, $A^n_j \in \hat{\mathcal{F}}^3_\infty(J)$ be symmetric with $A^n_0 \geq \eta I$, and $D^n \in \hat{\mathcal{F}}^3_\infty(J)$. Assume that $\|A^n_j\|_{W^{1,\infty}_{t,x}} \leq r$ and $\|D^n\|_{L^\infty_{t,x}} \leq r$, as well as $A^n_j \to A^\infty_j$ and $D^n \to D^\infty$ in $L^\infty_{t,x}$ as $n \to \infty$. Set $L_n = \sum_j A^n_j \partial_j + D^n$. We have functions $u_n \in C(\overline{J}, L^2_x)$ with $L_n u_n = f$ and $u_n(0) = u_0$. Then $u_n \to u_\infty$ in $C(\overline{J}, L^2_x)$ as $n \to \infty$.*

PROOF. For the given data there are functions $u_{0,m}$ in $\mathcal{H}^3_x$ and f_m in $\mathcal{Z}^3(J)$ converging to u_0 and f in L^2_x and $L^2_{t,x}$, respectively, as $m \to \infty$. For these data Theorem 1.9 provides solutions $u_{n,m} \in \tilde{\mathcal{G}}^3(J)$ of $L_n u_{n,m} = f_m$ and $u_{n,m}(0) = u_{0,m}$. Fixing $\gamma = \gamma_0(r,\eta)$ from Lemma 1.2 and (1.10), Proposition 1.4 now shows

$$\|u_n - u_{n,m}\|_{L^\infty_t L^2_x} \leq c\|u_n - u_{n,m}\|_{L^\infty_\gamma L^2_x} \leq c\big(\|u_0 - u_{0,m}\|^2_{L^2_x} + \|f - f_m\|^2_{L^2_{t,x}}\big).$$

with $c = c(r,\eta,T)$. The right-hand side tends to 0 as $m \to \infty$ uniformly for $n \in \mathbb{N} \cup \{\infty\}$. It is thus enough to take $u_0 \in \mathcal{H}^3_x$, $f \in \mathcal{Z}^3(J)$, and $u_n \in \tilde{\mathcal{G}}^3(J)$. We then compute

$$L_n(u_n - u_\infty) = L_\infty u_\infty - L_n u_\infty = \sum_{j=0}^{3} (A^\infty_j - A^n_j)\partial_j u_\infty + (D^\infty - D^n)u_\infty =: g_n.$$

Since $u_\infty \in \tilde{\mathcal{G}}^3(T)$, as above Lemma 1.2 yields

$$\|u_n - u_\infty\|_{L^\infty_t L^2_x} \leq c(\gamma,T)\,\|g_n\|_{L^\infty_\gamma L^2_x} \;\longrightarrow\; 0, \qquad n \to \infty. \qquad \square$$

1.4. The quasilinear problem on $\mathbb{R}^3$

In this section we treat the nonlinear system

$$L(u)u := \sum_{j=0}^{3} a_j(u)\partial_j u + d(u)u = f, \quad t \geq 0,\ x \in \mathbb{R}^3, \qquad u(0) = u_0, \quad (1.23)$$

under the assumptions

$$\begin{aligned} &a_j, d \in C^3(\mathbb{R}^3 \times \mathbb{R}^6, \mathbb{R}^{6\times 6}), \quad a_j = a_j^\top, \quad a_0 \geq \eta I, \quad \eta \in (0,1], && (1.24)\\ &\forall\, r > 0: \sup_{|\xi|\leq r} \max_{0\leq|\alpha|\leq 3} \|\partial_x^\alpha a_j(\cdot,\xi)\|_{L^\infty_x}, \|\partial_x^\alpha d(\cdot,\xi)\|_{L^\infty_x} < \infty, \quad j \in \{0,1,2,3\},\\ &u_0 \in \mathcal{H}^3_x,\ \ \forall\, T > 0: f \in \mathcal{Z}^3(T) = \mathcal{Z}^3(J) = L^2(J,\mathcal{H}^3_x) \cap \mathcal{H}^1(J,\mathcal{H}^2_x), \quad J = (0,T). \end{aligned}$$

One can also treat coefficients only defined for $(x,\xi) \in \mathbb{R}^3 \times U$ and an open subset $U \subseteq \mathbb{R}^6$, see Remark 1.20. This is already needed in the Kerr Example 1.1 if χ_3 is not non-negative. To simplify a bit, we focus on the case $U = \mathbb{R}^6$ in (1.24).

We look for solutions u of (1.23) in $C([0,T_+), \mathcal{H}^3_x) \cap C^1([0,T_+), \mathcal{H}^2_x)$ for a maximally chosen final time $T_+ \in (0,\infty]$. As indicated in the next section, solutions may blow up and so T_+ could be finite. The solutions will be constructed in a fixed-point argument on the space $\tilde{\mathcal{G}}^{k-}(J) = L^\infty(J,\mathcal{H}^k_x) \cap W^{1,\infty}(J,\mathcal{H}^{k-1}_x)$ endowed with its natural norm, where $k = 3$. The overall strategy of this section and many techniques are typical for quasilinear (or semilinear) evolution equations, though there are different (but related) approaches, see e.g. [**5**], [**7**], or [**28**].

We first state basic properties of substitution operators, which is remains valid for Lipschitz domains instead of $\mathbb{R}^3$ with the same proof. (Recall Remark 1.12 concerning $\tilde{\mathcal{F}}^3_\infty(J)$ and $\hat{\mathcal{H}}^2_\infty$.) We set $E_\gamma = L^\infty_\gamma(J, \mathcal{H}^2_x)$ for a moment.

Lemma 1.14. *Let a be as in (1.24) and $\gamma \geq 0$.*

a) Let $v \in \tilde{\mathcal{G}}^3(J)$ with $\|v\|_\infty \leq r$. Then $\|a(v)\|_{\tilde{F}^3_\infty(J)} \leq \kappa(r)(1 + \|v\|^3_{\tilde{\mathcal{G}}^3(J)})$.

b) Let $v, w \in L^\infty_t \mathcal{H}^2_x$ with norm $\leq r$. Then $\|a(v) - a(w)\|_{E_\gamma} \leq \kappa(r)\, \|v - w\|_{E_\gamma}$. *Here we can also replace $L^\infty_t \mathcal{H}^2_x$ and E_γ by $\tilde{\mathcal{G}}^2(J)$ and $\tilde{\mathcal{G}}^2_\gamma(J)$, respectively.*

c) Let $v_0 \in \mathcal{H}^2_x$ with $\|v_0\|_\infty \leq r_0$. Then $\|a(v_0)\|_{\hat{\mathcal{H}}^2_\infty} \leq \kappa_0(r_0)(1 + \|v_0\|^2_{\mathcal{H}^2_x})$.

d) Let $v_0, w_0 \in \mathcal{H}^2_x$ with norm $\leq r_0$. Then $\|a(v_0) - a(w_0)\|_{\mathcal{H}^2_x} \leq \kappa_0(r_0) \|v_0 - w_0\|^2_{\mathcal{H}^2_x}$.

Proof. We sketch the proof. (See §7.1 in [**51**] or §2 in [**52**] for more details.)

a) Take $\alpha \in \mathbb{N}^4_0$ with $1 \leq |\alpha| \leq 3$ and $\alpha_0 \in \{0, 1\}$. The latter refers to the time derivative. It is clear that the function $|(\partial^\beta a)(v)|$ is bounded by $c(r)$ for all $0 \leq |\beta| \leq 3$ where $\beta = (\beta_x, \beta_\xi) \in \mathbb{N}^3_0 \times \mathbb{N}^6_0$. Note that $\partial^\alpha a(v)$ is a linear combination of products of $(\partial^\beta a)(\cdot, v)$ and $j \in \{0, 1, 2, 3\}$ factors $\partial^{\gamma_i} v$ with $\beta_x + \gamma_1 + \cdots + \gamma_j = \alpha$. Since $v \in W^{1,\infty}_{t,x}$ by Sobolev's embedding, as in the proof of Lemma 1.8 one can estimate $\partial^\alpha a(v)$ in $L^\infty_t L^2_x$ if $j \geq 1$ and in $L^\infty_{t,x}$ if $j = 0$, both by $c(r)(1 + \|v\|^3_{\tilde{\mathcal{G}}^3(J)})$.

b) We start from the formula

$$a(v) - a(w) = \int_0^1 (\partial_\xi a)(\cdot, v + s(w - v))\, (w - v)\, \mathrm{d}s =: A(w - v).$$

Let $\varphi_s = v + s(w - v)$. We then compute

$$\begin{aligned}\nabla^2_x(a(v) - a(w)) = \int_0^1 (\partial_\xi a)(\cdot, \varphi_s)\, \nabla^2_x(w - v)\, \mathrm{d}s + \int_0^1 \nabla^2_x(\partial_\xi a)(\cdot, \varphi_s)\, (w - v)\, \mathrm{d}s \\ + 2\int_0^1 \nabla_x(\partial_\xi a)(\cdot, \varphi_s)\, \nabla_x(w - v)\, \mathrm{d}s \end{aligned} \tag{1.25}$$

The factor $\mathrm{e}^{-\gamma t}$ is put in front of $\nabla^j_x(w - v)$ on the right. We further have

$$\begin{aligned}\nabla^2_x(\partial_\xi a)(\cdot, \varphi_s) &= (\partial^2_x \partial_\xi a)(\cdot, \varphi_s) + (\partial_x \partial^2_\xi a)(\cdot, \varphi_s)\partial_x \varphi_s + (\partial^2_\xi a)(\cdot, \varphi_s)\partial^2_x \varphi_s \\ &\quad + (\partial^3_\xi a)(\cdot, \varphi_s)[\partial_x \varphi_s, \partial_x \varphi_s].\end{aligned}$$

Using Sobolev's embedding, one can then bound the second term on the right-hand side of (1.25) in $L^\infty_\gamma(J, \mathcal{H}^2_x)$ by $c(r)\|v - w\|_{E_\gamma}$. The other terms are handeld more easily. Parts c) and d) are treated similarly. □

As the space for the fixed-point argument we will use

$$\mathcal{E}(R, T) := \{v \in \tilde{\mathcal{G}}^{3-}(J) \mid \|v\|_{\tilde{\mathcal{G}}^{3-}(J)} \leq R,\ v(0) = u_0.\}$$

for suitable $R > \|u_0\|_{\mathcal{H}^3}$ and $T > 0$. This set is non-empty as it contains the constant function $t \mapsto v(t) = u_0$. It is crucial that $\mathcal{E}(R, T)$ is complete for a metric involving only two derivatives, which can be shown by a standard application of the Banach–Alaoglu theorem. For this we recall that $L_t^\infty L_x^2$ is the dual space of $L_t^1 L_x^2$, see Corollary 1.3.22 in [27]. (This is the reason to take L^∞ in time instead of C.)

LEMMA 1.15. *The space $\mathcal{E}(R,T)$ is complete with the metric $\|u-v\|_{L_t^\infty \mathcal{H}_x^2}$.*

PROOF. Let (u_n) be Cauchy in $\mathcal{E}(R,T)$ with this metric. Then (u_n) has a limit u in $C(\overline{J},\mathcal{H}_x^2)$. Take $\alpha \in \mathbb{N}_0^4$ with $\alpha_0 \le 1$ and $0 \le |\alpha| \le 3$. Applying Banach–Alaoglu iteratively, we obtain a subsequence (also denoted by (u_n)) such that $\partial^\alpha u_n$ tends to a function v_α weak* in $L_t^\infty L_x^2$ which also satisfies $\sum_{|\alpha|\le 3} \|v_\alpha\|_{L_t^\infty L_x^2}^2 \le R^2$. It remains to check that $v_\alpha = \partial^\alpha u$. To this end, take $\varphi \in \mathcal{H}_0^3(J\times\mathbb{R}^3)$. We compute

$$\langle \partial^\alpha \varphi, u\rangle = \lim_{n\to\infty} \langle \partial^\alpha\varphi, u_n\rangle = \lim_{n\to\infty} (-1)^{|\alpha|}\langle \varphi, \partial^\alpha u_n\rangle = (-1)^{|\alpha|}\langle\varphi, v_\alpha\rangle$$

in the duality pairing $L_t^1 L_x^2 \times L_t^\infty L_x^2$. There thus exists $\partial^\alpha u = v_\alpha$. □

In the next lemma we perform the core fixed-point argument.

LEMMA 1.16. *Let (1.24) hold and $\rho^2 \ge \|u_0\|_{\mathcal{H}_x^3}^2 + \|f(0)\|_{\mathcal{H}_x^2}^2 + \|f\|_{\mathcal{Z}^3(1)}^2$. Then there is a radius $R = R(\rho) > \rho$ given by (1.26), a time $T_0 = T_0(\rho) \in (0,1]$ given by (1.27), and a unique solution $u \in \mathcal{E}(R,T_0)$ of (1.23).*

PROOF. 1) Lemma 1.14 shows that $a_j(u_0)$ and $d(u_0)$ are bounded in $\hat{\mathcal{H}}_\infty^2$ by some $\kappa_0(\rho)$. This yields a constant $c_0 = c_0(\rho) \ge 1$ in (1.20), in the setting of Remark 1.12. We define

$$R^2 = R(\rho)^2 = \mathrm{e} c_0(\rho)\rho^2 + 1 > \rho^2. \tag{1.26}$$

Take $v, w \in \mathcal{E}(R,T)$ for some $T>0$. Let $a \in \{a_0,a_1,a_2,a_3,d\}$ and $\gamma \ge 0$. By Lemma 1.14 and $\mathcal{H}_x^2 \hookrightarrow L_x^\infty$ there is a constant $\kappa = \kappa(R)$ with

$$\|a(v)\|_{\tilde{F}_\infty^3(J)} \le \kappa \qquad \text{and} \qquad \|a(v)-a(w)\|_{L_\gamma^\infty \mathcal{H}_x^2} \le \kappa\|v-w\|_{L_\gamma^\infty \mathcal{H}_x^2}.$$

Let $c_1 = c_1(\kappa,\eta)$, $\tilde{c}_1 = \tilde{c}_1(\kappa,\eta)$, and $\gamma_1 = \max\{\gamma_1(\kappa,\eta), \tilde{\gamma}_1(\kappa,\eta)\}$ be given by Theorem 1.9 and Proposition 1.10. We fix

$$\gamma = \gamma(\rho) = \max\left\{\gamma_1, \mathrm{e} c_1\rho^2, \sqrt{\mathrm{e}\tilde{c}_1}\,\overline{c}\kappa R\right\}, \quad T_0 = T_0(\rho) = \min\{1,(2\gamma)^{-1}\}, \tag{1.27}$$

where the constant $\overline{c} > 0$ is introduced below.

2) Theorem 1.9 gives a solution $u \in \tilde{\mathcal{G}}^3(J_0)$ of $L(v)u = f$ and $u(0) = u_0$ satisfying

$$\|u(t)\|_{\mathcal{H}_x^3}^2 + \|\partial_t u(t)\|_{\mathcal{H}_x^2}^2 \le \mathrm{e}^{2\gamma T_0}\big(c_0(\|u_0\|_{\mathcal{H}_x^3}^2 + \|f(0)\|_{\mathcal{H}_x^2}^2) + c_1\gamma^{-1}\|f\|_{\mathcal{Z}^3(1)}^2\big) \le R^2$$

for $t \in [0,T_0]$. So the map $\Phi : v \mapsto u =: \hat{v}$ leaves invariant $\mathcal{E}(R,T_0)$. Observe that

$$L(v)(\hat{v}-\hat{w}) = (L(w)-L(v))\hat{w} = \sum_{j=0}^{3}(a_j(w)-a_j(v))\partial_j\hat{w} + (d(w)-d(v))\hat{w}.$$

The right-hand side at time t is bounded in $\mathcal{H}_x^2$ by $\overline{c}\kappa R\|v(t)-w(t)\|_{2,2}$ due to Lemma 1.14. Since $v(0) = w(0)$ and $T_0 \le 1$, Proposition 1.10 then implies

$$\|\Phi(v)-\Phi(w)\|_{L_t^\infty \mathcal{H}_x^2}^2 \le \mathrm{e}^{2\gamma T_0}\|\Phi(v)-\Phi(w)\|_{L_\gamma^\infty \mathcal{H}_x^2}^2 \tag{1.28}$$

$$\le \mathrm{e}\tilde{c}_1\gamma^{-1}\overline{c}^2\kappa^2 R^2 T_0 \,\|v-w\|^2_{L^\infty_\gamma \mathcal{H}^2_x} \le \tfrac{1}{2}\|v-w\|^2_{L^\infty_\gamma \mathcal{H}^2_x}.$$

The assertion now follows from the contraction mapping principle. □

The above result yields uniqueness only in the ball $\mathcal{E}(R,T_0)$, but the contraction estimate (1.28) itself will lead to a much more flexible uniqueness statement. Before showing it, we note that restrictions or translations of a solution $u \in \tilde{\mathcal{G}}^3(J)$ to (1.23) satisfy (obvious) variants of (1.23). Let $u \in \tilde{\mathcal{G}}^3(J)$ solve (1.23) and $v \in \tilde{\mathcal{G}}^3(J')$ with $v(T) = u(T)$ solve it on $J' = (T, T')$. Then the concatenation w of u and v belongs to $\tilde{\mathcal{G}}^3(0,T')$ and fulfills (1.23). (Use (1.23) to check $\partial_t w \in C([0,T'],\mathcal{H}^2_x)$.)

Lemma 1.17. *Let (1.24) hold, $\tilde{J} = (0,\tilde{T})$, $u \in \tilde{\mathcal{G}}^3(J)$ and $\tilde{u} \in \tilde{\mathcal{G}}^3(\tilde{J})$ solve (1.23) on J and $\tilde{J}$, respectively. Then $u = \tilde{u}$ on $J \cap \tilde{J} =: \hat{J}$.*

Proof. Let τ be the supremum of all $t \in [0, \sup \hat{J})$ for which $u = \tilde{u}$ on $[0,t]$. Note that $u(0) = u_0 = \tilde{u}(0)$. We suppose that $\tau < \sup \hat{J}$. Then $u = \tilde{u}$ on $[0,\tau]$ by continuity, and there exists a number $\overline{\delta} > 0$ with $J_{\overline{\delta}} := [\tau, \tau + \overline{\delta}] \subseteq \hat{J}$. Let $\overline{R}$ be the maximum of the norms of u and $\tilde{u}$ in $\tilde{\mathcal{G}}^3(J_{\overline{\delta}})$. Fix γ as in (1.27) (with $\overline{\kappa} = \kappa(\overline{R})$ and $\rho = 0$) and take $\delta \in (0, \overline{\delta}]$. As in (1.28), Proposition 1.10 yields a constant $\overline{c}_1 = \tilde{c}_1(\overline{R}) > 0$ with

$$\|u-\tilde{u}\|^2_{L^\infty_\gamma(J_\delta,\mathcal{H}^2_x)} \le \mathrm{e}\overline{c}_1\gamma^{-1}\overline{c}^2\overline{\kappa}^2\overline{R}^2\delta\,\|u-\tilde{u}\|_{L^\infty_\gamma(J_\delta,\mathcal{H}^2_x)}.$$

Choosing a sufficently small $\delta > 0$, we infer $u = \tilde{u}$ on J_δ. This fact contradicts the definition of τ, so that $\tau = \sup \hat{J}$ as asserted. □

We now use the above results to define a *maximal solution* u to (1.23) assuming (1.24). The *maximal existence time* is given by

$$T_+ = T_+(u_0, f) := \sup\{T \ge 0 \,|\, \exists\, u_T \in \tilde{\mathcal{G}}^3(T) \text{ solving (1.23) on } [0,T]\} \in (0,\infty].$$

Lemma 1.16 shows $T_+(u_0,f) > T_0(\rho)$ as we can restart the problem at time $t_0 = T_0(\rho)$ with the initial value $u_T(T)$. Moreover, by Lemma 1.17 the solutions u_t and u_T coincide on $[0,t]$ for $0 < t < T < T_+$. Setting $u(t) = u_T(t)$ for such times thus yields a unique solution u of (1.23) on $[0,T_+)$ which belongs to $\tilde{\mathcal{G}}^3(T)$ for each $T \in (0,T_+)$.

In the proof of our main result below, we need the following Moser-type estimates, which are still true if one replaces $\mathbb{R}^m$ by a Lipschitz domain in $\mathbb{R}^m$.

Lemma 1.18. *Let $k \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{N}_0^m$.*

a) For $v, w \in L^\infty(\mathbb{R}^m) \cap \mathcal{H}^k(\mathbb{R}^m)$ and $|\alpha| + |\beta| = k$, we have

$$\|\partial^\alpha v\, \partial^\beta w\|_2 \le c\,(\|v\|_\infty \|w\|_{k,2} + \|v\|_{k,2}\|w\|_\infty).$$

b) For $v, w \in W^{1,\infty}(\mathbb{R}^m) \cap \mathcal{H}^k(\mathbb{R}^m)$ with $\partial^\alpha v, \partial^\beta w \in L^2(\mathbb{R}^m)$ for $1 \le |\alpha| \le k$ and $|\alpha| + |\beta| = k+1$, we have

$$\|\partial^\alpha v \partial^\beta w\|_2 \le c\|\nabla v\|_\infty \sum_{j=1}^m \|\partial_j w\|_{k-1,2} + c\,\|\nabla w\|_\infty \sum_{j=1}^m \|\partial_j v\|_{k-1,2}.$$

Proof. We first recall the Gagliardo–Nirenberg inequality

$$\|\partial^\alpha \varphi\|_{2k/|\alpha|} \le c\, \|\varphi\|_\infty^{1-\frac{|\alpha|}{k}} \sum_{|\gamma|=k} \|\partial^\gamma \varphi\|_2^{\frac{|\alpha|}{k}}$$

for $\varphi \in L^\infty(\mathbb{R}^m)$ with $\partial^\gamma \varphi \in L^2(\mathbb{R}^m)$ for all $|\gamma| = k$, see [40].

Assertion a) is clear if $|\alpha|$ is 0 or k. So let $k \ge 2$ and $1 \le |\alpha| \le k-1$. Note that $\frac{|\beta|}{k} = 1 - \frac{|\alpha|}{k}$. The inequalities of Hölder (with $\frac{1}{2} = \frac{|\alpha|}{2k} + \frac{|\beta|}{2k}$), Gagliardo–Nirenberg and Young yield

$$\begin{aligned}\|\partial^\alpha v \partial^\beta w\|_2 &\le \|\partial^\alpha v\|_{2k/|\alpha|}\, \|\partial^\beta w\|_{2k/|\beta|} \le c \|v\|_\infty^{1-\frac{|\alpha|}{k}}\, \|v\|_{k,2}^{\frac{|\alpha|}{k}}\, \|w\|_\infty^{1-\frac{|\beta|}{k}}\, \|w\|_{k,2}^{\frac{|\beta|}{k}} \\ &= (\|v\|_\infty \|w\|_{k,2})^{1-\frac{|\alpha|}{k}} (\|w\|_\infty \|v\|_{k,2})^{\frac{|\alpha|}{k}} \lesssim \|v\|_\infty \|w\|_{k,2} + \|v\|_{k,2} \|w\|_\infty.\end{aligned}$$

In part b) we can assume that $k \ge 3$ and $2 \le |\alpha| \le k-1$. There are $i, j \in \{1, \dots, m\}$ with $\alpha = \alpha' + e_i$ and $\beta = \beta' + e_j$, where $|\alpha'| + |\beta'| = k-1$. From a) we deduce

$$\|\partial^\alpha v \partial^\beta w\|_2 = \|\partial^{\alpha'} \partial_i v\, \partial^{\beta'} \partial_j w\|_2 \lesssim \|\partial_i v\|_\infty\, \|\partial_j w\|_{k-1,2} + \|\partial_i v\|_{k-1,2}\, \|\partial_j w\|_\infty$$

and thus statement b). □

We state the core local wellposedness result for (1.23). Let $\mathcal{B}_T((u_0, f), r)$ be the closed ball in $\mathcal{H}^3_x \times \mathcal{Z}^3(T)$ with center (u_0, f) and radius $r > 0$.

Theorem 1.19. *Let (1.24) hold and $\rho^2 \ge \|u_0\|^2_{\mathcal{H}^3_x} + \|f(0)\|^2_{\mathcal{H}^2_x} + \|f\|^2_{\mathcal{Z}^3(1)}$. Then the following assertions are true.*

a) There is a unique solution $u = \Psi(u_0, f)$ of (1.23) on $[0, T_+)$, where $T_+ = T_+(u_0, f) \in (T_0(\rho), \infty]$ with $T_0(\rho) > 0$ from (1.27) and $u \in \tilde{\mathcal{G}}^3(T)$ for all $T \in (0, T_+)$.

b) Let $T_+ < \infty$. Then $\lim_{t\to T_+} \|u(t)\|_{\mathcal{H}^3_x} = \infty$ and $\overline{\lim}_{t\to T_+} \|u(t)\|_{W^{1,\infty}_x} = \infty$.

c) Take $T \in [0, T_+)$. Then there is a radius $\delta > 0$ such that for all $(v_0, g) \in \mathcal{B}_T((u_0, f), \delta)$ we have $T_+(v_0, f) > T$ and $\Psi : \mathcal{B}_T((u_0, f), \delta) \to \tilde{\mathcal{G}}^3(T)$ is continuous. Moreover, $\Psi : (\mathcal{B}_T((u_0, f), \delta), \|\cdot\|_{\mathcal{H}^2_x \times \mathcal{Z}^2(T)}) \to \tilde{\mathcal{G}}^2(T)$ is Lipschitz.

Proof. a)/b) Above we have shown part a). Let $T_+ < \infty$ and $u = \Psi(u_0, f)$.

1) Suppose there are $t_n \to T_+$ with $r := \sup_n \|u(t_n)\|_{3,2} < \infty$. Set $T = T_+ + 1$ and $\overline{\rho}^2 = r^2 + \|f\|^2_{\mathcal{Z}^3(T)} + \sup_n \|f(t_n)\|^2_{2,2} < \infty$. Let $\tau = T_0(\overline{\rho}) > 0$ be given by (1.27). Fix an index N such that $t_N + \tau > T_+$. Lemma 1.16 and a time shift yield a solution $v \in \tilde{\mathcal{G}}^3(t_N, t_N + \tau)$ of (1.23) with $v(t_N) = u(t_N)$. We thus obtain a solution on $[0, t_N + \tau]$. This fact contradicts the definition of T_+, and hence $\|u(t)\|_{3,2} \to \infty$ as $t \to T_+$.

2) Next, set $\omega = \sup_{0 \le t < T_+} \|u(t)\|_{1,\infty}$ and suppose that $\omega < \infty$. Let $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 3$. Using (1.9), we compute

$$\begin{aligned} L(u)\partial_x^\alpha u = \partial_x^\alpha f - \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \Big[\sum_{j=1}^3 \partial_x^\beta a_j(u) \partial_x^{\alpha-\beta} \partial_j u + \partial_x^\beta d(u) \partial_x^{\alpha-\beta} u \qquad (1.29) \\ + \partial_x^\beta a_0(u) \partial_x^{\alpha-\beta} \Big(a_0(u)^{-1} \Big(f - \sum_{j=1}^3 a_j(u) \partial_j u - d(u) u \Big)\Big)\Big] \end{aligned}$$

$$=: f_\alpha = \partial_x^\alpha f - g_\alpha.$$

In view of (the proofs of) Lemmas 1.8 and 1.14, the summands of f_α in the second line can be treated as the others (using Young's inequality for products of norms of f and u). Employing also Lemma 1.18 and $\mathcal{H}_x^3 \hookrightarrow W_x^{1,\infty}$, we can estimate

$$\begin{aligned}\|f_\alpha(t)\|_2 &\le c(\omega)\Big(\|f(t)\|_{\mathcal{H}_x^3} + 1 + \sum_{k=1}^{3} \sum_{|\gamma_i|\le 3, |\gamma_1|+\dots+|\gamma_k|\le 4} \|\partial_x^{\gamma_1} u \cdots \partial_x^{\gamma_k} u\|_2\Big)\\ &\le c(\omega)\big(\|f(t)\|_{\mathcal{H}_x^3} + 1 + (1+\omega^3)\|u(t)\|_{\mathcal{H}_x^3}\big).\end{aligned}$$

Take $\gamma \ge \gamma_0(\omega)$ in Proposition 1.4. For $t \in [0, T_+)$, this proposition and the above estimate yield (with $J_t = (0,t)$)

$$\|\partial_x^\alpha u\|^2_{L^2_\gamma(J_t, L^2_x)} + \tfrac{2\mathrm{e}^{-2\gamma t}}{\gamma}\|\partial_x^\alpha u(t)\|^2_{L^2_x} \le \tfrac{c(\omega)}{\eta\gamma}\|u_0\|^2_{\mathcal{H}^3_x} + \tfrac{c(\omega)}{\eta^2\gamma^2}\big[\|f\|^2_{L^2_\gamma \mathcal{H}^3_x} + 1 + \|u\|^2_{L^2_\gamma(J_t, \mathcal{H}^3_x)}\big].$$

We now sum over $|\alpha| \le 3$ and fix a large γ to absorb the last summand. It turns out that $\|u(t)\|_{3,2}$ is bounded for $t < T_+$ contradicting step 1), and hence part b) is shown.

c) The proof of assertion c) is more demanding. We first fix some constants, and then show continuity of Ψ at (u_0, f) on an interval $[0, b]$ assuming that we have solutions with uniform bounds on $[0, b]$. Using this fact and Lemma 1.16, we then prove inductively that solutions on $[0, T]$ exist and satisfy such bounds if we start in a certain ball around (u_0, f). Finally, we replace (u_0, f) by different data in this ball to obtain the asserted continuity statements.

1) Fix $T' \in (T, T_+)$ and set $J' = (0, T')$. We can extend maps g from $\mathcal{Z}^3(T)$ to $\mathcal{Z}^3(T')$ with norm bounded by $c_E\|g\|_{\mathcal{Z}^3(T)}$. Let $c_S \ge 1$ be the norm of the embedding $C([0,T'], \mathcal{H}^2_x) \hookrightarrow \mathcal{Z}^3(T')$, $\tilde\rho^2 \ge \|u_0\|^2_{3,2} + \|f\|^2_{\mathcal{Z}^3(T')} + \|f\|^2_{L^\infty(J', \mathcal{H}^2_x)}$, $\delta_0 = \tilde\rho / c_E \le \tilde\rho$, and $\tilde r \ge \max\{c_S\tilde\rho, \|u\|_{\tilde{\mathcal{G}}^3(T')}\}$. Below we take $R \ge \tilde r$, $b \le T'$, and $v \in \tilde{\mathcal{G}}^3(b)$ with norm less or equal R. Lemma 1.14 yields a constant $\overline{\kappa} = \overline{\kappa}(R)$ larger than the norms of $a_j(v)$ and $d(v)$ in $\hat{\mathcal{F}}^3_\infty(b)$ and of $a_j(v)(0)$ and $d(v)(0)$ in $\hat{\mathcal{H}}^2_\infty$.

2) Assume there are $b \in (0, T']$, $v_0 \in \mathcal{H}^3_x$ and $g \in \mathcal{Z}^3(T)$ such that $T_+(v_0, g) > b$. We write $v = \Psi(v_0, g) \in \tilde{\mathcal{G}}^3(b)$. Let $R \ge \|v\|_{\tilde{\mathcal{G}}^3(b)}$ with $R \ge \tilde r$. Observe that

$$L(u)(v-u) = g - f + (L(u) - L(v))v = g - f + \sum_{j=0}^{3}(a_j(u) - a_j(v))\partial_j v + (d(u) - d(v))v.$$

By Lemma 1.14, the function $(L(u) - L(v))v$ belongs to $\tilde{\mathcal{G}}^2_\gamma(b)$ with norm less than $c(\overline{\kappa})R\|v-u\|_{\tilde{\mathcal{G}}^2_\gamma(b)}$ for $\gamma \ge 0$. Proposition 1.10 yields

$$\|v-u\|_{\tilde{\mathcal{G}}^2_\gamma(b)} \le \tilde c(\overline{\kappa}, \eta, T')\big(\|u_0 - v_0\|^2_{\mathcal{H}^2_x} + \|f-g\|_{\mathcal{Z}^2_\gamma(b)} + \gamma^{-1}Rb\|v-u\|_{\tilde{\mathcal{G}}^2_\gamma(b)}\big)$$

for $\gamma \ge \tilde\gamma_1(\overline{\kappa}, \eta) \ge 1$. Fixing a large $\overline{\gamma}_1 = \overline{\gamma}_1(\overline{\kappa}, R, T', \eta) \ge \tilde\gamma_1(\overline{\kappa}, \eta)$, we thus obtain

$$\|v-u\|_{\tilde{\mathcal{G}}^2(b)} \le \tilde c(\overline{\kappa}, R, T', \eta)\big(\|u_0 - v_0\|^2_{\mathcal{H}^2_x} + \|f-g\|_{\mathcal{Z}^2(b)}\big). \tag{1.30}$$

3) Estimate (1.30) is related to Lipschitz continuity of Ψ in $\tilde{\mathcal{G}}^2$. The hard and core part of the proof is to check continuity of Ψ in $\tilde{\mathcal{G}}^3$ at (u_0, f), assuming apriori bounds. So let $(u_{0,n}, f_n) \in \mathcal{B}_T((u_0, f), \tilde{\delta})$ tend to (u_0, f) on $\mathcal{H}^3_x \times \mathcal{Z}^3(T)$ as $n \to \infty$, where $\tilde{\delta} > 0$. Hence, $f_n(0) \to f(0)$ in $\mathcal{H}^2_x$ and $f_n \to f$ in $\mathcal{Z}^3(T')$. Assume that $T_+(u_{0,n}, f_n) > b$ with $b \in (0, T']$ and that $u_n = \Psi(u_{0,n}, f_n)$ is bounded by some $R \geq \tilde{r}$ in $\tilde{\mathcal{G}}^3(b)$ for all $n \in \mathbb{N}$. Then u_n tends to u in $\tilde{\mathcal{G}}^2(b)$ as $n \to \infty$ by (1.30), and the coeffcients $a_j(u_n)$ and $d(u_n)$ satisfy the estimates of step 1) with a uniform $\overline{\kappa} = \overline{\kappa}(R)$.

The main idea is to split the n-dependence of the coefficients and the data. Let $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = 3$. As in (1.29) we write $L(u_n)\partial_x^\alpha u_n = \partial_x^\alpha f_n - g_{n,\alpha}$ and $L(u)\partial_x^\alpha u = \partial_x^\alpha f - g_\alpha$. Theorem 1.5 yields solutions $w_n, z_n \in C([0, b], L^2_x)$ of

$$\begin{aligned} L(u_n)w_n &= \partial_x^\alpha f - g_\alpha, \quad w_n(0) = \partial_x^\alpha u_0, \\ L(u_n)z_n &= \partial_x^\alpha f_n - \partial_x^\alpha f + g_\alpha - g_{n,\alpha}, \quad z_n(0) = \partial_x^\alpha u_{0,n} - \partial_x^\alpha u_0. \end{aligned}$$

By uniqueness, we have $w_n + z_n = \partial_x^\alpha u_n$ and hence

$$\partial_x^\alpha u_n - \partial_x^\alpha u = w_n - \partial_x^\alpha u + z_n.$$

Since $a_j(u_n) \to a_j(u)$ and $d(u_n) \to d(u)$ in $L^\infty_{t,x}$ as $n \to \infty$, Lemma 1.13 shows that $q_n := \|w_n - \partial_x^\alpha u\|_{L^\infty_t L^2_x}$ tends to 0. We thus have to prove $z_n \to 0$ in $L^\infty_t L^2_x$.

Choose $\gamma = \overline{\gamma}_1(R)$ as in step 2). For $t \in [0, b]$, Proposition 1.4 then implies

$$\begin{aligned} \|\partial_x^\alpha(u_n(t) - u(t))\|^2_{L^2_x} &\leq 2q_n^2 + 2\|z_n(t)\|^2_{L^2_x} \\ &\leq 2q_n^2 + c(R)\big(\|\partial_x^\alpha(u_{0,n} - u_0)\|^2_{L^2_x} + \|\partial_x^\alpha(f_n - f)\|^2_{L^2_{t,x}} + \|g_{n,\alpha} - g_\alpha\|^2_{L^2_{t,x}}\big). \end{aligned}$$

The estimation of $\|g_{n,\alpha} - g_\alpha\|$ is only sketched. Let $a \in \{a_j, a_0^{-1}, d\}$, $v \in \{u, u_n\}$, and $w \in \{u, u_n, f\}$. First, we look at summands of the type $\partial_x^\beta a(v(t))\partial_x^\gamma(u_n(t) - u(t))$ with $|\gamma| \leq 4 - |\beta|$, $|\gamma| \leq 3$ and $|\beta| \leq 3$. By Lemma 1.8 and the bounds on the coefficients these terms are bounded in L^2_x by $c(R)\|u_n(t) - u(t)\|_{3,2}$. Analogous summands with $f_n(t) - f(t)$ are treated similarly.

We next analyze terms like $W = \partial_x^\beta[a(u_n(t)) - a(u(t))]\,\partial_x^\gamma w(t)$. At first, we look at situations where we can estimate the first factor by $u - u_n$ in $L^\infty_t \mathcal{H}^2_x$ using Lemma 1.14. This works for $\beta = 0$ in L^∞_x for $|\gamma| \leq 3$, for $|\beta| = 1$ in L^6_x if $|\gamma| \leq 2$, and for $|\beta| = 2$ in L^2_x if $|\gamma| \leq 1$; and it yields terms as in the first case. If this does not work (which implies $w \in \{u, u_n\}$), we compute $\partial_x^\beta(a(u_n) - a(u))$ using the chain rule for each summand. For these terms we define

$$h_n(t) = \sum_{a \in \{a_j, d, a_0^{-1}\}} \sum_{k=1}^{3} \sum_{l_i=1}^{9} \|(\partial_{l_k} \cdots \partial_{l_1} a)(u_n(t)) - (\partial_{l_k} \cdots \partial_{l_1} a)(u(t))\|_{L^\infty_x}.$$

The L^2_x-norm of such W is then bounded by linear combinations of $c(R)$ times

$$h_n(t)\|\partial_x^{\gamma_1} v(t) \cdots \partial_x^{\gamma_{m-1}} v(t) \partial_x^{\gamma_m} w(t)\|_{L^2_x} + \|\partial_x^{\gamma_1} v(t) \cdots \partial_x^{\gamma_{m-1}} \varphi_n(t) \partial_x^{\gamma_m} w(t)\|_{L^2_x},$$

where $\varphi_n = u_n - u$, $m \in \{1, 2, 3, 4\}$, $|\gamma_i| \leq 3$, and $|\gamma_1| + \cdots + |\gamma_m| \leq 4$. This sum can be estimated by $c(R)(h_n(t) + \|u_n(t) - u(t)\|_{3,2})$ due to Sobolev embeddings and the bounds on u and u_n. We have shown that

$$\|g_{n,\alpha} - g_\alpha\|^2_{L^2((0,t),L^2_x)} \le c(R,T')\Big(\|f_n - f\|^2_{L^2_t\mathcal{H}^2_x} + \|u_n - u\|^2_{L^\infty_t\mathcal{H}^2_x} + \int_0^{T'} h_n(s)^2\,\mathrm{d}s + \int_0^t \sum_{|\gamma|=3} \|\partial_x^\gamma(u_n(s) - u(s))\|^2_{L^2_x}\,\mathrm{d}s\Big).$$

We write the last integrand as $\|\partial_x^3(u_n(s) - u(s))\|_2^2$. Note that $h_n(s)$ tends to 0 as $n \to \infty$ since $u_n \to u$ in $L^\infty_{t,x}$ and that it is bounded uniformly in s and n. By dominated convergence $\int_0^{T'} h_n^2\,\mathrm{d}s$ tends to 0. Summing up, we conclude that

$$\|\partial_x^3(u_n(t) - u(t))\|_2^2 \le c(R,T')\varepsilon_n + c(R,T')\int_0^t \|\partial_x^3(u_n(s) - u(s))\|_2^2\,\mathrm{d}s$$

for a null sequence (ε_n). By Gronwall, $\partial_x^3(u_n - u)$ tends to 0 in $C([0,b], L^2_x)$ as $n \to \infty$, and so $u_n \to u$ in $C([0,b],\mathcal{H}^3_x)$. Using (1.9) and Lemma 1.14, we infer $u_n \to u$ in $\tilde{\mathcal{G}}^3(b)$.

4) We now look for data to which we can apply steps 2) and 3). Let $(v_0, g) \in \mathcal{B}_T((u_0,f),\delta_0)$. We then obtain

$$\begin{aligned} \|v_0\|_{\mathcal{H}^3_x} &\le \|v_0 - u_0\|_{\mathcal{H}^3_x} + \|u_0\|_{\mathcal{H}^3_x} \le \delta_0 + \tilde{\rho} \le 2\tilde{\rho} \le 2\tilde{r}, \\ \|g\|_{\mathcal{Z}^3(T')} &\le \|g - f\|_{\mathcal{Z}^3(T')} + \|f\|_{\mathcal{Z}^3(T')} \le c_E\delta_0 + \tilde{\rho} \le 2\tilde{\rho} \le 2\tilde{r}, \\ \|g\|_{L^\infty(J',\mathcal{H}^2_x)} &\le c_S\|g\|_{\mathcal{Z}^3(T')} \le 2c_S\tilde{\rho} \le 2\tilde{r}. \end{aligned}$$

Lemma 1.16 thus yields a time $\tau = \tau(\tilde{r})$ and a solution $v \in \tilde{\mathcal{G}}^3(\tau)$ of (1.23) with data v_0 and g, where $\|v\|_{\tilde{\mathcal{G}}^3(\tau)} \le \tilde{R} = \tilde{R}(\tilde{r})$ and $\tilde{R} > 2\tilde{r}$. By parts a) and b), we have $v = \Psi(v_0, g)$ and $T_+(v_0, g) > \tau$. Fix $N \in \mathbb{N}$ with $(N-1)\tau \le T < N\tau$, set $t_k = k\tau$ for $k \in \{0, 1, \dots, N-1\}$ and $t_N = \min\{T', N\tau\}$.

Steps 2) and 3) show that (1.30) is true on $[0,\tau]$ for such v with a constant $\tilde{c} = \tilde{c}(\tilde{r})$ and that $\Psi : \mathcal{B}_T((u_0,f),\delta_0) \to \tilde{\mathcal{G}}^3(\tau)$ is continuous at (u_0, f). We can thus find a radius $\delta_1 \in (0, \delta_0]$ such that $\|v - u\|_{\tilde{\mathcal{G}}^3(\tau)} \le \tilde{r}$, and hence $\|v\|_{\tilde{\mathcal{G}}^3(\tau)} \le 2\tilde{r}$, for all $(v_0, g) \in \mathcal{B}_T((u_0, f), \delta_1)$.

5) We iterate the above argument. Assume that for some $k \in \{1, \dots, N-1\}$ and $\delta_k \in (0, \delta_0]$, we have $T_+(v_0, g) > t_k$ and $\|v - u\|_{\tilde{\mathcal{G}}^3(t_k)} \le \tilde{r}$ for all $(v_0, g) \in \mathcal{B}_T((u_0, f), \delta_k)$ and the map $\Psi : \mathcal{B}_T((u_0,f),\delta_k) \to \tilde{\mathcal{G}}^3(t_k)$ is continuous at (u_0, f). It follows $\|v\|_{\tilde{\mathcal{G}}^3(t_k)} \le 2\tilde{r}$. Since $\|v(t_k)\|_{3,2} \le 2\tilde{r}$, step 4) and a time shift provide a solution $\tilde{v} \in \tilde{\mathcal{G}}^3([t_k, t_{k+1}])$ of (1.23) with $\tilde{v}(t_k) = v(t_k)$ and norm less or equal $\tilde{R}$. We can thus extend v to a solution in $\tilde{\mathcal{G}}^3([0, t_{k+1}])$ bounded by $\tilde{R}$ and so $T_+(v_0, g) > t_{k+1}$. Because of this bound, steps 2) and 3) imply (1.30) on $[0, t_{k+1}]$ with $\tilde{c} = \tilde{c}(\tilde{r})$ for all $(v_0, g) \in \mathcal{B}_T((u_0,f),\delta_k)$ and the continuity of $\Psi : \mathcal{B}_T((u_0,f),\delta_k) \to \tilde{\mathcal{G}}^3(t_{k+1})$ at (u_0, f). Using the latter property, we find a radius $\delta_{k+1} \in (0, \delta_k]$ such that $\|v - u\|_{\tilde{\mathcal{G}}^3(t_{k+1})} \le \tilde{r}$ for $v = \Psi(v_0, g)$ and all $(v_0, g) \in \mathcal{B}_T((u_0, f), \delta_{k+1})$, and hence $\|v\|_{\tilde{\mathcal{G}}^3(t_{k+1})} \le 2\tilde{r}$.

Induction yields a radius $\delta = \delta_N$ such that for all $(v_0, g) \in \mathcal{B}_T((u_0, f), \delta)$ we have $T_+(v_0, g) > T$, the continuity of $\Psi : \mathcal{B}_T((u_0,f),\delta) \to \tilde{\mathcal{G}}^3(T)$ at (u_0, f), and $\|\Psi(v_0, g)\|_{\tilde{\mathcal{G}}^3(T)} \le 2\tilde{r}$. Moreover, (1.30) holds on $[0, T]$ for u and $v = \Psi(v_0, g)$.

6) Finally, we take any $(v_0, g), (w_0, h) \in \mathcal{B}_T((u_0, f), \delta)$ with corresponding solutions v and w. Replacing u by w in step 2), we then obtain the last assertion in c). Also step 3) can be repeated on $[0, T]$ for data converging to (w_0, h) in $\mathcal{B}_T((u_0, f), \delta)$. □

Observe that Theorem 1.7 yields finite speed of progation for a solution $u \in \tilde{\mathcal{G}}^3(T)$ of (1.23), setting $A_j = a_j(u)$ and $D = d(u)$. We comment on variants of Theorem 1.19.

Remark 1.20. One can easily extend Theorem 1.19 to negative times (e.g., by time reversion). Moreover, in (1.24) one can replace the domain $\mathbb{R}^3 \times \mathbb{R}^6$ of a_j and d by $\mathbb{R}^3 \times U$ for an open $U \subseteq \mathbb{R}^6$, restricting ξ in the supremum not to each closed ball $\overline{B}(0, r) \subseteq \mathbb{R}^6$ but to each compact subset of U. One further has to require that the closure K_0 of $u_0(\mathbb{R}^3)$ is contained in U, and the solution u has to take values in U. Theorem 1.19 is then valid with one modification. In part b) now $T_+ < \infty$ implies that $\limsup_{t<T_+} \|u(t)\|_{W^{1,\infty}_x} = \infty$ or that $u(t)$ leaves any compact subset of U as $t \to T_+$.

The proofs are very similar in this more general case. In the fixed-point argument one chooses a bounded open set V with $K_0 \subseteq V \subseteq \overline{V} \subseteq U$. Let $d > 0$ be the distance between V and ∂U. In $\mathcal{E}(R, T)$ one then also includes the condition that $\|v(t) - u_0\|_\infty \le d/2$ for all $t \in [0, T]$ which is preserved by limits in $L^\infty_t \mathcal{H}^2_x$. Other steps in the reasoning are modified accordingly. Compare Theorem 3.3 of [52]. ♢

As explained in Section 1.1, one can easily apply Theorem 1.19 to the Maxwell system (1.1) with material laws (1.3) and (1.4). We state the needed assumptions in a situation motivated by nonlinear optics.

Example 1.21. Let $\theta(x, E, H) = (\varepsilon_{\mathrm{lin}}(x)E + \varepsilon_{\mathrm{nl}}(x, E)E, \mu_{\mathrm{lin}}(x)H)$ and $J_e = \sigma(x, E)E + J_0$ in (1.3) and (1.4). Here we assume that $\varepsilon_{\mathrm{lin}}, \mu_{\mathrm{lin}} \in C^3_b(\mathbb{R}^3, \mathbb{R}^{3\times3}_{\mathrm{sym}})$ and $\sigma \in C^3(\mathbb{R}^6, \mathbb{R}^{3\times3})$ satisfy $\varepsilon_{\mathrm{lin}}, \mu_{\mathrm{lin}} \ge 2\eta I > 0$ and $\sup_{|\xi|\le r} \|\partial^\alpha_x \sigma(\cdot, \xi)\|_{L^\infty_x} < \infty$ for all $r \ge 0$ and $0 \le |\alpha| \le 3$, respectively. (The subscript b means that the functions and all occuring derivatives are bounded.) In Example 1.1 we had seen rather general isotropic nonlinear terms which fit to (1.24). A typical anistropic example is furnished by

$$\varepsilon_{\mathrm{nl}}(x, E) = \Big(\sum_{j,k=1}^{3} \chi_i^{jkl}(x) E_j E_k \Big)_{il}$$

for scalar coefficients $\chi_i^{jkl} \in C^3_b(\mathbb{R}^3)$, cf. [9]. Because of the triple sum in $\varepsilon_{\mathrm{nl}}(x, E)E$, the tensor $(\chi_i^{jkl})_{i,j,k,l}$ has to be symmetric in $\{j, k, l\}$. For (1.24) we also require symmetry in $\{i, l\}$, i.e., we can only prescribe χ_i^{jkl} for, say, $1 \le i \le j \le k \le l \le 3$. For $|E| < r$ and a suitable $r \in (0, \infty]$ and all $x, H \in \mathbb{R}^3$ we then obtain $\partial_{(E,H)}(x, E, H) \ge \eta I$. Rewriting the system as in (1.6), we see that hypothesis (1.24) (modified as in Remark 1.20 if $r < \infty$) is fulfilled. For initial fields in $\mathcal{H}^3_x$ with $|E_0| < r/2$ and a current density $J_0 \in \mathcal{Z}^3(T)$ for all $T > 0$, Theorem 1.19 and Remark 1.20 thus provide wellposedness in $\mathcal{H}^3_x$ of the Maxwell system (1.1) with the above material laws. ♢

1.5. Energy and blow-up

In the preceeding sections we have worked with the linear energy estimate which contains error terms caused by the time derivative of coefficients. (The space derivatives in C of (1.11) disappear in the Maxwell case.) These error terms have led to the $\mathcal{H}^3_x$ setting, which is quite inconvenient. The time dependence arises since we freeze a function in the nonlinearities of (1.23). One may wonder whether this is really necessary and whether it is not better to solve (1.23) based on a nonlinear energy identity. Actually, this can be done in the semilinear case where $D = \varepsilon(x)E$, $B = \mu(x)H$, and $J_e = \sigma(x,E)E$ under appropriate conditions on σ, cf. [**21**]. Below we see that this does not seem to work in the quasilinear case.

In this section we first establish an energy equality in the quasilinear case, without conductivity and for isotropic nonlinearities

$$D = \varepsilon_{\text{lin}}E + \beta_e(\cdot, |E|^2)E, \qquad B = \mu_{\text{lin}}H + \beta_m(\cdot, |H|^2)H, \tag{1.31}$$

Here ε_{lin} and μ_{lin} belong to $L^\infty(\mathbb{R}^3, \mathbb{R}^{3\times 3}_\eta)$ for some $\eta > 0$ and the maps $\beta_e, \beta_m : \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ are C^1, bounded in $x \in \mathbb{R}^3$ and non-decreasing in $s \in \mathbb{R}_{\geq 0}$. We set $u = (E, H)$ and

$$A_0 = \begin{pmatrix} \varepsilon_{\text{lin}} & 0 \\ 0 & \mu_{\text{lin}} \end{pmatrix}, \qquad \beta(|u|^2) = \begin{pmatrix} \beta_e(\cdot, |E|^2)I_{3\times 3} & 0 \\ 0 & \beta_m(\cdot, |H|^2)I_{3\times 3} \end{pmatrix},$$

$$M = \begin{pmatrix} 0 & \operatorname{curl} \\ -\operatorname{curl} & 0 \end{pmatrix} = -\sum_{j=1}^{3} S_j \partial_j, \qquad \mathcal{D}(M) = \mathcal{H}(\operatorname{curl}) \times \mathcal{H}(\operatorname{curl}),$$

where $\mathcal{H}(\operatorname{curl}) = \mathcal{H}(\operatorname{curl}, U) = \{v \in L^2(U, \mathbb{R}^3) \mid \operatorname{curl} v \in L^2(U, \mathbb{R}^3)\}$. The operator M is skew-adjoint in $L^2(\mathbb{R}^3, \mathbb{R}^6)$. Maxwell equations (1.1) then become

$$\partial_t[A_0 u(t) + \beta(|u(t)|^2)u(t)] = Mu(t), \quad t \geq 0, \qquad u(0) = u_0 = (E_0, H_0). \tag{1.32}$$

Omitting the argument x in the notation, we further define

$$b_j(s) = \int_0^s \beta_j(r)\,\mathrm{d}r, \qquad h_j(s) = s\beta_j(s) - \tfrac{1}{2} b_j(s).$$

We have $h_j(s) \geq \frac{s}{2}\beta_j(s)$ since β_j does not decrease and that $h_j'(s) = \frac{1}{2}\beta_j(s) + s\beta_j'(s)$, where $\beta_j' = \partial_2 \beta_j$. We now introduce the 'energy' for $u = (u_1, u_2)$ by

$$\mathcal{E}(u) = \int_{\mathbb{R}^3} \left[\tfrac{1}{2} A_0 u \cdot u + h_1(|u_1|^2) + h_2(|u_2|^2)\right] \mathrm{d}x$$

Note that $\mathcal{E}(u) \geq \frac{\eta}{2}\|u\|_2^2$ if $\beta_j \geq 0$. In the Kerr case $\varepsilon_{\text{lin}} = \mu_{\text{lin}} = 1$, $\beta_e(x, s) = \chi_3(x)s$ and $\beta_m = 0$, we obtain

$$\mathcal{E}_{\text{Kerr}}(E, H) = \int_{\mathbb{R}^3} \left[\tfrac{1}{2}|E(t)|^2 + \tfrac{3}{4}\chi_3 |E(t)|^4 + \tfrac{1}{2}|H(t)|^2\right] \mathrm{d}x.$$

Let $u \in \tilde{\mathcal{G}}^1(T)$ solve (1.32). The energy equality $\mathcal{E}(u(t)) = \mathcal{E}(u_0)$ for $t \in [0, T]$ follows from

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(u) = \int_{\mathbb{R}^3} \left[u \cdot \partial_t(A_0 u) + \beta(|u|^2)u \cdot \partial_t u + 2|u|^2 \beta'(|u|^2) u \cdot \partial_t u\right] \mathrm{d}x$$

$$= \int_{\mathbb{R}^3} \partial_t \big[A_0 u + \beta(|u|^2)u\big] \cdot u \,\mathrm{d}x = \int_{\mathbb{R}^3} Mu \cdot u \,\mathrm{d}x = 0. \tag{1.33}$$

In the Kerr case (with $\chi_3 \geq 0$) we can thus bound powers of p-norms of solutions. This is not enough control to pass to a weak limit in the nonlinearity when performing an approximation argument (which would typically produce a global solution). One would need an estimate involving derivatives. Such estimates are not known, and the next result on blow-up indicates that they do not hold.

We first stress that it is well known that the gradient of a solution to (1.32) may blow up in sup-norm in finite time, see [**36**]. However in the semilinear case one relies on estimates in $\mathcal{H}(\mathrm{curl})$, so we are interested in blow-up in this space (or at least in $\mathcal{H}^1$). Below we give such an example on a domain with periodic boundary conditions, taken from [**14**]. Such conditions arise if one truncates a fullspace problem with periodic coefficients to a periodicity cell. (See this paper for a weaker result on $\mathbb{R}^3$.) We work in the following more specific setting given by $D = (1+\alpha(|E|))E$ and $B = H$. We set $a(s) = (1+\alpha(|s|))s$ for $s \in \mathbb{R}$ and assume

$$\begin{aligned} &a \in C^2(\mathbb{R},\mathbb{R}), \quad \exists\, s_- < 0 < s_0 < s_+ : \; a' > 0 \;\text{ on }\; S := (s_-, s_+),\\ &q : S \to \mathbb{R};\; q(s) = \frac{a''(s)}{2a'(s)^{3/2}}, \;\text{ has a global maximum at }\; s = s_0, \\ &q \text{ is } C^1 \text{ near } s_0, \quad q(s) > 0 \;\text{ for }\; 0 < s \leq s_0. \end{aligned} \tag{1.34}$$

Let $\gamma > 2$ and $\alpha_0 > 0$. A simple example for (1.34) is furnished by any C^2-extension of $a : [0, s_+] \to \mathbb{R}$; $a(s) = s + \alpha_0 s^\gamma$, which is strictly growing on (s_-, s_+) for some $s_- < 0 < s_0 < s_+$ with

$$s_0 = \Big(\frac{2(\gamma-2)}{\alpha_0\gamma(\gamma+1)}\Big)^{\frac{1}{\gamma-1}}$$

in this case. We stress that the behavior of a for large s is arbitrary here.

Theorem 1.22. *Assume that (1.34) is true. Then there are numbers $M, T > 0$ and a divergence-free map $(E,H) \in C^1([0,T) \times [-M,M]^3)$ which solves (1.1) on $(-M,M)^3$ with periodic boundary conditions and the above material laws, and which satisfies*

$$\|\operatorname{curl} E(t)\|_{L^2_x} \to \infty \qquad \text{as} \;\; t \to T^-.$$

We look for a solution of the form

$$(E(t,x), B(t,x)) = (u(t,x_2), 0, 0, 0, 0, v(t,x_2)).$$

for $x \in (-M,M)^3$ and $t \in [0,T)$. Observe that such E and B are divergence-free. If u and v have support in $[0,T) \times (-M,M)$, then E and B fulfill periodic boundary conditions. Moreover, $(E,B) \in C^1$ satisfy (1.1) on $(-M,M)^3$ with the above material laws if and only if $(u,v) \in C^1$ solve

$$\partial_t a(u) = \partial_x v, \qquad \partial_t v = \partial_x u, \qquad (u(0), v(0)) = (u_0, v_0),$$

for $t \in [0,T)$ and $x \in \mathbb{R}$. This system can be rewritten as

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} + A(u,v)\partial_x \begin{pmatrix} u \\ v \end{pmatrix} = 0 \qquad \text{with} \;\; A(u,v) = \begin{pmatrix} 0 & -a'(u)^{-1} \\ -1 & 0 \end{pmatrix} \tag{1.35}$$

on $\mathbb{R}$. Here we assume that u takes values in S from (1.34). Since also $\partial_x u = \operatorname{curl} E$, the theorem thus follows from the next one-dimensional result.

The following proof uses a standard construction from Section 1.4 of [**36**]. However, it requires a rather detailed analysis to find a class of initial values for which we get the blow-up of $\partial_x u$ in L^2 instead of L^∞.

PROPOSITION 1.23. *Assume that (1.34) is true. Then there exist initial data $(u_0, v_0) \in C_c^1(\mathbb{R}, \mathbb{R}^2)$ and a C^1-solution (u, v) to (1.35) on $[0, T) \times \mathbb{R}$ for some $T \in (0, \infty)$ which is compactly supported and which satisfies $\|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})} \to \infty$ as $t \to T^-$.*

PROOF. 1) For $(u, v) \in S \times \mathbb{R}$, the matrix $A(u, v)$ has the eigenvalues and eigenvectors

$$\lambda_{1,2}(u, v) = \pm a'(u)^{-\frac{1}{2}}, \qquad w_{1,2}(u, v) = (\mp 1, a'(u)^{\frac{1}{2}}).$$

(Recall $S = (s_-, s_+)$, s_0 and q from (1.34).) These observations are a special case of the analysis in Section 3 of [**3**]. In the following we take $\lambda = \lambda_1$ and $w = w_1$ and drop the index 1. Fix $(\xi, \zeta) \in (s_0, s_+) \times \mathbb{R}$ such that

$$q(s) > 0 \qquad \text{for} \quad 0 < s \le \xi.$$

Observe that the interval $\xi - S = (\xi - s_+, \xi - s_-)$ contains $[0, \xi]$. The C^2-function $\phi : \xi - S \to S \times \mathbb{R}$

$$\phi_1(s) = \xi - s, \qquad \phi_2(s) = \zeta + \int_0^s a'(\xi - \tau)^{1/2} \, \mathrm{d}\tau,$$

solves the ordinary differential equation

$$\phi'(s) = w(\phi(s)), \quad s \in \xi - S, \qquad \phi(0) = (\xi, \zeta).$$

For later use, we note the identities

$$\nabla\lambda(\phi(s)) \cdot \phi'(s) = \nabla\lambda(\phi(s)) \cdot w(\phi(s)) = q(\xi - s), \qquad s \in \xi - S. \tag{1.36}$$

Let $\sigma_0 : \mathbb{R} \to [0, \xi]$ be C^1 and equal to ξ outside a compact set. There is a unique C^1-solution σ of the scalar partial differential equation

$$\begin{aligned} \partial_t \sigma(t, x) + \lambda(\phi(\sigma(t, x)))\partial_x \sigma(t, x) = 0, \qquad & t \ge 0, \ x \in \mathbb{R}, \\ \sigma(0, x) = \sigma_0(x), \qquad & x \in \mathbb{R}, \end{aligned} \tag{1.37}$$

on a bounded time interval $[0, \bar{t})$, where σ takes values in $\xi - S$. See e.g. Theorems 2.1 and 2.2 of [**36**]. We now define

$$\begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} = \phi(\sigma(t, x)).$$

It is easy to check that (u, v) is a C^1-solution of (1.35) on $[0, \bar{t}) \times \mathbb{R}$. We observe

$$\partial_x u = \phi_1'(\sigma)\partial_x \sigma = -\partial_x \sigma. \tag{1.38}$$

2) The method of characteristics yields the implicit formula

$$\begin{aligned} \sigma(t, x) &= \sigma_0(x - t\lambda(\phi(\sigma(t, x)))) = \sigma_0(y(t, x)), \\ y(t, x) &:= x - t\lambda(\phi(\sigma(t, x))) = x - ta'(\xi - \sigma(t, x))^{-1/2}, \end{aligned} \tag{1.39}$$

for the solution of (1.37) as long as

$$\begin{aligned} 1+&t\nabla\lambda(\phi(\sigma(t,x)))\cdot w(\phi(\sigma(t,x)))\sigma_0'(x-t\lambda(\phi(\sigma(t,x)))) \\ &= 1+t\sigma_0'(x-t\lambda(\phi(\sigma(t,x))))q(\xi-\sigma(t,x)) \quad > 0, \end{aligned} \tag{1.40}$$

see (1.36). Hence, σ is bounded. We now set

$$\gamma(t) \coloneqq \inf_{x\in\mathbb{R}} \sigma_0'(y(t,x))q(\xi-\sigma(t,x)) \quad \text{for} \quad t\in[0,\bar{t}).$$

Let $t_0 \geq 0$ be the supremum of $t\in[0,\bar{t})$ such that $\tau\gamma(\tau) > -1$ for all $\tau\in[0,t]$. In the following, we take $t\in[0,t_0)$ so that the inequality (1.40) is valid for all $x\in\mathbb{R}$. Equations (1.39) then imply

$$\begin{aligned} \partial_x\sigma(t,x) &= \sigma_0'(x-t\lambda(\phi(\sigma(t,x))))\Big(1-tq(\xi-\sigma(t,x))\partial_x\sigma(t,x)\Big), \\ \partial_x\sigma(t,x) &= \frac{\sigma_0'(y(t,x))}{1+tq(\xi-\sigma(t,x))\sigma_0'(y(t,x))}. \end{aligned}$$

In particular, $\partial_x\sigma$ is bounded on $[0,t_0-\delta]\times\mathbb{R}$ for each $\delta\in(0,t_0]$. The blow-up condition in Theorem 2.2 Annex of **[36]** (a variant of Theorem 1.19) thus yields $\bar{t}=t_0$. From formula (1.39) we further deduce $\partial_x\sigma(t,x)=\sigma_0'(y(t,x))\partial_x y(t,x)$ and therefore

$$\partial_x y(t,x) = \frac{1}{1+tq(\xi-\sigma(t,x))\sigma_0'(y(t,x))} > 0. \tag{1.41}$$

(In the case $\sigma_0'(y(t,x))=0$ the identity $\partial_x y(t,x)=1>0$ follows from (1.39).) Using also (1.39), we see that the map $x\mapsto y(t,x)$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$. This fact and (1.39) lead to the equation

$$\gamma(t) = \inf_{z\in\mathbb{R}} \sigma_0'(z)q(\xi-\sigma_0(z)) \eqqcolon \gamma_0.$$

3) We now fix a C^1-function $\sigma_0:\mathbb{R}\to[0,\xi]$ which is equal to ξ outside some compact set and satisfies

$$\sigma_0(0)=\xi-s_0, \qquad \sigma_0'(0)=\min_{z\in\mathbb{R}}\sigma_0'(z)<0.$$

In view of (1.34), we can determine

$$\gamma_0 = \sigma_0'(0)q(s_0) \quad \text{and} \quad t_0 = -\frac{1}{\gamma_0}. \tag{1.42}$$

Substituting $z=y(t,x)$ and using (1.41), we infer from (1.39) the identities

$$\begin{aligned} \|\partial_x\sigma(t,\cdot)\|_2^2 &= \int_{\mathbb{R}} |\partial_x\sigma(t,x)|^2\,\mathrm{d}x = \int_{\mathbb{R}} |\sigma_0'(y(t,x))\partial_x y(t,x)|^2\,\mathrm{d}x \\ &= \int_{\mathbb{R}} \frac{|\sigma_0'(z)|^2}{1+tq(\xi-\sigma_0(z))\,\sigma_0'(z)}\,\mathrm{d}z. \end{aligned}$$

Since q has a global maximum at s_0 while σ_0' has a global minimum at 0, we obtain the expansions

$$q(s)=q(s_0)-o_+(s-s_0), \qquad \sigma_0'(z)=\sigma_0'(0)+o_+(z), \qquad \sigma_0(z)=\xi-s_0+O(z),$$

where $o_+(z)$ denotes any nonnegative function with the property $o_+(z)/z \to 0$ as $z \to 0$. Hence, (1.42) yields

$$\begin{aligned} 1 + tq(\xi - \sigma_0(z))\,\sigma_0'(z) &= 1 + t\gamma_0 + t[q(s_0)o_+(z) + o_+(z)\,|\sigma_0'(0)| - o_+(z)^2] \\ &= 1 + t\gamma_0 + to_+(z) \end{aligned}$$

for small $|z|$. Fix a number $\delta_0 > 0$ such that the above identity is true and $|\sigma_0'(z)|^2 \geq \frac{1}{2}|\sigma_0'(0)|^2 =: c_0$ if $|z| \leq \delta_0$. For each $\epsilon > 0$ there exists a radius $\delta \in (0, \delta_0)$ with $0 \leq o_+(z) \leq \epsilon\delta$ for $z \in (-\delta, \delta)$. We can then estimate

$$\|\partial_x \sigma(t,\cdot)\|_2^2 \geq \int_{-\delta}^{\delta} \frac{|\sigma_0'(z)|^2}{1 + t\gamma_0 + to_+(z)}\,\mathrm{d}z \geq \int_{-\delta}^{\delta} \frac{c_0}{1 + t\gamma_0 + t\epsilon\delta}\,\mathrm{d}z = \frac{2c_0\delta}{1 + t\gamma_0 + t\epsilon\delta}.$$

Because of $t_0 = -1/\gamma_0 =: T$ in (1.42), it follows

$$\liminf_{t \to T^-} \|\partial_x \sigma(t,\cdot)\|_2^2 \geq \frac{2c_0}{T\epsilon}.$$

Since $\epsilon > 0$ is arbitrary, equation (1.38) finally implies that

$$\liminf_{t \to T^-} \|\partial_x u(t,\cdot)\|_2^2 = \liminf_{t \to T^-} \|\partial_x \sigma(t,\cdot)\|_2^2 = +\infty.$$

4) Note that $\sigma(t,x) = \sigma_0(y(t,x)) = \xi$ if $|y|$ is large enough. This fact holds for some $x_0 > 0$ and all $t \in [0,T)$ and $|x| \geq x_0$ because of (1.39) and the strict positivity of a' on $[0,\xi]$. So $u = \xi - \sigma$ has compact support. Fixing

$$\zeta = -\int_0^{\xi} a'(\xi - \tau)^{1/2}\,\mathrm{d}\tau,$$

also the function

$$v = \zeta + \int_0^{\sigma} a'(\xi - \tau)^{1/2}\,\mathrm{d}\tau$$

has compact support. □

CHAPTER 2

Local wellposedness on a domain

In this chapter we extend the results from the previous one to linear and quasilinear Maxwell systems on a spatial domain G, endowed with boundary conditions. The general theory of symmetric hyperbolic systems is much more sophisticated in this case. It uses Sobolev spaces of higher order and with weights encoding a loss of derivatives in normal direction, see [**26**] or [**49**]. Fortunately the Maxwell equations have a special structure which allows us to derive analogous theorems as on $\mathbb{R}^3$ using a similar approach. However, already in the half-space case $G = \mathbb{R}^3_+ := \{x \in \mathbb{R}^3 | x_3 > 0\}$ many new difficulties arise, which we describe and solve below (sketching or omitting some technical steps). The general case is treated via localization arguments and thus reduced to hyperbolic problems on $\mathbb{R}^3_+$. They still resemble the Maxwell system, but the resulting coefficients A_j, $j \in \{1, 2, 3\}$, are far more complicated than A_j^{co}. Here we can only indicate how one deals with the new situation. In a first section we start with a derivation of the boundary conditions and a discussion of the relevant trace operator and the compatibility conditions.

2.1. The Maxwell system on a domain

We continue to study the Maxwell equations

$$\partial_t D = \operatorname{curl} H - J_e, \qquad \partial_t B = -\operatorname{curl} E, \qquad t \geq 0,\ x \in G, \tag{2.1}$$

for $t \geq 0$ and $x \in G$, where $G \subseteq \mathbb{R}^3$ is open and bounded with a smooth boundary or $G = \mathbb{R}^3_+ = \mathbb{R}^2 \times \mathbb{R}_+$. As before, we can define solutions to these equations in $C(\overline{J}, L^2_x)$. Observe that the solutions still satisfy Gauß' laws (1.2). Below we will equip the system again with the material laws (1.3) and (1.4), or their linear variants. However, the derivation of the boundary conditions is independent of these laws.

We first establish the interface conditions for (2.1), arguing a bit informal. Let Σ be a surface in G, which is given by a chart $\varphi : U \to V$ with $\varphi(\Sigma) = V_0 \times \{0\}$. Given a point $x \in \Sigma$, we may choose φ with $\varphi'(x) = I$. Set $\psi = \varphi^{-1}$ and $U_\pm = \psi(V \cap \mathbb{R}^3_\pm)$ with $\mathbb{R}^3_- = \mathbb{R}^2 \times \mathbb{R}_-$. We equip Σ with the unit normal ν_Σ pointing into U_+, whereas ν and $\nu_\pm$ are the outer unit normal of U and $U_\pm$, respectively. Moreover, let $S \subseteq V_0$ be a line segment with direction p and $a > 0$ such that $Q = S \times [-a, a] \subseteq V$. Let ∂Q be oriented counter-clockwise and choose the normal n to Q with $\det[\mathrm{n}, p, e_3] > 0$. The surface $\Gamma = \psi(S \times [-a, a]) \subseteq U$ shall carry the induced orientation; i.e., its boundary (with a parametrization $\gamma = \gamma(\theta)$) winds positively around the unit normal ν_Γ of Γ. Note that ν_Γ is perpendicular to ν_Σ at a point x with $\varphi'(x) = I$. Let $\Gamma_\pm = \psi(Q \cap \mathbb{R}^3_\pm)$ be oriented accordingly. For a function f on $U_\pm$ we denote its trace on Σ by $f_\pm$ (assuming that it exists) and its jump across Σ by $[f] = f_+ - f_-$.

It is better to start from the more fundamental integral versions of the Maxwell equations and the Gauß' laws (1.2), namely

$$\begin{aligned} &\int_{\partial\Gamma} H\cdot \mathrm{d}\vec{s} = \int_\Gamma (\partial_t D + J_e)\cdot \nu_\Gamma \,\mathrm{d}\sigma, \qquad \int_{\partial\Gamma} E\cdot \mathrm{d}\vec{s} = -\int_\Gamma \partial_t B\cdot \nu_\Gamma\,\mathrm{d}\sigma,\\ &\int_{\partial U} D\cdot\nu\,\mathrm{d}\sigma = \int_U \rho_e\,\mathrm{d}x, \qquad \int_{\partial U} B\cdot \nu\,\mathrm{d}\sigma = 0, \end{aligned} \tag{2.2}$$

where we require that these traces and integrals exist. (Here $H\cdot \mathrm{d}\vec{s} = (H\circ\gamma)\cdot\gamma'\,\mathrm{d}\theta$ and σ is the surface measure.) If the fields belong to $\mathcal{H}^1(U)$, say, these equations follow from (2.1) and (1.2) by means of Stokes' and Gauß' theorems. To show the converse implication, after applying Stokes and Gauß again, one divides the integrals by the volume of U and Γ, respectively, and lets them tend to 0. (Note that ν_Γ can be any unit vector in $\mathbb{R}^3$ if one varies Σ and Γ.)

Let $\rho_\pm = \rho_e\restriction_{U_\pm}$ and $J_\pm = J_e\restriction_{U_\pm}$. We also allow for surface charges ρ_Σ and surface currents J_Σ concentrated on Σ, where $J_\Sigma \perp \nu_\Sigma$. Let D be regular on $U_\pm$ so that the jump $[D\cdot\nu_\Sigma] = [D]\cdot\nu_\Sigma$ is integrable on Σ. We then infer from (2.2) on U that

$$\begin{aligned} \int_{U_+}\rho_+\,\mathrm{d}x + \int_{U_-}\rho_-\,\mathrm{d}x + \int_\Sigma \rho_\Sigma\,\mathrm{d}\sigma &= \int_U \rho_e\,\mathrm{d}x = \int_{\partial U} D\cdot\nu\,\mathrm{d}\sigma\\ &= \int_{\partial U_+} D\cdot\nu_+\,\mathrm{d}\sigma + \int_{\partial U_-} D\cdot \nu_-\,\mathrm{d}\sigma + \int_\Sigma [D\cdot\nu_\Sigma]\,\mathrm{d}\sigma. \end{aligned}$$

By (2.2) on $U_\pm$, the first two terms on both sides cancel. We can replace V_0 by subsets V_0'. Dividing by the area of Σ and shrinking V_0', we see that $\rho_\Sigma = [D\cdot\nu_\Sigma]$ on Σ. In the same way one shows that $[B\cdot\nu_\Sigma] = 0$. Similarly, (2.2) also yields

$$\begin{aligned} \int_{\partial\Gamma_+} H\cdot \mathrm{d}\vec{s} + \int_{\partial\Gamma_-} H\cdot\mathrm{d}\vec{s} - \int_{\Gamma\cap\Sigma} [H]\cdot \mathrm{d}\vec{s} &= \int_{\partial\Gamma} H\cdot\mathrm{d}\vec{s}\\ = \int_{\Gamma_+} (\partial_t D + J_e)\cdot \nu_{\Gamma_+}\,\mathrm{d}\sigma + \int_{\Gamma_-} &(\partial_t D + J_e)\cdot\nu_{\Gamma_-}\,\mathrm{d}\sigma + \int_{\Gamma\cap\Sigma} J_\Sigma\cdot\nu_\Gamma\,\mathrm{d}s \end{aligned}$$

for $\mathrm{d}s = |\gamma'|\,\mathrm{d}\theta$. Choosing $\varphi'(x) = I$, the unit tangent vector of $\Gamma\cap\Sigma$ at x is equal to $\nu_\Sigma\times\nu_\Gamma$. We then deduce as above that $-J_\Sigma\cdot\nu_\Gamma = [H]\cdot(\nu_\Sigma\times\nu_\Gamma) = \nu_\Gamma\cdot[H\times\nu_\Sigma]$, and hence $[H\times\nu_\Sigma] = -J_\Sigma$ as n_Γ is an arbitrary tangent vector of Σ. Analogously one shows that $[E\times\nu_\Sigma] = 0$. We summarize the *interface conditions*

$$[E\times\nu_\Sigma] = 0, \qquad [D\cdot\nu_\Sigma] = \rho_\Sigma, \qquad [B\cdot\nu_\Sigma] = 0, \qquad [H\times\nu_\Sigma] = -J_\Sigma \tag{2.3}$$

for fields being sufficiently regular on $U_\pm$. (Cf. §I.4.2.4 in [**15**] or §1.7 in [**23**].)

As for Gauß' laws (1.2), the equations for D and B are redundant for solutions to (2.1) satisfying the interface conditions for E and H. More precisely, one has $[B(t)\cdot\nu_\Sigma] = [B(0)\cdot\nu_\Sigma]$, and $\rho_\Sigma(t)$ can be computed in terms of $[D(0)\cdot\nu_\Sigma] = \rho_\Sigma(0)$, J_e and J_Σ. See Lemma 8.1 in [**48**] and also §I.4.2.4 in [**15**] or our Lemma 2.4.

Arguably the basic set-up for the Maxwell system is $\mathbb{R}^3$ endowed with different material laws on subsets G and $\mathbb{R}^3\setminus\overline{G}$ (e.g., having vacuum $D = E$ and $H = B$ on $\mathbb{R}^3\setminus\overline{G}$) and equipped with initial conditions and interface conditions for

E and H on $\Sigma = \partial G$. In fact, one can extend the local wellposedness theory discussed in this chapter to this setting, see [48]. Here we treat the simpler situation that the traces at ∂G of the fields on $\mathbb{R}^3 \setminus \overline{G}$ are assumed to be 0. For the electric fields, this is reasonable in the case of a perfect conductor on $\mathbb{R}^3 \setminus \overline{G}$ which refers to the limit case of infinite conductivity σ so that $E = \frac{1}{\sigma} J_e = 0$ in Ohm's law, see §I.4.2.4+6 in [15] or §7.12 in [23]. In this setting we will derive a local wellposedness theory with the boundary condition of a *perfect coductor*

$$E \times \nu = 0 \qquad \text{on } \ \partial G. \tag{2.4}$$

It is usually combined with the condition

$$B \cdot \nu = 0 \qquad \text{on } \ \partial G, \tag{2.5}$$

which however turns out to be true if it holds at time 0 by Lemma 2.4.

Before we continue, we have to explain the meaning of the above equations for functions $E, B \in C(\overline{J}, L^2(G, \mathbb{R}^3))$ solving the Maxwell system. To this end, we first recall several known results about traces, see [1] and [16], for instance. Let $U \subseteq \mathbb{R}^m$ be an open subset with a Lipschitz boundary given by local graphs which yield a covering of U by finitely many charts $\varphi_j : U_j \to V_j$ with inverses ψ_j and parametrizations $F_j = \psi_j \restriction_{\{y_m = 0\}}$. Let ν be its outer unit normal. For $s \geq 0$ we have the fractional Sobolev spaces $\mathcal{H}^s(\mathbb{R}^m)$ consisting of $v \in L^2(\mathbb{R}^m)$ such that $|\xi|^s \mathcal{F} v$ belongs to $L^2(\mathbb{R}^m)$, where $|\xi|^s$ stands for the map $\xi \mapsto |\xi|^s$. They are endowed with the norm given $\|v\|_s^2 = \|v\|_2^2 + \||\xi|^s \mathcal{F} v\|_2^2$. Their dual spaces are denoted by $\mathcal{H}^{-s}(\mathbb{R}^m)$. The space $\mathcal{H}^s(U)$ contains the restrictions $v\restriction_U$ for $v \in \mathcal{H}^s(\mathbb{R}^m)$. For an open subset $\Gamma \subseteq \partial U$ and $s \in (0,1)$, we define

$$\mathcal{H}^s(\Gamma) = \big\{ v \in L^2(\Gamma, \sigma) \,\big|\, \forall j: \ v \circ F_j \in \mathcal{H}^s(\varphi_j(\Gamma \cap V_j)) \big\}.$$

Again we let $\mathcal{H}^{-s}(\Gamma)$ be the dual space. If ∂U is C^k, one can take here $s \in [0, k]$.

It is known that the trace operator $\operatorname{tr} : v \mapsto v\restriction_{\partial U}$ (defined on $\mathcal{H}^1(U) \cap C(\overline{U})$) extends to a continuous and surjective map from $\mathcal{H}^1(U)$ to $\mathcal{H}^{\frac{1}{2}}(\partial U)$. Its kernel is $\mathcal{H}_0^1(U)$. Here and in the treatment of div below, U can be a Lipschitz domain in $\mathbb{R}^m$. We discuss analogous results for the traces used in (2.4) and (2.5). To this end, we use the spaces

$$\begin{aligned} \mathcal{H}(\operatorname{div}, U) = \mathcal{H}(\operatorname{div}) &= \{ v \in L^2(U)^m \mid \operatorname{div} v \in L^2(U) \}, \\ \mathcal{H}(\operatorname{curl}, U) = \mathcal{H}(\operatorname{curl}) &= \{ v \in L^2(U, \mathbb{R}^3) \,\big|\, \operatorname{curl} v \in L^2(U, \mathbb{R}^3) \} \end{aligned}$$

endowed with their canonical norms. The closures of test functions in these spaces are denoted by $\mathcal{H}_0(\operatorname{div})$ and $\mathcal{H}_0(\operatorname{curl})$, respectively.

To work with curl, we need its basic integration by parts formula. We first treat a weak version. Let $w \in L^2(U)$ and $v \in C_c^\infty(U)$. With distributional derivatives we compute,

$$\begin{aligned} \int_U w \cdot \operatorname{curl} v \, \mathrm{d}x &= \int_U \big(w_1(\partial_2 v_3 - \partial_3 v_2) + w_2(\partial_3 v_1 - \partial_1 v_3) + w_3(\partial_1 v_2 - \partial_2 v_1) \big) \mathrm{d}x \\ &= \langle v, \operatorname{curl} w \rangle_{C_c^\infty(U)} \end{aligned}$$

By density, we deduce

$$\int_U w\cdot \operatorname{curl} v \,\mathrm{d}x = \langle v, \operatorname{curl} w\rangle_{\mathcal{H}_0^1(U)},$$
$$\int_U u\cdot \operatorname{curl} v \,\mathrm{d}x = \int_U \operatorname{curl} u\cdot v \,\mathrm{d}x \tag{2.6}$$

for $w \in L^2(U)$, $u \in \mathcal{H}(\mathrm{curl})$, and $v \in \mathcal{H}_0^1(U)$. To handle nonzero boundary terms, we express curl by means of div. Let $u, v \in \mathcal{H}^1(U)$. We set

$$\hat{u}_1 = (0, u_3, -u_2), \quad \hat{u}_2 = (-u_3, 0, u_1), \quad \hat{u}_3 = (u_2, -u_1, 0).$$

The divergence theorem and a straightforward computation then imply

$$\begin{aligned}\int_U \operatorname{curl} u\cdot v\,\mathrm{d}x &= \int_U \big(v_1 \operatorname{div}\hat{u}_1 + v_2\operatorname{div}\hat{u}_2 + v_3 \operatorname{div}\hat{u}_3\big)\,\mathrm{d}x \\ &= \sum_{j=1}^3 \Big(\int_U -\hat{u}_j\cdot\nabla v_j\,\mathrm{d}x + \int_{\partial G} \nu\cdot \hat{u}_j\, v_j\,\mathrm{d}\sigma\Big) \\ &= \int_U u\cdot \operatorname{curl} v\,\mathrm{d}x + \int_{\partial U} (\nu\times u)\cdot v\,\mathrm{d}\sigma \\ &= \int_U u\cdot \operatorname{curl} v\,\mathrm{d}x + \int_{\partial U} u\cdot (v\times \nu)\,\mathrm{d}\sigma.\end{aligned} \tag{2.7}$$

We can show the completeness of $\mathcal{H}(\mathrm{curl})$. Indeed, if (u_n) is Cauchy in $\mathcal{H}(\mathrm{curl})$, then it converges to some u and $\operatorname{curl} u_n$ to some w in L^2. For $v \in \mathcal{H}_0^1(U)$, formula (2.6) now yields

$$\int_U w\cdot v\,\mathrm{d}x = \lim_{n\to\infty}\int_U \operatorname{curl} u_n\cdot v\,\mathrm{d}x = \lim_{n\to\infty}\int_U u_n\cdot\operatorname{curl} v\,\mathrm{d}x = \int_U u\cdot \operatorname{curl} v\,\mathrm{d}x.$$

This means that $\operatorname{curl} u = w$ in $\mathcal{H}^{-1}(U)$ and thus in $L^2(U)$. The completeness of $\mathcal{H}(\mathrm{div})$ is shown analogously.

We now state the basic trace theorems for $\mathcal{H}(\mathrm{div})$ and $\mathcal{H}(\mathrm{curl})$. To this end we define the *normal trace* $\mathrm{tr}_{\mathrm{no}} : v \mapsto (v\cdot\nu)\restriction_{\partial U}$ on $\mathcal{H}(\mathrm{div}) \cap C(\overline{U})$ and the *tangential trace* $\mathrm{tr}_{\mathrm{ta}} : v \mapsto (v\times\nu)\restriction_{\partial U}$ on $\mathcal{H}(\mathrm{curl})\cap C(\overline{U})$.

THEOREM 2.1. *Let $U \subseteq \mathbb{R}^m$ be open having a Lipschitz boundary as described above. Then the following assertions are true.*

a) The space $C_c^\infty(\overline{U}) := C_c^\infty(\mathbb{R}^m)\restriction_{\overline{U}}$ is dense in $\mathcal{H}(\mathrm{div})$.

b) The normal trace extends to a continuous and surjective map $\mathrm{tr}_{\mathrm{no}} : \mathcal{H}(\mathrm{div}) \to \mathcal{H}^{-\frac{1}{2}}(\partial U)$ with kernel $\mathcal{H}_0(\mathrm{div})$.

c) For $v \in \mathcal{H}(\mathrm{div})$ and $\varphi \in \mathcal{H}^1(U)$, we have Gauß' formula

$$\int_U v\cdot\nabla\varphi\,\mathrm{d}x = -\int_U \varphi \operatorname{div} v\,\mathrm{d}x + \langle \mathrm{tr}\,\varphi, \mathrm{tr}_{\mathrm{no}}\, v\rangle_{\mathcal{H}^{1/2}(\partial U)}. \tag{2.8}$$

THEOREM 2.2. *Let $U \subseteq \mathbb{R}^3$ be open having a Lipschitz boundary as described above. Then the following assertions are true.*

a) The space $C_c^\infty(\overline{U})$ is dense in $\mathcal{H}(\mathrm{curl})$.

b) The tangential trace has a continuous extension $\mathrm{tr}_{\mathrm{ta}} : \mathcal{H}(\mathrm{curl}) \to \mathcal{H}^{-\frac{1}{2}}(\partial U)$ with kernel $\mathcal{H}_0(\mathrm{curl})$.

c) For $v \in \mathcal{H}(\mathrm{curl})$ and $\varphi \in \mathcal{H}^1(U,\mathbb{R}^3)$ we have

$$\int_U v \cdot \operatorname{curl} \varphi \,\mathrm{d}x = \int_U \operatorname{curl} v \cdot \varphi \,\mathrm{d}x + \langle \operatorname{tr} \varphi, \operatorname{tr}_{\mathrm{ta}} v\rangle_{\mathcal{H}^{1/2}(\partial U)}. \tag{2.9}$$

If $v \in \mathcal{H}_0(\mathrm{curl})$ and $\varphi \in L^2(U,\mathbb{R}^3)$, we obtain $\mathcal{H}_0(\mathrm{curl})^* \hookrightarrow \mathcal{H}^{-1}(U,\mathbb{R}^3)$ and

$$\int_U \operatorname{curl} v \cdot \varphi \,\mathrm{d}x = \langle v, \operatorname{curl} \varphi\rangle_{\mathcal{H}_0(\mathrm{curl})}. \tag{2.10}$$

In Theorem 2.4 in [**10**] the range of $\operatorname{tr}_{\mathrm{ta}}$ is determined if $\partial U \in C^2$, say. We only show the results for curl. Those for div are similarly proven, see Theorem IX.1.1 in [**16**]. The core step of the prof is the density statement in a), which relies on the following description of $\mathcal{H}_0(\mathrm{curl})$.

LEMMA 2.3. *Let $U \subseteq \mathbb{R}^3$ be open having a Lipschitz boundary as described above. Assume that $u \in \mathcal{H}(\mathrm{curl})$ satisfies*

$$\int_U u \cdot \operatorname{curl} \phi \,\mathrm{d}x = \int_U \operatorname{curl} u \cdot \phi \,\mathrm{d}x$$

for all $\phi \in C_c^\infty(\overline{U})^3$. Then u belongs to $\mathcal{H}_0(\mathrm{curl})$.

PROOF. We proceed in several steps to approximate the given map u in $\mathcal{H}(\mathrm{curl}, U)$ by test functions in G. For the approximation, we first use the assumption to extend u to an element in $\mathcal{H}(\mathrm{curl}, \mathbb{R}^3)$.

1) Set $v = \operatorname{curl} u \in L^2(U)$. Let $\tilde{u}, \tilde{v} \in L^2(\mathbb{R}^3)$ be the 0-extensions of u and v, respectively. Take $\varphi \in C_c^\infty(\mathbb{R}^3)^3$. The assumption then yields

$$\int_{\mathbb{R}^3} \tilde{u} \cdot \operatorname{curl} \varphi \,\mathrm{d}x = \int_G u \cdot \operatorname{curl} \varphi \,\mathrm{d}x = \int_G \operatorname{curl} u \cdot \varphi \,\mathrm{d}x = \int_{\mathbb{R}^3} \tilde{v} \cdot \varphi \,\mathrm{d}x.$$

By density, this equation is true for $\varphi \in \mathcal{H}^1(\mathbb{R}^3)^3$ so that $\operatorname{curl} \tilde{u} = \tilde{v}$ in $\mathcal{H}^{-1}(\mathbb{R}^3)$, and hence $\tilde{u}$ belongs to $\mathcal{H}(\mathrm{curl}, \mathbb{R}^3)$.

2) We next restrict the problem to compactly supported $\tilde{u}$ if G is unbounded. Take a cut-off function $\chi \in C_c^\infty(\mathbb{R}^3)$ with $0 \le \chi \le 1$, $\chi = 1$ on $B(0,1)$, and support in $B(0,2)$. For $a > 0$ the map $\chi_a(x) \coloneqq \chi(\frac{1}{a}x)$ satisfies $|\chi_a| \le 1$, $|\nabla \chi_a| \le \|\nabla \chi\|_\infty / a$, $\operatorname{supp} \chi_a \subseteq B(0, 2a)$, and tends pointwise to $\mathbb{1}$ as $a \to \infty$. So $\chi_a \tilde{u}$ converges to $\tilde{u}$ in $L^2(\mathbb{R}^3)$ by dominated convergence. We further obtain

$$\operatorname{curl}(\chi_a \tilde{u}) = \chi_a \operatorname{curl} \tilde{u} + \nabla \chi_a \times \tilde{u} \;\longrightarrow\; \operatorname{curl} \tilde{u}$$

in $L^2(\mathbb{R}^3)$ as $a \to \infty$. Hence, the restriction of $\chi_a \tilde{u}$ to U tends to u in $\mathcal{H}(\mathrm{curl}, U)$. It is thus enough to show that $\chi_a \tilde{u}\restriction_U$ belongs to $\mathcal{H}_0(\mathrm{curl}, U)$ for all a. Therefore, we assume that $\tilde{u}$ has compact support without loss of generality.

3) In this main step, we require U to be strictly starlike; i.e., θU is contained in U for all $\theta \in [0, 1)$. For maps $v \in L^2(\mathbb{R}^3)$ we set $D_\theta v = v(\theta\, \cdot)$ for $\theta \in [\frac{1}{2}, 1)$. The operators D_θ are uniformly bounded on $L^2(\mathbb{R}^3)$ and converge strongly to I as $\theta \to 1$ on $C_c(\mathbb{R}^3)$ and thus on $L^2(\mathbb{R}^3)$. The function $\tilde{u}_\theta = D_\theta \tilde{u}$ has a (compact) support S_θ in U. Since $\partial_j \tilde{u}_{\theta,k}(x) = \theta(\partial_j \tilde{u}_k)(\theta x)$, we obtain

$$\operatorname{curl} \tilde{u}_\theta = \theta D_\theta \operatorname{curl} \tilde{u} \;\longrightarrow\; \operatorname{curl} \tilde{u}$$

in $L^2(\mathbb{R}^3)$ as $\theta \to 1$. Hence, the restrictions of $\tilde{u}_\theta$ to U tend to u in $\mathcal{H}(\mathrm{curl}, U)$.

4) Let $\delta < \operatorname{dist}(S_\theta, \partial U)$. Then the mollified maps $R_\delta \tilde{u}_\theta{\restriction}_U$ belong to $C_c^\infty(U)$ and tend to $\tilde{u}_\theta$ in $L^2(U)$. Moreover, the restrictions of $\operatorname{curl} R_\delta \tilde{u}_\theta = R_\delta \operatorname{curl} \tilde{u}_\theta$ to U converge to $\operatorname{curl} \tilde{u}_\theta$ in $L^2(U)$. Combined with step 3), we have shown the lemma for strictly starlike U.

5) The general case can be treated by localization. See the proof of Lemma IX.1.1 in [**16**]. □

We can now show the trace theorem for curl.

Proof of Theorem 2.2. 1) To show the density statement, take $u \in \mathcal{H}(\operatorname{curl})$ such that

$$\forall\, \phi \in C_c^\infty(\overline{U}): \quad 0 = \langle u, \phi \rangle_{\operatorname{curl}} := \int_U \big(u \cdot \phi + \operatorname{curl} u \cdot \operatorname{curl} \phi\big)\, \mathrm{d}x.$$

Using (2.6), we infer $\operatorname{curl}\operatorname{curl}\phi = -u$ in $\mathcal{H}^{-1}(U)$ and hence $v := \operatorname{curl} u$ belongs to $\mathcal{H}(\operatorname{curl})$ with $\operatorname{curl} v = -u$. Lemma 2.3 now implies that v is an element of $\mathcal{H}_0(\operatorname{curl})$. We can thus approximate it by $\psi_n \in C_c^\infty(U)^3$ in $\mathcal{H}(\operatorname{curl})$. For all $w \in \mathcal{H}(\operatorname{curl})$, formula (2.6) implies

$$\begin{aligned} \langle u, w \rangle_{\operatorname{curl}} &= \langle -\operatorname{curl} v, w \rangle_{L^2} + \langle v, \operatorname{curl} w \rangle_{L^2} \\ &= \lim_{n\to\infty} \big(\langle -\operatorname{curl} \psi_n, w \rangle_{L^2} + \langle \psi_n, \operatorname{curl} w \rangle_{L^2}\big) = 0, \end{aligned}$$

so that $u = 0$ and assertion a) is true.

2) We extend $\operatorname{tr}_{\mathrm{ta}}$ to $\mathcal{H}(\operatorname{curl})$ by means of (2.7) for $v \in C_c^\infty(\overline{U})$ and $\varphi \in \mathcal{H}^1(U)$. This formula yields

$$\Big| \int_{\partial U} (v \times \nu) \cdot \varphi \, \mathrm{d}\sigma \Big| \le \|v\|_2 \, \|\operatorname{curl} \varphi\|_2 + \|\operatorname{curl} v\|_2 \, \|\varphi\|_2 \le c\, \|v\|_{\mathcal{H}(\operatorname{curl})} \, \|\varphi\|_{\mathcal{H}^1}.$$

Let $R: \mathcal{H}^{\frac12}(\partial U) \to \mathcal{H}^1(U)$ be a right inverse of tr and ϱ be its norm. Writing $Y = \mathcal{H}^{\frac12}(\partial U)$ and $\psi = \varrho R \phi$, we then estimate

$$\begin{aligned} \|\operatorname{tr}_{\mathrm{ta}} v\|_{\mathcal{H}^{-\frac12}(\partial U)} &= \sup_{\|\phi\|_Y \le 1} |\langle \phi, \operatorname{tr}_{\mathrm{ta}} v \rangle_Y| \le \varrho \sup_{\|\varrho^{-1}\psi\|_{\mathcal{H}^1(U)} \le 1} |\langle \operatorname{tr} \tfrac{1}{\varrho}\psi, \operatorname{tr}_{\mathrm{ta}} v \rangle_Y| \\ &= \varrho \sup_{\|\varrho^{-1}\psi\|_{\mathcal{H}^1(U)} \le 1} |\langle \operatorname{tr} \tfrac{1}{\varrho}\psi, \operatorname{tr}_{\mathrm{ta}} v \rangle_{L^2}| \le c\varrho\, \|v\|_{\mathcal{H}(\operatorname{curl})}. \end{aligned}$$

We can thus extend $\operatorname{tr}_{\mathrm{ta}}$ to a continuous map from $\mathcal{H}(\operatorname{curl})$ to $\mathcal{H}^{-\frac12}(\partial U)$

3) By continuity and density, $\mathcal{H}_0(\operatorname{curl})$ is contained in the kernel of $\operatorname{tr}_{\mathrm{ta}}$ and (2.9) follows from (2.7). Let $u \in \mathrm{N}(\operatorname{tr}_\tau)$. For $\varphi \in \mathcal{H}^1(U)$, formula (2.9) yields

$$\int_U u \cdot \operatorname{curl} \varphi \, \mathrm{d}x = \int_U \operatorname{curl} u \cdot \varphi \, \mathrm{d}x,$$

so that u belongs to $\mathcal{H}_0(\operatorname{curl})$. Hence, also b) is shown. Let $\varphi \in L^2(U)$. For $v \in \mathcal{H}_0^1(U)$ we define the functional $\Phi(v) = \langle v, \operatorname{curl} \varphi \rangle_{\mathcal{H}_0^1(U)}$. Since $\mathcal{H}_0^1(U)$ is dense in $\mathcal{H}_0(\operatorname{curl})$, equation (2.6) implies that Φ can be extended to $\mathcal{H}_0(\operatorname{curl})^*$, so that curl actually maps $L^2(U)$ into $\mathcal{H}_0(\operatorname{curl})^* \hookrightarrow \mathcal{H}^{-1}(U)$ and satisfies (2.10). □

These results justify the boundary condition (2.5) in view of (1.2), but not yet (2.4) since we only require $u = (E, H) \in C(\overline{J}, L^2_x)$. We first fix our assumptions for the linear problem $Lu = \sum_{j=0}^3 A_j \partial_j u + Du = f$ on G as

$$\begin{aligned} &A_0 \in W^{1,\infty}_{t,x} = W^{1,\infty}(J \times G, \mathbb{R}^{6\times 6}), \quad A_0 = A_0^\top \geq \eta I > 0, \quad J = (0, T), \\ &A_j = A_j^{\mathrm{co}} \ \text{ for } \ j \in \{1, 2, 3\}, \quad D \in L^\infty_{t,x} = L^\infty(J \times G, \mathbb{R}^{6\times 6}), \\ &u_0 \in L^2(G, \mathbb{R}^6) = L^2_x, \quad f \in L^2_{t,x} = L^2(J \times G, \mathbb{R}^6). \end{aligned} \tag{2.11}$$

(The matrices A_j^{co} were defined in (1.5).) Let $Lu = f$ for $u \in C(\overline{J}, L^2_x)$. We thus obtain

$$\sum_{j=0}^{3} \partial_j (A_j u) = f - Du + \partial_t A_0 \, u \ \in \ L^2_{t,x}.$$

By Theorem 2.1, the function $\sum_{j=0}^3 \mathrm{n}_j A_j u$ has a trace in $\mathcal{H}^{-1/2}(\partial(J \times G))$, where n is the outer unit normal of $J \times G$. Restricting to the subset $\{0\} \times G$ with $\mathrm{n} = -e_0$ this gives a meaning to the initial condition $u(0) = u_0$ as $A_0(0)$ is Lipschitz and invertible. On the lateral boundary $J \times \partial G$ with $\mathrm{n} = (0, \nu)$, by means of the comments before (1.5) we infer that $(-\nu \times u^2, \nu \times u^1)$ has a trace in $\mathcal{H}^{-1/2}(J \times \partial G)$, and so (2.4) for $E = u^1$ is well defined. We denote the latter trace also by $\mathrm{tr}_{\mathrm{ta}}$. See §2.1 in **[51]** for a detailed exposition in which several basic properties are shown that are used below without further notice.

We now check that condition (2.5) is preserved for $\mathcal{H}^1$-solutions of the Maxwell system with (2.4).

Lemma 2.4. *Let $B, E \in C^1(\overline{J}, L^2_x) \cap C(\overline{J}, \mathcal{H}^1_x)$ satisfy $\partial_t B = -\operatorname{curl} E$ and $\mathrm{tr}_{\mathrm{ta}} E = 0$. Then $\mathrm{tr}_{\mathrm{no}} B(t) = \mathrm{tr}_{\mathrm{no}} B(0)$ for all $t \in \overline{J}$.*

Proof. Let $t \in \overline{J}$ and $\varphi \in \mathcal{H}^2(G)$. The assumption implies $\operatorname{div} \partial_t B = 0$ so that $\mathrm{tr}_{\mathrm{no}} \partial_t B(t)$ exists in $\mathcal{H}^{-\frac{1}{2}}(\partial G)$, and the same is true for $\operatorname{curl} E(t)$. Using also (2.8) and (2.9), we thus obtain

$$\begin{aligned} \partial_t \langle B(t) \cdot \nu, \varphi \rangle_{L^2(\partial G)} &= \langle \mathrm{tr}_{\mathrm{no}} (\partial_t B)(t), \varphi \rangle_{\mathcal{H}^{-1/2}(\partial G)} = \langle -\,\mathrm{tr}_{\mathrm{no}} \operatorname{curl} E(t), \varphi \rangle_{\mathcal{H}^{-1/2}(\partial G)} \\ &= -\int_G \operatorname{div} \operatorname{curl} E(t) \, \varphi \, \mathrm{d}x - \int_G \operatorname{curl} E(t) \cdot \nabla \varphi \, \mathrm{d}x \\ &= -\int_G E(t) \cdot \operatorname{curl} \nabla \varphi \, \mathrm{d}x + \langle \mathrm{tr}_{\mathrm{ta}} E(t), \nabla \varphi \rangle_{\mathcal{H}^{-1/2}(\partial G)} = 0 \end{aligned}$$

omitting tr in front of φ. The result follows by density. □

2.2. The linear problem on $\mathbb{R}^3_+$ in L^2

We treat the linear Maxwell equations (2.1) on $G = \mathbb{R}^3_+$ with the boundary condition (2.4) of a perfect conductor for $\nu = -e_3$. As in Eample 1.6, we rewrite them as the symmetric hyperbolic system

$$\begin{aligned} Lu = \sum_{j=0}^{3} A_j \partial_j u + Du = f, \quad & t \geq 0, \ x \in \mathbb{R}^3_+, \\ Bu = -E \times e_3 = 0, \quad & t \geq 0, \ x \in \partial \mathbb{R}^3_+, \end{aligned} \tag{2.12}$$

$$u(0) = u_0, \quad x \in \mathbb{R}^3_+$$

assuming hypothesis (2.11) for $G = \mathbb{R}^3_+$. We first look for a solution $u = (E, H) \in C(\overline{J}, L^2_x)$, using the notation of (2.11) for $G = \mathbb{R}^3_+$, proceeding as in [7], [11] or [45], for instance. The trace operator B is defined as in the previous section and will be identified with the matrix $B^{\mathrm{co}} = (S_3 \ 0) \in \mathbb{R}^{3\times 6}$, where the matrices $S_j \in \mathbb{R}^{3\times 3}$ were introduced before (1.5). We proceed similar to Section 1.2 starting with an energy estimate. Later it turns out to be important that we require a bit less than $\mathcal{H}^1$ in the lemma. We set $v_{\mathrm{ta}} = (v_1, v_2, v_4, v_5)$ for the tangential components of a function $v : \mathbb{R}^3_+ \to \mathbb{R}^6$, and $v_{\mathrm{no}} = (v_3, v_6)$ for the normal ones.

Lemma 2.5. *Assume that* (2.11) *is true for* $G = \mathbb{R}^3_+$ *and that* $u \in C(\overline{J}, L^2_x)$ *solves* (2.12) *and has derivatives* $\partial_j u, \partial_3 u_{\mathrm{ta}}$ *in* $L^2_{t,x}$ *for* $j \in \{0, 1, 2\}$. *Let* $C^+ := \frac{1}{2}\partial_t A_0 - D$, $\gamma \geq \gamma_0^{+\prime}(L) := \max\{1, 4\,\|C^+\|_\infty/\eta\}$, $t \in \overline{J}$, *and* $L^2_\gamma = L^2_\gamma(0,t)$. *We then obtain*

$$\tfrac{\gamma\eta}{4}\|u\|^2_{L^2_\gamma L^2_x} + \tfrac{\eta}{2}\mathrm{e}^{-2\gamma t}\|u(t)\|^2_{L^2_x} \leq \tfrac{1}{2}\|A_0(0)\|_\infty\,\|u_0\|^2_{L^2_x} + \tfrac{1}{2\gamma\eta}\|f\|^2_{L^2_\gamma L^2_x}. \tag{2.13}$$

Proof. Let $v = e_{-\gamma}u$ and $g = e_{-\gamma}f$. By assumption, $\partial_j u$ for $j \in \{0, 1, 2\}$ and $\partial_3 A_3^{\mathrm{co}} u$ belong to $L^2_{t,x}$, and hence u_{ta} has a trace on $\{x_3 = 0\}$ in $L^2(J \times \mathbb{R}^2)$. It is 0 for u_1 and u_2 by the boundary condition. As in Lemma 1.2, the equation $\gamma A_0 v + Lv = g$ yields

$$\langle g, v\rangle = \gamma\langle A_0 v, v\rangle + \sum_{j=0}^{3}\langle A_j \partial_j v, v\rangle + \langle Dv, v\rangle$$

for the scalar products in $L^2((0,t), L^2_x)$. For $j \in \{1, 2, 3\}$ the summand with A_j is equal to $\int \frac{1}{2}\partial_j(A_j v \cdot v)\,\mathrm{d}(s,x)$ since A_j is constant and symmetric. The integral in x_j then vanishes by the above properties and since $A_3^{\mathrm{co}} u \cdot u = (u_5 u_1, -u_4 u_2, 0, -u_2 u_4, u_1 u_5, 0)$ has trace 0 on $\{x_3 = 0\}$. For $j = 0$, one obtains $2A_0\partial_t v \cdot v = \partial_t(A_0 v \cdot v) - \partial_t A_0 v \cdot v$. Integrating in t, we derive

$$\gamma\langle A_0 v, v\rangle + \frac{1}{2}\int_{\mathbb{R}^3_+} A_0(t)v(t)\cdot v(t)\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^3_+} A_0(0)u_0 \cdot u_0\,\mathrm{d}x + \langle C^+ v, v\rangle + \langle g, v\rangle.$$

The assertion now follows as in Lemma 1.2. □

We use (2.13) only for

$$\gamma \geq \gamma_0^+(r, \eta) := \max\{1, 6r/\eta\} \geq \gamma_0^{+\prime}(L) \tag{2.14}$$

assuming that $\|\partial_t A_0\|_\infty, \|D\|_\infty \leq r$. As in (1.11) the above proof yields the energy equality

$$\begin{aligned}&\int_{\mathbb{R}^3_+} A_0(t)u(t)\cdot u(t)\,\mathrm{d}x \qquad (2.15)\\ &\quad = \int_{\mathbb{R}^3_+} A_0(0)u_0 \cdot u_0\,\mathrm{d}x + 2\int_0^t\int_{\mathbb{R}^3_+} \big(C^+(s)u(s) + f(s)\big)\cdot u(t)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

For the existence result we need analogous estimates for the (formal) adjoint

$$L^\circ = -\sum_{j=0}^{3} A_j\partial_j + D^T - \partial_t A_0$$

in backward time and on the time interval $\mathbb{R}$. To this end, we extend the coefficents A_0 constantly and D by 0 to $t \in \mathbb{R}$.

Proposition 2.6. *Let (2.11) be true for $G = \mathbb{R}^3_+$ with $\|\partial_t A_0\|_\infty, \|D\|_\infty \le r$. Extend A_0 constantly and D by 0 to $t \in \mathbb{R}$. Let $\gamma \ge \gamma_0^+(r,\eta)$, see (2.14).*

a) Let $v \in C(\overline{J}, L^2_x)$ with $\partial_j v, \partial_3 v_{\mathrm{ta}} \in L^2(J, L^2_x)$ for $j \in \{0,1,2\}$ satisfy $L^\circ v = f$, $Bv = 0$ and $v(T) = v_0$. For the weight $\tilde{e}_\gamma(t) = \mathrm{e}^{\gamma(t-T)}$, $t \in \overline{J}$, and $L^2_{t,x} = L^2((t,T), L^2_x)$ we obtain

$$\tfrac{\gamma\eta}{4}\|\tilde{e}_\gamma v\|^2_{L^2_{t,x}} + \tfrac{\eta}{2}\mathrm{e}^{2\gamma(t-T)}\|v(t)\|^2_{L^2_x} \le \tfrac{1}{2}\|A_0(T)\|_\infty \|v_0\|^2_{L^2_x} + \tfrac{1}{2\gamma\eta}\|\tilde{e}_\gamma f\|^2_{L^2_{t,x}}. \tag{2.16}$$

b) Let $h, v \in L^2_{-\gamma}(\mathbb{R}, L^2_x)$ with $\partial_j v, \partial_3 v_{\mathrm{ta}} \in L^2_{-\gamma}(\mathbb{R}, L^2_x)$ for $j \in \{0,1,2\}$ satisfy $L^\circ v = h$ and $Bv = 0$. We then have

$$\tfrac{\gamma\eta}{4}\|v\|^2_{L^2_{-\gamma}(\mathbb{R},L^2_x)} \le \tfrac{1}{2\gamma\eta}\|h\|^2_{L^2_{-\gamma}(\mathbb{R},L^2_x)}. \tag{2.17}$$

The same estimate holds if we replace $-\gamma$ by γ and L° by L.

Proof. Assertion a) can be reduced to Lemma 2.5 as in step 1) of the proof of Theorem 1.5. For b), we first show the addendum. For $t \in \mathbb{R}$, as in Lemma 2.5 and with $L^2_{\gamma,t} = L^2_\gamma(-\infty, t)$ we derive

$$\tfrac{\gamma\eta}{4}\|v\|^2_{L^2_{\gamma,t}L^2_x} + \tfrac{\eta}{2}\mathrm{e}^{-2\gamma t}\|v(t)\|^2_{L^2_x} \le \tfrac{1}{2\gamma\eta}\|h\|^2_{L^2_{\gamma,t}L^2_x} \le \tfrac{1}{2\gamma\eta}\|h\|^2_{L^2_\gamma L^2_x}. \tag{2.18}$$

On the left we can drop the second summand and then let $t \to \infty$ using Fatou's lemma. Transforming $t \mapsto -t$ as in Theorem 1.5, estimate (2.17) follows from (2.18). □

Also in the present setting the duality argument from Theorem 1.5 provides a solution of (2.12) in $L^2_{t,x}$. However, the regularization argument does not work anymore in x_3. To obtain uniqueness and a continuous solution satisfying the energy estimate, we first pass to a problem with $t \in \mathbb{R}$ so that we can use regularization in t instead. We set $\|v\|^2_{\mathcal{H}^1_\gamma} = \|v\|^2_{L^2_\gamma \mathcal{H}^1_x} + \|\partial_t v\|^2_{L^2_\gamma L^2_x}$.

Proposition 2.7. *Let (2.11) be true for $G = \mathbb{R}^3_+$ with $\|\partial_t A_0\|_\infty, \|D\|_\infty \le r$.*

a) Then we have a solution $u \in L^2(J, L^2_x)$ of (2.12).

b) Let $\gamma \ge \gamma_0^+(r,\eta)$, see (2.14), and $\tilde{f} \in L^2_\gamma(\mathbb{R}, L^2_x)$. Then there is a function $u \in L^2_\gamma(\mathbb{R}, L^2_x) \cap C(\mathbb{R}, L^2_x)$ satisfying $Lu = \tilde{f}$ and $Bu = 0$. Let $\tilde{f}$ also have support in $\mathbb{R}_{\ge 0}$. Then u solves (2.12) on J with $u_0 = 0$ and $\tilde{f}$, and it fulfills (2.13) and (2.15).

Proof. a) We proceed as in Theorem 1.5 and define for $v \in V := \{v \in \mathcal{H}^1(J \times \mathbb{R}^3_+) \mid Bv = 0,\ v(T) = 0\}$ the functional

$$\ell_0 : L^\circ V \to \mathbb{R}; \quad \ell_0(L^\circ v) = \langle v, f\rangle_{L^2_{t,x}} + \langle v(0), A_0(0)u_0\rangle_{L^2_x}.$$

Estimate (2.16) with $\gamma = \gamma_0^+(r,\eta)$ shows that ℓ_0 is well defined and that

$$|\ell_0(L^\circ v)| \le \|f\|_{L^2_{t,x}} \|v\|_{L^2_{t,x}} + \|A_0(0)u_0\|_{L^2_x} \|v(0)\|_{L^2_x} \le c\|L^\circ v\|_{L^2_{t,x}}.$$

As a result, ℓ_0 can be extended to a functional on $L^2_{t,x}$ which in turn is represented by a function $u \in L^2(J, L^2_x)$ satisfying $\ell_0(L^\circ v) = \langle L^\circ v, u\rangle_{L^2_{t,x}}$ for all $v \in V$; i.e.,

$$\begin{aligned}&\langle v, f\rangle_{L^2_{t,x}} + \langle v(0), A_0(0)u_0\rangle_{L^2_x} \qquad (2.19)\\ &\quad = \langle v, Du\rangle_{L^2_{t,x}} - \sum_{j=0}^{2} \int_0^T \int_{\mathbb{R}^3_+} \partial_j(A_j v)\cdot u \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_{\mathbb{R}^3_+} \partial_3 v \cdot A_3^{\mathrm{co}} u \, \mathrm{d}x \, \mathrm{d}t.\end{aligned}$$

First, for $v \in \mathcal{H}^1_0(J \times \mathbb{R}^3_+)$ this formula yields

$$\langle v, f\rangle_{L^2_{t,x}} = \langle v, Du\rangle_{L^2_{t,x}} + \sum_{j=0}^{3} \langle v, A_j\partial_j u\rangle_{\mathcal{H}^1_0(J\times\mathbb{R}^3_+)} = \langle v, Lu\rangle_{\mathcal{H}^1_0(J\times\mathbb{R}^3_+)},$$

so that $Lu = f$ in $\mathcal{H}^{-1}_{t,x}$. Since $f \in L^2_{t,x}$, from (1.9) we deduce that $\partial_t u$ belongs to $L^2_t\mathcal{H}^{-1}_x$ and from $Lu = f$ that $\partial_3 u_{\mathrm{ta}}$ is contained in $L^2_{x_3}(\mathbb{R}_+, \mathcal{H}^{-1}(J \times \mathbb{R}^2))$.

In a second step, we take $v = \phi v_0$ for some $v_0 \in C_c^\infty(\mathbb{R}^3_+)$ and $\phi \in C^1([0,T])$ with $\phi(0) = 1$ and $\phi(T) = 0$. Equation (2.19) now implies

$$\begin{aligned}&\langle v, Lu\rangle_{L^2_{t,x}} + \langle v_0, A_0(0)u_0\rangle_{L^2_x}\\ &\quad = \langle v, Du\rangle_{L^2_{t,x}} + \sum_{j=0}^{3} \int_0^T \langle v(s), A_j(s)\partial_j u(s)\rangle_{\mathcal{H}^1_0(\mathbb{R}^3_+)} \, \mathrm{d}s + \langle v_0, A_0(0)u(0)\rangle_{L^2_x}\\ &\quad = \langle v, Lu\rangle_{L^2_{t,x}} + \langle v_0, A_0(0)u(0)\rangle_{L^2_x}.\end{aligned}$$

As $C_c^\infty(\mathbb{R}^3_+)$ is dense in $\mathcal{H}^1_0(\mathbb{R}^3_+)$, it follows $A_0(0)u_0 = A_0(0)u(0)$ in $\mathcal{H}^{-1}_x$, and so $u(0) = u_0$.

Finally, let $v \in C_c^\infty(J \times \mathbb{R}^2 \times \mathbb{R}_{\ge 0})$. Identity (2.19) then leads to

$$\langle v, Lu\rangle_{L^2_{t,x}} = \langle v, Du\rangle_{L^2_{t,x}} + \sum_{j=0}^{2} \langle v, A_j\partial_j u\rangle_{\mathcal{H}^1_0(J\times\mathbb{R}^2, L^2(\mathbb{R}_+))} - \int_0^T \int_{\mathbb{R}^3_+} \partial_3 v \cdot A_3^{\mathrm{co}} u \, \mathrm{d}x \, \mathrm{d}t.$$

We now choose v such that only $v_5 \ne 0$. Write $\Gamma = \partial\mathbb{R}^3_+ = \{x_3 = 0\}$. Combined with the identity

$$-\int_0^\infty \langle \partial_3 v_5, u_1\rangle_{L^2(J\times\mathbb{R}^2)} \, \mathrm{d}x_3 = \int_0^\infty \langle v_5, \partial_3 u_1\rangle_{\mathcal{H}^1_0(J\times\mathbb{R}^2)} \mathrm{d}x_3 + \langle \mathrm{tr}_\Gamma\, v_5, \mathrm{tr}_\Gamma\, u_1\rangle_{\mathcal{H}^1_0(J\times\mathbb{R}^2)}$$

the above equation in display yields $\langle v, Lu\rangle = \langle v, Lu\rangle + \langle \mathrm{tr}_\Gamma\, v_5, \mathrm{tr}_\Gamma\, u_1\rangle$ so that the last term is equal to 0. Again by density we conclude $\mathrm{tr}_\Gamma\, u_1 = 0$ in $\mathcal{H}^{-1}(J \times \mathbb{R}^2)$, and similarly $\mathrm{tr}_\Gamma\, u_2 = 0$. Therefore $u \in L^2(J, L^2_x)$ solves (2.12).

b) 1) Let $\tilde{f} \in L^2_\gamma(\mathbb{R}, L^2_x)$ for a fixed $\gamma \ge \gamma_0^+(r,\eta)$. We proceed as above on the time interval $\mathbb{R}$, setting $V = \{v \in \mathcal{H}^1_{-\gamma}(\mathbb{R} \times \mathbb{R}^3_+) \,|\, Bv = 0\}$ and $\ell_0(L^\circ v) = \langle v, \tilde{f}\rangle_{L^2_{t,x}}$ for $v \in V$. We note that L^2_γ is the dual of $L^2_{-\gamma}$ via the L^2-scalar product. Estimate (2.17) then implies that ℓ_0 is welldefined and bounded. As in part a) we can then represent ℓ_0 by a function $u \in L^2_\gamma(\mathbb{R}, L^2_x)$ and show that $Lu = \tilde{f}$ and $Bu = 0$.

2) Let $R_{1/n}$ be a mollifier in (t, x_1, x_2) for $n \in \mathbb{N}$. Then $R_{1/n}u$ is an element of $\mathcal{H}^1_\gamma(\mathbb{R} \times \mathbb{R}^2, L^2(\mathbb{R}_+))$, and it satisfies $BR_{1/n}u = 0$ and

$$LR_{\frac{1}{n}}u = R_{\frac{1}{n}}\tilde{f} + [A_0, R_{\frac{1}{n}}]\partial_t u + [D, R_{\frac{1}{n}}]u =: \tilde{f}_n. \tag{2.20}$$

By Proposition 1.3 and dominated convergence, the functions $\tilde{f}_n$ tend to $\tilde{f}$ in $L^2_\gamma(\mathbb{R}, L^2_x)$ as $n \to \infty$. Moreover,

$$\partial_3 A_3^{\text{co}} R_{\frac{1}{n}} u = \tilde{f}_n - \sum_{j=0}^{2} A_j \partial_j R_{\frac{1}{n}} u - D R_{\frac{1}{n}} u \tag{2.21}$$

is contained in $L^2_\gamma L^2_x$. So $(R_{1/n}u)_n$ is Cauchy in $C_{b,\gamma}L^2_x \cap L^2_\gamma L^2_x$ by (2.18) and (2.20). Since $R_{1/n}u \to u$ in $L^2_\gamma L^2_x$, we conclude that u belongs to $C(\mathbb{R}, L^2_x)$ and fulfills (2.18).

3) Let $\tilde{f}$ have support in $\mathbb{R}_{\geq 0}$. Using (2.18) for u, we estimate

$$\int_{-\infty}^{0} \|u(s)\|^2_{L^2_x}\,\mathrm{d}s \leq \int_{-\infty}^{0} \mathrm{e}^{-2\gamma s}\|u(s)\|^2_{L^2_x}\,\mathrm{d}s \leq \frac{2}{\gamma^2\eta^2}\|\tilde{f}\|^2_{L^2_\gamma(\mathbb{R},L^2_x)} \leq \frac{2}{\gamma^2\eta^2}\|\tilde{f}\|^2_{L^2_{t,x}}.$$

Letting $\gamma \to \infty$, we infer that u vanishes for $t \leq 0$. Inequality (2.18) then shows (2.13) with $u_0 = 0$. Moreover, the functions $R_{1/n}u$ from step 2) satisfy the energy equality (2.15) with $u_0 = 0$, and thus also u by approximation. □

The above proof also yields uniqueness of solutions to (2.12) even in $L^2_{t,x}$.

PROPOSITION 2.8. *Assume that* (2.11) *is true for* $G = \mathbb{R}^3_+$. *Let* $u, v \in L^2(J, L^2_x)$ *solve* (2.12). *Then* $u = v$.

PROOF. The function $w = u - v \in L^2_{t,x}$ solves (2.12) with $u_0 = 0$ and $f = 0$. Extend w by 0 to $\mathbb{R}$. We then have $w \in \mathcal{H}^1((0,T), \mathcal{H}^{-1}_x)$ by (1.9) and $Lw = 0$ on $(-\infty, T)$. Take times $0 < t_0 < t_1 < T$ and a function $\theta \in C^\infty(\mathbb{R})$ being 1 on $(-\infty, t_0]$ and 0 on $[t_1, \infty)$. The map $\tilde{w} = \theta w$ has support in $\overline{J}$ and satisfies $L\tilde{w} = \theta' A_0 w =: g \in L^2_t L^2_x$, where $\operatorname{supp} g \subseteq [t_0, t_1]$. As in parts 2) and 3) of the proof of Proposition 2.7 b), we then check that $w = 0$ on $[0, t_0]$ (using the weight $\mathrm{e}^{-\gamma(t-t_0)}$ and replacing the time 0 by t_0). Here $t_0 < T$ is arbitrary. □

As a final preliminary step we show the desired result if $f = 0$. Recall that we have extended A_0 and D to $\mathbb{R}$.

LEMMA 2.9. *Assume that* (2.11) *is true and* $f = 0$. *Then there is a unique solution* $u \in C(\overline{J}, L^2_x)$ *of* (2.12), *and it satisfies* (2.13) *and* (2.15) *with* $f = 0$.

PROOF. Proposition 2.7 provides a solution $u \in L^2_{t,x}$ on $(0, T+1)$.

1) First, let u_0 be 0 outside a compact set in $\mathbb{R}^3_+$. Then extend it by 0 to $\mathbb{R}^3$. Theorem 1.5 (and a backward version) yield a solution $\tilde{u} \in C(\mathbb{R}, L^2_x)$ of $L\tilde{u} = 0$ with $\tilde{u}(0) = u_0$. There is a time $\tau > 0$ such that $\tilde{u}(t)$ is supported in $\mathbb{R}^3_+$ for all $t \in [-\tau, \tau]$ due to the finite speed of propagation, see Theorem 1.7. So the restriction v of $\tilde{u}$ to $[-\tau, \tau] \times \mathbb{R}^3_+$ solves (2.12) with $f = 0$. Proposition 2.8 shows

that $u = v$ on $[0,\tau]$, and hence u belongs to $C([0,\tau], L^2_x)$. We extend u by $\tilde{u}|_{\mathbb{R}^3_+}$ continuously to $t < 0$.

As in the proof of Proposition 2.7, we set $u_n = R_{1/n}u$ on $[0,T]$ for a mollifier in $(t, x_1, x_2) = (t, x')$. Then the functions $u_n \in C(\overline{J}, L^2_x)$ tend to u in $L^2(J, L^2_x)$ and satisfy $Bu_n = 0$, $Lu_n =: f_n \to 0$ in $L^2(J, L^2_x)$ as well as (2.13) and (2.15) on J. (See (2.20) and (2.21) and use Lemma 2.5.) Moreover, we have

$$u_n(0,x) = \int_{-1/n}^{1/n} \int_{B(x',1/n)} \rho_{1/n}(-s, x'-y')u(s,y',x_3)\,\mathrm{d}y'\,\mathrm{d}s$$

which tends to u_0 in L^2_x by the time continuity of u. For $t \in \overline{J}$ and $\gamma = \gamma_0^+$, see (2.14), estimate (2.13) then yields

$$\|u_n(t) - u_m(t)\|^2_{L^2_x} \le c\big(\|u_n(0) - u_m(0)\|^2_{L^2_x} + \|f_n - f_m\|^2_{L^2_{t,x}}\big) \longrightarrow 0$$

for $n, m \to \infty$ so that (u_n) is Cauchy in $C(\overline{J}, L^2_x)$. As a result, u belongs to this space and fulfills (2.13) and (2.15) with $f = 0$.

2) Let $u_0 \in L^2_x$. Set $u_{0,n} = \mathbb{1}_{K_n}u_0$ for compact sets $K_n \subseteq \mathbb{R}^3_+$ with $\bigcup_{n\in\mathbb{N}} K_n = \mathbb{R}^3_+$. Step 1) provides a map $u_n \in C(\overline{J}, L^2_x)$ with $Lu_n = 0$, $Bu_n = 0$ and $u_n(0) = u_{0,n}$ which satisfies (2.13) and (2.15). This estimate then implies that (u_n) is Cauchy in $C(\overline{J}, L^2_x)$, and hence the limit u has the asserted properties. □

We now obtain the basic linear well-posedness result in L^2_x. (The additional factor 2 could be avoided using $R_{1/n}$ as above.)

THEOREM 2.10. *Let (2.11) be true for $G = \mathbb{R}^3_+$ with $\|\partial_t A_0\|_\infty, \|D\|_\infty \le r$. Then there is a unique solution $u \in C(\overline{J}, L^2_x)$ of (2.12). It satisfies (2.15) and (2.13) with a factor 2 on the right-hand side for $\gamma \ge \gamma_0^+(r,\eta)$ from (2.14).*

PROOF. Uniqueness was shown in Corollary 2.6. Proposition 2.7 and Lemma 2.9 provide functions $v, w \in C(\overline{J}, L^2_x)$ satisfying $Lv = f$, $Bv = 0$, $v(0) = 0$, as well as $Lw = 0$, $Bw = 0$, $w(0) = u_0$. Then $u = v + w \in C(\overline{J}, L^2_x)$ solves (2.12). Since v and w fulfill (2.13) and (2.15) for the respective data, the last assertion also follows. □

2.3. The linear problem on $\mathbb{R}^3_+$ in $\mathcal{H}^3$

On $G = \mathbb{R}^3$ we have reduced the wellposedness of the linear problem in $\mathcal{H}^3$ to that in L^2 by means of the transformation $v \mapsto (I - \Delta)^{3/2}v$. For the Maxwell system on domains such a procedure does not seem to work anymore because of the boundary condition (2.4). (See **[32]** for cases where one can proceed in such a way also in the presence of (simpler) boundary conditions.) Instead we will first derive apriori estimates for $\mathcal{H}^3$-solutions and then show by regularization arguments that the L^2-solution of Theorem 2.10 is actually an $\mathcal{H}^3$-solution if the data satisfy natural assumptions.

In our reasoning we will mix space and time regularity so that we need the same number of derivatives in space and in time. We thus look for solutions in

$\mathcal{G}^3(J \times \mathbb{R}^3_+)$, where we set

$$\mathcal{G}^m(J \times G) = \bigcap_{k=0}^{m} C^k(\overline{J}, \mathcal{H}^{m-k}(G, \mathbb{R}^6)),$$
$$\mathcal{G}^{m-}(J \times G) = \bigcap_{k=0}^{m} W^{k,\infty}(J, \mathcal{H}^{m-k}(G, \mathbb{R}^6))$$

for $m \in \mathbb{N}_0$ and an open bounded set $G \subseteq \mathbb{R}^3$ with smooth boundary or $G \in \{\mathbb{R}^3_+, \mathbb{R}^3\}$. These spaces are endowed with their canonical norms. Sometimes we only write $\mathcal{G}^m(T)$ if G is clear from the context and $J = (0,T)$. For the coefficients we use

$$\mathcal{F}^m(J \times G) = \{A \in W^{1,\infty}(J \times G, \mathbb{R}^{6\times 6}) \,|\, \forall \alpha \in \mathbb{N}_0^4 \text{ with } 1 \le |\alpha| \le m : \\ \partial^\alpha A \in L^\infty(J, L^2_x)\},$$

These spaces are endowed with their natural norms, and the same symbols also denote spaces with different range spaces. The spaces $\hat{\mathcal{H}}^m_x = \hat{\mathcal{H}}^m(G)$ are defined as on $\mathbb{R}^3$ after (1.18). As before, the subscript 'sym' means that the functions take values in symmetric matrices, 'η' that they are bounded from below by ηI in addition, and 'γ' refers to norms with weight $\mathrm{e}^{-\gamma t}$. To obtain solutions in $\mathcal{G}^3$, we strengthen hypotheses (2.11) to

$$A_0, D \in \mathcal{F}^3(J \times G, \mathbb{R}^{6\times 6}), \quad A_0 = A_0^\top \ge \eta I > 0, \quad J = (0,T), \tag{2.22}$$
$$A_j = A_j^{\mathrm{co}} \text{ for } j \in \{1,2,3\}, \quad u_0 \in \mathcal{H}^k(G, \mathbb{R}^6) = \mathcal{H}^k_x, \quad f \in \mathcal{H}^k(J \times G, \mathbb{R}^6) = \mathcal{H}^k_{t,x},$$
$$\|A_0\|_{\mathcal{F}^3}, \|D\|_{\mathcal{F}^3} \le r, \quad \|A_0(0)\|_{\hat{\mathcal{H}}^2_x}, \|D(0)\|_{\hat{\mathcal{H}}^2_x}, \|\partial_t^l A_0(0)\|_{\mathcal{H}^{2-l}_x}, \|\partial_t^l D(0)\|_{\mathcal{H}^{2-l}_x} \le r_0$$

for all $l \in \{1,2\}$, some $k \in \{1,2,3\}$, and constants $r \ge r_0 \ge 1$. We note that the product and inversion rules from Lemma 1.8 remain true on the present spatial domain G and for $\mathcal{G}^3$, since G admits an extension operator and the additional time derivatives can be treated similarly, see §2 of [**53**] or §2.2 of [**51**].

If one has a solution $u \in C(\overline{J}, \mathcal{H}^1_x)$ of (2.12), the initial value $u(0) = u_0$ must satisfy the boundary condition $Bu_0 = 0$ by continuity. If u even belongs to $\mathcal{G}^3$, also $u^1 = \partial_t u(0)$ and $u^2 = \partial_t^2 u(0)$ have to fulfill $Bu^j = 0$, where we put $u^0 = u_0$. In view of (2.12) and (2.22), the following (linear) *compatibility conditions* (of order 3) are thus neccessary for the existence of a solution $u \in \mathcal{G}^3$.

$$Bu^j = 0 \quad \text{for} \quad j \in \{0,1,2\}, \quad \text{where:}$$
$$u^1 := A_0(0)^{-1}\Big[f(0) - D(0)u_0 - \sum_{j=1}^3 A_j \partial_j u_0\Big], \tag{2.23}$$
$$u^2 := A_0(0)^{-1}\Big[\partial_t f(0) - \partial_t D(0)u_0 - D(0)u^1 - \partial_t A_0(0)u^1 - \sum_{j=1}^3 A_j \partial_j u^1\Big].$$

The function u^3 is defined analogously applying ∂_t^2 to $Lu = f$. Assuming (2.22), the product and inversion rules easily yield

$$\|u^j\|_{\mathcal{H}^{k-j}_x} \le c(r_0, \eta)\left(\|u_0\|_{\mathcal{H}^k_x} + \|f(0)\|_{\mathcal{H}^{k-1}_x} + \dots + \|\partial_t^{j-1} f(0)\|_{\mathcal{H}^{k-j}_x}\right) \tag{2.24}$$

for $k\in\{1,2,3\}$ and $j\in\{0,\dots,k\}$, cf. Lemma 2.3 in [**53**] or Lemma 2.33 of [**51**].

We start with the apriori estimates for the time and tangential derivatives. We write $\mathcal{H}^k_{\mathrm{ta}}(J\times G)$ for functions $g\in L^2_{t,x}$ with $\partial^\alpha g\in L^2_{t,x}$ for all $\alpha=(\alpha_0,\dots,\alpha_3)\in\mathbb{N}_0^4$ with $|\alpha|\le k$ and $\alpha_3=0$. Analogously, we define $\mathcal{G}^k_{\mathrm{ta},\gamma}$ and $\mathcal{H}^k_{\mathrm{ta}}(G)=\mathcal{H}^k_{\mathrm{ta},x}$.

Lemma 2.11. *Let (2.22) be true for $G=\mathbb{R}^3_+$ and some $k\in\{1,2,3\}$, where we only require $f\in\mathcal{H}^3_{\mathrm{ta}}(J\times G)$. Assume that $u\in\mathcal{G}^k(J\times G)$ solves (2.12). Then there exist constants $\tilde\gamma_k^+=\tilde\gamma_k^+(r,\eta)\ge\gamma_0^+(r,\eta)$, see (2.14), $c_k^+=c_k^+(r,\eta)$ and $c_{k,0}^+=c_{k,0}^+(r_0,\eta)$ such that u satisfies*

$$\|u\|^2_{\mathcal{G}^k_{\mathrm{ta},\gamma}}+\gamma\|u\|^2_{\mathcal{H}^k_{\mathrm{ta},\gamma}}\le c^+_{k,0}\big[\|u_0\|^2_{\mathcal{H}^k_x}+\|f(0)\|^2_{\mathcal{H}^{k-1}_x}+\dots+\|\partial_t^{k-1}f(0)\|^2_{L^2_x}\big]$$
$$+\frac{c_k^+}{\gamma}\big(\|f\|^2_{\mathcal{H}^k_{\mathrm{ta},\gamma}}+\|u\|^2_{\mathcal{G}^k_\gamma}\big)$$

for all $\gamma\ge\tilde\gamma_k^+$.

Proof. Take $\alpha\in\mathbb{N}_0^4$ with $|\alpha|\le k$ and $\alpha_3=0$ and apply ∂^α to (2.12). We then have $B\partial^\alpha u=0$, $\partial^\alpha u(0)=\partial^{(\alpha_1,\alpha_2)}u^{\alpha_0}$ and

$$L\partial^\alpha u=\partial^\alpha f-\sum_{0<\beta\le\alpha}\tbinom{\alpha}{\beta}\big(\partial^\beta A_0\partial^{\alpha-\beta}\partial_t u+\partial^\beta D\partial^{\alpha-\beta}u\big).\tag{2.25}$$

Combined with (2.24) and Lemma 1.8, the energy inequality (2.13) applied to the above equation yields the assertion. □

The extra term involving u on the right-hand side will be absorbed below. The above argument fails for the normal derivative ∂_3 since ∂_3 destroys the boundary condition $Bu=0$. However, the equation (2.12) directly allows to bound $\partial_3 u_{\mathrm{ta}}$ in terms of u and $\partial_j u$ for $j\in\{0,1,2\}$. For instance, the first line of (2.12) yields

$$\partial_3 H_2=\partial_2 H_3-(A_0\partial_t u)_1-(Du)_1+f_1.\tag{2.26}$$

Here we mix space and time regularity which forces us to use the solution space $\mathcal{G}^3$. The remaining derivatives $\partial_3 u_3$ and $\partial_3 u_6$ (and higher-order analogues) can be treated using $\operatorname{div}\operatorname{curl}=0$. We stress that we do not employ boundary conditions in these two steps.

Proposition 2.12. *Let (2.22) be true for $G=\mathbb{R}^3_+$ and some $T'>0$ and $k\in\{1,2,3\}$. Let $T\in(0,T']$ and $u\in\mathcal{G}^k(J\times G)$ solve (2.12). Then there exist constants $\gamma_k^+=\gamma_k^+(r,\eta,T')\ge\tilde\gamma_k^+(r,\eta)$, see Lemma 2.11, $C_k^+=C_k^+(r,\eta,T')$ and $C_{k,0}^+=C_{k,0}^+(r_0,\eta)$ such that*

$$\|u\|^2_{\mathcal{G}^k_\gamma}\le(C^+_{k,0}+TC^+_k)\mathrm{e}^{kC_1^+T}\big(\|u_0\|^2_{\mathcal{H}^k_x}+\|f(0)\|^2_{\mathcal{H}^2_x}+\dots+\|\partial_t^{k-1}f(0)\|^2_{L^2_x}\big)$$
$$+\frac{C_k^+}{\gamma}\mathrm{e}^{kC_1^+T}\|f\|^2_{\mathcal{H}^k_\gamma}\qquad\textit{for all }\gamma\ge\gamma_k^+.\tag{2.27}$$

Proof. 1) Let $k=1$ and $t\in[0,T]$. To carry out the argument indicated above, we first note that $\|A_0(t)\|_{L^\infty_x}\le r_0+rt$ since $A_0(t)=A_0(0)+\int_0^t\partial_t A_0(s)\,\mathrm{d}s$ and analogously for D and f. So (2.12) and Lemma 2.11 yield

$$\|\partial_3 u_{\mathrm{ta}}\|_{\mathcal{G}^0_\gamma(t)}\le c(r_0+Tr)\|(u,\partial_{t,x_1,x_2}u)\|_{\mathcal{G}^0_\gamma(t)}+\|f(0)\|_{L^2_x}\tag{2.28}$$

$$+ \sup_{\tau\in[0,t]} \int_0^\tau \mathrm{e}^{-\gamma(\tau-s)}\mathrm{e}^{-\gamma s}\|\partial_t f(s)\|_{L^2_x}\,\mathrm{d}s,$$
$$\le c(r_0+Tr)\|(u,\partial_{t,x_1,x_2}u)\|_{\mathcal{G}^0_\gamma(t)} + \|f(0)\|_{L^2_x} + \tfrac{c}{\sqrt{\gamma}}\|\partial_t f\|_{L^2_\gamma((0,t),L^2_x)}$$
$$\le (c_0(r_0,\eta)+Tc(r,\eta))\big(\|u_0\|_{\mathcal{H}^1_x} + \|f(0)\|_{L^2_x}\big) + \tfrac{c(r,\eta,T')}{\sqrt{\gamma}}(\|f\|_{\mathcal{H}^1_{\gamma,\mathrm{ta}}(t)} + \|u\|_{\mathcal{G}^1_\gamma(t)}),$$

using also Young's inequality. We next treat $\partial_3 u_{\mathrm{no}}$ with $u_{\mathrm{no}} = (u_3,u_6)$. To simplify notation, we assume in this proof that $D=0$ and $A_0 = \mathrm{diag}(a^e,a^m)$, where a^j maps into $\mathbb{R}^{3\times3}_\eta$, and we write $f=(f^e,f^m)$. (Compare Proposition 3.3 in [**53**].) Equation (2.12) then leads to

$$\partial_t(a^e\nabla_x E) = \partial_t a^e\nabla_x E + a^e\nabla_x(a_e^{-1}\operatorname{curl} H + a_e^{-1}f^e)$$
$$= \partial_t a^e\nabla_x E + a^e\nabla_x a_e^{-1}(\operatorname{curl} H + f^e) + \nabla_x f^e + \nabla_x \operatorname{curl} H$$

in $\mathcal{H}^{-1}_x$. The first two summands in the last line are denoted by Λ and are bounded pointwise by $c(r)(|\nabla_x u| + |f|)$. Observe that the trace $\mathrm{sp}(\nabla_x \operatorname{curl} H)$ is equal to $\operatorname{div}\operatorname{curl} H = 0$. Integrating in time, we thus obtain

$$a^e_{33}(t)\partial_3 E_3(t) = \mathrm{sp}(a^e(0)\nabla E_0) - \sum_{(j,k)\neq(3,3)} a^e_{jk}(t)\partial_k E_j(t) + \int_0^t \mathrm{sp}(\nabla f^e + \Lambda)\,\mathrm{d}s. \tag{2.29}$$

Let $\partial' u = (u,\partial_1 u,\partial_2 u,\partial_3 u_{\mathrm{ta}})$. Since $a^e_{33}\ge\eta$, as in (2.28) we derive

$$\|\partial_3 E_3(t)\|_{L^2_x} \le c(r_0,\eta)\|u_0\|_{\mathcal{H}^1_x} + c\tfrac{r_0+rt}{\eta}\|\partial' u(t)\|_{L^2_x} + c(r,\eta)\int_0^t \|(f(s),u(s))\|_{\mathcal{H}^1_x}\mathrm{d}s$$
$$\le c(r_0)\|u_0\|_{\mathcal{H}^1_x} + (c_0(r)+tc(r))\mathrm{e}^{\gamma t}\|\partial' u(t)\|_{\mathcal{G}^0_\gamma(t)} + \frac{c(r)}{\sqrt{\gamma}}\mathrm{e}^{\gamma t}\|f\|_{\mathcal{H}^1_\gamma(t)}$$
$$+ c_1(r)\int_0^t \|\partial_3 u_{\mathrm{no}}(s)\|_{L^2_x}\,\mathrm{d}s, \tag{2.30}$$

dropping the dependence on η in the constants. We now multiply this inequality by $\mathrm{e}^{-\gamma t}$, and add the analogous one for $\partial_3 H_3$ as well as Lemma 2.11 and (2.28). It follows

$$\mathrm{e}^{-\gamma t}\|(u(t),\partial_{t,x}u(t))\|_{L^2_x} \le (c(r_0)+Tc(r))\big(\|u_0\|_{\mathcal{H}^1_x} + \|f(0)\|_{L^2_x}\big) + \tfrac{c(r,T')}{\sqrt{\gamma}}\|f\|_{\mathcal{H}^1_\gamma(t)}$$
$$+ \tfrac{c(r,T')}{\sqrt{\gamma}}\|u\|_{\mathcal{G}^1_\gamma(t)} + c(r)\int_0^t \mathrm{e}^{-\gamma s}\|\partial_3 u_{\mathrm{no}}(s)\|_{L^2_x}\,\mathrm{d}s. \tag{2.31}$$

Taking the supremum $t\le t'$, we educe this estimate with t' instead of t and $\|u\|_{\mathcal{G}^1_\gamma(t')}$ on the left-hand side. We now absorb the norm of u on the right-hand side choosing sufficiently large γ. Going back to the form (2.31) of the estimate, Gronwall's inequality yields (2.27) for $k=1$.

2) Let $k=3$, the variant for $k=2$ is shown by a modification of the proof.

a) We first take $\alpha\in\mathbb{N}_0^4$ with $|\alpha|\le 2$ and $\alpha_3 = 0$. Equation (2.25) and Lemma 1.8 show that $L\partial^\alpha u = f_\alpha$ with $\|f_\alpha\|_{\mathcal{H}^1_{\gamma,\mathrm{ta}}} \le c(r)\,(\|f\|_{\mathcal{H}^3_{\gamma,\mathrm{ta}}} + \|u\|_{\mathcal{H}^3_\gamma})$. Using also (2.24), we can bound $\partial^\alpha u(0)$ in $\mathcal{H}^1_x$ by

$$c(r_0)(\|u_0\|_{\mathcal{H}^3_x} + \|f(0)\|^2_{\mathcal{H}^2_x} + \|\partial_t f(0)\|_{\mathcal{H}^1_x} + \|\partial_t^2 f(0)\|_{L^2_x}) =: c(r_0)\kappa(u_0,f).$$

Lemma 2.11 and (2.28) thus imply

$$\|u\|_{\mathcal{G}^3_{\mathrm{ta},\gamma}} + \|\partial_3\partial^\alpha u_{\mathrm{ta}}\|_{\mathcal{G}^0_\gamma(t)} \le (c(r_0) + Tc(r))\kappa(u_0, f) + \tfrac{c(r,T')}{\sqrt{\gamma}}\big(\|f\|_{\mathcal{H}^3_{\gamma,\mathrm{ta}}} + \|u\|_{\mathcal{G}^3_\gamma}\big).$$

We next employ (2.30) with $\partial^\alpha u$ and f_α instead of u and f. Using also the above estimates, we derive (2.31) for $\partial^\alpha u$ and f, replacing $\|u\|_{\mathcal{G}^1_\gamma(t)}$ by $\|u\|_{\mathcal{G}^3_\gamma(t)}$ on the right. Note that we still work with L and so the constant $c_1(r)$ in (2.30) is unchanged. Gronwall's inquality thus implies (2.27) for $k = 3$ up to the term $c(r, T')\gamma^{-1}\mathrm{e}^{c_1(r)t}\|u\|^2_{\mathcal{G}^3_\gamma}$ on the right-hand side, if restrict ourselves to derivatives of u and f with $\alpha_3 \le 1$.

b) In a next step, in a) we choose α with $|\alpha| \le 2$ and $\alpha_3 = 1$. Proceeding as in step a) and using it, we first obtain

$$\|u\|_{\mathcal{G}^3_{\mathrm{ta},\gamma}(t)} + \|\partial_3 u\|_{\mathcal{G}^2_{\mathrm{ta},\gamma}(t)} + \|\partial_3^2 u_{\mathrm{ta}}\|_{\mathcal{G}^1_{\mathrm{ta},\gamma}(t)}$$
$$\le (c(r_0){+}Tc(r))\mathrm{e}^{c_1(r)t}\kappa(u_0, f) + \tfrac{c(r,T')}{\sqrt{\gamma}}\big[\|\partial_3 f\|_{\mathcal{H}^2_{\gamma,\mathrm{ta}}} {+}\, \mathrm{e}^{c_1(r)t}(\|f\|_{\mathcal{H}^3_{\gamma,\mathrm{ta}}} + \|u\|_{\mathcal{G}^3_\gamma})\big].$$

In (2.30) we now insert $\partial_3\partial_{\mathrm{ta}}u$ and $\partial_{3j}f$ with $j \in \{0, 1, 2\}$. As in (2.31), we derive from the above estimate and Gronwall's inquality the assertion up to the error term for all derivatives except ∂_3^3 and $k = 2$ in the exponent. The step for ∂_3^3 can be performed analogously. Finally we absorb $c(r, T')\gamma^{-1}\|u\|^2_{\mathcal{G}^3_\gamma}$ by the left-hand side, choosing large $\gamma \ge \gamma_3^+(r, \eta, T')$. □

Using the above estimate, we now show in several steps that the solution $u \in \mathcal{G}^0(J)$ of (2.12) given by Theorem 2.10 actually belongs to $\mathcal{G}^3(J)$ if (2.22) is true with $k = 3$. For $k = 1$, we first show that $u \in C^1(\overline{J}, L^2_x)$ by solving an equation formally satisfied by $\partial_t u$. In this step, we need the compatibility condition to obtain time regularity. The tangential regularity is then derived by means of mollifiers as in Proposition 2.7. The normal regularity finally follows from (2.12) and (2.29).

Lemma 2.13. *Let (2.22) be true for $G = \mathbb{R}^3_+$ and $k = 1$, and let $u \in \mathcal{G}^0(J \times G)$ solve (2.12). Assume that $Bu_0 = 0$. Then u belongs to $C^1(\overline{J}, L^2_x)$.*

Proof. 1) Define $u^1 \in L^2_x$ as in (2.23). We look for $v \in C(\overline{J}, L^2_x)$ solving

$$L'v := \sum_{j=0}^{3} A_j\partial_j v + (D + \partial_t A_0)v = \partial_t f - \partial_t D\Big(u_0 + V(t)\Big), \quad t \ge 0,\ x \in \mathbb{R}^3_+,$$
$$Bv = 0, \quad t \ge 0,\ x \in \partial\mathbb{R}^3_+, \tag{2.32}$$
$$v(0) = u^1, \qquad x \in \mathbb{R}^3_+,$$

where we have set $V(t) = \int_0^t v(s)\,\mathrm{d}s$. If we already knew that u belonged to $C^1(\overline{J}, L^2_x)$, then $v = \partial_t u$ would satisfy (2.32). However, we can solve this problem directly using a simple fixed-point argument. Indeed, take $w \in \mathcal{G}^0(J)$ and replace v by w on the right-hand side of the evolution equation in (2.32). Theorem 2.10 then yields a solution $v \in \mathcal{G}^0(J)$ of the resulting problem. For $\gamma \ge \gamma_0^+$ and $c = c(r, \eta)$, we further obtain

$$\|v - \overline{v}\|^2_{\mathcal{G}^0_\gamma} \le \tfrac{c}{\gamma}\,\|W - \overline{W}\|^2_{L^2_\gamma L^2_x} \le \tfrac{cT^2}{2\gamma^2}\,\|w - \overline{w}\|^2_{\mathcal{G}^0_\gamma},$$

where $\overline{v} \in \mathcal{G}^0(J)$ solves (2.32) for $\overline{w} \in \mathcal{G}^0(J)$ instead of v on the right. Fixing a large γ, we obtain a unique fixed point $v \in \mathcal{G}^0(J)$ solving (2.32).

2) The function $w \coloneqq u_0 + V \in C^1(\overline{J}, L^2_x)$ satisfies $w(0) = u_0$ and $Bw = 0$ due to the compatibility condition $Bu_0 = 0$. Observe that $\partial_t v \in L^2_t \mathcal{H}^{-1}_x$ by (2.32) and hence

$$A_0(t)v(t) = A_0(0)u^1 + \int_0^t \big(\partial_t A_0(s)v(s) + A_0(s)\partial_t v(s)\big)\,\mathrm{d}s$$

in $\mathcal{H}^{-1}_x$ for $t \in [0,T]$. Similarly, we have

$$D(t)w(t) = D(0)u_0 + \int_0^t \big(\partial_t D(s)w(s) + D(s)v(s)\big)\,\mathrm{d}s.$$

These identities, (2.32) and (2.23) imply

$$\begin{aligned} Lw(t) &= (A_0 v)(t) + \sum_{j=1}^3 A_j^{\mathrm{co}} \partial_j (u_0 + V(t)) + D(t)w(t) \\ &= A_0(0)u^1 + \sum_{j=1}^3 A_j^{\mathrm{co}} \partial_j u_0 + D(0)u_0 + \int_0^t \big(L'v(s) + \partial_t D(s)w(s)\big)\,\mathrm{d}s \\ &= f(0) + \int_0^t \partial_t f(s)\,\mathrm{d}s = f(t). \end{aligned}$$

The uniqueness statement of Theorem 2.10 now yields $u = w \in C^1(\overline{J}, L^2_x)$. □

Lemma 2.14. *Let (2.22) be true for $G = \mathbb{R}^3_+$ and $k = 1$, and let $u \in C^1(\overline{J}, L^2_x)$ solve (2.12). Then u belongs to $C(\overline{J}, \mathcal{H}^1_{\mathrm{ta},x})$.*

Proof. Let $R_{1/n}$ be a mollifier with respect to (x_1, x_2) and set $u_n = R_{1/n}u$ for $n \in \mathbb{N}$. This function belongs to $C(\overline{J}, \mathcal{H}^1_{\mathrm{ta},x}) \cap C^1(\overline{J}, L^2_x)$ and tends to u in $C(\overline{J}, L^2_x)$ by the properties of u. As in (2.20), we have $BR_{1/n}u = 0$ and

$$Lu_n = R_{\frac{1}{n}} f + [A_0, R_{\frac{1}{n}}]\partial_t u + [D, R_{\frac{1}{n}}]u =: f_n.$$

Hence, $\partial_3 A_3^{\mathrm{co}} u_n$ is contained in $L^2_{t,x}$. We can thus apply ∂_j for $j \in \{1,2\}$ to $Lu_n = f_n$ resulting in

$$L\partial_j u_n = R_{\frac{1}{n}}\big(\partial_j f - \partial_j A_0 \partial_t u - \partial_j D u\big) + [A_0, R_{\frac{1}{n}}]\partial_j \partial_t u + [D, R_{\frac{1}{n}}]\partial_j u =: g_n.$$

In view of the regularity of u and the data, Proposition 1.3 implies that g_n tends to $\partial_j f - \partial_j A_0 \partial_t u - \partial_j D u$ in $L^2_{t,x}$ and $\partial_j u_n(0)$ to $\partial_j u_0$ in L^2_x as $n \to \infty$. Hence, $(\partial_j u_n)_n$ is Cauchy in $C(\overline{J}, L^2_x)$ by (2.13). This means that (u_n) converges to u in $C(\overline{J}, \mathcal{H}^1_{\mathrm{ta},x})$. □

The next lemma on normal regularity does not involve boundary conditions. If $Bu = 0$, then u satisfies its regularity assumptions thanks to the previous lemmas.

Lemma 2.15. *Let (2.22) be true for $G = \mathbb{R}^3_+$ and $k = 1$. Assume that $u \in C^1(\overline{J}, L^2_x) \cap C(\overline{J}, \mathcal{H}^1_{\mathrm{ta},x})$ solves $Lu = f$. Then u belongs to $\mathcal{G}^1(J)$.*

PROOF. Observe that $\partial_3 u_{\mathrm{ta}}$ is contained in $C(\overline{J}, L^2_x)$ by (2.26) and the assumptions. We want to regularize u in x_3-direction and then use (2.30) to pass to the limit. To this end, we simplify a bit and restrict ourselves to the special case $A_0 = \operatorname{diag}(a^e, a^m)$ as in the proof of Proposition 2.12. More importantly, for technical reasons we shift the functions on $\mathbb{R}^3_+$ downwards by $S_\delta v(x) = v(x', x_3 + \delta)$ for $x = (x', x_3) \in \mathbb{R}^3$ with $x_3 > -\delta$ and $\delta > 0$. This destroys the boundary condition, which is not needed fortunately. We then extend the function $S_\delta v$ by 0 to $\mathbb{R}^3$ and apply the mollifier $R_{1/n}$ in x_3. Afterwards we will restrict to $x \in \mathbb{R}^3_+$ again. This allows us to justify the calculations below, see the proof of Lemma 4.1 in [**53**] for the details.

We write $u_\delta = S_\delta u$, $f_\delta = S_\delta f$, and L_δ for the operator with shifted coefficients $S_\delta A_0$ and $S_\delta D$. One has $L_\delta u_\delta = f_\delta$, and hence $\partial_3 u_{\delta,\mathrm{ta}}$ is contained in $C_t L^2_x$. We compute

$$L_\delta R_{\frac{1}{n}} u_\delta = R_{\frac{1}{n}} f_\delta + [S_\delta A_0, R_{\frac{1}{n}}]\partial_t u_\delta + [S_\delta D, R_{\frac{1}{n}}] u_\delta =: g_{\delta,n}$$

for $n \in \mathbb{N}$ and $\delta > 0$ with $\frac{1}{n} < \delta$. Then $R_{1/n} u_\delta$ belongs to $\mathcal{G}^1(J \times \mathbb{R}^3_+)$, $g_{\delta,n}$ tends to f_δ in $\mathcal{H}^1_{t,x}$ as $n \to \infty$, and $u^0_{\delta,n} := R_{1/n} S_\delta u_0$ to $S_\delta u_0$ in $\mathcal{H}^1_x$ by our assumptions and Proposition 1.3. We can now derive the analogue of formula (2.29) for $R_{\frac{1}{n}} u_\delta$ as before. As in (2.30) it follows

$$\begin{aligned} \|\partial_3 (R_{\frac{1}{n}} - R_{\frac{1}{m}}) S_\delta u_{\mathrm{no}}(t)\|_{L^2_x} \le c \Big[&\|u^0_{\delta,n} - u^0_{\delta,m}\|_{\mathcal{H}^1_x} + \|(R_{\frac{1}{n}} - R_{\frac{1}{m}})\partial' S_\delta u\|_{\mathcal{G}^0(t)} \\ &+ \|g_{\delta,n} - g_{\delta,m}\|_{\mathcal{H}^1_{t,x}} \Big] + c\int_0^t \|\partial_3 (R_{\frac{1}{n}} - R_{\frac{1}{m}}) S_\delta u_{\mathrm{no}}\|_{L^2_x} \,\mathrm{d}s, \end{aligned}$$

where $u_{\mathrm{no}} = (u_3, u_6)$ and $\partial' u = (\partial_1 u, \partial_2 u, \partial_3 u_{\mathrm{ta}})$. In view of the above comments, the terms in brackets tend to 0 as $n, m \to \infty$, and hence the same is true for the left-hand side due to Gronwall's inequality.

As $(R_{1/n} S_\delta u_{\mathrm{no}})_n$ has the limit $S_\delta u_{\mathrm{no}}$ in $\mathcal{G}^0$, we infer that $\partial_3 S_\delta u_{\mathrm{no}} = S_\delta \partial_3 u_{\mathrm{no}}$ is an element of $\mathcal{G}^0(J \times \mathbb{R}^3_+)$. The strong continuity of $(S_\delta)_\delta$ on $L^2(\mathbb{R}^3_+)$ then implies that also $\partial_3 u_{\mathrm{no}}$ belongs to $\mathcal{G}^0(J \times \mathbb{R}^3_+)$, so that u is an element of $\mathcal{G}^1$. □

We can now show the linear wellposedness result in $\mathcal{H}^3(\mathbb{R}^3_+)$.

THEOREM 2.16. *Let (2.22) be true for $G = \mathbb{R}^3_+$ and $k = 3$. Assume that the compatibility conditions (2.23) hold. Then there is a unique solution $u \in \mathcal{G}^3(J \times \mathbb{R}^3_+)$ of (2.12). It satisfies (2.27).*

PROOF. 1) Theorem 2.10 provides a unique solution u in $\mathcal{G}^0(J)$. If we can prove that u belongs to $\mathcal{G}^3(J)$, then it satisfies (2.27) by Proposition 2.12. Lemmas 2.13, 2.14 and 2.15 already show that u is an element of $\mathcal{G}^1$.

2) For the iteration steps, we also assume that $\partial_t A_0 \in \mathcal{F}^3(J)$. Let $\tilde{L}$ be the operator with $\tilde{D} = D + \partial_t A_0 \in \mathcal{F}^3(J)$ instead of D. We then have $\partial_t u \in \mathcal{G}^0(J)$, $B\partial_t u = 0$, and

$$\tilde{L}\partial_t u = \partial_t f - \partial_t D u = \tilde{f} \in \mathcal{H}^1_{t,x}.$$

As $\partial_t u(0) = u^1 \in \mathcal{H}^1_x$ and $Bu^1 = 0$ by (2.24) and (2.23), step 1) yields $\partial_t u \in \mathcal{G}^1$.

Due to this regularity, $L\partial_j u = \partial_j f - \partial_j A_0 \partial_t u - \partial_j D u =: f_j$ belongs to $\mathcal{H}^1_{t,x}$, and we have $\partial_j u(0) = \partial_j u_0 \in \mathcal{H}^1_x$ for $j \in \{1,2,3\}$. If $j \neq 3$ also the boundary conditions $B\partial_j u = 0$ are preserved. Lemmas 2.14 and 2.15 thus show that $\partial_j u$ is an element of $C(\overline{J}, \mathcal{H}^1_x)$ for $j = \{1, 2\}$. In particular, $\partial_3 u$ is contained in $C^1(\overline{J}, L^2_x) \cap C(\overline{J}, \mathcal{H}^1_{\mathrm{ta},x})$, and hence in $\mathcal{G}^1(J)$ due to Lemma 2.15 and $f_3, \partial_3 u_0 \in \mathcal{H}^1$. Therefore u belongs to $\mathcal{G}^2(J)$.

3) To show $u \in \mathcal{G}^3(J)$, we proceed similarly as in step 2), writing $\hat{L}$ for the operator with D replaced by $\hat{D} = D + 2\partial_t A_0 \in \mathcal{F}^3(J)$. Because of step 2) and the assumption on A_0, the function $\partial_t^2 u \in \mathcal{G}^0(J)$ satisfies

$$\hat{L}\partial_t^2 u = \partial_t^2 f - \partial_t^2 A_0 \partial_t u - \partial_t^2 D u - 2\partial_t D \partial_t u \ \in \mathcal{H}^1_{t,x}$$

and $B\partial_t^2 u = 0$. Starting from $\tilde{L}\partial_t u = \tilde{f}$ and $\partial_t u(0) = u^1$, we compute

$$\partial_t^2 u(0) = A_0(0)^{-1}\Big[\partial_t f(0) - \partial_t D(0) u_0 - D(0)u^1 - \partial_t A_0(0) u^1 - \sum_{j=1}^{3} A_j \partial_j u^1\Big] = u^2,$$

see (2.23). Using also (2.24), we infer $u^2 \in \mathcal{H}^1_x$ and $Bu^2 = 0$. So $\partial_t^2 u$ belongs to $\mathcal{G}^1(J)$ by step 1).

We next look at $\partial_{jt} u \in \mathcal{G}^0(J)$ for $j \in \{1, 2, 3\}$. By the above established properties of u and the regularity of A_0, the map

$$\tilde{L}\partial_{jt} u = \partial_{jt} f - \partial_j A_0 \partial_t^2 u - \partial_{jt} A_0 \partial_t u - \partial_{jt} D u - \partial_t D \partial_j u - \partial_j D \partial_t u$$

is an element of $\mathcal{H}^1_{t,x}$ and $\partial_{jt} u(0) = \partial_j u^1$ of $\mathcal{H}^1_x$. Because of $B\partial_{jt} u = 0$ and $B\partial_j u^1 = 0$ if $j \neq 3$, step 1) yields $\partial_{jt} u \in \mathcal{G}^1(J)$ in this case. As in step 2), $\partial_t u$ thus belongs to $C(\overline{J}, \mathcal{H}^2_x)$ by Lemma 2.15.

Finally, we treat $\partial_{jk} u \in \mathcal{G}^0(J)$ for $j, k \in \{1, 2, 3\}$. Again the right-hand side

$$L\partial_{jk} u = \partial_{jk} f - \partial_{jk} A_0 \partial_t u - \partial_j A_0 \partial_{kt} u - \partial_k A_0 \partial_{jt} u - \partial_{jk} D u - \partial_j D \partial_k u - \partial_k D \partial_j u$$

is contained in $\mathcal{H}^1_{t,x}$ and $\partial_{jk} u(0) = \partial_{jk} u_0$ in $\mathcal{H}^1_x$. For $j, k \leq 2$, we deduce that $\partial_{jk} u \in \mathcal{G}^1(J)$ again from step 1). Hence, $\partial_{j3} u$ belongs to $C^1(\overline{J}, L^2_x) \cap C(\overline{J}, \mathcal{H}^1_{\mathrm{ta},x})$ and thus to $\mathcal{G}^1(J)$ by Lemma 2.15. The remaining property $\partial_{33} u \in C(\overline{J}, \mathcal{H}^1_x)$ is shown analogously.

4) We still have to remove the extra assumption $\partial_t A_0 \in \mathcal{F}^3(J)$. To this end, one has to regularize A_0 and to approximate u_0 in $\mathcal{H}^3_x$ so that the compatibility conditions (2.23) remain valid. This technical step is omitted, see Lemma 4.8 in [**53**]. □

As in Remarks 1.11 and 1.12, we list variants of the above theorem, which can be shown analogously.

Remark 2.17. Theorem 2.16 remains valid if we replace in (2.22) the differentiation order 3 by $m \in \mathbb{N}$ throughout, and impose corresponding variants of the compatibility conditions (2.23). If $m = 2$, the second-order derivatives of A_0 also have to belong to $L^\infty_t L^3_x$. On the other hand, in (2.11) we can replace $\mathcal{F}^3 = \mathcal{F}^3(J \times \mathbb{R}^3_+)$ by $\hat{\mathcal{F}}^3_\infty = \hat{\mathcal{F}}^3 + W^{3,\infty}_{t,x}$ and $\hat{\mathcal{H}}^2_x = \hat{\mathcal{H}}^2_x(\mathbb{R}^3_+)$ by $\hat{\mathcal{H}}^2_\infty = \hat{\mathcal{H}}^2_x + W^{2,\infty}$. We use this notation below also for other domains G. ◊

2.4. The quasilinear problem on $\mathbb{R}^3_+$

We now treat the nonlinear Maxwell system

$$\begin{aligned} a_0(u)\partial_t u + \sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j u + d(u)u &= f, \quad t \ge 0,\ x \in G, \\ Bu = E \times \nu &= 0, \quad t \ge 0,\ x \in \partial G, \\ u(0) &= u_0, \quad x \in G. \end{aligned} \tag{2.33}$$

on $G = \mathbb{R}^3_+$ under the hypothesis

$$\begin{aligned} &a_0, d \in C^3(G \times \mathbb{R}^6, \mathbb{R}^{6\times 6}), \qquad a_0 = a_0^\top \ge \eta I, \\ &\forall\, r > 0 : \sup_{|\xi|\le r} \max_{0\le|\alpha|\le 3} \|\partial_x^\alpha a_0(\cdot,\xi)\|_{L_x^\infty}, \|\partial_x^\alpha d(\cdot,\xi)\|_{L_x^\infty} < \infty, \\ &u_0 \in \mathcal{H}^3(G,\mathbb{R}^6), \qquad \forall\, T > 0 : f \in \mathcal{H}^3((0,T)\times G, \mathbb{R}^6) = \mathcal{H}^3_{t,x}(T), \\ &\rho^2 \ge \|u_0\|^2_{\mathcal{H}^3_x} + \|f\|^2_{\mathcal{H}^3_{t,x}} + \|f(0)\|^2_{\mathcal{H}^2_x} + \|\partial_t f(0)\|^2_{\mathcal{H}^1_x} + \|\partial_t^2 f(0)\|^2_{L^2_x}. \end{aligned} \tag{2.34}$$

We state a version of Lemma 1.14 on G in the framework of $\mathcal{G}^3$ and $\mathcal{F}^3$. The proof is similar and thus omitted, see §2 of [**52**] or §7.1 in [**51**].

Lemma 2.18. *Let a be as in (2.34) and $\gamma \ge 0$.*

a) Let $v \in \mathcal{G}^3(J)$ with $\|v\|_\infty \le r$. Then $\|a(v)\|_{\tilde{F}^m_\infty(J)} \le \kappa(r)(1 + \|v\|^3_{\mathcal{G}^3(J)})$.

b) Let $v, w \in \mathcal{G}^2(J)$ with norm $\le r$. Then $\|a(v) - a(w)\|_{\mathcal{G}^2_\gamma(J)} \le \kappa(r)\|v - w\|_{\mathcal{G}^2_\gamma(J)}$.

c) Let $v_0 \in \mathcal{H}^2_x$ with $\|v_0\|_\infty \le r_0$. Then $\|a(v_0)\|_{\hat{\mathcal{H}}^2_\infty} \le \kappa_0(r_0)(1 + \|v_0\|^2_{\mathcal{H}^2_x})$.

d) Let $v_0, w_0 \in \mathcal{H}^2_x$ with norm $\le r_0$. Then $\|a(v_0) - a(w_0)\|_{\mathcal{H}^2_x} \le \kappa_0(r_0)\|v_0 - w_0\|^2_{\mathcal{H}^2_x}$.

We need a nonlinear variant of the compatibility conditions (2.23), derived similarly:

$$\begin{aligned} &Bu^j = 0 \ \text{ for } j \in \{0,1,2\}, \quad u^1 \coloneqq a_0(u_0)^{-1}\Big(f(0) - d(u_0)u_0 - \sum_{j=1}^{3} A_j\partial_j u_0\Big), \\ &u^2 \coloneqq a_0(u_0)^{-1}\Big(\partial_t f(0) - \partial_\xi a_0(u_0)[u^1,u^1] - d(u_0)u^1 - \partial_\xi d(u_0)[u^1,u_0] \\ &\qquad - \sum_{j=1}^{3} A_j\partial_j u^1\Big), \end{aligned} \tag{2.35}$$

where $u^0 \coloneqq u_0$ and the derivatives in $\xi \in \mathbb{R}^6$ act on the vectors in brackets bilinearly. We set $u^j \eqqcolon S_j(u_0, f, a_0, d)$. Lemma 2.18 (and analogous versions for $\partial_\xi a_0$ and $\partial_\xi d$) imply that S_j satisfy estimates as in (2.24) and related Lipschitz bounds, see Lemma 2.4 in [**52**] or Lemma 7.7 in [**51**].

We now state the local wellposedness result on $G = \mathbb{R}^3_+$, using the data manifold

$$\mathcal{D}_{T,a_0,d}((u_0,f),r) = \big\{(u_0,f) \in \mathcal{H}^3_x \times \mathcal{H}^3_{t,x}(T) \;\big|\; \|u_0\|^2_{\mathcal{H}^3_x} + \|f\|^2_{\mathcal{H}^3_x} \le r^2,\ (2.35) \text{ true}\big\}.$$

It is endowed with the metric of $\mathcal{H}^3_x \times \mathcal{H}^3_{t,x}(T)$ unless something else is stated.

THEOREM 2.19. *Let* (2.34) *and* (2.35) *hold with* $G = \mathbb{R}^3_+$. *The following assertions are true.*

a) There is a maximal existence time $T_+ = T_+(u_0, f) \in (T_0(\rho), \infty]$ *and a unique (maximal) solution* $u = \Psi(u_0, f) \in \mathcal{G}^3([0, T_+))$ *of* (2.33). *(For* $T_0(\rho)$ *see the proof.)*

b) Let $T_+ < \infty$. *Then* $\lim_{t\to T_+} \|u(t)\|_{\mathcal{H}^3_x} = \infty$ *and* $\sup_{t<T_+} \|u(t)\|_{W^{1,\infty}_x} = \infty$.

c) Let $T \in [0, T_+)$. *Then there is a radius* $\delta > 0$ *such that for all* (v_0, g) *in* $\mathcal{D}_{T,a_0,d}((u_0, f), \delta)$ *we have* $T_+(v_0, f) > T$ *and* $\Psi : \mathcal{D}_T((u_0, f), \delta) \to \mathcal{G}^3(T)$ *is continuous. Moreover,* $\Psi : (\mathcal{D}_T((u_0, f), \delta), \|\cdot\|_{\mathcal{H}^2_x \times \mathcal{H}^2_{t,x}(T)}) \to \mathcal{G}^2(T)$ *is Lipschitz.*

PROOF. The arguments are only sketched, see [**53**] for full proofs in a more general setting. As a fixed-point space we employ

$$\mathcal{E}(R, T) = \big\{v \in \mathcal{G}^{3-}(J) \,\big|\, \|v\|_{\mathcal{G}^{3-}(J)} \le R,\ \partial_j^t v(0) = S_j(u_0, f, a_0, d),\ j \in \{0, 1, 2\}\big\}.$$

One can check as before that this set is complete for the metric induced by the norm of $\mathcal{G}^2(T)$. (Compared to Lemma 1.15, we need the time derivatives in the norm because of the new initial conditions.) It is now more difficult to show that $\mathcal{E}(R, T)$ is non-empty for sufficiently large R, see Lemma 2.6 in [**53**]. For $v \in \mathcal{E}(R, T)$ one sets $A_0 = a_0(v)$ and $D = d(v)$ with a corresponding linear operator $L(v)$. Observe that the nonlinear compatibility conditions (2.35) for a_0 and d coincide with the linear ones (2.23) for A_0 and D because of the initial conditions in $\mathcal{E}(R, T)$.

We then choose R, γ and T_0 depending on ρ as in Lemma 1.16. Here we have also to control first and second time derivatives of A_0 and D at $t = 0$ since these bounds enter the higher-order energy estimate (2.27) via (2.22). Theorem 2.16 (and Remark 2.17) now yield a unique solution $u = \Phi(v) \in \mathcal{G}^3(T_0)$ of $L(v) = f$ with $Bu = 0$ and $u(0) = u_0$. The constants are arranged so that $\|u\|_{\mathcal{G}^3} \le R$ and Φ is strictly contractive. The compatibility conditions of this equation also imply that u belongs to $\mathcal{E}(R, T_0)$. So we have solved (2.33) on $[0, T_0]$.

The proofs of Lemma 1.17 as well as of assertion a) and the first part of b) of Theorem 1.19 follow a general pattern so that these arguments and statements can easily be extended to the present situation. The second part of assertion b) is more involved. One proceeds as in the proof of Theorem 1.19 and estimates $\partial_x^\alpha u$ in terms of the data and $\omega = \sup_{t<T_+} \|u(t)\|_{W^{1,\infty}_x}$. This works as before if $\alpha_3 = 0$. Using the resulting inequality in this case, one then follows the iteration steps of the proof of Theorem 2.16. The analogous difficulty occurs in the core step 3) of the proof of Theorem 1.19 c), the other steps do not change much. □

We discuss variants of the above theorem in Remark 2.23 in greater generality.

2.5. The main wellposedness result

We now treat the Maxwell system on an open and bounded set $G \subseteq \mathbb{R}^3$ with a smooth boundary. Again we first look at the linear problem

$$Lu = \sum_{j=0}^{3} A_j \partial_j u + Du = f, \quad t \ge 0,\ x \in G,$$

$$Bu = E \times \nu = 0, \quad t \ge 0,\ x \in \partial G, \qquad (2.36)$$
$$u(0) = u_0, \quad x \in G,$$

assuming hypothesis (2.22), and the nonlinear system (2.33) under asumption (2.34). We further assume the compatibility conditions (2.23) respectively (2.35) on G are satisfied.

Below we state the wellposedness theorems on the spatial domain G. We cannot give full proofs (since they are too lengthy and technical), but rather explain main features and differences to the case $G = \mathbb{R}^3_+$. We start with the localization procedure which is the core point.

Localization. In principle, we follow a standard localization procedure. One covers ∂G by finitely many charts $\varphi_i : U_i \to V_i$ and adds another open set U_0 with $\overline{U_0} \subseteq G$ so that $U_0, U_1, \dots, U_N$ cover $\overline{G}$. Let $\varphi_0 : U_0 \to U_0$ be the identity, $\{\theta_0, \cdots, \theta_N\}$ be a smooth partition of unity for this cover and $V_i^+ = \{x \in V_i \,|\, x_3 > 0\}$ be the range $\varphi_i(U_i \cap G)$ for $i \ge 1$, where we put $\bar{V}_i^+ = \{x \in V_i \,|\, x_3 \ge 0\}$. Set $\psi_i = \varphi_i^{-1}$ and $\Phi_i : L^2(U_i) \to L^2(V_i)$; $\Phi_i u = u \circ \psi_i$, with inverse $\Phi_i^{-1} v = v \circ \varphi_i$.

First, let u solve (2.22). One looks at the transformed function $\Phi_i(\theta_i u) \in L^2(J \times V_i^+)$. After extension by 0, the map $\Phi_0(\theta_0 u)$ solves the original problem (1.8) on $\mathbb{R}^3$. For $i \ge 1$ and $v \in L^2(J \times V_i^+)$, we compute

$$\begin{aligned}
\tilde{L}^i v &:= \Phi_i L \Phi_i^{-1} v = \Phi_i\Big(A_0(\partial_t v) \circ \varphi_i + \sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j (v \circ \varphi_i) + Dv \circ \varphi_i\Big) \\
&= \Phi_i A_0 \partial_t v + \Phi_i D v + \sum_{j=1}^{3} A_j^{\mathrm{co}} \Phi_i\Big(\sum_{k=1}^{3} (\partial_k v) \circ \varphi_i \, \partial_j \varphi_{i,k}\Big) \\
&= \Phi_i A_0 \partial_t v + \Phi_i D + \sum_{k=1}^{3} \Big(\sum_{j=1}^{3} A_j^{\mathrm{co}} \Phi_i(\partial_j \varphi_{i,k})\Big) \partial_k v \\
&=: \sum_{k=0}^{3} \tilde{A}_k^i \partial_k v + \tilde{D}^i v, \qquad (2.37)
\end{aligned}$$

where $\varphi_{i,k}$ is the k-th component of φ_i. Note that $\tilde{A}_k^i$ is symmetric and $\tilde{A}_0^i \ge \eta I$.

One can check that there is a constant $\tau > 0$ and for each $i \ge 1$ an index $k(i) \in \{1, 2, 3\}$ such that $|\partial_{k(i)} \varphi_{i,3}| \ge \tau$ on U_i, see Lemma 5.1 in [**51**]. To simplify, we only look at the case that $k(i) = 3$ and $\partial_3 \varphi_{i,3} \ge \tau$. Since $U_i \cap \partial G$ equals $U_i^0 := \{x \in U_i \,|\, \varphi_{i,3}(x) = 0\}$, the vector $\nabla \varphi_{i,3}(x)$ is orthogonal to ∂U_i^0 at x and hence given by $\nabla \varphi_{i,3}(x) = -\kappa_i(x)\nu(x)$ for the smooth function $\kappa_i = -\nu \cdot \nabla \varphi_{i,3}$. On U_i^0 the boundary condition $Bu = E \times \nu = 0$ thus is equivalent to $\kappa_i B u = \kappa_i E \times \nu = 0$. Using this reformulation, the transformation then yields the new boundary condition

$$\tilde{B}^i v := \Phi_i(\kappa_i B \Phi_i^{-1} v) = \Phi_i(\kappa_i B) v$$

on $\bar{V}_i^+$. The coefficients $\tilde{A}_j^i$, $\tilde{D}^i$, and $\tilde{B}^i$ are extended to $\mathbb{R}^3_+$ or its closure keeping their properties. (This extension is omitted below, cf. Chapter 5 of [**51**].)

The matrices $\tilde{A}_j^i$ for $j \in \{1,2,3\}$ and $\tilde{B}^i$ now depend (smoothly) on the space variable x. Moreover, $\tilde{A}_j^i$ has twelve instead of four non-zero entries, whereas $\tilde{B}^i$ still has just two of them. For instance, we have

$$\tilde{A}_3^i = \begin{pmatrix} 0 & -\tilde{S}_3^i \\ \tilde{S}_3^i & 0 \end{pmatrix}, \qquad \tilde{S}_3^i = \begin{pmatrix} 0 & -\Phi_i\partial_3\varphi_{i,3} & \Phi_i\partial_2\varphi_{i,3} \\ \Phi_i\partial_3\varphi_{i,3} & 0 & -\Phi_i\partial_1\varphi_{i,3} \\ -\Phi_i\partial_2\varphi_{i,3} & \Phi_i\partial_1\varphi_{i,3} & 0 \end{pmatrix}.$$

In the calculations of this chapter, these changes lead to plenty of additional commutators, which are partly hard to control and which make the iteration arguments much more complicated. Even worse is the change from A_3^{co} to $\tilde{A}_3^i$ since the form of A_3^{co} plays a crucial role in the above treatment of normal regularity. It is not clear how to extend the corresponding arguments to the transformed operator.

Instead, one passes to the function $v^i = R_i^{-1}\Phi_i(\theta_i u)$ for invertible matrices $R_i(x) = \operatorname{diag}(\hat{R}_i(x), \hat{R}_i(x))$ that are defined using φ_i. Let L^i be the operator on $\mathbb{R}^3_+$ with coefficients $A_j^i = R_i^\top \tilde{A}_j^i R_i$ and $D^i = R_i^\top \tilde{D}^i R_i - \sum_{j=1}^3 A_j^i \partial_j R_i^{-1} R_i$ and set $B^i = \hat{R}_i^\top \tilde{B}^i R_i$ as well as $v_0^i = R_i^{-1}\Phi_i(\theta_i u_0)$. We then infer $v^i(0) = v_0^i$, $B^i v^i = 0$, and

$$\begin{aligned} L^i v^i &= \sum_{j=0}^3 A_j^i \partial_j (R_i^{-1}\Phi_i(\theta_i u)) + R_i^\top \tilde{D}^i \Phi_i(\theta_i u) - \sum_{j=1}^3 A_j^i \partial_j R_i^{-1} \Phi_i(\theta_i u) \\ &= \sum_{j=0}^3 R_i^\top \tilde{A}_j^i \partial_j \Phi_i(\theta_i u) + R_i^\top \tilde{D}^i \Phi_i(\theta_i u) = R_i^\top \tilde{L}^i \Phi_i(\theta_i u) \\ &= R_i^\top \Phi_i(L(\theta_i u)) = R_i^\top \Phi_i(\theta_i f) + R_i^\top \Phi_i\Big(\sum_{j=1}^3 A_j^{\mathrm{co}} \partial_j \theta_i u\Big) =: f^i(f,u). \end{aligned}$$

We now choose R_3^i so that $A_3^i = A_3^{\mathrm{co}}$ and $B^i = B^{\mathrm{co}}$, namely

$$R_i = \operatorname{diag}(\hat{R}_i, \hat{R}_i) \quad \text{with} \qquad \hat{R}_i = \frac{1}{\sqrt{\Phi_i\partial_3\varphi_{i,3}}} \begin{pmatrix} 1 & 0 & \Phi_i\partial_1\varphi_{i,3} \\ 0 & 1 & \Phi_i\partial_2\varphi_{i,3} \\ 0 & 0 & \Phi_i\partial_3\varphi_{i,3} \end{pmatrix}.$$

A computation shows $\hat{R}_i^\top \tilde{S}_3^i \hat{R}_i = S_3$, and hence $R_i^\top \tilde{A}_3^i R_i = A_3^{\mathrm{co}}$. We write $B = (B_1 \; 0_{3\times 3})$ and recall that $B_1 E = E \times \nu = -\sum_j \nu_j S_j E$. It follows

$$\Phi_i(\kappa_i B_1) = -\sum_{j=1}^3 \Phi_i(\kappa_i \nu_j) S_j = \sum_{j=1}^3 \Phi_i(\partial_j \varphi_{i,3}) S_j = \tilde{S}_3^i.$$

Therefore we obtain $B^i = \hat{R}_i^\top \tilde{B}_i R_i = B^{\mathrm{co}}$.

We only sketch the remaining steps, see Chapter 5 of [**51**] for details. One can check that the new coefficients and data satisfy hypothesis (2.22) with A_1^{co} and A_2^{co} replaced by $A_1^i, A_2^i \in \mathcal{F}^3_{\mathrm{sym}}$ and the compatibility conditions (2.23) on $G = \mathbb{R}^3_+$. Moreover, the relevant norms of the transformed functions on $\mathbb{R}^3_+$ are bounded by a constant $c(G)$ times the same norms of the corresponding functions on G.

So the apriori estimates of Theorems 2.16 and 1.9 for v^i on $\mathbb{R}^3_+$ respectively $\theta_0 u$ on $\mathbb{R}^3$ yield analogous inequalities for u on G. This also shows uniqueness of solutions. To construct a solution, one solves the transformed problems on V_i^+ and V_0 and glues the solutions together. For this, we need another set of cut-off functions $\sigma_i \in C_c^\infty(U_i)$ which are 1 on the support of θ_i. We have included the original solution u into f^i to compensate for error terms with θ_i when deriving the transformed system. This forces us to set up a fixed-point argument on the space of functions v in $\mathcal{G}^3(J \times G)$ which satisfy $\partial_t^j v(0) = u^j$ for $j = \{0,1,2\}$ and u^j from (2.23) given by the data. The fixed point then solves the problem.

Main wellposedness results. Besides Theorem 1.9 on $\mathbb{R}^3$, the above reasoning requires a variant of Theorem 2.16 on $\mathbb{R}^3_+$ for the modified coefficients A_1^i and A_2^i, where A_0^i and D^i have the same properties as before and $A_3 = A_3^{\mathrm{co}}$ and $B = B^{\mathrm{co}}$ do not change. This modification has quite unexpected consequences, as many estimates and iteration arguments become much more involved since the additional commutators intertwine our three steps (time, tangential and normal regularity) to a larger extent. However, with some effort these problems can be solved.

We first state the linear wellposedness result for L^2-solutions which follows by localization from Theorem 2.10. The result is surely older, but it is also a special case of the more general Theorem 1.4 of [**19**], which is actually devoted to boundary regularity. (The validity of (2.15) in this case is derived in Lemma 4.2 of [**34**] based on [**53**].)

THEOREM 2.20. *Assume that* (2.11) *is true for* G *with* $\|\partial_t A_0\|_\infty, \|D\|_\infty \le r$. *Then there is a unique solution* $u \in C(\overline{J}, L_x^2)$ *of* (2.36)*, and it satisfies* (2.15) *on* G *as well as* (2.13) *with a factor 2 on the right-hand side.*

Theorem 1.1 of [**53**] yields the linear wellposedness theorem in $\mathcal{G}^3(J \times G)$.

THEOREM 2.21. *Let* (2.22) *be true for* $k = 3$ *and the compatibility conditions* (2.23) *hold. Then there is a unique solution* $u \in \mathcal{G}^3(J \times G)$ *of* (2.36)*. It satisfies* (2.27) *on* G.

Remark 2.17 remains valid on G (after replacing $\mathbb{R}^3_+$ by G). The first part of the proof of the quasilinear result on G is close to that of Theorem 2.19 on $\mathbb{R}^3_+$, now based on Theorem 2.21. In Theorem 2.19 for $G = \mathbb{R}^3_+$, when proving the blow-up condition in $W_x^{1,\infty}$ and of the continuous dependence on data, we have omitted steps in which arguments from the derivation of the apriori estimate are extended to the nonlinear level. On G the procedure is even more involved since one has to perform the localization procedure also for the nonlinear problem. So we skip these arguments, too. The following local wellposedness result is proved in Theorem 5.3 of [**52**]. We use the notation from Theorem 2.19.

THEOREM 2.22. *Let* (2.34) *and the compatibility conditions* (2.35) *hold. Then the following assertions are true.*

a) There is a maximal existence time $T_+ = T_+(u_0, f) \in (T_0(\rho), \infty]$ *and a unique (maximal) solution* $u = \Psi(u_0, f) \in \mathcal{G}^3([0, T_+))$ *of* (2.33).

b) Let $T_+ < \infty$. *Then* $\lim_{t \to T_+} \|u(t)\|_{\mathcal{H}_x^3} = \infty$ *and* $\sup_{t < T_+} \|u(t)\|_{W_x^{1,\infty}} = \infty$.

c) Let $T \in [0, T_+)$. Then there is a radius $\delta > 0$ such that for all (v_0, g) in $\mathcal{D}_{T,a_0,d}((u_0, f), \delta)$ we have $T_+(v_0, f) > T$ and $\Psi : \mathcal{D}_T((u_0, f), \delta) \to \mathcal{G}^3(T)$ is continuous. Moreover, $\Psi : (\mathcal{D}_T((u_0, f), \delta), \|\cdot\|_{\mathcal{H}^2_x \times \mathcal{H}^2_{t,x}(T)}) \to \mathcal{G}^2(T)$ is Lipschitz.

Example 1.21 directly carries over from $\mathbb{R}^3$ to G. In Theorem 5.3 of [**52**] more general results were shown which we sketch below, cf. Remark 1.20.

Remark 2.23. a) In [**52**] one allows for unbounded domains G having a 'uniformly smooth' boundary (e.g. a compact, smooth one). Theorem 5.3 in [**52**] actually deals with solutions on an interval (T_-, T_+) containing 0.

b) One also obtains solutions in $\mathcal{G}^m$ for data in $\mathcal{H}^m$ or C^m with $m \geq 3$ and $m \in \mathbb{N}$, assuming higher-oder compatibility conditions.

c) One can admit nonlinearities a_0 and d taking values in an open subset $U \subseteq \mathbb{R}^6$. The necessary modifications are like those described in Remark 1.20.

d) Also boundary data $Bu = g$ from the space $\bigcap_{j=0}^m \mathcal{H}^j(J, \mathcal{H}^{m+\frac{1}{2}-j}(G))$ can be included. The corresponding linear result in $\mathcal{G}^0(J)$ is taken from [**19**] where it is assumed that the coefficients are constant outside a compact set. This leads to a restriction on a_0 and d in [**52**]: They have to converge if $|x| \to \infty$ (if G is unbounded).

e) For the linear system on G , [**51, 52**] show finite speed of propagation. $\diamondsuit$

The local wellposedness theory for general hyperbolic systems would require much more regularity for the above theorem, see [**26**] or [**49**]. In [**48**] we establish results analogous to Theorem 2.22 for corresponding interface problems, and in [**47**] for so-called absorbing boundary conditions. For these, in [**42**] an existence result was proven (without uniqueness or continuous dependence on data).

CHAPTER 3

Exponential decay caused by conductivity

In this last chapter we use the wellposedness Theorem 2.22 to show global existence and exponential decay to 0 for small initial data in the presence of a strictly positive conductivity σ. The result is taken from [**34**]. Its proof is based on a standard procedure for quasilinear problems, going back to [**37**] at least. Besides local wellposedness, it uses the energy estimates for $\partial_t^k u$ with $k \in \{0,1,2,3\}$ including the dissipation terms $\|\sigma^{1/2}\partial_t^k E(t)\|_{L_x^2}^2$. One further needs an observability-type estimate for the time-differentiated linear problem (inspired by [**20**] in our case) to control the norms of $\partial_t^k u$ in $\mathcal{H}_x^{3-k}$ by the dissipation terms, globally in time. This can only be done up to error terms which are small, but only in the stronger topology of $\mathcal{G}^3$. Astonishingly, a variant of the apriori estimates from Chapter 2 allows us to bound space by time derivatives, again globally in time. Our presentation is based on our paper [**34**]. We also show results on Helmholtz decompositions on G following Section X1.1 in [**16**], which are used in the derivation of the observability-type estimate.

3.1. Introduction and theorem on decay

We study the special case of the Maxwell system (1.1) given by

$$\begin{aligned} \partial_t\big(\varepsilon(E)E\big) &= \operatorname{curl} H - \sigma E, && t \ge 0,\ x \in G, \\ \partial_t\big(\mu(H)H\big) &= -\operatorname{curl} E, && t \ge 0,\ x \in G, \\ \operatorname{tr}_{\mathrm{ta}} E = E \times \nu &= 0, && t \ge 0,\ x \in \Gamma, \\ E(0) = E_0, \quad H(0) &= H_0, && x \in G, \end{aligned} \tag{3.1}$$

on an open, bounded, simply connected domain $G \subseteq \mathbb{R}^3$ with smooth boundary $\partial G = \Gamma$. As before, we also use the equivalent version

$$\begin{aligned} \varepsilon^{\mathrm{d}}(E)\partial_t E &= \operatorname{curl} H - \sigma E, && t \ge 0,\ x \in G, \\ \mu^{\mathrm{d}}(H)\partial_t H &= -\operatorname{curl} E, && t \ge 0,\ x \in G, \\ \operatorname{tr}_{\mathrm{ta}} E = E \times \nu &= 0, && t \ge 0,\ x \in \Gamma, \\ E(0) = E_0, \quad H(0) &= H_0, && x \in G, \end{aligned} \tag{3.2}$$

for energy estimates, with the differentiated coefficients

$$\varepsilon^{\mathrm{d}}(\cdot,\xi) := \varepsilon(\cdot,\xi) + \Big(\sum_{l=1}^{3} \partial_{\xi_k}\varepsilon_{jl}(\cdot,\xi)\xi_l\Big)_{jk}, \qquad \mu^{\mathrm{d}} \ \text{ analogously defined.}$$

We modify our assumptions (2.34) and impose the hypothesis

$$\begin{aligned}&\varepsilon, \mu, \varepsilon^{\mathrm{d}}, \mu^{\mathrm{d}}, \sigma \in C^3(\overline{G} \times \mathbb{R}^3, \mathbb{R}^{3\times 3}_{\mathrm{sym}}),\\ &\sigma \geq \eta I > 0, \quad \varepsilon(\cdot, 0), \mu(\cdot, 0), \varepsilon^{\mathrm{d}}(\cdot, 0), \mu^{\mathrm{d}}(\cdot, 0) \geq 2\eta I \quad \text{on } G,\end{aligned} \tag{3.3}$$

thus assuming that σ is uniformly positive definite. The material laws in Example 1.21 on G fulfill the above condititions for ε and μ. By continuity, we can fix a constant $\kappa > 0$ such that

$$\varepsilon(\cdot, \xi), \mu(\cdot, \xi), \varepsilon^{\mathrm{d}}(\cdot, \xi), \mu^{\mathrm{d}}(\cdot, \xi) \geq \eta \qquad \text{if } \ |\xi| \leq 2\kappa. \tag{3.4}$$

The initial fields shall also satisfy the magnetic divergence and boundary conditions now. Together with the simple connectedness of G, these conditions exclude non-zero H_0 in the kernel of curl, see ?? for the case $\mu = 1$, which would produce a constant-in-time solution $(E, H) = (0, H_0)$ of our system (3.1). Let C_S be the norm of the Sobolev embedding $\mathcal{H}^2(G) \hookrightarrow L^\infty(G)$. We assume

$$\begin{aligned}&E_0, H_0 \in \mathcal{H}^3(G, \mathbb{R}^3), \quad \|E_0\|^2_{\mathcal{H}^3_x} + \|H_0\|^2_{\mathcal{H}^3_x} \leq r^2, \quad \text{where } \ r \leq \kappa / C_S, \qquad (3.5)\\ &\operatorname{div}(\mu(H_0)H_0) = 0, \quad \operatorname{tr}_{\mathrm{no}}(\mu(H_0)H_0) = 0, \quad \operatorname{tr}_{\mathrm{ta}} E_0 = \operatorname{tr}_{\mathrm{ta}} E^1 = \operatorname{tr}_{\mathrm{ta}} E^2 = 0,\\ &E^1 := \varepsilon^{\mathrm{d}}(E_0)^{-1}[\operatorname{curl} H_0 - \sigma E_0], \qquad H^1 := -\mu^{\mathrm{d}}(H_0)^{-1} \operatorname{curl} E_0,\\ &E^2 := \varepsilon^{\mathrm{d}}(E_0)^{-1}\big[\operatorname{curl} H^1 - \sigma E^1 - (\nabla_E \varepsilon^{\mathrm{d}}(E_0)E^1) \cdot E^1\big].\end{aligned}$$

Note that the initial data are bounded by κ. In view of (3.4), Theorem 2.22 and Remark 2.23 provide a unique local solution $u = (E, H) \in \mathcal{G}^3(J_+)$ of (3.1) with a maximal existence time $T_+ = T_+(E_0, H_0)$ and $J_+ = [0, T_+)$. Moreover, (1.2) and Lemma 2.4 show

$$\operatorname{div}\big(\mu(H(t))H(t)\big) = 0, \qquad \operatorname{tr}_{\mathrm{no}}(\mu(H(t))H(t)) = 0, \tag{3.6}$$

$$\operatorname{div}\big(\varepsilon(E(t))E(t)\big) = \operatorname{div}\big(\varepsilon(E_0)E_0\big) - \int_0^t \operatorname{div}\big(\sigma E(s)\big)\,\mathrm{d}s \tag{3.7}$$

for $t \in J_+$. We state the decay result for small data, see Theorem 2.2 in [**34**].

Theorem 3.1. *Let (3.3) and (3.5) hold. Then there exist a radius $r > 0$ in (3.5) and numbers $M, \omega > 0$ such that $T_+(E_0, H_0) = \infty$ and*

$$\max_{k \in \{0,1,2,3\}} \|\partial_t^k (E(t), H(t))\|_{\mathcal{H}^{3-k}_x} \leq M \mathrm{e}^{-\omega t} \qquad \textit{for all } \ t \geq 0.$$

The theorem is proved at the end of Section 3.3. In [**43**] we prove such a result for boundary damping $H \times \nu + (\zeta(E \times \nu)) \times \nu = 0$ with $\zeta(x) \geq \eta$ on ∂G, where G is strictly starlike. These theorems are the first decay results for quasilinear Maxwell systems on domains. On $\mathbb{R}^3$ one has global existence for small data and certain material laws exploiting dispersive estimates, see §11.6 of [**44**] (with polynomial decay), [**35**] or [**50**]. In [**6**] convergence to equilibria is shown for a class of hyperbolic systems with damping on $\mathbb{R}^m$ (not including the Maxwell system).

There are some decay results for linear Maxwell systems with conductivity. In [**21**], [**33**], [**38**] and [**41**], for instance, isotropic constitutive relations and (semilinear) strictly positive conductivity were considered, whereas matrix-valued coefficients were investigated only recently in [**20**], see also [**18**] for related results on boundary obervability. Partially positive conductivities were treated in

[39] in some cases, as well as in [41] for constant $\varepsilon, \mu > 0$ and in [20] without decay rates.

We discuss the background of the proof which employs time-differentiated versions of (3.1). For the sake of brevity, we set

$$\widehat{\varepsilon}_k = \begin{cases} \varepsilon(E), & k = 0, \\ \varepsilon^{\mathrm{d}}(E), & k \in \{1,2,3\}, \end{cases} \qquad \widehat{\mu}_k = \begin{cases} \mu(H), & k = 0, \\ \mu^{\mathrm{d}}(H), & k \in \{1,2,3\}. \end{cases} \tag{3.8}$$

For $k \in \{0,1,2,3\}$, we then obtain the system

$$\begin{aligned} \partial_t(\widehat{\varepsilon}_k \partial_t^k E) &= \operatorname{curl} \partial_t^k H - \sigma \partial_t^k E - \partial_t f_k, && t \in J_+,\ x \in G, \\ \partial_t(\widehat{\mu}_k \partial_t^k H) &= -\operatorname{curl} \partial_t^k E - \partial_t g_k, && t \in J_+,\ x \in G, \\ \operatorname{tr}_{\mathrm{ta}} \partial_t^k E = 0, &\quad \operatorname{tr}_{\mathrm{no}}(\widehat{\mu}_k \partial_t^k H) = -\operatorname{tr}_{\mathrm{no}} g_k, && t \in J_+,\ x \in \Gamma, \end{aligned} \tag{3.9}$$

with the commutator terms

$$\begin{aligned} &f_0 = f_1 = 0, \quad f_2 = \partial_t \varepsilon^{\mathrm{d}}(E)\, \partial_t E, \quad f_3 = \partial_t^2 \varepsilon^{\mathrm{d}}(E)\, \partial_t E + 2\partial_t \varepsilon^{\mathrm{d}}(E)\, \partial_t^2 E, \\ &g_0 = g_1 = 0, \quad g_2 = \partial_t \mu^{\mathrm{d}}(H)\, \partial_t H, \quad g_3 = \partial_t^2 \mu^{\mathrm{d}}(H)\, \partial_t H + 2\partial_t \mu^{\mathrm{d}}(H)\, \partial_t^2 H. \end{aligned} \tag{3.10}$$

Note that the electric boundary condition remains unchanged. The magnetic one is well-defined in $\mathcal{H}^{-1/2}(\Gamma)$ by Theorem 2.1 and the divergence relations

$$\operatorname{div}(\mu^{\mathrm{d}}(H)\partial_t^k H) = -\operatorname{div} g_k, \qquad \operatorname{div}(\varepsilon^{\mathrm{d}}(E)\partial_t^k E) = -\operatorname{div}(\sigma \partial_t^{k-1} E + f_k), \tag{3.11}$$

which follow from (3.6) and from (3.7) for $k \in \{1,2,3\}$. Estimate (3.16) below shows that all maps $\partial_t f_k, \partial_t g_k, \operatorname{div} f_k, \operatorname{div} g_k$ belong to $L^\infty(J, L^2(G))$ for $T < T_+$.

For the energy estimate, it is useful to consider the equivalent version of (3.9)

$$\begin{aligned} \varepsilon^{\mathrm{d}}(E)\, \partial_t \partial_t^k E &= \operatorname{curl} \partial_t^k H - \sigma \partial_t^k E - \tilde{f}_k, && t \in J_+,\ x \in G, \\ \mu^{\mathrm{d}}(H)\, \partial_t \partial_t^k H &= -\operatorname{curl} \partial_t^k E - \tilde{g}_k, && t \in J_+,\ x \in G, \\ \operatorname{tr}_{\mathrm{ta}} \partial_t^k E &= 0, && t \in J_+,\ x \in \Gamma, \end{aligned} \tag{3.12}$$

for $k \in \{0,1,2,3\}$, which is based on (3.2) and has the new commutator terms

$$\tilde{f}_k = \sum_{j=1}^{k} \binom{k}{j} \partial_t^j \varepsilon^{\mathrm{d}}(E)\, \partial_t^{k+1-j} E, \qquad \tilde{g}_k = \sum_{j=1}^{k} \binom{k}{j} \partial_t^j \mu^{\mathrm{d}}(H)\, \partial_t^{k+1-j} H,$$

where we put $\tilde{f}_0 = \tilde{g}_0 = 0$. We further introduce the quantities

$$\begin{aligned} e_k(t) &= \tfrac{1}{2} \max_{0 \le j \le k} \big(\|\widehat{\varepsilon}_k^{1/2} \partial_t^j E(t)\|_{L^2_x}^2 + \|\widehat{\mu}_k^{1/2} \partial_t^j H(t)\|_{L^2_x}^2 \big), \qquad e = e_3, \\ d_k(t) &= \max_{0 \le j \le k} \|\sigma^{1/2} \partial_t^j E(t)\|_{L^2_x}^2, \qquad d = d_3, \\ z_k(t) &= \max_{0 \le j \le k} \big(\|\partial_t^j E(t)\|_{\mathcal{H}_x^{k-j}}^2 + \|\partial_t^j H(t)\|_{\mathcal{H}_x^{k-j}}^2 \big), \qquad z = z_3, \end{aligned} \tag{3.13}$$

for $k \in \{0,1,2,3\}$ and $t \in J_+$. The choice of weights simplifies some estimates below. Here e_k is related to energy and d_k to dissipation. We stress that d_k only contains the electric field and that d_k and e_k only involve time derivatives, in contrast to z_k.

To control the norms of (E, H) and the above quantities, we set $\delta_0 = \min\{1, \kappa/C_S\}$ and take $\delta \in (0, \delta_0]$, to be fixed in the proof of Theorem 3.1.

Theorem 2.22 then yields a radius $r(\delta) \in (0, \delta]$ such that for all $r \in (0, r(\delta)]$ and (E_0, H_0) as in (3.5) we have $T_+ > 1$ and $z(t) \le \delta^2$ for $t \in [0, 1]$. Given such (E_0, H_0), we now introduce the final time

$$T_*(E_0, H_0) = T_* = \sup\big\{T \in [1, T_+) \,\big|\, \forall\, t \in [0, T] : z(t) \le \delta^2\big\}. \tag{3.14}$$

The blow-up condition in Theorem 2.22 implies that $T_+ > T_*$, and hence

$$z(T_*) = \max_{0\le k\le 3} \big(\|\partial_t^k E(T_*)\|^2_{\mathcal{H}_x^{3-k}} + \|\partial_t^k H(T_*)\|^2_{\mathcal{H}_x^{3-k}}\big) = \delta^2 \quad (\text{if} \;\; T_* < \infty) \tag{3.15}$$

by continuity. We will *suppose* that $T_* < \infty$. For sufficiently small $\delta > 0$ (and thus $r > 0$), below we then show that $z(T_*) < \delta^2$. This contradiction to (3.15) then establishes $T_* = \infty$. The exponential decay in Theorem 3.1 will be a by-product of this argument, see the end of Section 3.3.

In the following we always look at solutions with data (E_0, H_0) as in (3.5) for some $r \in (0, r(\delta)]$ and a corresponding solution $u = (E, H)$ of (3.1) on $J_* = [0, T_*)$, which thus satisfies $z(t) \le \delta^2 \le 1$ for all $t \in J_*$. The constants c, c_k, C or C_k below do not depend on $s, t \in J_*$, T_*, $\delta \in (0, \delta_0]$, $r \in (0, r(\delta_0)]$, or (E_0, H_0) satisfying (3.5).

Using Lemmas 1.8 and 2.18 and formula (3.14), one can estimate the above commutator terms by

$$\begin{aligned}
\|\widehat{\varepsilon}_k(t)\|_{L^\infty_x}, \|\widehat{\mu}_k(t)\|_{L^\infty_x}, \|\widehat{\varepsilon}_k^{-1}(t)\|_{L^\infty_x}, \|\widehat{\mu}_k^{-1}(t)\|_{L^\infty_x} &\le c,\\
\|\partial^\alpha \widehat{\varepsilon}_j(t)\|_{L^2_x}, \|\partial^\alpha \widehat{\mu}_j(t)\|_{L^2_x} &\le c(z_k^{1/2}(t) + \delta_{\alpha_0=0}),\\
\max_{k\in\{2,3\}, j\in\{0,1\}} \big(\|\partial_t^j f_k(t)\|_{\mathcal{H}_x^{4-j-k}} + \|\partial_t^j g_k(t)\|_{\mathcal{H}_x^{4-j-k}}\big) &\le cz(t),\\
\|f_2(t)\|_{L^2_x}, \|g_2(t)\|_{L^2_x}, \|f_3(t)\|_{L^2_x}, \|g_3(t)\|_{L^2_x} &\le ce_2^{1/2}(t),\\
\|\tilde{f}_k(t)\|_{\mathcal{H}_x^{3-k}}, \|\tilde{g}_k(t)\|_{\mathcal{H}_x^{3-k}} &\le cz(t)
\end{aligned} \tag{3.16}$$

for $j, k \in \{0, 1, 2, 3\}$, $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = k > 0$, $t \in J_*$, where we set $\delta_{\alpha_0=0} = 1$ if $\alpha_0 = 0$ and $\delta_{\alpha_0=0} = 0$ if $\alpha_0 > 0$. The second summand in the second line of (3.16) arises if all derivatives in ∂^α are applied to the x-variable of ε or μ.

3.2. Helmholtz decompositions

In this section we discuss kernel and range of div and curl for a bounded open subset $G \subseteq \mathbb{R}^3$ with C^2-boundary. To simplify a bit, we assume that G is simply connected in the main results. (See [**10**] or [**16**] for the general case.) We obtain on one hand spaces between which curl acts bijectively, and on the other hand decompositions of a given L^2-map into gradient and curl fields. Both types of results will often be used in the proof of Theorem 3.1, but they also play a key role in many areas of analysis and its applications. We follow the treatment in Section IX.1 of [**16**], see also [**4**] and [**10**].

We introduce subspaces of $\mathcal{H}(\mathrm{div})$ and $\mathcal{H}(\mathrm{curl})$ on G, where $\Gamma_1, \cdots, \Gamma_N$ are the components of $\Gamma = \partial G$ and N denotes the kernel of $\mathrm{div}, \mathrm{curl} : L^2_x \to \mathcal{H}_x^{-1}$.

$$\mathrm{N}_0(\mathrm{curl}) = \{v \in \mathrm{N}(\mathrm{curl}) \,|\, \mathrm{tr}_{\mathrm{ta}}\, v = 0\}, \quad \mathrm{N}_0(\mathrm{div}) = \{v \in \mathrm{N}(\mathrm{div}) \,|\, \mathrm{tr}_{\mathrm{no}}\, v = 0\},$$

$$\mathrm{N}^\Gamma(\mathrm{div}) = \big\{v \in \mathrm{N}(\mathrm{div}) \,\big|\, \forall\, j : \textstyle\int_{\Gamma_j} \mathrm{tr}_{\mathrm{no}}\, v \,\mathrm{d}\sigma = 0\big\}, \quad \mathcal{N} = \mathrm{N}(\mathrm{div}) \cap \mathrm{N}_0(\mathrm{curl}),$$

$$\mathcal{H}^1_{\mathrm{ta}0}(G) = \{v \in \mathcal{H}^1(G)^3 \mid \mathrm{tr}_{\mathrm{ta}}\, v = 0\} = \mathcal{H}(\mathrm{div}) \cap \mathcal{H}_0(\mathrm{curl}).$$

The last identity is shown in Theorem XI.1.3 of [**16**], compare also Proposition 3.11 below. The first three spaces are endowed with the L^2-norm, and we use the $\mathcal{H}^1$-norm for $\mathcal{H}^1_{\mathrm{ta0}}(G)$ and other subspaces of $\mathcal{H}^1_x$.

We start with the basic observation that smooth functions in the kernel of curl and div are locally given by gradient and curl fields, respectively. Later on we show global variants of this fact.

LEMMA 3.2. *Let $G \subseteq \mathbb{R}^3$ be open and $Q \subseteq \overline{Q} \subseteq G$ be a cuboid.*

a) Let $u \in \mathrm{N}(\mathrm{curl}) \cap C^1(G)^3$. Then there is a map $\varphi \in C^2(Q)$ with $u = \nabla\varphi$.

a) Let $u \in \mathrm{N}(\mathrm{div}) \cap C^1(G)^3$. Then there is a map $w \in C^1(Q)^3$ with $u = \mathrm{curl}\, w$.

PROOF. a) Let a be a corner of Q. For $x \in Q$ we set

$$\varphi(x) = \int_{a_3}^{x_3} u_3(x_1, x_2, \xi_3)\,\mathrm{d}\xi_3 + \int_{a_2}^{x_2} u_2(x_1, \xi_2, a_3)\,\mathrm{d}\xi_2 + \int_{a_1}^{x_1} u_1(\xi_1, a_2, a_3)\,\mathrm{d}\xi_1.$$

The assumption $\mathrm{curl}\, u = 0$ yields $\partial_1 u_2 = \partial_2 u_1$ and $\partial_1 u_3 = \partial_3 u_1$, so that $\partial_1 \varphi = u_1$. The other components are treated similarly.

b) Analogously, we define

$$v_1(x) = \int_{a_2}^{x_2} u_3(x_1, \xi_2, x_3)\,\mathrm{d}\xi_2 - \int_{a_3}^{x_3} u_3(x_1, x_2, \xi_3)\,\mathrm{d}\xi_3$$

and v_1^a with u replaced by

$$u^a(x) = (u_1(a_1, x_2, x_3), u_2(x_1, a_2, x_3), u_3(x_1, x_2, a_3)).$$

The components v_2, v_2^a, v_3 and v_3^a are given by circular permutations of the indices. Using $\mathrm{div}\, u = 0$, one computes $\mathrm{curl}\, v = -3u + u^a$ and $\mathrm{curl}\, v = -2u^a$. So $w = -\frac{1}{3}v + \frac{1}{6}v^a$ satisfies $\mathrm{curl}\, w = u$. □

To show our decay Theorem 3.1, we will need div-curl estimates that control the $\mathcal{H}^1$-norm by the norms in $\mathcal{H}(\mathrm{div})$ and $\mathcal{H}(\mathrm{curl})$ plus boundary terms, see Theorem XI.1.3 of [**16**] and our Proposition 3.11. We start with the simple result on $\mathbb{R}^3$.

LEMMA 3.3. *The space $\mathcal{H}(\mathrm{div}, \mathbb{R}^3) \cap \mathcal{H}(\mathrm{curl}, \mathbb{R}^3)$ (endowed with $\|\cdot\|_{\mathrm{div}} + \|\cdot\|_{\mathrm{curl}}$) is equal to $\mathcal{H}^1(\mathbb{R}^3)^3$ with equivalent norms.*

PROOF. The proof of Theorem 2.2 shows that $C_c^\infty(\mathbb{R}^3)^3$ is dense in $\mathcal{H}(\mathrm{div}, \mathbb{R}^3) \cap \mathcal{H}(\mathrm{curl}, \mathbb{R}^3)$. Hence it is enough to prove the equivalence of the norms for test functions v. Observe that $\mathrm{curl}\,\mathrm{curl}\, v = \nabla \mathrm{div}\, v - \Delta v$. Integration by parts, see (2.6) and (2.8), thus yields

$$\begin{aligned}\int_{\mathbb{R}^3} \big(|\mathrm{curl}\, v|^2 + |\mathrm{div}\, v|^2\big)\,\mathrm{d}x &= \int_{\mathbb{R}^3} \big(v \cdot \mathrm{curl}\,\mathrm{curl}\, v - v \cdot \nabla \mathrm{div}\, v\big)\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^3} v \cdot \Delta v\,\mathrm{d}x = \int_{\mathbb{R}^3} |\nabla v|^2\,\mathrm{d}x. \qquad \square\end{aligned}$$

The intersection of the kernels of div and curl will play an important role below. We first show that it contains only smooth functions.

COROLLARY 3.4. *Let $G \subseteq \mathbb{R}^3$ be open. Then the space $\mathrm{N}(\mathrm{div}) \cap \mathrm{N}(\mathrm{curl})$ is contained in $C^\infty(G)^3$.*

PROOF. We choose a bounded open set $U \subseteq \overline{U} \subseteq G$ and a cut-off function $\varphi \in C_c^\infty(G)$ with $\varphi = 1$ on U. Take $v \in \mathrm{N}(\mathrm{div}) \cap \mathrm{N}(\mathrm{curl})$. Let $\tilde{v}$ be the 0-extension of φv to $\mathbb{R}^3$. We then have

$$\operatorname{curl} \tilde{v} = \begin{cases} \nabla\varphi \times v, & \text{on } G, \\ 0, & \text{on } \mathbb{R}^3 \setminus G, \end{cases} \qquad \operatorname{div} \tilde{v} = \begin{cases} \nabla\varphi \cdot v, & \text{on } G, \\ 0, & \text{on } \mathbb{R}^3 \setminus G. \end{cases}$$

Hence, $\tilde{v}$ belongs to $\mathcal{H}(\mathrm{div}, \mathbb{R}^3) \cap \mathcal{H}(\mathrm{curl}, \mathbb{R}^3)$ which is equal to $\mathcal{H}^1(\mathbb{R}^3)^3$ by Lemma 3.3. As a result, v is an element of $\mathcal{H}^1_{\mathrm{loc}}(G)^3$. Since $\partial_j \operatorname{curl} v = \operatorname{curl} \partial_j v$ and $\partial_j \operatorname{div} v = \operatorname{div} \partial_j v$, we can iterate the procedure obtaining $v \in \mathcal{H}^k_{\mathrm{loc}}(G)^3$ for all k and then $v \in C^\infty(G)^3$ by Sobolev's embedding. □

Our analysis is based on the following functional analytic tool due to Peetre. Recall that a bounded operator T between Banach spaces is an isomorphism onto its range if it satisfies the lower bound $\|x\| \leq c\|Tx\|$ for all x. Peetre's lemma admits also a compact perturbation in this context.

LEMMA 3.5. *Let X, Y and Z be Banach spaces, $T \in \mathcal{B}(X, Y)$, and $K \in \mathcal{B}(X, Z)$, such that K is compact and there is a constant $c > 0$ with*

$$\forall\, x \in X : \qquad \|x\|_X \leq c\big(\|Tx\|_Y + \|Kx\|_Z\big) \tag{3.17}$$

Then the kernel $\mathrm{N}(T)$ has a finite dimension, the range $\mathrm{R}(T)$ is closed and the restriction $T : \tilde{X} \to \mathrm{R}(T)$ is an isomorphism, where $\tilde{X}$ is a closed subspace of X with $X = \tilde{X} \oplus \mathrm{N}(T)$.

PROOF. 1) On $\mathrm{N}(T)$ we have $\|x\|_X \leq c\|Kx\|_Z$ so that the range $\mathrm{R}(K)$ is closed and $\tilde{K} : \mathrm{N}(T) \to \mathrm{R}(K)$ is an isomorphism. Since $\tilde{K}$ is also compact, the set $K(B_{\mathrm{N}(T)}(0,1))$ has a compact closure, which contains a ball of $\mathrm{R}(K)$. It follows that $\mathrm{R}(K)$, and thus $\mathrm{N}(T)$, are finite dimensional.

2) There is a closed subspace $\tilde{X}$ with $X = \tilde{X} \oplus \mathrm{N}(T)$. Set $\tilde{T} = T\restriction_{\tilde{X}}$. For $\tilde{x} \in \tilde{X}$ and $x_0 \in \mathrm{N}(T)$, we have $T(\tilde{x} + x_0) = \tilde{T}\tilde{x}$ and so $\tilde{T} : \mathrm{N}(T) \to \mathrm{R}(T)$ is bijective and continuous. We claim that there is a constant $\tilde{c} > 0$ with $\|\tilde{x}\| \leq c\|\tilde{T}\tilde{x}\|$ for all $\tilde{x} \in \tilde{X}$. This lower bound then implies the result.

To show the claim, we suppose that there are vectors $\tilde{x}_n \in \tilde{X}$ of norm 1 such that $(\tilde{T}\tilde{x}_n)$ tends to 0 in Y as $n \to \infty$. Compactness yields a converging subsequence $(K\tilde{x}_{n_k})_k$. The estimate (3.17) then shows that $(\tilde{x}_{n_k})$ tends to some $\tilde{x}$ in $\tilde{X}$. This vector also has norm 1 which contradicts $\tilde{T}\tilde{x} = \lim_k \tilde{T}\tilde{x}_{n_k} = 0$. □

We now determine the range of ∇ and show its closedness in three settings, where we set $\mathcal{A}^1(G) = \{\varphi \in \mathcal{H}^1(G) \,|\, \Delta\varphi = 0\}$. The orthogonality in the direct sums refers to the usual L^2 scalar product.

PROPOSITION 3.6. *Let $G \subseteq \mathbb{R}^3$ be open and bounded with a Lipschitz boundary. Then the follwing assertion hold.*

a) The ranges $\nabla\mathcal{H}^1(G)$, $\nabla\mathcal{H}^1_0(G)$, and $\nabla\mathcal{A}^1(G)$ are closed in $L^2(G)^3$.

b) $L^2(G)^3 = \nabla\mathcal{H}^1(G) \oplus_\perp \mathrm{N}_0(\mathrm{div}) = \nabla\mathcal{H}^1_0(G) \oplus_\perp \nabla\mathcal{A}^1(G) \oplus_\perp \mathrm{N}_0(\mathrm{div})$.

c) $L^2(G)^3 = \nabla\mathcal{H}^1_0(G) \oplus_\perp \mathrm{N}(\mathrm{div})$.

Proof. a) We use Lemma 3.5 with $X = \mathcal{H}^1(G)$, $Y = L^2(G)^3$, $Z = L^2(G)$, $T = \nabla$, and $K = I$. Then (3.17) holds and K is compact since G is bounded. So $\nabla\mathcal{H}^1(G)$ is closed. This argument also applies to $\nabla\mathcal{H}^1_0(G)$ and $\nabla\mathcal{A}^1(G)$.

b) Let $v \in L^2(G)^3$ be perpendicular to $\nabla\mathcal{H}^1(G)$. For all $\varphi \in C_c^\infty(G)$, we infer

$$0 = \int_G v \cdot \nabla\varphi \,\mathrm{d}x = \langle \varphi, \operatorname{div} v\rangle_{\mathcal{H}^1_0}$$

so that v belongs to N(div). Theorem 2.1 then yields $\mathcal{H}^{1/2}(\Gamma) = \operatorname{tr}\mathcal{H}^1(G)$ and

$$0 = \int_G v \cdot \nabla\varphi \,\mathrm{d}x = \langle \operatorname{tr}\varphi, \operatorname{tr}_{\mathrm{no}} v\rangle_{\mathcal{H}^{1/2}(\Gamma)} \tag{3.18}$$

for every $\varphi \in \mathcal{H}^1(G)$. This means that $\operatorname{tr}_{\mathrm{no}} v = 0$ in $\mathcal{H}^{-\frac{1}{2}}(\Gamma)$, and so v is an element of $\mathrm{N}_0(\mathrm{div})$ thanks to Theorem 2.1 b). Conversely, let $v \in \mathrm{N}_0(\mathrm{div})$. Because of $\operatorname{tr}_{\mathrm{no}} v = 0$ equation (3.18) now yields that $v \perp \nabla\mathcal{H}^1(G)$. We have proven the first part of part b). Assertion c) is shown similarly.

Finally, let $v = \nabla\varphi$ and $w = \nabla\psi$ for some $\varphi \in \mathcal{H}^1_0(G)$ and $\psi \in \mathcal{A}^1(G)$. Using again (2.8), we derive

$$\int_G v \cdot w \,\mathrm{d}x = \int_G \nabla\varphi \cdot \nabla\psi \,\mathrm{d}x = -\int_G \varphi\Delta\psi \,\mathrm{d}x = 0,$$

and thus $\nabla\mathcal{H}^1_0(G) \perp \nabla\mathcal{A}^1(G)$. Further, take $u = \nabla\chi$ for some $\chi \in \mathcal{H}^1(G)$. Theorem 8.3 in [**24**] provides a function $\psi \in \mathcal{H}^1(G)$ with $\Delta\psi = 0$ and $\operatorname{tr}\psi = \operatorname{tr}\chi$. Hence, $\varphi := \psi - \chi$ belongs to $\mathcal{H}^1_0(G)$ and so $\nabla\mathcal{H}^1(G) = \nabla\mathcal{H}^1_0(G) \oplus_\perp \nabla\mathcal{A}^1(G)$. □

To invert curl, we now determine its kernel. Here we use the simple connectedness of G.

Proposition 3.7. *Let $G \subseteq \mathbb{R}^3$ be open, bounded and simply connected with a Lipschitz boundary. We then have* $\mathrm{N}(\mathrm{curl}) = \nabla\mathcal{H}^1(G)$ *and* $\mathrm{N}(\mathrm{curl}) \cap \mathrm{N}_0(\mathrm{div}) = \{0\}$.

Proof. The inclusion $\nabla\mathcal{H}^1(G) \subseteq \mathrm{N}(\mathrm{curl})$ follows from the identity $\operatorname{curl}\nabla = 0$. For the converse, take $w \in \mathrm{N}(\mathrm{curl})$ with $w \perp \nabla\mathcal{H}^1(G)$. We have to show that $w = 0$. Proposition 3.6 yields $w \in \mathrm{N}_0(\mathrm{div})$ and so w is smooth due to Corollary 3.4. For each open cuboid $Q \subseteq \overline{Q} \subseteq G$, Lemma 3.2 provides a potential $q \in C^2(Q)$ with $\nabla q = w$. We thus obtain

$$0 = \int_Q w \cdot \nabla\varphi \,\mathrm{d}x = \int_Q \nabla q \cdot \nabla\varphi \,\mathrm{d}x = -\int_Q \varphi\Delta q \,\mathrm{d}x$$

for every $\varphi \in C_c^\infty(Q)$ (extended by 0 to an element of $H^1(G)$). This means that $\Delta q = 0$ on G, and so q is real analytic thanks to Theorem 2.2.10 of [**22**]. By simple connectedness and using analytic continuation, see Theorem 16.15 in [**46**], we can extend q from some Q to an analytic function on G satisfying $\Delta q = 0$ and $\nabla q = w$ on G.

Moreover, q belongs to $L^2(G)$ since $\nabla q \in L^2(G)$ and G is bounded. We also have $\operatorname{tr}_{\mathrm{no}} \nabla q = \operatorname{tr}_{\mathrm{no}} w = 0$. By means of (2.8), we then compute

$$0 = \int_G q \operatorname{div}\nabla q \,\mathrm{d}x = -\int_G |\nabla q|^2 \,\mathrm{d}x.$$

Hence, $w = \nabla q$ vanishes, which yields the first assertion as $\nabla\mathcal{H}^1(G)$ is closed due to Proposition 3.6. The second assertion then follows from this proposition. □

In the next main result we show the invertibility of curl on vector fields in a subspace of $\mathrm{N}(\mathrm{tr}_{\mathrm{no}})$, where we also determine the range of curl in this setting and obtain several decompositions. Recall that $\mathcal{N} = \mathrm{N}(\mathrm{div}) \cap \mathrm{N}_0(\mathrm{curl})$. We set

$$\mathcal{A}_c^1(G) = \{\varphi \in \mathcal{A}_1(G) \,|\, \forall j: \ \mathrm{tr}\,\varphi \ \text{ is constant on } \Gamma_j\}.$$

THEOREM 3.8. *Let $G \subseteq \mathbb{R}^3$ be open, bounded and simply connected with a C^2-boundary. Set $V = \mathcal{H}^1(G)^3 \cap \mathrm{N}_0(\mathrm{div})$. The following assertions are true.*

a) The range $\mathrm{curl}\,\mathcal{H}^1(G)^3 = \mathrm{curl}\,V$ *is equal to* $\mathrm{N}^\Gamma(\mathrm{div})$ *and closed in* $L^2(G)^3$.

b) We have $\mathcal{N} = \nabla\mathcal{A}_c^1(G)$, $\mathrm{N}(\mathrm{div}) = \mathrm{curl}\,\mathcal{H}^1(G)^3 \oplus_\perp \mathcal{N}$, *and*

$$\mathrm{N}_0(\mathrm{curl}) = \nabla\mathcal{H}_0^1(G) \oplus_\perp \mathcal{N}, \quad L^2(G)^3 = \mathrm{N}^\Gamma(\mathrm{div}) \oplus_\perp \nabla\mathcal{H}_0^1(G) \oplus_\perp \mathcal{N}. \tag{3.19}$$

c) The map $\mathrm{curl} : \mathcal{H}^1(G)^3 \cap \mathrm{N}_0(\mathrm{div}) \to \mathrm{N}^\Gamma(\mathrm{div})$ *is invertible.*

PROOF. 1) We start with the first equality in assertion a). We clearly have $\mathrm{curl}\,V \subseteq \mathrm{curl}\,\mathcal{H}^1(G)^3$. Let $w \in \mathcal{H}^1(G)^3$. Proposition 3.6 yields the decomposition $w = \nabla\varphi + v$ for some $\varphi \in \mathcal{H}^1(G)$ and $v \in \mathrm{N}_0(\mathrm{div})$. We deduce that $\Delta\varphi = \mathrm{div}\,w \in L^2(G)$ and hence, by Theorem 2.1, there exists the Neumann trace $\partial_\nu\varphi := \mathrm{tr}_{\mathrm{no}}\nabla\varphi = \mathrm{tr}_{\mathrm{no}}\,w \in \mathcal{H}^{\frac{1}{2}}(\Gamma)$. Elliptic regularity, see Proposition 5.7.7 in [**55**], shows that φ belongs to $\mathcal{H}^2(G)$ and thus v to $\mathcal{H}^1(G)^3$. Since $\mathrm{curl}\,w = \mathrm{curl}\,v$, we have proven $\mathrm{curl}\,\mathcal{H}^1(G)^3 = \mathrm{curl}\,V$.

To show the closedness, take $v \in V$. Since $\mathrm{div}\,v = 0$, estimate (1.31) in Section IX.1 of [**16**], cf. Lemma 3.3, yields

$$\int_G |\nabla v|^2 \,\mathrm{d}x \le \int_G |\mathrm{curl}\,v|^2 \,\mathrm{d}x + c\int_\Gamma |v|^2 \,\mathrm{d}\sigma.$$

For $X = V$, $Y = L^2(G)^3$, $Z = L^2(\Gamma)^3$, $T = \mathrm{curl}$ and $K = \mathrm{tr}$, Lemma 3.5 then implies the closedness of $\mathrm{curl}\,V$. (Recall that $\mathrm{tr}\,\mathcal{H}^1(G) = \mathcal{H}^{\frac{1}{2}}(\Gamma)$ is compactly embedded into $L^2(\Gamma)$.)

2) We next determine the complement of $\mathrm{curl}\,\mathcal{H}^1$. Take $w \in L^2(G)^3$ with $w \perp \mathrm{curl}\,\mathcal{H}^1(G)^3$. For every $v \in \mathcal{H}_0^1(G)^3$, formula (2.6) yields

$$0 = \int_G w \cdot \mathrm{curl}\,v \,\mathrm{d}x = \langle v, \mathrm{curl}\,w\rangle_{\mathcal{H}_0^1}$$

and hence $\mathrm{curl}\,w = 0$. We can now use (2.9) to compute

$$0 = \int_G w \cdot \mathrm{curl}\,v \,\mathrm{d}x - \int_G v \cdot \mathrm{curl}\,w \,\mathrm{d}x = \langle \mathrm{tr}\,v, \mathrm{tr}_{\mathrm{ta}}\,w\rangle_{\mathcal{H}^{1/2}(\Gamma)} \tag{3.20}$$

for $v \in \mathcal{H}^1(G)^3$, which yields $\mathrm{tr}_{\mathrm{ta}}\,w = 0$ in $\mathcal{H}^{-\frac{1}{2}}(\Gamma)^3$ and thus $w \in \mathrm{N}_0(\mathrm{curl})$.

Conversely, let $w \in \mathrm{N}_0(\mathrm{curl})$. Now (3.20) implies that $w \perp \mathrm{curl}\,\mathcal{H}^1(G)^3$. We have shown that $L^2(G)^3 = \mathrm{curl}\,\mathcal{H}^1(G)^3 \oplus_\perp \mathrm{N}_0(\mathrm{curl})$.

3) We next decompose $\mathrm{N}_0(\mathrm{curl})$. So let $w \in \mathrm{N}_0(\mathrm{curl})$. Proposition 3.7 provides a potential $\varphi \in \mathcal{H}^1(G)$ with $\nabla\varphi = w$. Note that $\nabla\varphi \in \mathrm{N}_0(\mathrm{curl})$. We want to show that $\mathrm{tr}\,\varphi$ is constant on each component Γ_j of Γ. To this end, let $F : U_0 \to \Gamma \cap U$ be a parametrization on a connected open set $U_0 \subseteq \mathbb{R}^2$ and $F = \psi\restriction_{U_0}$ for an inverse chart ψ. We then have $0 = \mathrm{tr}_{\mathrm{ta}}\nabla(\varphi \circ \psi) = \nabla_\xi(\varphi \circ F)$

in $\mathcal{H}^{-\frac{1}{2}}(U_0)$. Let $\chi = \varphi \circ F$ and $\chi_\delta = \rho_\delta * \chi \in C^\infty(U_0)$ for a mollifier ρ in ξ. We then obtain

$$\partial_{\xi_j} \chi_\delta = \rho_\delta * \partial_{\xi_j} \chi = 0, \tag{3.21}$$

so that χ_δ is constant. The same is true for $\operatorname{tr}\varphi$ on $\Gamma \cap U$ by approximation, and hence there are constants with $c_j = \operatorname{tr}\varphi$ on each Γ_j.

Theorem 8.3 in [**24**] yields a function $p \in \mathcal{A}^1(G)$ with $\operatorname{tr} p = c_j$ on Γ_j. The map $\varphi_0 := \varphi - p$ then belongs to $\mathcal{H}^1_0(G)$ and thus $\mathrm{N}_0(\mathrm{curl}) \subseteq \nabla\mathcal{H}^1_0(G) \oplus_\perp \nabla\mathcal{A}^1(G)$, where the orthogonality was shown in Proposition 3.6. As in (3.21) one sees that the sum belongs to $\mathrm{N}(\mathrm{tr}_{\mathrm{ta}})$, and it follows

$$(\operatorname{curl}\mathcal{H}^1(G)^3)^\perp = \mathrm{N}_0(\mathrm{curl}) = \nabla\mathcal{H}^1_0(G) \oplus_\perp \nabla\mathcal{A}^1(G) =: \nabla\mathcal{H}^1_c(G). \tag{3.22}$$

4) By (3.22), we have $\int_G v\cdot\nabla\varphi\,\mathrm{d}x = 0$ for all $v \in \operatorname{curl}\mathcal{H}^1(G)^3$ and $\varphi \in \mathcal{H}^1_c(G)$. Since $v \in \mathrm{N}(\mathrm{div})$, formula (2.8) implies that

$$0 = \int_G \varphi \operatorname{div} v\,\mathrm{d}x = \langle \operatorname{tr}\varphi, \operatorname{tr}_{\mathrm{no}} v\rangle_{\mathcal{H}^{1/2}(\Gamma)} = \sum_{j=1}^N \langle c_j \mathbb{1}, \operatorname{tr}_{\mathrm{no}} v\rangle_{\mathcal{H}^{1/2}(\Gamma_j)}.$$

Choosing $c_j = \delta_{ij}$ we obtain $v \in \mathrm{N}^\Gamma(\mathrm{div})$. The above computation also shows that $\mathrm{N}^\Gamma(\mathrm{div})$ is contained in $(\nabla\mathcal{H}^1_c(G))^\perp = \mathrm{N}_0(\mathrm{curl})$. Assertion a) thus holds.

5) We have $\mathrm{N}(\mathrm{div}) \perp \nabla\mathcal{H}^1_0(G)$ due to Proposition 3.6 c). Hence, (3.22) yields $\mathcal{N} = \nabla\mathcal{A}^1(G)$. This fact and (3.22) imply (3.19). The remaining part of b) now follows from $\mathrm{N}(\mathrm{div}) = (\nabla\mathcal{H}^1_0(G))^\perp$, (3.19) and statement a).

We have shown that the continuous map $\operatorname{curl} : V \to \mathrm{N}^\Gamma(\mathrm{div})$ is surjective, and it is injective thanks to Proposition 3.7. □

We next invert curl on vector fields in a subspace of $\mathrm{N}(\mathrm{tr}_{\mathrm{ta}})$.

THEOREM 3.9. *Let $G \subseteq \mathbb{R}^3$ be open, bounded and simply connected with a C^2-boundary. Set $W = \mathcal{H}^1_{\mathrm{ta}0}(G) \cap \mathrm{N}^\Gamma(\mathrm{div})$. Then $\operatorname{curl}\mathcal{H}^1_{\mathrm{ta}0}(G) = \operatorname{curl} W$ is closed in $L^2(G)^3$ and $\operatorname{curl} : W \to \mathrm{N}_0(\mathrm{div})$ is invertible.*

PROOF. 1) We first look at the space $W_1 = \mathcal{H}^1_{\mathrm{ta}0}(G) \cap \mathrm{N}(\mathrm{div})$ which contains W and satisfies $\operatorname{curl} W_1 \subseteq \operatorname{curl}\mathcal{H}^1_{\mathrm{ta}0}(G)$. For the converse, take $u \in \mathcal{H}^1_{\mathrm{ta}0}(G)$. Theorems 8.3 and 8.12 in [**24**] provide a function $\varphi \in \mathcal{H}^2(G) \cap \mathcal{H}^1_0(G)$ with $\Delta\varphi = \operatorname{div} u$. The field $w = u - \nabla\varphi$ thus belongs to $\mathcal{H}^1(G)^3 \cap \mathrm{N}(\mathrm{div})$. As in step 3) of the previous proof, we see that $\mathrm{tr}_{\mathrm{ta}}\nabla\varphi = 0$ implying $w \in W_1$ and $\operatorname{curl} W_1 = \operatorname{curl}\mathcal{H}^1_{\mathrm{ta}0}(G)$. Next, Theorem 3.8 implies the decomposition $\mathrm{N}(\mathrm{div}) = \mathrm{N}^\Gamma(\mathrm{div}) \oplus_\perp \mathcal{N}$. Since $\mathcal{N} \subseteq \mathrm{N}_0(\mathrm{curl})$ we conclude $\operatorname{curl}\mathcal{H}^1_{\mathrm{ta}0}(G) = \operatorname{curl} W$.

For $w \in W$, inequality (1.29) of Section IX.1 of [**16**] yields

$$\int_G |\nabla w|^2\,\mathrm{d}x \le \int_G |\operatorname{curl} w|^2\,\mathrm{d}x + c\int_\Gamma |w|^2\,\mathrm{d}\sigma.$$

We now deduce the closedness of $\operatorname{curl} W$ from Lemma 3.5 as in step 1) of the proof of Theorem 3.8.

2) To compute $\operatorname{curl} W$, let $v \perp \operatorname{curl}\mathcal{H}^1_{\mathrm{ta}0}(G)$. Formula (2.6) then yields

$$0 = \langle u, \operatorname{curl} v\rangle_{\mathcal{H}^1_0} = \int_G v \cdot \operatorname{curl} u\,\mathrm{d}x$$

for all $u \in \mathcal{H}_0^1(G)^3$, and so $\operatorname{curl} v = 0$. Conversely, take $v \in \mathrm{N}(\operatorname{curl})$. For each $u \in \mathcal{H}_{\mathrm{ta}0}^1(G)$, we compute

$$0 = \int_G u \cdot \operatorname{curl} v \,\mathrm{d}x = \int_G v \cdot \operatorname{curl} u \,\mathrm{d}x$$

using (2.9). This means that $\mathrm{N}(\operatorname{curl})$ is the complement of $\operatorname{curl} \mathcal{H}_{\mathrm{ta}0}^1(G)$ in L^2. Since $\mathrm{N}(\operatorname{curl}) = \nabla \mathcal{H}^1(G)^3$ by Proposition 3.7, we infer $\operatorname{curl} \mathcal{H}_{\mathrm{ta}0}^1(G) = \mathrm{N}_0(\operatorname{div})$ from Proposition 3.6 b).

3) It remains to check injectivity of curl on W. Theorem 3.8 shows that $W_1 \cap \mathrm{N}(\operatorname{curl}) = \mathcal{H}^1(G)^3 \cap \mathcal{N}$ and $\mathcal{N} = \nabla \mathcal{A}_c^1(G)$. The latter space is contained in $\mathcal{H}^1(G)^3$ because of Theorems 8.3 and 8.12 in [24], and hence $W_1 \cap \mathrm{N}(\operatorname{curl}) = \mathcal{N}$. Take $w \in W_1$ with $w \perp \mathcal{N}$. For all $\psi \in \mathcal{A}_c^1(G)$ we then compute

$$0 = \int_G w \cdot \nabla\psi \,\mathrm{d}x = \sum\nolimits_{j=1}^N \int_{\Gamma_j} c_j \operatorname{tr}_{\mathrm{no}} w \,\mathrm{d}\sigma$$

by means of (2.8) and $\operatorname{div} w = 0$, where $\psi = c_j$ on Γ_j. Choosing $c_j = \delta_{ij}$, we conclude that w belongs to $\mathrm{N}^\Gamma(\operatorname{div})$ and thus to W. Equation (2.8) also shows that $w \in W$ is perpendicular to $\nabla \mathcal{A}_c^1(G)$; i.e,

$$W = W_1 \otimes_\perp \mathcal{N} = W_1 \otimes_\perp (W_1 \cap \mathrm{N}(\operatorname{curl})) \tag{3.23}$$

and $\operatorname{curl} : W \to \mathrm{N}_0(\operatorname{div})$ is bijective. □

Combining the above results, we obtain two Helmholtz decompositions.

Corollary 3.10. *Let $G \subseteq \mathbb{R}^3$ be open, bounded and simply connected with a C^2-boundary. Then the following assertions are true*

*a) $L^2(G)^3 = \nabla\mathcal{H}^1(G) \oplus_\perp \operatorname{curl} \mathcal{H}_{\mathrm{ta}0}^1(G) = \nabla\mathcal{H}^1(G) \oplus_\perp \operatorname{curl} W$.
Hence, for each $v \in L^2(G)^3$ there maps $\varphi \in \mathcal{H}^1(G)$ and $w \in W$ such that $v = \nabla\varphi + \operatorname{curl} w$, where w is uniquely determined and φ is unique up to constants.*

*b) $L^2(G)^3 = \nabla\mathcal{H}_0^1(G) \oplus_\perp \nabla\mathcal{A}_c^1(G) \oplus_\perp \operatorname{curl}\mathcal{H}^1(G)^3 = \nabla\mathcal{H}_0^1(G) \oplus_\perp \nabla\mathcal{A}_c^1(G) \oplus_\perp \operatorname{curl} V$.
Hence, for each $v \in L^2(G)^3$ there are maps $\varphi \in \mathcal{H}_0^1(G)$, $p \in \nabla\mathcal{A}_c^1(G)$ and $w \in v$ such that $v = \nabla\varphi + \nabla p + \operatorname{curl} w$, where w and φ are uniquely determined and p is unique up to constants.*

Proof. Assertion a) follows from Proposition 3.6 and Theorem 3.9, and part b) from Theorem 3.8 b) and c). The decomposition is unique because of the direct product and the injectivity of curl on W and V, and since the kernel of ∇ consists of constants as G is connected. □

One can bound the $\mathcal{H}^1$-norm of a field v by its norms in $\mathcal{H}(\operatorname{curl}) \cap \mathcal{H}(\operatorname{div})$ and the $\mathcal{H}^{1/2}$-norm of $\operatorname{tr}_{\mathrm{ta}} v$ or $\operatorname{tr}_{\mathrm{no}} v$, see Corollary XI.1.1 of [16] and also our Lemma 3.3. In the proof of Proposition 3.17 below, we need a version of this result with regular, matrix-valued coefficients a (which does not directly follow from the case $a = I$ unless a is scalar). It is stated in Remark 4 of [20] with a brief indication of a proof. We present a (different) proof inspired by Lemma 4.5.5 of [13].

PROPOSITION 3.11. *Let G be bounded with $\partial G \in C^2$, $a \in W^{1,\infty}(G, \mathbb{R}^{3\times 3}_\eta)$ for some $\eta > 0$ and let $v \in \mathcal{H}(\mathrm{curl})$ fulfill $\operatorname{div}(av) \in L^2(G)$ and $\operatorname{tr}_{\mathrm{no}}(av) \in \mathcal{H}^{1/2}(\Gamma)$. Then v belongs to $\mathcal{H}^1(G)^3$ and satisfies*

$$\|v\|_{\mathcal{H}^1_x} \leq c\big(\|v\|_{\mathcal{H}(\mathrm{curl})} + \|\operatorname{div}(av)\|_{L^2_x} + \|\operatorname{tr}_{\mathrm{no}}(av)\|_{\mathcal{H}^{1/2}(\Gamma)}\big) =: c\kappa(v).$$

PROOF. There exists a finite partition of unity $\{\chi_i\}_i$ on $\overline{G}$ such that the support of each χ_i is contained in a simply connected subset of $\overline{G}$ with a connected smooth boundary. Since each χ_i is scalar, we obtain the estimate

$$\|\chi_i v\|_{L^2_x} + \|\mathrm{curl}(\chi_i v)\|_{L^2_x} + \|\operatorname{div}(a\chi_i v)\|_{L^2_x} + \|\operatorname{tr}_{\mathrm{no}}(a\chi_i v)\|_{\mathcal{H}^{1/2}(\Gamma)} \leq c\kappa(v).$$

We can thus assume that Γ is connected and G simply connected. In this case, $\operatorname{curl} v$ is an element of $\mathrm{N}^\Gamma(\mathrm{div})$ and so Theorem 3.8 c) yields a map $w \in \mathcal{H}^1(G)^3 \cap \mathrm{N}_0(\mathrm{div})$ with $\operatorname{curl} w = \operatorname{curl} v$ and $\|w\|_{\mathcal{H}^1_x} \leq c\|\operatorname{curl} v\|_{L^2_x}$. As the difference $v - w$ belongs to $\mathrm{N}(\mathrm{curl})$, it is represented by $v - w = \nabla\varphi$ for a function $\varphi \in \mathcal{H}^1(G)$ thanks to Proposition 3.7. Here we can assume that $\int_G \varphi \,\mathrm{d}x = 0$ and so $\|\varphi\|_2 \lesssim \|\nabla\varphi\|_2 \lesssim \|v\|_2 + \|w\|_2$ by Poincaré's inequality. We further have

$$\begin{aligned} \operatorname{div}(a\nabla\varphi) &= \operatorname{div}(av) - \operatorname{div}(aw) \ \in L^2(G), \\ \operatorname{tr}_{\mathrm{no}}(a\nabla\varphi) &= \operatorname{tr}_{\mathrm{no}}(av) - \operatorname{tr}_{\mathrm{no}}(aw) \ \in \mathcal{H}^{1/2}(\Gamma), \end{aligned}$$

because of the assumptions and $w \in \mathcal{H}^1(G)^3$. Due to Proposition 5.7.7 in [**55**], φ is thus an element of $\mathcal{H}^2(G)$ satisfying

$$\|\varphi\|_{\mathcal{H}^2_x} \leq c\big(\|v\|_{L^2_x} + \|\operatorname{div}(av)\|_{L^2_x} + \|\operatorname{tr}_{\mathrm{no}}(av)\|_{\mathcal{H}^{1/2}(\Gamma)} + \|w\|_{\mathcal{H}^1_x}\big) \leq c\kappa(v).$$

The assertion now follows from the equation $v = w + \nabla\varphi$. □

3.3. Energy and observability-type inequalities

Wev now go back to the proof of Theorem 3.1. We first establish an energy inequality for $\partial_t^k u$ involving dissipation. The error term $z^{3/2}$ is caused by various commutators with $\varepsilon(E)$ and $\mu(H)$.

PROPOSITION 3.12. *We assume the conditions of Theorem 3.1 except for the simple connectedness of G. For $0 \leq s \leq t < T_*$ and $k \in \{0, 1, 2, 3\}$, we obtain the inequality*

$$e_k(t) + \int_s^t d_k(\tau)\,\mathrm{d}\tau \leq e_k(s) + c_1 \int_s^t z^{3/2}(\tau)\,\mathrm{d}\tau. \tag{3.24}$$

We first give the direct proof for the case $k = 0$. Since (E, H) even belongs to $\mathcal{G}^3$, the system (3.1) and the integration by parts formula (2.9) yield

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_G \big(\varepsilon(E(t))E(t)\cdot E(t) + \mu(H(t))H(t)\cdot H(t)\big)\,\mathrm{d}x \\
&= \frac{1}{2}\int_G \Big[\partial_t(\varepsilon(E)E)\cdot E + \varepsilon(E)E\cdot\big[\varepsilon(E)^{-1}\partial_t(\varepsilon(E)E)\big] + \varepsilon(E)E\cdot\big[\partial_t\varepsilon(E)^{-1}\varepsilon(E)E\big] \\
&+\partial_t(\mu(H)H)\cdot H + \mu(H)H\cdot\big[\mu(H)^{-1}\partial_t(\mu(H)H)\big] + \mu(H)H\cdot\big[\partial_t\mu(H)^{-1}\mu(H)H\big]\Big]\mathrm{d}x \\
&= \int_G \big[\operatorname{curl} H\cdot E - \sigma E\cdot E - \operatorname{curl} E\cdot H - \tfrac{1}{2}\partial_t\varepsilon(E)\,E\cdot E - \tfrac{1}{2}\partial_t\mu(H)\,H\cdot H\big]\,\mathrm{d}x
\end{aligned}$$

$$= -\int_G \left[\sigma E\cdot E + \tfrac{1}{2}\partial_t\varepsilon(E)\,E\cdot E + \tfrac{1}{2}\partial_t\mu(H)\,H\cdot H\right]\mathrm{d}x.$$

We thus obtain the energy equality

$$e_0(t) + \int_s^t d_0(\tau)\,\mathrm{d}\tau = e_0(t) - \frac{1}{2}\int_s^t\int_G \big(\partial_t\varepsilon(E)\,E\cdot E + \partial_t\mu(H)\,H\cdot H\big)\,\mathrm{d}x\,\mathrm{d}\tau.$$

Combined with estimate (3.16), we derive (3.24) for the case $k = 0$.

For $k \in \{1,2,3\}$ in Proposition 3.12, we have different coefficients in the energy e_k defined in (3.13) and more error terms. In this case, (3.24) follows from Lemma 3.13 below, the system (3.12) and the estimates (3.16). This lemma provides an energy identity in a more general situation to be encountered later.

Take coefficients $a, b \in W^{1,\infty}(J\times G, \mathbb{R}^{3+3}_\eta)$ for some $T, \eta > 0$ and data $v_0, w_0 \in L^2_x$, $\varphi, \psi \in L^2_{t,x}$, and $\omega \in L^2(J, \mathcal{H}^{1/2}(\Gamma))^3$ with $\nu\cdot\omega = 0$. Theorem 1.4 of [**19**] yields a unique solution $(v, w) \in \mathcal{G}^0(J)$ with $\mathrm{tr}_{\mathrm{ta}}(v, w) \in L^2(J, \mathcal{H}^{-1/2}(\Gamma))^6$ of the linear system

$$\begin{aligned}
a\partial_t v &= \operatorname{curl} w - \sigma v + \varphi, && t\in J,\ x\in G,\\
b\partial_t v &= -\operatorname{curl} v + \psi, && t\in J,\ x\in G,\\
\mathrm{tr}_{\mathrm{ta}}\, v &= \omega, && t\in J,\ x\in \Gamma,\\
v(0) &= v_0, \quad w(0) = w_0, && x\in G.
\end{aligned}$$

(Theorem 2.20 deals with the case $\omega = 0$ without the regularity of $\mathrm{tr}_{\mathrm{ta}}\, w$.) For $\omega = 0$ and $G = \mathbb{R}^3_+$, the next lemma is a part of Theorem 2.10. In the present form it follows from Theorem 1.1 of [**53**] (a version of Theorem 2.21 with boundary inhomogeneities), using approximation arguments omitted here, see Lemma 4.2 in [**34**].

Lemma 3.13. *Under the assumptions above, for* $0 \le s \le t \le T$ *we have*

$$\begin{aligned}
\frac{1}{2}\int_G \big(a(t)v(t)\cdot v(t) + b(t)w(t)\cdot w(t)\big)\mathrm{d}x + \int_s^t\int_G \sigma v\cdot v\,\mathrm{d}x\,\mathrm{d}\tau\\
= \frac{1}{2}\int_G \big(a(0)v_0\cdot v_0 + b(0)w_0\cdot w_0\big)\mathrm{d}x + \int_s^t\int_\Gamma \omega\cdot \mathrm{tr}_{\mathrm{ta}}\, w\,\mathrm{d}x\,\mathrm{d}\tau\\
+ \int_s^t\int_G \big(\tfrac{1}{2}\partial_t a\, v\cdot v + \tfrac{1}{2}\partial_t b\, w\cdot w + \varphi\cdot v + \psi\cdot w\big)\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}$$

In the next proposition we control the energy by the dissipation, i.e., $\partial_t^k E$ by $\partial_t^k H$. Following [**20**], our aproach is based on a Helmholtz decomposition. Our result is a variant of Proposition 2 in [**20**] where the case of time-independent ε and μ and less regular solutions was treated.

Lemma 3.14. *Let the assumptions of Theorem 3.1 be satisfied and let* (E, H) *solve* (3.1). *Then there exist functions* w *in* $C^3\big(J_+, \mathcal{H}^1_{\mathrm{ta}0}(G)\cap \mathrm{N}^\Gamma(\mathrm{div})\big)\cap C^4(J_+, L^2(G))^3$, p *in* $C^3(J_+, \mathcal{H}^1_0(G))$ *and* h *in* $C^3(J_+, \mathcal{N})$ *with*

$$\partial_t^k E = -\partial_t^{k+1} w + \nabla\partial_t^k p + \partial_t^k h, \qquad \widehat{\mu}_k\partial_t^k H = \operatorname{curl}\partial_t^k w - g_k \tag{3.25}$$

for $k\in\{0,1,2,3\}$, *cf.* (3.8) *and* (3.10). *The sum for* $\partial_t^k E$ *is orthogonal in* L^2_x.

PROOF. Let $t \in J_+$. Equation (3.6) implies that the function $\mu(H(t))H(t)$ is contained in $\mathrm{N}_0(\mathrm{div})$. Since G is simply connected, Theorem 3.9 then yields a vector field $w(t)$ in $\mathcal{H}^1_{\mathrm{ta}0}(G) \cap \mathrm{N}^\Gamma(\mathrm{div})$ satisfying $\operatorname{curl} w(t) = \mu(H(t))H(t)$. Moreover, the map w belongs to $C^3(J_+, \mathcal{H}^1_{\mathrm{ta}0}(G) \cap \mathrm{N}^\Gamma(\mathrm{div}))$ because of $(E,H) \in \mathcal{G}^3$ and Theorem 3.9. Differentiating $\operatorname{curl} w = \mu(H)H$ in t, we deduce

$$\operatorname{curl} \partial_t^k w = \partial_t^k(\mu(H)H) = \mu^{\mathrm{d}}(H)\partial_t^k H + g_k$$

for $k \in \{1,2,3\}$ which shows the second part of (3.25). Comparing this relation for $k=1$ with (3.2), we infer $\operatorname{curl}(E + \partial_t w) = 0$. Morever, $E + \partial_t w$ belongs to the kernel of $\mathrm{tr}_{\mathrm{ta}}$. From (3.19) we obtain functions $p(t) \in \mathcal{H}^1_0(G)$ and $h(t) \in \mathcal{N}$ such that

$$E(t) = -\partial_t w(t) + \nabla p(t) + h(t)$$

for $t \in J_+$ with orthogonal sums. This fact and $(E,H) \in \mathcal{G}^3$ imply the remaining regularity assertions. Differentiating the above identity in t, we prove (3.25). $\square$

We can now show the desired observability-type estimate. Let us explain this name. For solutions of (3.1) with $\sigma = 0$, $\varepsilon = \varepsilon(x)$ and $\mu = \mu(x)$, Lemma 3.13 shows the energy equality $e_0(t) = e_0(0)$ for $t \geq 0$. Take $\sigma = 1$ in the definition of d_0. Then the next inequality can still be shown with modified constants and $z = 0$, implying $(t - 2c_3')e_0(0) \leq c_2' \int_0^t \|E(\tau)\|^2_{L^2_x}\,\mathrm{d}\tau$. Hence, the initial fields can be determined by observing the electric field alone until $t > 2c_3'$.

PROPOSITION 3.15. *Let the conditions of Theorem 3.1 be satisfied. For* $0 \leq s \leq t < T_*$ *and* $k \in \{0,1,2,3\}$, *we can estimate*

$$\int_s^t e_k(\tau)\,\mathrm{d}\tau \leq c_2 \int_s^t d_k(\tau)\,\mathrm{d}\tau + c_3(e_k(t) + e_k(s)) + c_4 \int_s^t z^{3/2}(\tau)\,\mathrm{d}\tau.$$

PROOF. Let $k \in \{0,1,2,3\}$. To simplify, take $s = 0$. Equality (3.25) yields

$$\int_{G_t} \widehat{\mu}_k \partial_t^k H \cdot \partial_t^k H \,\mathrm{d}(x,\tau) = \int_{G_t} \operatorname{curl} \partial_t^k w \cdot \partial_t^k H \,\mathrm{d}(x,\tau) - \int_{G_t} g_k \cdot \partial_t^k H \,\mathrm{d}(x,\tau), \quad (3.26)$$

where $G_t = G \times (0,t)$. Using that $\partial_t^k w \in C(J_+, \mathcal{H}^1_{\mathrm{ta}0}(G))$ by Lemma 3.14, we apply (2.10), insert the first line of the system (3.9), and integrate by parts in t. It follows

$$\begin{aligned}
\int_{G_t} \operatorname{curl} \partial_t^k w \cdot \partial_t^k H \,\mathrm{d}(x,\tau) &= \langle \partial_t^k w, \operatorname{curl} \partial_t^k H\rangle_{L^2((0,t),\mathcal{H}_0(\mathrm{curl}))} \quad (3.27)\\
&= \langle \partial_t^k w, \partial_t(\widehat{\varepsilon}_k \partial_t^k E)\rangle_{L^2((0,t),\mathcal{H}_0(\mathrm{curl}))} + \int_{G_t} \partial_t^k w \cdot (\sigma \partial_t^k E + \partial_t f_k)\,\mathrm{d}(x,\tau)\\
&= \int_G \partial_t^k w(t) \cdot \widehat{\varepsilon}_k(t) \partial_t^k E(t)\,\mathrm{d}x - \int_G \partial_t^k w(0) \cdot \widehat{\varepsilon}_k(0) \partial_t^k E(0)\,\mathrm{d}x\\
&\quad - \int_{G_t} \partial_t^{k+1} w \cdot \widehat{\varepsilon}_k \partial_t^k E \,\mathrm{d}(x,\tau) + \int_{G_t} \partial_t^k w \cdot (\sigma \partial_t^k E + \partial_t f_k)\,\mathrm{d}(x,\tau).
\end{aligned}$$

Since $\partial_t^k w(t) \in \mathcal{H}^1_{\mathrm{ta}0}(G)^3 \cap \mathrm{N}^\Gamma(\mathrm{div})$, Theorem 3.9 yields the Poincaré-type estimate $\|\partial_t^k w(\tau)\|_{L^2_x} \leq c \|\operatorname{curl} \partial_t^k w(\tau)\|_{L^2_x}$. From (3.25) and (3.16), we then infer the bound

$$\|\partial_t^k w(\tau)\|_{L^2_x} \leq c \|\operatorname{curl} \partial_t^k w(\tau)\|_{L^2_x} = c \|\hat{\mu}_k \partial_t^k H(\tau) + g_k(\tau)\|_{L^2_x} \leq c e_k^{1/2}(\tau). \quad (3.28)$$

The orthogonality in the first part of (3.25) gives $\|\partial_t^{k+1} w(\tau)\|_{L^2_x} \le \|\partial_t^k E(\tau)\|_{L^2_x}$. For any $\theta > 0$, these inequalities along with (3.27) and (3.16) lead to the estimate

$$\Big| \int_{G_t} \operatorname{curl} \partial_t^k w \cdot \partial_t^k H \,\mathrm{d}(x,\tau) \Big| \le c(e_k(t) + e_k(0)) + c \int_{G_t} |\partial_t^k E|^2 \,\mathrm{d}(x,\tau) \qquad (3.29)$$

$$+ \theta \int_{G_t} |\partial_t^k w|^2 \mathrm{d}(x,\tau) + c_\theta \int_{G_t} |\partial_t^k E|^2 \,\mathrm{d}(x,\tau) + c \int_0^t z^{\frac{3}{2}}(\tau) \,\mathrm{d}\tau.$$

As in (3.28), we further compute

$$\begin{aligned} \int_{G_t} |\partial_t^k w|^2 \,\mathrm{d}(x,\tau) &\le c \int_{G_t} \operatorname{curl} \partial_t^k w \cdot \widehat{\mu}_k^{-1} \operatorname{curl} \partial_t^k w \,\mathrm{d}(x,\tau) \\ &= c \int_{G_t} \operatorname{curl} \partial_t^k w \cdot (\partial_t^k H + \widehat{\mu}_k^{-1} g_k) \,\mathrm{d}(x,\tau) \\ &\le c \Big| \int_{G_t} \operatorname{curl} \partial_t^k w \cdot \partial_t^k H \,\mathrm{d}(x,\tau) \Big| + c \int_0^t z^{\frac{3}{2}}(\tau) \,\mathrm{d}\tau. \end{aligned}$$

Fixing a small number $\theta > 0$, the term with $|\partial_t^k w|^2$ in equation (3.29) can now be absorbed by the left-hand side and by the integral of $z^{3/2}$. So we arrive at

$$\Big| \int_{G_t} \operatorname{curl} \partial_t^k w \cdot \partial_t^k H \,\mathrm{d}(x,\tau) \Big| \le c(e_k(t) + e_k(0)) + c \int_0^t d_k(\tau) \,\mathrm{d}\tau + c \int_0^t z^{\frac{3}{2}}(\tau) \,\mathrm{d}\tau,$$

also using that $d_k(t)$ is equivalent to $\max_{j \le k} \|\partial_t^j E(t)\|^2_{L^2_x}$. This fact, equation (3.26), the last inequality, and the estimates (3.16) yield the claim. □

Combining Propositions 3.12 and 3.15, we arrive at the following energy bound.

PROPOSITION 3.16. *Under the conditions of Theorem 3.1, we have*

$$e_k(t) + \int_s^t e_k(s) ds \le C_1 e_k(s) + C_2 \int_s^t z^{3/2}(\tau) \,\mathrm{d}\tau$$

for $0 \le s \le t < T_*$ *and* $k \in \{0, 1, 2, 3\}$.

PROOF. We multiply the inequality in Proposition 3.15 by $\alpha = \min\{\frac{1}{c_2}, \frac{1}{2c_3}\}$ and add it to (3.24), obtaining

$$e_k(t) + 2\alpha \int_s^t e_k(\tau) \,\mathrm{d}\tau \le 3 e_k(s) + 2(c_1 + \alpha c_4) \int_s^t z^{3/2}(\tau) \,\mathrm{d}\tau. \qquad \square$$

For $z = 0$, from Corollary 3.16 one could easily infer exponential decay by a standard argument, see below. The extra term can be made small since $z^{1/2}(\tau) \le \delta$ for $\tau < T_*$ by (3.14). However, z involves space derivatives so that it cannot be absorbed by e that does not contain them. This gap is closed by the next surprising result proved in the next section. It then allows us to show Theorem 3.1.

PROPOSITION 3.17. *We impose the conditions of Theorem 3.1 except for the simple connectedness of G. Then the solutions (E, H) to (3.1) satisfy*

$$z(t) + \int_s^t z(\tau) \,\mathrm{d}\tau \le c_5 \big(z(s) + e(t) + z^2(t) \big) + c_6 \int_s^t \big(e(\tau) + z^{3/2}(\tau) \big) \,\mathrm{d}\tau$$

for all $0 \le s \le t < T_*$.

PROOF OF THEOREM 3.1. Proposition 3.17 and Corollary 3.16 show that

$$z(t)+\int_s^t z(\tau)\,\mathrm{d}\tau \le (c_5+C_1(c_5+c_6))z(s)+c_5z^2(t)+(c_6+C_2(c_5+c_6))\int_s^t z^{3/2}(\tau)\,\mathrm{d}\tau.$$

Fixing a sufficiently small radius $\delta \in (0,\delta_0]$, we can now absorb the superlinear terms involving z^2 and $z^{3/2}$ by the left-hand side and hence obtain

$$z(t)+\int_s^t z(\tau)\,\mathrm{d}\tau \le Cz(s), \qquad \text{for all } \ 0\le s\le t<T_*$$

and some constant $C>0$. Since then $z(\tau)\ge C^{-1}z(t)$, we infer that

$$(1+(t-s)C^{-1})z(t)\le Cz(s). \tag{3.30}$$

The differentiated Maxwell system (3.12) and the bounds from (3.16) yield

$$z(0)\le c_0\|(E_0,H_0)\|_{\mathcal{H}^3}^2 \le c_0r^2$$

for a constant $c_0>0$. We now fix the radius

$$r := \min\Big\{r(\delta), \frac{\delta}{\sqrt{2c_0C}}\Big\},$$

where $r(\delta)$ was introduced before (3.14).

We suppose that $T_*<\infty$, yielding $z(T_*)=\delta^2$ by (3.15). Because of (3.30), the number $z(t)$ is bounded by $Cz(0)\le \delta^2/2$ for $t<T_*$ and by continuity also for $t=T_*$. This contradiction shows that $T_*=\infty$ and hence $T_+=\infty$.

In particular, (3.30) is true for all $t\ge s\ge 0$. Fixing the time $T>0$ with $C^2/(C+T)=1/2$, we derive $z(nT)\le \frac{1}{2}z((n-1)T)$ for $n\in\mathbb{N}$ and then $z(nT)\le 2^{-n}z(0)$ by induction. With (3.30) one then obtains the asserted exponential decay. □

3.4. Time regularity controls space regularity

In the proof of Proposition 3.17, we want to avoid the localization procedure since we need global-in-time estimates. This can be done using a new coordinate system near $\Gamma=\partial G$. (Possibly, one could derive the apriori estimates in §2.3 in a similar way; but for the regularization this is not clear because of the mollifier arguments.)

For a fixed distance $\varrho>0$, on the collar $\Gamma_\varrho=\{x\in\overline{G}\mid \operatorname{dist}(x,\Gamma)<\varrho\}$, we can find smooth functions $\tau^1,\tau^2,\nu:\Gamma_\varrho\to\mathbb{R}^3$ such that the vectors $\{\tau^1(x),\tau^2(x),\nu(x)\}$ form a basis of $\mathbb{R}^3$ for each point $x\in\Gamma_\varrho$, ν extends the outer unit normal at Γ, and $\{\tau^1,\tau^2\}$ span the tangential planes at Γ. For $\xi,\zeta\in\{\tau^1,\tau^2,\nu\}$, $v\in\mathbb{R}^3$ and $a\in\mathbb{R}^{3\times3}$, we set

$$\partial_\xi=\sum\nolimits_j \xi_j\partial_j, \quad v_\xi=v\cdot\xi, \quad v^\xi=v_\xi\xi, \quad v^\tau=v_{\tau^1}\tau^1+v_{\tau^2}\tau^2, \quad a_{\xi\zeta}=\xi^\top a\zeta.$$

We state several calculus formulas needed below, assuming that the functions involved are sufficiently regular. We can switch between the derivatives of the coefficient v_ξ and the component v^ξ up to a zero-order term since

$$\partial_\zeta v^\xi=\partial_\zeta v_\xi\xi+v_\xi\partial_\zeta\xi.$$

The commutator of tangential derivatives and traces

$$\partial_\tau \operatorname{tr}_{\mathrm{ta}} v = \partial_\tau(v \times \nu) = \operatorname{tr}_{\mathrm{ta}} \partial_\tau v + v \times \partial_\tau \nu \qquad \text{on} \quad \Gamma$$

is also of lower order. Similarly, the directional derivatives commute

$$\partial_\xi \partial_\zeta v = \sum\nolimits_{j,k} \xi_j \partial_j(\zeta_k \partial_k v) = \partial_\zeta \partial_\xi v + \sum\nolimits_{j,k} \xi_j \partial_j \zeta_k \, \partial_k v - \zeta_k \partial_k \xi_j \, \partial_j v$$

up to a first-order operator with bounded coefficients.

The gradient of a scalar function φ is expanded as

$$\nabla \varphi = \sum\nolimits_\xi \xi \cdot \nabla \varphi \, \xi = \sum\nolimits_\xi \xi \partial_\xi \varphi,$$

so that $\partial_j = \sum_\xi \xi_j \partial_\xi$ for $j \in \{1,2,3\}$. Due to the formulas before (1.5) we have

$$\operatorname{curl} = \sum\nolimits_j S_j \partial_j = \sum\nolimits_{j,\xi} S_j \xi_j \partial_\xi =: \sum\nolimits_\xi S(\xi) \partial_\xi.$$

Since the kernel of $S(\nu)$ is spanned by ν, we can write $S(\nu)v = S(\nu)v^\tau$, and the restriction of $S(\nu)$ to $\operatorname{span}\{\tau^1, \tau^2\}$ has an inverse $R(\nu)$.

We now provide the tools that allow us transfer to the arguments of Proposition 2.12 from $\mathbb{R}^+_3$ to the present setting. We first isolate the normal derivative of the tangential components of v in the equation $\operatorname{curl} v = f$. Starting from the above expansion

$$\operatorname{curl} v = S(\nu)(\partial_\nu v)^\tau + S(\tau^1)\partial_{\tau^1} v + S(\tau^2) \partial_{\tau^2} v,$$

we obtain

$$\partial_\nu v^\tau = \sum\nolimits_i (\partial_\nu \tau^i \, v_{\tau^i} + \tau^i \partial_\nu \tau^i \cdot v) + R(\nu)\Big(f - \sum\nolimits_i S(\tau^i) \partial_{\tau^i} v\Big), \tag{3.31}$$

where the first sum only contains zero-order terms.

In order to recover the normal derivative of the normal component of v, we resort to the divergence operator. The divergence of a vector field v can be expressed as

$$\operatorname{div} v = \sum\nolimits_j \partial_j \sum\nolimits_\xi v_\xi \xi_j = \sum\nolimits_\xi \big(\partial_\xi v_\xi + \operatorname{div}(\xi) v_\xi\big).$$

Letting $\varphi = \operatorname{div}(av)$ for a matrix-valued function a, we derive

$$\begin{aligned}
\operatorname{div}(av) &= \sum\nolimits_{\xi,\zeta} \partial_\xi(\xi^\top a \zeta v_\zeta) + \sum\nolimits_\xi \operatorname{div}(\xi)\, \xi^\top a v \\
&= \sum\nolimits_{\xi,\zeta} (a_{\xi\zeta} \partial_\xi v_\zeta + \partial_\xi a_{\xi\zeta} v_\zeta) + \sum\nolimits_\xi \operatorname{div}(\xi)\, \xi^\top a v, \\
a_{\nu\nu} \partial_\nu v_\nu &= \varphi - \sum_{(\xi,\zeta) \neq (\nu,\nu)} a_{\xi\zeta} \partial_\xi v_\zeta - \sum\nolimits_{\xi,\zeta} \partial_\xi a_{\xi\zeta} v_\zeta - \sum\nolimits_\xi \operatorname{div}(\xi)\, \xi^\top a v \\
&=: \varphi - D(a) v,
\end{aligned} \tag{3.32}$$

where $D(a)v$ contains all tangential derivatives and normal derivatives of tangential components of v plus zero-order terms. Next, let $a \in W^{1,\infty}(J \times G, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ be positive definite, $v \in C^1(\overline{J}, \mathcal{H}^1_x)$, and $\psi \in L^2_{t,x}$. In view of (3.7), we look at the equation

$$\operatorname{div}\big(a(t)v(t)\big) = \operatorname{div}\big(a(0)u(0)\big) - \int_0^t \big(\operatorname{div}(\sigma u(s)) + \psi(s)\big)\, \mathrm{d}s \tag{3.33}$$

for $0 \le t \le T$. We set $\gamma = \sigma_{\nu\nu}/a_{\nu\nu}$ and $\Gamma(t,s) = \exp(-\int_s^t \gamma(\tau)\,\mathrm{d}\tau)$. Equations (3.32) and (3.33) yield

$$a_{\nu\nu}(t)\partial_\nu v_\nu(t) = \operatorname{div}\big(a(0)v(0)\big) - D\big(a(t)\big)v(t) - \int_0^t \big(\gamma(s)a_{\nu\nu}(s)\partial_\nu v_\nu(s) + D\big(\sigma\big)v(s) + \psi(s)\big)\,\mathrm{d}s,$$

cf. (2.29). Differentiating with respect to t and solving the resulting ODE, we derive

$$\begin{aligned} &a_{\nu\nu}(t)\partial_\nu v_\nu(t) \\ &\quad = \Gamma(t,0)a_{\nu\nu}(0)\partial_\nu n_\nu(0) - \int_0^t \Gamma(t,s)\big(D(\sigma)v(s) + \psi(s) + \partial_s\big(D(a(s))v(s)\big)\big)\,\mathrm{d}s \\ &\quad = \Gamma(t,0)\operatorname{div}(a(0)v(0)) - D(a(t))v(t) \\ &\qquad + \int_0^t \Gamma(t,s)\big(\gamma(s)D\big(a(s)\big)v(s) - D(\sigma)v(s) - \psi(s)\big)\,\mathrm{d}s. \end{aligned} \tag{3.34}$$

Before tackling the (quite demanding) proof of Proposition 3.17, we describe our reasoning. We have to bound $\partial_t^k E$ and $\partial_t^k H$ in $\mathcal{H}_x^{3-k}$ for $k \in \{0,1,2\}$ by the L_x^2-norms of $\partial_t^j E$ and $\partial_t^j H$ for $j \in \{0,1,2,3\}$.

The $\mathcal{H}_x^1$-norm of $\partial_t^k H$ with $k \in \{0,1,2\}$ can easily be estimated by means of the curl-div estimates from Proposition 3.11 since we control curl, divergence and normal trace of $\partial_t^k H$ via the time differentiated Maxwell system (3.9) and (3.11). Aiming at higher space regularity, we can apply the above strategy to tangential derivatives of $\partial_t^k H$ only, whereas normal derivatives destroy the boundary conditions in (3.9). Here we proceed as in Proposition 2.12: The tangential components of normal derivatives are read off the differentiated Maxwell system using the expansion (3.31) of the curl-operator, while the normal components are bounded employing the divergence condition (3.11) and formula (3.32). In these arguments we have to restrict ourselves to fields localized near the boundary. The localized fields in the interior can be controlled more easily since the boundary conditions become trivial for them.

The electric fields E have less favorable divergence properties because of the conductivity term in (3.9). Instead of Proposition 3.11, we thus employ the energy bound of the system (3.36) derived by differentiating the Maxwell equations in time and tangential directions. The normal derivatives are again treated by the curl-div-strategy indicated in the previous paragraph. However, to handle the extra divergence term in (3.11) caused by the conductivity, we need the more sophisticated divergence formula (3.34) relying on an ODE derived from (3.11).

This program is carried out by iteration on the space regularity. In each step one has to start with the magnetic fields in order to use their better properties when estimating the electric ones.

Proof of Proposition 3.17. Let (E,H) be a solution of (3.1) on $J_* = [0,T_*)$ satisfying $z(t) \le \delta^2$ and the equations (3.6) and (3.7). Take $k \in \{0,1,2\}$ and $0 \le t < T_*$, where we let $s = 0$ for simplicity. To localize the fields, we choose smooth scalar functions χ and $1-\chi =: \vartheta$ on $\overline{G}$ having compact support in $G\setminus\Gamma_{\varrho/2}$ and Γ_ϱ, respectively. The proof is divided into several steps following the above outline.

1) Estimate of $\partial_t^k H$ in $\mathcal{H}_x^1$. The time differentiated Maxwell system (3.9) and (3.11) combined with estimates (3.16) yield

$$\begin{aligned}\big\|\operatorname{curl}\partial_t^k H(t)\big\|_{L_x^2} &\le c e_{k+1}^{1/2}(t) + c z(t)\delta_{k2},\\ \big\|\operatorname{div}(\widehat{\mu}_k \partial_t^k H(t))\big\|_{L_x^2} &\le c z(t)\delta_{k2},\\ \big\|\operatorname{tr}_{\mathrm{no}}(\widehat{\mu}_k\partial_t^k H(t))\big\|_{\mathcal{H}^{1/2}(\Gamma)} &\le c z(t)\delta_{k2},\end{aligned}$$

where $\delta_{k2}=1$ for $k=2$ and $\delta_{k2}=0$ for $k\in\{0,1\}$. Proposition 3.11 thus implies

$$\begin{aligned}\big\|\partial_t^k H(t)\big\|_{\mathcal{H}_x^1}^2 &\le c e_{k+1}(t) + c z^2(t)\delta_{k2},\\ \int_0^t \big\|\partial_t^k H(s)\big\|_{\mathcal{H}_x^1}^2 \,\mathrm{d}s &\le c\int_0^t (e_{k+1}(s)+z^2(s)\delta_{k2})\,\mathrm{d}s.\end{aligned} \tag{3.35}$$

We stress the core fact that the inhomogeneities in (3.9) and (3.11) are quadratic in (E,H) and can thus be bounded by z via (3.16).

2) Estimates in the interior for E and H. We look at the localized fields $\partial_t^k(\chi E)$ and $\partial_t^k(\chi H)$ whose support $\operatorname{supp}\chi$ is strictly separated from the boundary. Hence, their spatial derivatives satisfy the boundary conditions of the Maxwell system so that we can treat the electric fields via energy bounds and the magnetic ones via the curl-div estimates.

a) Let $\alpha\in\mathbb{N}_0^3$ with $|\alpha|\le 3-k$. We apply $\partial_x^\alpha\chi$ to the Maxwell system (3.12), deriving the equations

$$\begin{aligned}\varepsilon^{\mathrm{d}}(E)\,\partial_t\partial_x^\alpha\partial_t^k(\chi E) &= \operatorname{curl}\partial_x^\alpha\partial_t^k(\chi H) - \sigma\partial_x^\alpha\partial_t^k(\chi E) + \partial_x^\alpha([\chi,\operatorname{curl}]\partial_t^k H)\\ &\quad - \sum_{0\le\beta<\alpha}\binom{\alpha}{\beta}\partial_x^{\alpha-\beta}(\sigma+\varepsilon^{\mathrm{d}}(E))\,\partial_x^\beta\partial_t^k(\chi E) - \partial_x^\alpha(\chi\tilde{f}_k),\\ \mu^{\mathrm{d}}(H)\,\partial_t\partial_x^\alpha\partial_t^k(\chi H) &= -\operatorname{curl}\partial_x^\alpha\partial_t^k(\chi E) - \partial_x^\alpha([\chi,\operatorname{curl}]\partial_t^k E) - \partial_x^\alpha(\chi\tilde{g}_k)\\ &\quad - \sum_{0\le\beta<\alpha}\binom{\alpha}{\beta}\partial_x^{\alpha-\beta}\mu^{\mathrm{d}}(H)\,\partial_x^\beta\partial_t^k(\chi H),\\ \operatorname{tr}_{\mathrm{ta}}\partial_x^\alpha\partial_t^k(\chi E) &= 0, \qquad \operatorname{tr}_{\mathrm{no}}\partial_x^\alpha\partial_t^k(\chi H)=0.\end{aligned} \tag{3.36}$$

Note that the commutator $m := [\chi,\operatorname{curl}]$ is merely a multiplication operator. Lemma 3.13 and the inequalities (3.16) thus yield

$$\begin{aligned}&\|\partial_x^\alpha\partial_t^k(\chi E)(t)\|_{L_x^2}^2 + \int_0^t \|\partial_x^\alpha\partial_t^k(\chi E)(s)\|_{L_x^2}^2\,\mathrm{d}s\\ &\quad\le c z(0) + c\int_0^t \big(z^{3/2}(s) + \|\partial_t^k(\chi E(s))\|_{\mathcal{H}_x^{|\alpha|-1}}^2 + \|\partial_t^k(\chi H(s))\|_{\mathcal{H}_x^{|\alpha|-1}}^2\big)\,\mathrm{d}s\\ &\qquad + c\int_{G_t}\big(\partial_x^\alpha(m\partial_t^k H)\cdot\partial_x^\alpha\partial_t^k(\chi E)) - \partial_x^\alpha(m\partial_t^k E)\cdot\partial_x^\alpha\partial_t^k(\chi H)\big)\,\mathrm{d}(x,s),\end{aligned}$$

where $G_t = (0,t) \times G$. The former part of the last line can be estimated by

$$\frac{1}{4}\int_0^t \big\|\partial_x^\alpha \partial_t^k (\chi E)(s)\big\|_{L_x^2}^2 \,\mathrm{d}s + c\int_0^t \big\|\tilde{\chi}\partial_t^k H(s)\big\|_{\mathcal{H}_x^{|\alpha|}}^2 \,\mathrm{d}s$$

with another cut-off function $\tilde{\chi} \in C_c^\infty(G \setminus \Gamma_{\varrho/2})$ that is equal to 1 on $\operatorname{supp}\chi$. The first summand is absorbed by the left-hand side, while the second one only involves H and can be treated separately. The latter part of the integral on G_t is similarly bounded by

$$\theta\int_0^t \big\|\partial_t^k E(s)\big\|_{\mathcal{H}_x^{|\alpha|}}^2 \,\mathrm{d}s + c(\theta)\int_0^t \big\|\tilde{\chi}\partial_t^k H(s)\big\|_{\mathcal{H}_x^{|\alpha|}}^2 \,\mathrm{d}s$$

for an arbitrary (small) $\theta > 0$. It follows

$$\begin{aligned}\big\|\partial_x^\alpha \partial_t^k (\chi E)(t)\big\|_{L_x^2}^2 &+ \int_0^t \big\|\partial_x^\alpha \partial_t^k (\chi E)(s)\big\|_{L_x^2}^2 \,\mathrm{d}s \\ &\leq cz(0) + c\int_0^t (z^{3/2}(\tau) + \big\|\partial_t^k(\chi E(s)\big\|_{\mathcal{H}_x^{|\alpha|-1}}^2)\,\mathrm{d}s \\ &\quad + \theta\int_0^t \big\|\partial_t^k E(s)\big\|_{\mathcal{H}_x^{|\alpha|}}^2 \,\mathrm{d}s + c(\theta)\int_0^t \big\|\tilde{\chi}\partial_t^k H(s)\big\|_{\mathcal{H}_x^{|\alpha|}}^2 \,\mathrm{d}s.\end{aligned} \tag{3.37}$$

b) To treat H, we only need $|\alpha| \leq 2-k$. Equations (3.6) and (3.11) yield

$$\begin{aligned}&\operatorname{div}\big(\widehat{\mu}_k \partial_x^\alpha \partial_t^k(\chi H)\big) \\ &= \partial_x^\alpha([\operatorname{div},\chi]\widehat{\mu}_k\partial_t^k H) - \sum_{0\leq\beta<\alpha}\binom{\alpha}{\beta}\operatorname{div}\big(\partial_x^{\alpha-\beta}\widehat{\mu}_k\,\partial_x^\beta(\partial_t^k(\chi H)\big) - \partial_x^\alpha(\chi\operatorname{div} g_k).\end{aligned} \tag{3.38}$$

Recalling formulas (3.36) and (3.16), we deduce

$$\begin{aligned}&\big\|\operatorname{curl}\partial_x^\alpha\partial_t^k(\chi H(t))\big\|_{L_x^2} + \big\|\widehat{\mu}_k\operatorname{div}\partial_x^\alpha\partial_t^k(\chi H(t))\big\|_{L_x^2} \\ &\quad \leq c\Big(z(t) + \big\|\partial_t^k\tilde{\chi}H(t)\big\|_{\mathcal{H}_x^{|\alpha|}} + \big\|\partial_t^{k+1}(\chi E(t))\big\|_{\mathcal{H}_x^{|\alpha|}} + \big\|\partial_t^k(\chi E(t))\big\|_{\mathcal{H}_x^{|\alpha|}}\Big)\end{aligned}$$

Proposition 3.11 now implies the inequalities

$$\big\|\partial_t^k\chi H(t)\big\|_{\mathcal{H}_x^{|\alpha|+1}}^2 \leq c\big[z^2(t) + \big\|\partial_t^k\tilde{\chi}H(t)\big\|_{\mathcal{H}_x^{|\alpha|}}^2 + \max_{j\leq k+1}\|\partial_t^j(\chi E(t))\|_{\mathcal{H}_x^{|\alpha|}}^2\big], \tag{3.39}$$

$$\int_0^t\big\|\partial_t^k\chi H(s)\big\|_{\mathcal{H}_x^{|\alpha|+1}}^2\mathrm{d}s \leq c\int_0^t\big[z^2(s) + \big\|\partial_t^k\tilde{\chi}H(s)\big\|_{\mathcal{H}_x^{|\alpha|}}^2 + \max_{j\leq k+1}\big\|\partial_t^j(\chi E(s))\big\|_{\mathcal{H}_x^{|\alpha|}}^2\big]\mathrm{d}s.$$

Here, we can replace χ by $\tilde{\chi}$ from inequality (3.37) and $\tilde{\chi}$ by a function $\breve{\chi} \in C_c^\infty(G \setminus \Gamma_{\varrho/2})$ which is equal to 1 on $\operatorname{supp}\tilde{\chi}$.

We set $y_j(t) = \max_{0\leq k\leq 3-j}\big\|\partial_t^k\chi(E(t),H(t))\big\|_{\mathcal{H}_x^j}^2$. The estimates (3.35), (3.37) and (3.39) iteratively imply

$$\begin{aligned}y_j(t) + \int_0^t y_j(s)\,\mathrm{d}s &\leq cz(0) + c\big(e(t) + z^2(t)\big) + c(\theta)\int_0^t (e(s) + z^{3/2}(s))\,\mathrm{d}s \\ &\quad + \theta\max_{1\leq l\leq j}\max_{0\leq k\leq 3-l}\int_0^t \|\partial_t^k E(s)\|_{\mathcal{H}_x^l}^2\,\mathrm{d}s\end{aligned} \tag{3.40}$$

for any $\theta > 0$ and $j \in \{1,2,3\}$.

3) Boundary-collar estimate of $\partial_t^k E$ in $\mathcal{H}_x^1$. We write $\vartheta = 1-\chi$ and $\partial_\tau = (\partial_{\tau^1}, \partial_{\tau^2})$. Let $\alpha \in \mathbb{N}_0^2$ with $0 < |\alpha| \le 3-k$. (For the later use, also higher-order space derivatives are treated.)

a) We localize the system near the boundary by including the cut-off ϑ into the equations (3.12), and then apply ∂_τ^α to the resulting system. The localized tangential-time derivatives of (E,H) thus satisfy

$$\begin{aligned}
\varepsilon^{\mathrm{d}}(E)\,\partial_t\partial_\tau^\alpha\partial_t^k(\vartheta E) &= \operatorname{curl}\partial_\tau^\alpha\partial_t^k(\vartheta H) - \sigma\partial_\tau^\alpha\partial_t^k(\vartheta E) + [\partial_\tau^\alpha, \operatorname{curl}]\partial_t^k(\vartheta H)\\
&\quad + \partial_\tau^\alpha([\vartheta,\operatorname{curl}]\partial_t^k H) - \sum_{0\le\beta<\alpha}\binom{\alpha}{\beta}\partial_\tau^{\alpha-\beta}(\sigma+\varepsilon^{\mathrm{d}}(E))\,\partial_\tau^\beta\partial_t^k(\vartheta E) - \partial_\tau^\alpha(\vartheta\tilde f_k),\\
\mu^{\mathrm{d}}(H)\,\partial_t\partial_\tau^\alpha\partial_t^k(\vartheta H) &= -\operatorname{curl}\partial_\tau^\alpha\partial_t^k(\vartheta E) - \partial_\tau^\alpha([\vartheta,\operatorname{curl}]\partial_t^k E) - [\partial_\tau^\alpha,\operatorname{curl}]\partial_t^k(\vartheta E)\\
&\quad - \sum_{0\le\beta<\alpha}\binom{\alpha}{\beta}\partial_\tau^{\alpha-\beta}\mu^{\mathrm{d}}(H)\,\partial_\tau^\beta\partial_t^k(\vartheta H) - \partial_\tau^\alpha(\vartheta\tilde g_k),\\
\operatorname{tr}_{\mathrm{ta}}\partial_\tau^\alpha\partial_t^k(\vartheta E) &= [\partial_\tau^\alpha, \operatorname{tr}_{\mathrm{ta}}]\partial_t^k(\vartheta E) =: \omega.
\end{aligned} \tag{3.41}$$

The commutators $[\partial_\tau^\alpha, \operatorname{curl}]$ are differential operators of order $|\alpha|$ with bounded coefficients, whereas $[\partial_\tau^\alpha, \operatorname{tr}_{\mathrm{ta}}]$ is of order $|\alpha|-1$ on the boundary and hence a bounded operator from $\mathcal{H}^{|\alpha|-1/2}(\Gamma)$ to $\mathcal{H}^{1/2}(\Gamma)$. We now use the energy identity in Lemma 3.13 with $a = \varepsilon^{\mathrm{d}}(E)$, $b = \mu^{\mathrm{d}}(H)$, $v = \partial_\tau^\alpha\partial_t^k(\vartheta E)$, and $w = \partial_\tau^\alpha\partial_t^k(\vartheta H)$. The commutator terms, the sums, and the summands with $\tilde f_k$ and $\tilde g_k$ yield the inhomogeneities φ and ψ, respectively. From Lemma 3.13 we deduce the inequality

$$\begin{aligned}
&\|\partial_\tau^\alpha\partial_t^k(\vartheta E)(t)\|_{L_x^2}^2 + \int_0^t \|\partial_\tau^\alpha\partial_t^k(\vartheta E)(s)\|_{L_x^2}^2\,\mathrm{d}s\\
&\quad\le cz(0) + c\int_{G_t}\big(|\partial_t a v\cdot v| + |\partial_t b w\cdot w| + |\varphi\cdot v| + |\psi\cdot w|\big)\,\mathrm{d}(s,x)\\
&\qquad + c\int_{\Gamma_t}|\omega\cdot\operatorname{tr}_{\mathrm{ta}} w|\,\mathrm{d}(s,x).
\end{aligned}$$

Several terms on the right-hand side are super-quadratic in (E,H) and can be bounded by $cz^{3/2}$ due to (3.16). The quadratic ones need more care. The summands in $\varphi\cdot v$ and $\psi\cdot w$ containing the commutators are less or equal to

$$\theta\int_0^t \|\partial_t^k E(s)\|_{\mathcal{H}_x^{|\alpha|}}^2\,\mathrm{d}s + c(\theta)\int_0^t \|\tilde\vartheta\partial_t^k H(s)\|_{\mathcal{H}_x^{|\alpha|}}^2\,\mathrm{d}s$$

with any (small) constant $\theta > 0$ and a cut-off $\tilde\vartheta \in C_c^\infty(\Gamma_\varrho)$ being equal to 1 on $\operatorname{supp}\vartheta$. The boundary integral is estimated by the same expression, where we use the dual paring $\mathcal{H}^{1/2}(\Gamma)\times\mathcal{H}^{-1/2}(\Gamma)$ and that ∂_{τ^i} belongs to $\mathcal{B}(\mathcal{H}^{1/2}(\Gamma), \mathcal{H}^{-1/2}(\Gamma))$. The sums over β give rise to the terms

$$\frac14\int_0^t \|\partial_\tau^\alpha(\partial_t^k\vartheta E(s))\|_{L_x^2}^2\,\mathrm{d}s + c\int_0^t \|\vartheta\partial_t^k E(s)\|_{\mathcal{H}_x^{|\alpha|-1}}^2\,\mathrm{d}s + c\int_0^t \|\vartheta\partial_t^k H(s)\|_{\mathcal{H}_x^{|\alpha|}}^2\,\mathrm{d}s$$

plus super-quadratic terms. We thus arrive at

$$\|\partial_\tau^\alpha\partial_t^k(\vartheta E)(t)\|_{L_x^2}^2 + \int_0^t \|\partial_\tau^\alpha\partial_t^k(\vartheta E)(s)\|_{L_x^2}^2\,\mathrm{d}s \tag{3.42}$$

$$\leq cz(0) + c(\theta)\int_0^t \big(\big\|\vartheta\partial_t^k E(s)\big\|^2_{\mathcal{H}_x^{|\alpha|-1}} + \big\|\tilde\vartheta\partial_t^k H(s)\big\|^2_{\mathcal{H}_x^{|\alpha|}}\big)\,\mathrm{d}s + \theta\int_0^t \big\|\partial_t^k E(s)\big\|^2_{\mathcal{H}_x^{|\alpha|}}\,\mathrm{d}s + c\int_0^t z^{3/2}(s)\,\mathrm{d}s.$$

b) To finalize the $\mathcal{H}_x^1$-estimate for E, we must control the normal derivatives. As in Proposition 2.12, we first treat their tangential component using the second equation in (3.12). Combined with formula (3.31) and estimate (3.16) it implies

$$\big\|\partial_\nu(\partial_t^k(\vartheta E(t))^\tau\big\|^2_{L_x^2} \leq c\big(e_{k+1}(t) + z^2(t) + \big\|\partial_\tau\partial_t^k(\vartheta E(t))\big\|^2_{L_x^2}\big). \tag{3.43}$$

For the normal component we use the div-relations, where we also consider higher tangential derivatives for later use. We first look at the case $k \in \{1,2\}$ and apply $\partial_\tau^\alpha\vartheta$ to equation (3.11) with $|\alpha| \leq 2-k$. It follows

$$\begin{aligned}\operatorname{div}\big(\varepsilon^{\mathrm{d}}(E)\partial_\tau^\alpha\partial_t^k(\vartheta E)\big) &= -D(\varepsilon^{\mathrm{d}}(E),\alpha)\partial_t^k E - \operatorname{div}(\sigma\partial_\tau^\alpha(\vartheta\partial_t^{k-1}E)) \\ &\quad - D(\sigma,\alpha)\partial_t^{k-1}E - \partial_\tau^\alpha(\vartheta\operatorname{div} f_k\big).\end{aligned} \tag{3.44}$$

Here we abbreviate the commutator terms

$$D(a,\alpha)v \coloneqq \partial_\tau^\alpha\big([\vartheta,\operatorname{div}](av)\big) + [\partial_\tau^\alpha,\operatorname{div}](\vartheta a v) + \sum_{0\leq\beta<\alpha}\binom{\alpha}{\beta}\operatorname{div}\big(\partial_\tau^{\alpha-\beta}a\,\partial_\tau^\beta(\vartheta v)\big)$$

for a matrix-valued function a and a vector function v. Observe that $D(a,\alpha)$ is a differential operator of order $|\alpha|$ and that $|D(a,0)v| \leq c|v|$. Below we treat the equality (3.44) by means of formula (3.32). For $k=0$, the divergence equation contains a time integral and initial data which are handled using identity (3.34). To avoid terms which grow linearly in time, we have to derive another equation from (3.1), namely,

$$\begin{aligned}\partial_t(\varepsilon(E)\partial_\tau^\alpha(\vartheta E)) &= \operatorname{curl}\partial_\tau^\alpha(\vartheta H) - \sigma\partial_\tau^\alpha(\vartheta E) - [\operatorname{curl},\partial_\tau^\alpha](\vartheta H) - \partial_\tau^\alpha([\operatorname{curl},\vartheta]H) \\ &\quad - \sum_{0\leq\beta<\alpha}\binom{\alpha}{\beta}\partial_\tau^{\alpha-\beta}(\sigma+\varepsilon(E))\,\partial_\tau^\beta(\vartheta E).\end{aligned}$$

Writing h for the sum of the three errors terms, we infer the divergence relation

$$\begin{aligned}\operatorname{div}\big(\varepsilon(E(t))\partial_\tau^\alpha(\vartheta E(t))\big) &= \operatorname{div}\big(\varepsilon(E_0)\partial_\tau^\alpha(\vartheta E_0)\big) \\ &\quad - \int_0^t\big(\operatorname{div}(\sigma\partial_\tau^\alpha(\vartheta E(s))) + \operatorname{div} h(s)\big)\,\mathrm{d}s.\end{aligned} \tag{3.45}$$

c) To control $\partial_\nu E_\nu$, we use equation (3.45) with $\alpha = 0$ and identity (3.34), where we put $a = \varepsilon(E)$, $v = \vartheta E$, and $\psi = \operatorname{div} h$. The function $\gamma = \sigma_{\nu\nu}/a_{\nu\nu}$ is bounded from below by $\gamma_0 = c\eta > 0$. We then get the estimate

$$\begin{aligned}\big\|\partial_\nu(\vartheta E(t))_\nu\big\|^2_{L_x^2} &\leq c\mathrm{e}^{-\gamma_0 t}z(0) + c\big[\|E(t)\|^2_{L_x^2} + \|\partial_\tau(\vartheta E(t))\|^2_{L_x^2} + \|\partial_\nu(\vartheta E(t))^\tau\|^2_{L_x^2}\big] \\ &\quad + c\int_0^t \mathrm{e}^{-\gamma_0(t-s)}\Big[\|E(s)\|^2_{L_x^2} + \|\partial_\tau(\vartheta E(s))\|^2_{L^2} + \|\partial_\nu(\vartheta E(s))^\tau\|^2_{L_x^2} + \|H(s)\|^2_{\mathcal{H}_x^1}\Big]\mathrm{d}s.\end{aligned}$$

This bound together with equations (3.42), (3.43) and (3.35) implies

$$\big\|\partial_\nu(\vartheta E(t))_\nu\big\|^2_{L_x^2} + \int_0^t \big\|\partial_\nu(\vartheta E(s))_\nu\big\|^2_{L_x^2}\,\mathrm{d}s \tag{3.46}$$

$$\leq c\big(z(0)+e(t)+z^2(t)\big)+\theta\int_0^t \|E(s)\|^2_{\mathcal{H}^1_x}\,\mathrm{d}s+c(\theta)\int_0^t \big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s,$$

where the (small) number $\theta>0$ comes from (3.42). Combining (3.42), (3.43), (3.46) and (3.35), we conclude

$$\begin{aligned}&\big\|\vartheta E(t)\big\|^2_{\mathcal{H}^1_x}+\int_0^t \big\|\vartheta E(s)\big\|^2_{\mathcal{H}^1_x}\,\mathrm{d}s\\ &\quad\leq c\big(z(0)+e(t)+z^2(t)\big)+\theta\int_0^t \|E(s)\|^2_{\mathcal{H}^1_x}\,\mathrm{d}s+c(\theta)\int_0^t \big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s.\end{aligned}$$

For $k\in\{1,2\}$, we proceed similarly using equation (3.44) with $\alpha=0$ and formula (3.32) for the normal component. Here the term $\|\partial_t^{k-1}\vartheta E(t)\|^2_{\mathcal{H}^1_x}$ appears on the right-hand side, which can be treated iteratively. We thus show the inequality

$$\begin{aligned}&\big\|\partial_t^k\vartheta E(t)\big\|^2_{\mathcal{H}^1_x}+\int_0^t \big\|\partial_t^k\vartheta E(s)\big\|^2_{\mathcal{H}^1_x}\,\mathrm{d}s \qquad (3.47)\\ &\quad\leq c\big(z(0)+e(t)+z^2(t)\big)+\theta\int_0^t \big\|\partial_t^k E(s)\big\|^2_{\mathcal{H}^1_x}\,\mathrm{d}s+c(\theta)\int_0^t \big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s\end{aligned}$$

for $k\in\{0,1,2\}$. Both in (3.47) and (3.37) for $|\alpha|=1$, we now choose a sufficiently small $\theta>0$. Together with (3.35) for $k\in\{0,1,2\}$, we derive the first-order bound

$$\begin{aligned}&\big\|\partial_t^k(E(t),H(t))\big\|^2_{\mathcal{H}^1_x}+\int_0^t \big\|\partial_t^k(E(s),H(s))\big\|^2_{\mathcal{H}^1_x}\,\mathrm{d}s \qquad (3.48)\\ &\qquad\leq c\big(z(0)+e(t)+z^2(t)\big)+c\int_0^t \big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s.\end{aligned}$$

4) Estimate in $\mathcal{H}^2_x$. While the bound of H in $\mathcal{H}^1_x$ was entirely based on the curl-div-estimates of Proposition 3.11, this is only partly possible in $\mathcal{H}^2_x$ or $\mathcal{H}^3_x$ since normal derivatives violate the boundary conditions. We thus have to employ the curl-div strategy of step 3) also for H. Let $k\in\{0,1\}$.

a) We first control tangential space-time derivatives of H in $\mathcal{H}^1_x$ by means of Proposition 3.11, which yields

$$\|v\|_{\mathcal{H}^1_x}\leq c\big(\|v\|_{\mathcal{H}(\mathrm{curl})}+\|\operatorname{div}(\widehat{\mu}_k v)\|_{L^2_x}+\|\operatorname{tr}_{\mathrm{no}}(\widehat{\mu}_k v)\|_{\mathcal{H}^{1/2}(\Gamma)}\big) \qquad (3.49)$$

for $v=\partial_\tau\partial_t^k\vartheta H$. The curl-term appears in the first equation in (3.41) with $|\alpha|=1$. From equations (3.9), (3.6) and (3.11) we further deduce

$$\begin{aligned}\operatorname{tr}_{\mathrm{no}}\big(\widehat{\mu}_k v\big)&=[\operatorname{tr}_{\mathrm{no}},\partial_\tau](\partial_t^k\vartheta H)-\operatorname{tr}_{\mathrm{no}}\big(\partial_\tau\widehat{\mu}_k\,\partial_t^k(\vartheta H)\big),\\ \operatorname{div}\big(\widehat{\mu}_k v\big)&=\partial_\tau([\operatorname{div},\vartheta]\widehat{\mu}_k\partial_t^k H)-[\partial_\tau,\operatorname{div}](\widehat{\mu}_k\partial_t^k(\vartheta H))-\operatorname{div}(\partial_\tau\widehat{\mu}_k\,\partial_t^k(\vartheta H)).\end{aligned}$$

The commutator $[\partial_\tau,\operatorname{div}]$ is of order one and the others are of order zero. By means of (3.16), we then estimate

$$\begin{aligned}\|\operatorname{div}\big(\widehat{\mu}_k\partial_\tau\partial_t^k(\vartheta H(t))\big)\|_{L^2_x}&\leq c\|\partial_t^k H(t)\|_{\mathcal{H}^1_x},\\ \|\operatorname{curl}\big(\partial_\tau\partial_t^k\vartheta H(t)\big)\|_{L^2_x}&\leq c\big(\|\partial_t^{k+1}E(t)\|_{\mathcal{H}^1_x}+\|\partial_t^k(E(t),H(t))\|_{\mathcal{H}^1_x}+z(t)\big),\\ \|\operatorname{tr}_{\mathrm{no}}\big(\widehat{\mu}_k\partial_\tau\partial_t^k(\vartheta H(t))\big)\|_{\mathcal{H}^{1/2}(\Gamma)}&\leq c\|\partial_t^k H(t)\|_{\mathcal{H}^1_x}. \qquad (3.50)\end{aligned}$$

Since $k+1 \le 2$, inequalities (3.48), (3.49) and (3.50) now imply

$$\begin{aligned}\big\|\partial_\tau \partial_t^k(\vartheta H(t))\big\|_{\mathcal{H}^1_x}^2 + \int_0^t \big\|\partial_\tau \partial_t^k(\vartheta H(s))\big\|_{\mathcal{H}^1_x}^2 \,\mathrm{d}s \qquad (3.51)\\ \le c\big(z(0)+e(t)+z^2(t)\big) + c\int_0^t \big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s.\end{aligned}$$

b) To treat $\partial_\nu \partial_t^k H$ in $\mathcal{H}^1_x$, we first solve in the first equation of (3.41) with $\alpha = 0$ for the tangential component $\partial_\nu(\partial_t^k \vartheta H(t))^\tau$ using (3.31). It follows

$$\|\partial_\nu \partial_t^k(\vartheta H(t))^\tau\|_{\mathcal{H}^1_x} \le c\big[\|\partial_\tau \partial_t^k(\vartheta H(t))\|_{\mathcal{H}^1_x} + \|\partial_t^k(E(t),H(t))\|_{\mathcal{H}^1_x} + \|\partial_t^{k+1}E(t)\|_{\mathcal{H}^1_x}\big].$$

Formulas (3.48) and (3.51) now allow us to bound the tangential component by

$$\begin{aligned}\big\|\partial_\nu \partial_t^k(\vartheta H(t))^\tau\big\|_{\mathcal{H}^1_x}^2 + \int_0^t \big\|\partial_\nu \partial_t^k(\vartheta H(s))^\tau\big\|_{\mathcal{H}^1_x}^2 \,\mathrm{d}s \qquad (3.52)\\ \le c\big(z(0)+e(t)+z^2(t)\big) + c\int_0^t \big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s.\end{aligned}$$

As to the normal component, we apply identity (3.32) to the divergence equation (3.38) with $\alpha = 0$ and ϑ instead of χ. The $\mathcal{H}^1_x$-norm of $\partial_\nu \partial_t^k(\vartheta H(t))_\nu$ is thus controlled by that of $\partial_t^k \vartheta H(t)$, $\partial_\tau \partial_t^k(\vartheta H(t))$, and $\partial_\nu \partial_t^k(\vartheta H(t))^\tau$. Estimates (3.48), (3.51), and (3.52) then yield

$$\begin{aligned}\big\|\partial_\nu \partial_t^k(\vartheta H(t))_\nu\big\|_{\mathcal{H}^1_x}^2 + \int_0^t \big\|\partial_\nu \partial_t^k(\vartheta H(s))_\nu\big\|_{\mathcal{H}^1_x}^2 \,\mathrm{d}s \qquad (3.53)\\ \le c\big(z(0)+e(t)+z^2(t)\big) + c\int_0^t \big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s.\end{aligned}$$

Collecting the inequalities (3.51), (3.52), (3.53), (3.39) and (3.48), we arrive at the $\mathcal{H}^2_x$-estimate for the fields H and $\partial_t H$

$$\begin{aligned}\big\|\partial_t^k H(t)\big\|_{\mathcal{H}^2_x}^2 + \int_0^t \big\|\partial_t^k H(s)\big\|_{\mathcal{H}^2_x}^2 \,\mathrm{d}s \qquad (3.54)\\ \le c\big(z(0)+e(t)+z^2(t)\big) + c\int_0^t \big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s.\end{aligned}$$

c) We now turn our attention to E. Let $|\alpha| = 2$. The L^2_x-norm of the tangential derivative $\partial_\tau^\alpha(\vartheta \partial_t^k E)$ is already controlled via inequalities (3.42), (3.48), and (3.54) up to the term

$$\theta \int_0^t \big\|\partial_t^k E(s)\big\|_{\mathcal{H}^2_x}^2 \,\mathrm{d}s.$$

The second equation in (3.41) with $|\alpha| = 1$ and formula (3.31) lead to the estimate

$$\begin{aligned}\big\|\partial_\nu\big[\partial_\tau \partial_t^k(\vartheta E(t))\big]^\tau\big\|_{L^2_x} \le c\big(\|\partial_\tau^2 \partial_t^k(\vartheta E(t))\|_{L^2_x} + \|\partial_t^k(E(t),H(t))\|_{\mathcal{H}^1_x}\\ + \|\partial_t^{k+1}E(t)\|_{\mathcal{H}^1_x} + z(t)\big).\end{aligned}$$

Combined with the tangential bound and the $\mathcal{H}^1_x$-result (3.48), we obtain

$$\big\|\partial_\nu\big(\partial_\tau \partial_t^k(\vartheta E(t)\big)^\tau\big\|_{L^2_x}^2 + \big\|\partial_\tau^\alpha \partial_t^k(\vartheta E(t))\big\|_{L^2_x}^2$$

$$+\int_0^t \Big(\big\|\partial_\nu\big(\partial_\tau\partial_t^k(\vartheta E(s))\big)^\tau\big\|_{L^2}^2+\big\|\partial_\tau^\alpha\partial_t^k(\vartheta E(s))\big\|_{L^2_x}^2\Big)\mathrm{d}s \tag{3.55}$$
$$\le c(z(0)+e(t)+z^2(t))+\theta\int_0^t\|\partial_t^k E(s)\|_{\mathcal{H}^2_x}^2\,\mathrm{d}s+c(\theta)\int_0^t\big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s.$$

d) For the normal component and $k=0$, we look at the divergence relation (3.45) with $|\alpha|=1$. As in (3.46), we deduce from (3.34) the estimate

$$\big\|\partial_\nu\big(\partial_\tau(\vartheta E(t))\big)_\nu\big\|_{L^2_x}^2+\int_0^t\big\|\partial_\nu\big(\partial_\tau(\vartheta E(s))\big)_\nu\big\|_{L^2_x}^2\,\mathrm{d}s \tag{3.56}$$
$$\le c(z(0)+e(t)+z^2(t))+\theta\int_0^t\|E(s)\|_{\mathcal{H}^2_x}^2\,\mathrm{d}s+c(\theta)\int_0^t\big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s.$$

The two above inqualities imply

$$\big\|\partial_\tau(\vartheta E(t))\big\|_{\mathcal{H}^1_x}^2+\int_0^t\big\|\partial_\tau(\vartheta E(s))\big\|_{\mathcal{H}^1_x}^2\,\mathrm{d}s \tag{3.57}$$
$$\le c\big(z(0)+e(t)+z^2(t)\big)+\theta\int_0^t\|E(s)\|_{\mathcal{H}^2_x}^2\,\mathrm{d}s+c(\theta)\int_0^t\big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s.$$

To treat the case $k=1$, we start from the divergence equation (3.44) with $|\alpha|=1$ and use formula (3.32). Employing also estimates (3.55), (3.57) and (3.16), we get

$$\big\|\partial_\nu\big(\partial_\tau(\vartheta\partial_t E(t))\big)_\nu\big\|_{L^2_x}^2+\int_0^t\big\|\partial_\nu\big(\partial_\tau(\vartheta\partial_t E(s))\big)_\nu\big\|_{L^2_x}^2\,\mathrm{d}s \tag{3.58}$$
$$\le c(z(0)+e(t)+z^2(t))+\theta\int_0^t\big(\|E(s)\|_{\mathcal{H}^2_x}^2+\|\partial_t E(s)\|_{\mathcal{H}^2_x}^2\big)\,\mathrm{d}s$$
$$+c(\theta)\int_0^t\big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s.$$

Together with inequality (3.55), this relation leads to

$$\|\partial_\tau\partial_t(\vartheta E(t))\|_{\mathcal{H}^1_x}^2+\int_0^t\|\partial_\tau\partial_t(\vartheta E(s))\|_{\mathcal{H}^1_x}^2\,\mathrm{d}s \tag{3.59}$$
$$\le c(z(0)+e(t)+z^2(t))+\theta\int_0^t\big(\|E(s)\|_{\mathcal{H}^2_x}^2+\|\partial_t E(s)\|_{\mathcal{H}^2_x}^2\big)\,\mathrm{d}s$$
$$+c(\theta)\int_0^t\big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s.$$

e) It remains to control the term $\partial_\nu^2(\partial_t^k\vartheta E)$. We first replace the derivative ∂_τ^α by ∂_ν in system (3.41). The resulting second equation, the curl-formula (3.31) and estimates (3.16) imply

$$\big\|\partial_\nu\big(\partial_\nu\partial_t^k(\vartheta E(t))\big)^\tau\|_{L^2_x}\le c\big[\big\|\partial_\tau\partial_\nu\partial_t^k(\vartheta E(t))\big\|_{L^2_x}+\max_{j\le 2}\big\|\partial_t^j(E(t),H(t))\big\|_{\mathcal{H}^1_x}+z(t)\big].$$

The right-hand side can be estimated via inequalities (3.48) and (3.59).

For the normal component, we employ the modifications of the divergence relations (3.45) and (3.44) with ∂_ν instead of ∂_τ^α. We then estimate $\partial_\nu\big(\partial_\nu\partial_t^k(\vartheta E(t))\big)_\nu$ for $k\in\{0,1\}$ as in inequalities (3.56) and (3.58). Here and in (3.40), (3.57) and (3.59), we take a small $\theta > 0$ to absorb the $\mathcal{H}^2$-norms of $\partial_t^k E$ on the right-hand side. Using also (3.54) for the magnetic field, for $k \in \{0,1\}$ we derive the desired bound in $\mathcal{H}_x^2$

$$\big\|\partial_t^k(E(t),H(t))\big\|_{\mathcal{H}_x^2}^2 + \int_0^t \|\partial_t^k(E(s),H(s))\|_{\mathcal{H}_x^2}^2\,\mathrm{d}s \qquad (3.60)$$
$$\leq c\big(z(0)+e(t)+z^2(t)\big) + c\int_0^t \big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s.$$

5) Estimate in $\mathcal{H}_x^3$. Since the reasoning is similar to step 4), we will omit some details here. Let $k=0$.

a) We again begin with the magnetic field H. We first look at the tangential derivative $\partial_\tau^\alpha(\vartheta E)$ with $|\alpha|=2$, where we proceed as in (3.51) using Proposition 3.11. For $\xi,\zeta \in \{\nu,\tau^1,\tau^2\}$, differentiating the divergence relation (3.6) we obtain

$$\begin{aligned}\operatorname{div}\big(\mu(H)\partial_\xi\partial_\zeta(\vartheta H)\big) &= \partial_\xi\partial_\zeta([\operatorname{div},\vartheta]\mu(H)H) - [\partial_\xi\partial_\zeta,\operatorname{div}](\mu(H)\vartheta H)\\ &\quad - \operatorname{div}(\partial_\zeta\mu(H)\,\partial_\xi(\vartheta H)) - \operatorname{div}(\partial_\xi\mu(H)\,\partial_\zeta(\vartheta H))\\ &\quad - \operatorname{div}(\partial_\xi\partial_\zeta\mu(H)\,\vartheta H).\end{aligned}$$

Similary, the magnetic boundary condition in (3.6) yields

$$\operatorname{tr}_{\mathrm{no}}\big(\mu(H)\partial_\tau^\alpha(\vartheta H)\big) = [\operatorname{tr}_{\mathrm{no}},\partial_\tau^\alpha](\mu(H)\vartheta H) + \operatorname{tr}_{\mathrm{no}}\sum_{0\leq\beta<\alpha}\binom{\alpha}{\beta}\partial_\tau^{\alpha-\beta}\mu(H)\,\partial_\tau^\beta(\vartheta H).$$

Employing (3.16), we deduce the estimates

$$\begin{aligned}\|\operatorname{curl}\partial_\tau^\alpha(\vartheta H(t))\|_{L_x^2} &\leq c\big(\|\partial_t E(t)\|_{\mathcal{H}_x^2} + \|(E(t),H(t))\|_{\mathcal{H}_x^2} + z(t)\big),\\ \|\operatorname{div}\big(\mu(H(t))v\big)\|_{L_x^2} &\leq c\|H(t)\|_{\mathcal{H}_x^2},\\ \|\operatorname{tr}_{\mathrm{no}}\big(\mu(H(t))v\big)\|_{\mathcal{H}^{1/2}(\Gamma)} &\leq c\|H(t)\|_{\mathcal{H}_x^2}\end{aligned}$$

from (3.41) and the above formulas. The second-order bound (3.60) and Proposition 3.11 thus imply

$$\big\|\partial_\tau^\alpha(\vartheta H(t))\big\|_{\mathcal{H}_x^1}^2 + \int_0^t \|\partial_\tau^\alpha(\vartheta H(s))\|_{\mathcal{H}_x^1}^2\,\mathrm{d}s \qquad (3.61)$$
$$\leq c\big(z(0)+e(t)+z^2(t)\big) + c\int_0^t \big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s.$$

To include one normal derivative, we first use (3.41) with $|\alpha|=1$ and the curl-formula (3.31). We can then bound the $\mathcal{H}_x^1$-norm of $\partial_\nu(\partial_\tau(\vartheta H(t)))^\tau$ by

$$\|\partial_\tau^2(\vartheta H(t))\|_{\mathcal{H}_x^1} + \max_{j\leq 1}\big\|\partial_t^j(E(t),H(t))\big\|_{\mathcal{H}_x^2} + z(t).$$

The normal component is treated as in (3.53), based on the divergence relation (3.38) with $|\alpha|=1$, χ replaced by ϑ, and ∂_x^α by ∂_τ. By means of (3.32) and (3.16), the $\mathcal{H}_x^1$-norm of the function $\partial_\nu(\partial_\tau(\vartheta H(t)))_\nu$ is thus controlled by that of $\partial_\nu(\partial_\tau(\vartheta H(t)))^\tau$ and $\partial_\tau^2(\vartheta H(t))$ plus lower order terms. Combining these inequalities with (3.60) and (3.61), we infer

$$\begin{aligned} \big\|\partial_\nu\partial_\tau(\vartheta H(t))\big\|_{\mathcal{H}^1_x}^2 + \int_0^t \|\partial_\nu\partial_\tau(\vartheta H(s))\|_{\mathcal{H}^1_x}^2 \,\mathrm{d}s \\ \le c\big(z(0)+e(t)+z^2(t)\big) + c\int_0^t \big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s. \end{aligned} \tag{3.62}$$

In this reasoning we can replace ∂_τ by ∂_ν, arriving at

$$\begin{aligned} \big\|\partial_\nu^2(\vartheta H(t))\big\|_{\mathcal{H}^1_x}^2 + \int_0^t \|\partial_\nu^2(\vartheta H(s))\|_{\mathcal{H}^1_x}^2 \,\mathrm{d}s \\ \le c\big(z(0)+e(t)+z^2(t)\big) + c\int_0^t \big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s. \end{aligned} \tag{3.63}$$

Together with (3.39) and (3.60), the estimates (3.61), (3.62) and (3.63) lead to

$$\begin{aligned} \|H(t)\|_{\mathcal{H}^3_x}^2 + \int_0^t \|H(s)\|_{\mathcal{H}^3_x}^2 \,\mathrm{d}s \\ \le c\big(z(0)+e(t)+z^2(t)\big) + c\int_0^t \big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s. \end{aligned} \tag{3.64}$$

b) We finally tackle E in $\mathcal{H}^3_x$. The third-order tangential derivatives $\partial_\tau^\alpha(\vartheta E)$ were already treated in estimate (3.42) with $k=0$, where the lower order-terms on the right-hand side are now dominated by (3.60) and (3.64). Let $|\beta|=2$. The second equation in (3.41) with $|\alpha|=2$ and the curl-formula (3.31) allow us to bound $\partial_\nu(\partial_\tau^\beta(\vartheta E))^\tau$ in the same fashion. The normal component $\partial_\nu(\partial_\tau^\beta(\vartheta E))_\nu$ can also be estimated via equations (3.45) and (3.34) as in (3.46). We thus arrive at

$$\begin{aligned} \big\|\partial_\tau^\beta(\vartheta E(t))\big\|_{\mathcal{H}^1_x}^2 + \int_0^t \big\|\partial_\tau^\beta(\vartheta E(s))\big\|_{\mathcal{H}^1_x}^2 \,\mathrm{d}s \\ \le c\big(z(0)+e(t)+z^2(t)\big) + \theta\int_0^t \big\|E(s)\big\|_{\mathcal{H}^3_x}^2\,\mathrm{d}s + c(\theta)\int_0^t \big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s. \end{aligned} \tag{3.65}$$

We replace the tangential derivative ∂_τ^α by $\partial_\nu\partial_\tau$ in system (3.41). The second equation therein and formula (3.31) provide control of the tangential component $\partial_\nu(\partial_\nu\partial_\tau(\vartheta E))^\tau$ in L^2_x via inequalities (3.65) and (3.60). The related normal component can then be handled through the formula (3.34) and the divergence identity (3.45) with $\partial_\nu\partial_\tau$ instead of ∂_τ^α. In this way we show the inequality

$$\begin{aligned} \big\|\partial_\tau(\vartheta E(t))\big\|_{\mathcal{H}^2_x}^2 + \int_0^t \big\|\partial_\tau(\vartheta E(s))\big\|_{\mathcal{H}^2_x}^2 \,\mathrm{d}s \\ \le c\big(z(0)+e(t)+z^2(t)\big) + \theta\int_0^t \big\|E(s)\big\|_{\mathcal{H}^3_x}^2\,\mathrm{d}s + c(\theta)\int_0^t \big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s. \end{aligned}$$

The remaining term $\partial_\nu^3(\vartheta E)$ is managed analogously, resulting in

$$\begin{aligned} \big\|\vartheta E(t)\big\|_{\mathcal{H}^3_x}^2 + \int_0^t \big\|\vartheta E(s)\big\|_{\mathcal{H}^3_x}^2 \,\mathrm{d}s \\ \le c\big(z(0)+e(t)+z^2(t)\big) + \theta\int_0^t \big\|E(s)\big\|_{\mathcal{H}^3_x}^2\,\mathrm{d}s + c(\theta)\int_0^t \big(e(s)+z^{3/2}(s)\big)\,\mathrm{d}s. \end{aligned}$$

Fixing a sufficiently small number $\theta > 0$, the above inequalities and the interior estimate (3.40) combined with (3.48) and (3.60) lead to the bound

$$\big\|E(t)\big\|^2_{\mathcal{H}^3_x} + \int_0^t \big\|E(s)\big\|^2_{\mathcal{H}^3_x}\,\mathrm{d}s \le c\big(z(0) + e(t) + z^2(t)\big) + c\int_0^t \big(e(s) + z^{3/2}(s)\big)\,\mathrm{d}s.$$

The above equation and (3.64) now furnish our last result

$$\begin{aligned} \big\|(E(t),H(t))\big\|^2_{\mathcal{H}^3_x} &+ \int_0^t \big\|(E(s),H(s))\big\|^3_{\mathcal{H}^3_x}\,\mathrm{d}s \\ &\le c\big(z(0) + e(t) + z^2(t)\big) + c\int_0^t \big(e(s) + z^{3/2}(s)\big)\mathrm{d}s. \end{aligned} \tag{3.66}$$

Proposition 3.17 now follows from formulas (3.48), (3.60) and (3.66). □

Bibliography

[1] R.A. Adams and J.J.F. Fournier, *Sobolev Spaces.* 2nd edition. Academic Press, 2003.

[2] G.P. Agrawal, *Nonlinear Fiber Optics.* 6th edition. Academic Press, 2019.

[3] D. Aregba-Driollet and B. Hanouzet, Kerr-Debye relaxation shock profiles for Kerr equations. *Commun. Math. Sci.* **9**:1 (2011), 1–31.

[4] F. Assous, P. Ciarlet and S. Labrunie, *Mathematical Foundations of Computational Electromagnetism.* Springer, 2018.

[5] H. Bahouri, J.Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations.* Springer, 2011.

[6] K. Beauchard and E. Zuazua, Large time asymptotics for partially dissipative hyperbolic systems. *Arch. Ration. Mech. Anal.* **199**:1 (2011), 177–227.

[7] S. Benzoni-Gavage and D. Serre, *Multidimensional Hyperbolic Partial Differential Equations.* Oxford University Press, 2007.

[8] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer, 2011.

[9] P.N. Butcher and D. Cotter, *The Elements of Nonlinear Optics.* Cambridge University Press, 1990.

[10] M. Cessenat, *Mathematical Methods in Electromagnetism.* World Scientific, 1996

[11] J. Chazarain and A. Piriou, *Introduction to the Theory of Linear Partial Differential Equations.* North-Holland, 1982.

[12] F. Colombini, E. De Giorgi and S. Spagnolo, Sur les équations hyperboliques avec des coefficients qui ne dépendent que du temps. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **6**:3 (1979), 511–559.

[13] M. Costabel, M. Dauge and S. Nicaise, *Corner Singularities and Analytic Regularity for Linear Elliptic Systems. Part I*, 2010. See http://hal.archives-ouvertes.fr/hal-00453934/en/

[14] P. D'Ancona, S. Nicaise and R. Schnaubelt, Blow-up for nonlinear Maxwell equations. *Electron. J. Differential Equations* (2018), Paper No. 73.

[15] R. Dautray and J.L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology. Vol. 1.* Physical origins and classical methods, with the collaboration of P. Bénilan, M. Cessenat, A. Gervat, A. Kavenoky and H. Lanchon. Springer, 1990.

[16] R. Dautray and J.L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology. Vol. 3.* Spectral theory and applications, with the collaboration of M. Artola and M. Cessenat. Springer, 1990.

[17] W. Dörfler, M. Hochbruck, J. Köhler, A. Rieder, R. Schnaubelt and C. Wieners, *Wave Phenomena: Mathematical Analysis and Numerical Approximation.* Oberwolfach Seminars **49**, Birkhäuser, 2023.

[18] M. Eller, Continuous observability for the anisotropic Maxwell system. *Appl. Math. Optim.* **55**:2 (2007), 185–201.

[19] M. Eller, On symmetric hyperbolic boundary problems with nonhomogeneous conservative boundary conditions. *SIAM J. Math. Anal.* **44**:3 (2012), 1925–1949.

[20] M. Eller, Stability of the anisotropic Maxwell equations with a conductivity term. *Evol. Equ. Control Theory* **8**:2 (2019), 343–357.

[21] M. Eller, J.E. Lagnese and S. Nicaise, Decay rates for solutions of a Maxwell system with nonlinear boundary damping. *Comput. Appl. Math.* **21**:1 (2002), 135–165.

[22] L.C. Evans, *Partial Differential Equations.* American Mathematical Society, 1998.

[23] M. Fabrizio and M. Morro, *Electromagnetism of Continuous Media.* Oxford University Press, 2003.
[24] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order.* Springer, 2001.
[25] L. Grafakos, *Modern Fourier Analysis.* 2nd edition. Springer, 2009.
[26] O. Guès, Problème mixte hyperbolique quasi-linéaire caractéristique. *Comm. Partial Differential Equations* **15**:5 (1990), 595–645.
[27] T. Hytönen, J. van Neerven, M. Veraar and L. Weis, *Analysis in Banach Spaces. Vol. I. Martingales and Littlewood-Paley Theory.* Springer, 2016.
[28] M. Ifrim and D. Tataru, Local well-posedness for quasilinear problems: a primer. Preprint 2020, arXiv:2008.05684
[29] J.D. Jackson, *Classical Electrodynamics.* 3rd edition. John Wiley & Sons, 1999.
[30] T. Kato, Quasi-linear equations of evolution, with applications to partial differential equations. In: William N. Everitt (Ed.), *Spectral Theory and Differential Equations* (Proceedings Dundee, 1974). Springer, 1975, pp. 25–70.
[31] T. Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems. *Arch. Rational Mech. Anal.* **58**:3 (1975), 181–205.
[32] T. Kato, *Abstract Differential Equations and Nonlinear Mixed Problems.* Scuola Normale Superiore, Pisa; Accademia Nazionale dei Lincei, Rome, 1985.
[33] V. Komornik, Boundary stabilization, observation and control of Maxwell's equations. *PanAmer. Math. J.* **4**:4 (1994), 47–61.
[34] I. Lasiecka, M. Pokojovy and R. Schnaubelt, Exponential decay of quasilinear Maxwell equations with interior conductivity. *NoDEA Nonlinear Differential Equations Appl.* **26**:6 (2019), Paper No. 51, 34 pp.
[35] S. Lucente and G. Ziliotti, Global existence for a quasilinear Maxwell system. *Rend. Istit. Mat. Univ. Trieste* **31**:suppl. 2 (2000), 169–187.
[36] A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables.* Springer, 1984.
[37] J.E. Muñoz Rivera and R. Racke, Mildly dissipative nonlinear Timoshenko systems—global existence and exponential stability. *J. Math. Anal. Appl.* **276**:1 (2002), 248–278.
[38] S. Nicaise and C. Pignotti, Boundary stabilization of Maxwell's equations with space-time variable coefficients. *ESAIM Control Optim. Calc. Var.* **9** (2003), 563–578.
[39] S. Nicaise and C. Pignotti, Internal stabilization of Maxwell's equations in heterogeneous media. *Abstr. Appl. Anal.* **7** (2005), 791–811.
[40] L. Nirenberg, On elliptic partial differential equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (3) **13** (1959), 115–162.
[41] K.D. Phung, Contrôle et stabilisation d'ondes électromagnétiques. *ESAIM Control Optim. Calc. Var.* **5** (2000), 87–137.
[42] R.H. Picard and W.M. Zajączkowski, Local existence of solutions of impedance initial-boundary value problem for non-linear Maxwell equations. *Math. Methods Appl. Sci.* **18**:3 (995), 169–199.
[43] M. Pokojovy and R. Schnaubelt, Boundary stabilization of quasilinear Maxwell equations. *J. Differential Equations* **268**:2 (2020), 784–812.
[44] R. Racke, *Lectures on Nonlinear Evolution Equations.* Vieweg, 1992.
[45] J. Rauch, $\mathcal{L}_2$ is a continuable initial condition for Kreiss' mixed problems. *Comm. Pure Appl. Math.* **25** (1972), 265–285.
[46] W. Rudin, *Real and Complex Analysis.* 3rd edition. McGraw-Hill, 1987.
[47] R. Schnaubelt and M. Spitz: Local wellposedness of quasilinear Maxwell equations with absorbing boundary conditions. *Evol. Equ. Control Theory* **10**:1 (2021), 155–198.
[48] R. Schnaubelt and M. Spitz, Local wellposedness of quasilinear Maxwell equations with conservative interface conditions. *Commun. Math. Sci.* **20** (2022), 2265-2313.
[49] P. Secchi, Well-posedness of characteristic symmetric hyperbolic systems. *Arch. Rational Mech. Anal.* **134**:2 (1996), 155–197.
[50] J. Speck, The nonlinear stability of the trivial solution to the Maxwell-Born-Infeld system. *J. Math. Phys.* **53**:8, No. 083703 (2012), 83 pp.

[51] M. Spitz, *Local Wellposedness of Nonlinear Maxwell Equations.* Ph.D. thesis, Karlsruhe Institute of Technology, 2017. https://doi.org/10.5445/IR/1000078030

[52] M. Spitz, Local wellposedness of nonlinear Maxwell equations with perfectly conducting boundary conditions. *J. Differential Equations* **266**:8 (2019), 5012–5063.

[53] M. Spitz, Regularity theory for nonautonomous Maxwell equations with perfectly conducting boundary conditions. *J. Math. Anal. Appl.* **506**:1, No. 125646 (2022), 43 pp.

[54] M.E. Taylor, *Pseudodifferential Operators and Nonlinear PDE.* Birkhäuser, 1991.

[55] M.E. Taylor, *Partial Differential Equations I. Basic Theory.* 2nd edition. Springer, 2011.