SEMILINEAR OBSERVATION SYSTEMS

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Abstract. In this paper, we introduce locally Lipschitz observation systems for nonlinear semigroups and show that they can be represented by an 'admissible' nonlinear output operator defined on a suitable subspace. In the semilinear case, this concept fits well to the Lebesgue extension known from linear system theory. Also in the semilinear case, we show robustness of exact observability near equilibria under locally small Lipschitz perturbations. Finally, we apply our results to a damped nonlinear beam equation and a semilinear thermo-elastic system.

1. Introduction

In distributed parameter systems pointwise or boundary observations lead to unbounded (and even non closable) output operators. To treat such situations in a unified and efficient way, the concepts of admissible observation operators and of observation systems have been introduced in the linear case by Salamon and Weiss in [17] and [20]. An operator $C \in \mathcal{L}(D(A), Y)$ is called admissible for a C_0-semigroup $T = (T(t))_{t\geq 0}$ with generator A if the output map $x \mapsto C(T(\cdot)x)$, initially defined on $D(A)$, can be extended to a continuous map Ψ from X to $L^2_{loc}(\mathbb{R}_+, Y)$. The pair (T, ψ) is then an observation system; i.e., it holds $(\Psi x)(\cdot + \tau) = \Psi T(\tau)x$ for all $x \in X$ and $\tau \geq 0$. Conversely, for any observation system (T, Ψ) there is an admissible output operator $C \in \mathcal{L}(D(A), Y)$ such that $\Psi x = CT(\cdot)x$ for every $x \in D(A)$. Moreover, there exists the 'Lebesgue extension' C_L of C satisfying $T(t)x \in D(C_L)$ for a.e. $t \geq 0$ and $\Psi x = C_L T(\cdot)x$ for all $x \in X$, see [20] and also [6, 10, 17].

In this paper we extend this successful linear theory to general nonlinear locally Lipschitz semigroups $S = (S(t))_{t\geq 0}$ (see Definition 2.3) and densely defined nonlinear output operators C. In particular, for such semigroups S we define locally Lipschitz observation systems Ψ and locally Lipschitz admissible observation operators in Section 3. We further prove that such observation systems Ψ can be represented by $\Psi x = \widetilde{C}(S(\cdot)x)$ for a (possibly nonlinear) admissible observation operator $\widetilde{C}$, see Theorem 3.6.

We then focus on the semilinear observation system

$$\begin{aligned} \dot{u}(t) &= Au(t) + F(u(t)), \quad u(0) = x \in X, \ t \geq 0, && (1.1) \\ y(t) &= C(u(t)), && (1.2) \end{aligned}$$

where A is assumed to be the generator of a linear C_0-semigroup T on a Banach space X, C is a nonlinear unbounded operator from a domain $D(C)$ to another

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Banach space Y and F is a locally Lipschitz continuous nonlinear operator from X into itself. Throughout we assume that F has linear growth.

It is well known, see e.g. [16], that the state equation (1.1) has a global unique mild solution given by $u(\cdot;x)$ for every $x \in X$. Moreover, by $S(t)x = u(t;x)$ one defines a semigroup S of locally Lipschitz continuous operators. One now looks for sufficient conditions for the admissibility of C for S. As an important special case, we assume that C is an admissible linear output operators for T. In this situation one can in fact construct a nonlinear observation system (S, Ψ_F) given by (3.5), which is the integrated version of (1.1)–(1.2). Moreover, the system is (S, Ψ_F) represented by the Lebesgue extension C_L of C with respect to T, see Theorem 3.7.

In Section 4 we then define and study local exact observability of locally Lipschitz observation systems. Again, in the case of the semilinear system (1.1)–(1.2) with a linear admissible operator C, it is desirable to have criteria of the observability of the system in terms of the linear system given by T and C. In fact, in Theorem 4.4 we show that near an equilibrium x_0 of (1.1) the linear system is exactly observable on $[0, \tau]$ if and only if (1.1)–(1.2) is locally exactly observable on $[0, \tau]$, provided that the Lipschitz constant of F near x_0 is sufficiently small. This property holds if $F \in C^1(X)$ with $F'(x_0) = 0$.

Similar robustness results for admissibility and exact observability were shown for globally Lipschitz F in the paper [3] by two of the authors. In that paper also additional regularity properties of F or T were assumed which were needed to treat the variation of constants formula related to (1.1). In the present we could discard these extra assumptions by using an estimate for the convolution $f \mapsto C_L T * f$ established in [18] for admissible C, see (3.6).

If the system is linear and X is a reflexive Banach space (e.g. Hilbert space), then the concept of controllability is dual to the concept of observability. For nonlinear systems the situation is more involved. Consequently, most publication study exact controllability and exact observability separately. There are various publications in the literature on the controllability of specific semilinear systems. We refer the reader to [2, 4, 5, 11, 22] and the references therein. On the other side, to our knowledge, there are only few results on observability of semilinear systems with linear (or nonlinear) observation operators. In particular, Mangnusson established in [15] a robustness result for exact observability near an equilibrium. He allowed for a larger class of nonlinearities in (1.1), but considered only (nonlinear) observation operators defined on X. In contrast, we focus on observation operators defined only on subspaces.

The paper is organized as follows. In Section 2, we discuss background material on locally Lipschitz semigroups and semilinear equations as well as the basic notions and results on linear observation systems. In Section 3 and 4 we prove the above indicated results on admissibility and observability, respectively. Finally, in Section 5, we apply our results to a damped semilinear beam equation and a semilinear thermo-elastic system.

2. Background

In this section we introduce notations and assumptions used throughout the paper. We further discuss known results about semilinear evolution equations and linear observation systems. We denote by X and Y Banach spaces (the state and the observation space, respectively). The family $T = (T(t))_{t\geq 0}$ of linear operators is a C_0-semigroup on X with generator $(A, D(A))$. We can fix constants $M, \omega > 0$ such that

$$\|T(t)\| \leq Me^{\omega t} \tag{2.1}$$

holds for all $t \geq 0$. We denote by $\mathcal{L}(E,G)$ the space of bounded linear operators between two Banach spaces E and G and put $\mathcal{L}(E) := \mathcal{L}(E,E)$. Moreover, the (nonlinear) operator $F : X \longrightarrow X$ is always assumed to be locally Lipschitz continuous; that is, for each $r > 0$ there exists a constant $L(r) \geq 0$ such that

$$\|F(x) - F(y)\| \leq L(r)\|x - y\|,$$

for all x, $y \in X$ with $\|x\| \leq r$ and $\|y\| \leq r$.

It is well-known (see e.g. Theorem 6.1.4 in [16]) that, under the above assumptions, for every $x \in X$ there is a maximal $t(x) \in (0, \infty]$ such that the problem (1.1) admits a unique mild solution $u = u(\cdot; x) \in C([0, t(x)), X)$ given by the variation of constant formula

$$u(t) = T(t)x + \int_0^t T(t - \sigma)F(u(\sigma))d\sigma. \tag{2.2}$$

Moreover, if $t(x) < \infty$ then $\lim_{t\to t(x)} \|u(t)\| = \infty$. For our investigations it suffices to consider mild solutions. The question whether they are in fact classical solutions of (1.1) is discussed, e.g., in [16, Chapter 6]. In this paper we work in the situation of global solvability assuming that

(H) $\|F(x)\| \leq a\|x\| + b$ holds for all $x \in X$ and some constants $a, b \geq 0$.

Under this condition of linear growth, the formula (2.2) and Gronwall's inequality easily yield the next result.

Proposition 2.1. *Let A generate a C_0-semigroup T satisfying (2.1) and $F : X \to X$ be locally Lipschitz such that (H) holds. Then the problem (1.1) has a unique global mild solution in $C([0,\infty), X)$ for each $x \in X$. Moreover, u is exponentially bounded in the sense that*

$$\|u(t)\| \leq \frac{Mb}{\omega}e^{\omega t} + Me^{(\omega + aM)t}\|x\| \qquad \textit{for all } t \geq 0. \tag{2.3}$$

Remark 2.2. *If we assume that F is globally Lipschitz continuous, then it has linear growth and thus (1.1) has a unique global mild solution for each $x \in X$.*

Definition 2.3. *A family $S = (S(t))_{t\geq 0}$ of locally Lipschitz operators from X into itself is called a* semigroup of locally Lipschitz operators *on X if it satisfies the following conditions:*

(a) *$S(t + s)x = S(t)S(s)x$ and $S(0)x = 0$ for all $t, s \geq 0$ and $x \in X$.*
(b) *For each $x \in X$, the X-valued function $S(\cdot)x$ is continuous on $[0, \infty)$.*

(c) *For every* $r > 0$ *and* $t_0 > 0$ *there exists a constant* $L(t_0, r) > 0$ *such that for all* $x, y \in X$ *with* $\|x\|, \|y\| \le r$ *we have*

$$\|S(t)x - S(t)y\| \le L(t_0, r)\|x - y\| \qquad \textit{for all} \quad t \in [0, t_0]. \tag{2.4}$$

Let $u(\cdot; x)$ be the solution of 1.1 for a given $x \in X$, where we assume that (H) holds. We define $S(t)x := u(t; x)$ for all $x \in X$ and $t \ge 0$. The operators $S(t)$ then map X into itself and satisfy the properties stated in Definition 2.3. In fact, the first property follows from the uniqueness of mild solutions. The second one is an immediate consequence of the continuity of $t \mapsto u(t; x)$. The last property can be shown using (2.2), (2.3), the local Lipschitz continuity of F and Gronwall's inequality. Hence, the output function in (1.2) is formally given by

$$y(t) = C(S(t)x).$$

Of course, this expression only makes sense if $S(t)x$ belongs to the domain $D(C)$ of C. We note that, in general, $D(C)$ is not invariant under $S(t)$. Such problems already occur in the linear case. To motivate our approach, we first recall the linear theory before proceeding with the nonlinear one.

Let T be a C_0-semigroup on X with generator $(A, D(A))$ and $C : D(A) \to Y$ be a linear bounded operator, where $D(A)$ is endowed with its graph norm. (The operator C could be unbounded and even non closable in X.) We consider the observation system

$$\begin{aligned} \dot{u}(t) &= Au(t), \qquad & u(0) = x, \\ y(t) &= Cu(t), \qquad & t \ge 0. \end{aligned} \tag{2.5}$$

Since $T(t) : D(A) \to D(A)$ is bounded, the output $y = Cu : \mathbb{R}_+ \to Y$ is well-defined and continuous. Wellposedness of (2.5) should mean that the map $x \mapsto y$ is continuous from X to $L^2([0, t], Y)$ for each $t \ge 0$. Correspondingly, Weiss introduced in [20] the concept of *admissibility* of C (for T), which says that the estimate

$$\int_0^{t_0} \|CT(t)x\|^2 dt \le \gamma(t_0)^2 \|x\|^2, \tag{2.6}$$

holds for some (hence all) $t_0 > 0$, all $x \in D(A)$ and locally bounded constants $\gamma(t_0) > 0$. Due to (2.6), the linear operator $\Psi : D(A) \to L^2_{loc}(R_+, Y)$ defined by

$$(\Psi x)(\tau) = CT(\tau)x, \qquad \tau \ge 0, \tag{2.7}$$

can be extended to a continuous linear operator from X to $L^2_{loc}(R_+, Y)$ denoted again by Ψ. Here and below, $L^2_{loc}(\mathbb{R}_+, Y)$ is endowed with its canonical metric, i.e., functions f_n converge to f in $L^2_{loc}(\mathbb{R}_+, Y)$ as $n \to \infty$ if the restrictions $f_n|[0, t_0]$ converge to $f|[0, t_0]$ in $L^2([0, t_0], Y)$ as $n \to \infty$, for every $t_0 > 0$. Thus, a linear operator $\Phi : X \to L^2_{loc}(\mathbb{R}_+, Y)$ is continuous if and only if there are constants $c(t_0)$ such that $\|\Phi x\|_{L^2([0,t_0],Y)} \le c(t_0)\|x\|$ for all $x \in X$ and $t_0 > 0$. It further holds

$$(\Psi x)(\cdot + \tau) = \Psi T(\tau)x \qquad \text{on} \;\; \mathbb{R}_+ \quad \text{for all } \tau \ge 0. \tag{2.8}$$

If a linear operator $\Psi : X \to L^2_{loc}(\mathbb{R}_+, Y)$ is continuous and satisfies (2.8), we say that (T, Ψ) is a *linear observation system* on X and Y.

Therefore, for an admissible C, the system (2.5) possesses the (extended) observation Ψx for every $x \in X$, where $\Psi : X \to L^2_{loc}(\mathbb{R}_+, Y)$ is the extension of the operator defined in (2.7) and (T, Ψ) is an observation system. Conversely, for any given observation system (T, Ψ), Weiss constructed in Theorem 3.3 of [20] an admissible observation operator $C \in \mathcal{L}(D(A), Y)$ such that (2.7) holds. Moreover, he defined the *Lebesgue extension* of C by

$$C_L x := \lim_{\tau \searrow 0} C \frac{1}{\tau} \int_0^\tau T(t) x dt, \tag{2.9}$$

with domain

$$D(C_L) := \{x \in X : \text{ the limit in (2.9) exists}\}.$$

This definition makes sense for any $C \in \mathcal{L}(D(A), Y)$ without assuming admissibility. It then holds $D(A) \subset D(C_L) \subset X$, $Cy = C_L y$ for all $y \in D(A)$, as well as $T(t)x \in D(C_L)$ and $\Psi x(t) = C_L T(t)x$ for all $x \in X$ and a.e. $t \geq 0$, see Section 4 in [20]. We note that in [21] another extension of C (the Yosida extension C_Λ) was introduced. However, for our purposes the Lebesgue extension is more appropriate.

3. Locally Lipschitz observation systems

We start with our basic definitions.

Definition 3.1. *A* locally Lipschitz observation system *on the Banach spaces X and Y is a pair (S, Ψ) (resp. (T, Ψ)), where $S := (S(t))_{t \geq 0}$ (resp. $T := (T(t))_{t \geq 0}$) is a semigroup of locally Lipschitz operators (resp. a linear C_0-semigroup) on X and Ψ is a family of (possibly nonlinear) operators from X to $L^2_{loc}([0, \infty), Y)$ such that for every t_0, $r > 0$ there exists a constant $k(r, t_0) > 0$ such that*

$$\begin{aligned} &(\Psi x)(\cdot + \tau) = \Psi S(\tau) x \quad (\textit{resp. } (\Psi x)(\cdot + \tau) = \Psi T(\tau) x) \quad \textit{on } \mathbb{R}_+, \\ &\|\Psi x - \Psi y\|_{L^2([0,t_0],Y)} \leq k(r, t_0) \|x - y\|, \end{aligned} \tag{3.1}$$

for all $\tau \geq 0$ and $x, y \in X$ with $\|x\|, \|y\| \leq r$.

Definition 3.2. *Let S (resp. T) be a semigroup of locally Lipschitz operators (resp. a linear C_0-semigroup) on X and let $C : D(C) \to Y$ be a (possibly nonlinear) operator with dense domain $D(C)$ in X. We say that C is a* locally Lipschitz admissible observation operator *for S (resp. T) if, for every $x \in D(C)$, it holds $S(t)x \in D(C)$ (resp. $T(t)x \in D(C)$) for a.e. $t \geq 0$, the function $C(S(\cdot)x) : \mathbb{R}_+ \to Y$ (resp. $CT(\cdot)x : \mathbb{R}_+ \to Y$) is strongly measurable and if for every $t_0 > 0$ and every $r > 0$ there is a constant $\gamma(r, t_0) > 0$ such that*

$$\int_0^{t_0} \|CS(t)x - CS(t)y)\|_Y^2 dt \leq \gamma(r, t_0)^2 \|x - y\|^2, \tag{3.2}$$

$$(\textit{resp.} \int_0^{t_0} \|CT(t)x - CT(t)y)\|_Y^2 dt \leq \gamma(r, t_0)^2 \|x - y\|^2) \tag{3.3}$$

for all x, $y \in D(C)$ with $\|x\|$, $\|y\| < r$.

Remark 3.3. *In the linear case with $D(A) = D(C)$ the above concepts coincide with those recalled in the previous section. In the nonlinear setting we want to avoid the use of the generator A which is a more complicated concept in this case.*

Let C be locally Lipschitz admissible for S (resp. T). Then the map $\Psi : D(C) \to L^2_{loc}(\mathbb{R}_+, Y)$, $x \mapsto CS(\cdot)x$ (resp. $x \mapsto CT(\cdot)x$), possesses a locally Lipschitz continuous extension from X to $L^2_{loc}(\mathbb{R}_+, Y)$. In fact, let $x \in X$ and $t_0 > 0$. Since $D(C)$ is dense, there exist $x_n \in D(C)$ converging to x in X as $n \to \infty$. Estimate (3.2) (resp. (3.3)) imply that Ψx_n is a Cauchy sequence which therefore converges to some z in the complete metric space $L^2_{loc}(\mathbb{R}_+, Y)$. If $x_n' \in D(C)$ converge to x in X, then $\Psi x_n'$ also converges to z in $L^2_{loc}(\mathbb{R}_+, Y)$ because of (3.2) and (3.3). So we can extend Ψ to map from X to $L^2_{loc}(\mathbb{R}_+, X)$ denoted by the same symbol. Let $t_0, r > 0$ and x, $y \in X$ with $\|x\|$, $\|y\| < r$. There are $x_n \in D(C)$ and $y_n \in D(C)$ converging to x and y, respectively. Using (3.2) and (3.3) we can then estimate

$$\begin{aligned} \|\Psi x - \Psi y\|_{L^2([0,t_0],Y)} &= \lim_{n\to\infty} \|\Psi x_n - \Psi y_n\|_{L^2([0,t_0],Y)} \\ &\le \gamma(r, t_0) \lim_{n\to\infty} \|x_n - y_n\| = \gamma(r, t_0)\|x - y\|. \end{aligned}$$

Hence, Ψ is locally Lipschitz continuous on X. We further obtain

$$\Psi x(\tau + \cdot) = \lim_{n\to\infty} \Psi x_n(\tau + \cdot) = \lim_{n\to\infty} \Psi S(\tau)x_n = \Psi S(\tau)x$$

in $L^2_{loc}(\mathbb{R}_+, X)$. We state this result as a lemma.

Lemma 3.4. *Let C be a locally Lipschitz admissible observation operator for S (resp. T). There there exists a locally Lipschitz continuous extension $\Psi : X \longrightarrow L^2_{loc}([0, \infty), Y)$ of the map $x \mapsto CS(\cdot)x$ (resp. $x \mapsto CT(\cdot)x$) defined on $D(C)$. Moreover, (S, Ψ) (resp. (T, Ψ)) is a locally Lipschitz observation system.*

For a given locally Lipschitz observation system we can now construct a pointwise representation in terms of an observation operator.

Definition 3.5. *For a locally Lipschitz observation system (S, Ψ) (resp. (T, Ψ)) we define*

$$\widetilde{C}x = \lim_{\tau\downarrow 0} \frac{1}{\tau} \int_0^\tau (\Psi x)(t)dt, \tag{3.4}$$

for $x \in D(\widetilde{C}) := \{x \in X :$ the limit in (3.4) exists in $Y\}$.

The next representation result extends Theorem 4.5 of [20] to locally Lipschitz observation systems.

Theorem 3.6. *Let (S, Ψ) (resp. (T, Ψ)) be a locally Lipschitz observation system, and let $\widetilde{C} : D(\widetilde{C}) \to Y$ be the nonlinear operator defined by (3.4). Then, for all $x \in X$ and $t \ge 0$ we have $S(t)x \in D(\widetilde{C})$ (resp. $T(t)x \in D(\widetilde{C})$) if and only if*

$$\frac{1}{\tau} \int_0^\tau (\Psi x)(t + s)\, ds \quad \text{converges as} \quad \tau \searrow 0.$$

If this is the case, then the limit equals $\widetilde{C}S(t)x$ (resp. $\widetilde{C}T(t)x$). We thus obtain $(\Psi x)(t) = \widetilde{C}S(t)x$ (resp. $(\Psi x)(t) = \widetilde{C}T(t)x$) for almost every $t \ge 0$, namely for all Lebesgue points $t \ge 0$ of Ψx.

Proof. The theorem follows from the identity

$$\frac{1}{\tau}\int_0^\tau (\Psi S(t)x)(r)\,dr = \frac{1}{\tau}\int_0^\tau (\Psi x)(t+r)dr$$

and the fact that this limit exists for almost every $t \geq 0$ since Ψx is locally integrable. This proof also works in the linear case. ☐

In particular, $\widetilde{C}$ is an locally Lipschitz admissible observation operator for S (resp. T). According to Lemma 3.4, $\widetilde{C}$ and S (resp. T) generate an observation system $(S, \widetilde{\Psi})$ (resp. $(T, \widetilde{\Psi})$). It is easy to see that, in fact, $\Psi = \widetilde{\Psi}$. We say that the operator $\widetilde{C}$ *represents* the observation system (S, Ψ) (resp. (T, Ψ)).

In a second step we now consider the special case of the semilinear system (1.1) and (1.2), where we assume in addition that C is linear. So let (T, Ψ) be a linear observation system with observation operator C and Lebesgue extension C_L and $(S(t))_{t\geq 0}$ the semigroup of locally Lipschitz operators solving (1.1) in the mild sense. Recall from Section 2 that $\Psi x = C_L T(\cdot)x$.

In order to describe the output of (1.1) and (1.2), we define

$$\Psi_F x = \Psi x + C_L \mathbb{K} F(S(\cdot)x) \tag{3.5}$$

for all $x \in X$, where $\mathbb{K}f(t) := \int_0^t T(t-s)f(s)ds$ for $f \in L^1_{loc}(\mathbb{R}_+, X)$ and $t \geq 0$. Observe that $F(S(\cdot)x)$ is locally bounded due to (H) and (2.3). We recall from Proposition 2.11 in [18] (and its proof) that $\mathbb{K}f(t) \in D(C_L)$ for a.e. $t \geq 0$, $C_L\mathbb{K}f : \mathbb{R}_+ \to Y$ is strongly measurable and

$$\|C_L\mathbb{K}f\|_{L^2([0,t_0],Y)} \leq c(t_0)t_0^{\frac{1}{2}}\,\|f\|_{L^2([0,t_0],X)} \tag{3.6}$$

for all $f \in L^2_{loc}(\mathbb{R}_+, X)$ and $t_0 > 0$, where $c(t_0) = \gamma(t_0+1)$ and γ is given by (2.6). (Hence, $c : \mathbb{R}_+ \to \mathbb{R}_+$ is locally bounded.) We now show that (Ψ_F, S) is a locally Lipschitz observation system represented by C_L.

Theorem 3.7. *Let (T, Ψ) be a linear observation system with observation operator $C \in \mathcal{L}(D(A), Y)$, $F : X \to X$ be locally Lipschitz, and $S(\cdot)$ solve (1.1). Assume that (H) holds. Define Ψ_F as in (3.5). Then, (Ψ_F, S) is a locally Lipschitz observation system which is represented by the Lebesgue extension C_L.*

Proof. Let $t_0 > 0$ and $r > 0$, and take $x, y \in X$ with $\|x\|, \|y\| \leq r$. Using the assumptions, (3.6) and (2.4), we can estimate

$$\begin{aligned}
\|\Psi_F x - \Psi_F y\|_{L^2([0,t_0],Y)} &\leq \|\Psi(x-y)\|_{L^2([0,t_0],Y)} \\
&\quad + \|C_L\mathbb{K}[F(S(\cdot)x) - F(S(\cdot)y)]\|_{L^2([0,t_0],X)} \\
&\leq c\|x-y\| + c(t_0)t_0^{\frac{1}{2}}\|F(S(\cdot)x) - F(S(\cdot)y)\|_{L^2([0,t_0],X)} \\
&\leq c\|x-y\| + c(r,t_0)t_0^{\frac{1}{2}}\|S(\cdot)x - S(\cdot)y\|_{L^2([0,t_0],X)} \\
&\leq c(r,t_0)\|x-y\|.
\end{aligned}$$

Let $t \geq 0$. For a.e. $\tau \geq 0$, the formulas (3.5) and (2.2) lead to

$$\begin{aligned}
(\Psi_F x)(t+\tau) &= C_L T(\tau)T(t)x + C_L \textstyle\int_t^{t+\tau} T(t+\tau-s)F(S(s)x))\,ds \\
&\quad + C_L T(\tau)\textstyle\int_0^t T(t-s)F(S(s)x))\,ds \\
&= C_L T(\tau)S(t)x + C_L \textstyle\int_0^\tau T(\tau-s)F(S(s)S(t)x))\,ds \\
&= (\Psi_F(S(t)x))(\tau).
\end{aligned}$$

So we have shown that (Ψ_F, S) is a locally Lipschitz observation system. For the second assertion, let $x \in X$ and $t \in (0,1]$. Equation (3.5) yields

$$\frac{1}{t}\int_0^t (\Psi_F x)(s)\,ds = \frac{1}{t}\int_0^t (\Psi x)(s)\,ds + \frac{1}{t}\int_0^t C_L \mathbb{K} F(S(\cdot)x)(s)\,ds.$$

The second integral on the right hand side is denoted by $J(t)$. From Hölder's inequality and estimate (3.6) we deduce that

$$\|J(t)\| \le t^{-\frac{1}{2}} \|C_L \mathbb{K} F(S(\cdot)x)\|_{L^2([0,t],Y)} \le c\,\|F(S(\cdot)x)\|_{L^2([0,t],X)} \longrightarrow 0$$

as $t \to 0$. We then conclude that $D(\tilde{C}) = D(C_L)$ and $\tilde{C}x = C_L x$, where $\tilde{C}$ represents Ψ_F. □

4. Exact observability

Again we start with our basic definitions in the linear and the nonlinear case.

Definition 4.1. *Let $C \in \mathcal{L}(D(A), Y)$ be an admissible observation operator for the linear C_0-semigroup T with generator A. The system (2.5) is called* exactly observable in time $\tau > 0$ *if there is a constant $\kappa > 0$ such that*

$$\|CT(\cdot)x\|_{L^2([0,\,\tau],Y)} \ge \kappa \|x\| \qquad \text{for all } \ x \in D(A). \tag{4.1}$$

Definition 4.2. *Let $C : D(C) \to Y$ be an locally Lipschitz admissible observation operator for the semigroup S of locally Lipschitz operators solving* (1.1). *The system (1.1) and (1.2) is called* locally exact observable in time $\tau > 0$ at $x_0 \in D(C)$ *(or on $B(x_0, r_0)$) if there are numbers $r_0, \kappa > 0$ such that*

$$\|CS(\cdot)x - CS(\cdot)y\|_{L^2([0,\tau],Y)} \ge \kappa\|x - y\| \tag{4.2}$$

for all $x, y \in D(C)$ with $\|x_0 - x\| \le r_0$ and $\|x_0 - y\| \le r_0$.

Remark 4.3. *One can see that the linear system (2.5) is exactly observable if and only if is locally exact observable at some x_0, see the proof of Theorem 4.4 below.*

We now establish a robustness result for exact observability in the semilinear case. Observe that x_0 is fixed point for the semilinear problem (1.1), i.e., $S(t)x_0 = x_0$ holds for all ≥ 0, if and only if $x_0 \in D(A_0)$ and $Ax_0 = -F(x_0)$. In particular, $x_0 = 0$ is a fixed point for (1.1) if and only if $F(0) = 0$.

Theorem 4.4. *Let $C \in \mathcal{L}(D(A), Y)$ be an admissible linear observation operator for the C_0-semigroup T with generator A. Let $F : X \to X$ be locally Lipschitz and S be the nonlinear semigroup solving 1.1. Assume that (H) holds. Let $x_0 \in D(A)$ satisfy $Ax_0 = -F(x_0)$ and denote by $L_0(r)$ the Lipschitz constant of F on the ball $B(x_0, r)$ in X. Then there are constants $L_1, L_2 > 0$ such that the following assertions hold.*

(a) *If the linear system (2.5) is exactly observable in time $\tau > 0$ and if there is an $\tilde{r} > 0$ with $L_0(\tilde{r}) < L_1$, then the nonlinear system (1.1) and (1.2) is locally exact observable in time τ.*
(b) *If the nonlinear system (1.1) and (1.2) is locally exact observable in time $\tau > 0$ on the ball $B(x_0, r_0)$ and there is an $\tilde{r} \in (0, r_0)$ with $L_0(\tilde{r}) < L_2$, then the linear system (2.5) is exactly observable in time τ.*

Proof. We first establish certain Lipschitz estimates for S near x_0. Fix an $R > 0$ and take any $r \in (0, R)$. Let $\rho \in (0, r)$, $x, y \in B(x_0, \rho)$, and $t \in [0, \tau]$. Let $t_1 > 0$ be the supremum of $t \in [0, \tau]$ such that $\|S(s)x - x_0\| < r$ for all $s \in [0, t]$. The formula (2.2) and estimate (2.1) then imply the inequality

$$\|S(t)x - x_0\| = \|S(t)x - S(t)x_0\| \leq Me^{\omega\tau}\|x - x_0\| + Me^{\omega\tau}\int_0^t L_0(r)\|S(s)x - x_0\|\, ds$$

for all $0 \leq t < t_1$. From Gronwall's inequality it follows that

$$\|S(t)x - x_0\| \leq Me^{\omega\tau}\exp(Me^{\omega\tau}L_0(r)\tau)\rho$$

for all $0 \leq t < t_1$. Choosing a sufficiently small $\rho = \rho(r) > 0$ we thus obtain $\|S(t_1)x - x_0\| < r$ so that $t_1 = \tau$ and $S(t)x \in B(x_0, r)$ for all $t \in [0, \tau]$. Using again (2.2), we can now deduce the Lipschitz estimate

$$\|S(t)x - S(t)y\| \leq Me^{\omega\tau}\exp(Me^{\omega\tau}L_0(r)\tau)\|x - y\| =: k(R)\|x - y\|$$

if $\|x - x_0\|$, $\|y - x_0\| \leq \rho(r) < r$ and $t \in [0, \tau]$.

We now assume that the system (2.5) is exactly observable in time $\tau > 0$ with constant $\kappa > 0$. Formula (3.5) yields

$$C_L T(t)(x - y) = C_L S(t)x - C_L S(t)y - C_L\int_0^t T(t - \sigma)[F(S(\sigma)x) - F(S(\sigma)y)]\, d\sigma.$$

Using (3.6) and the above estimates, we then deduce that

$$\begin{aligned}
\|C_L\ \ & T(\cdot)x - C_L T(\cdot)y\|_{L^2([0,\tau],Y)} \\
\leq\ \ & \|C_L S(\cdot)x - C_L S(\cdot)y\|_{L^2([0,\tau],Y)} + c(\tau)\,\|F(S(\cdot)x) - F(S(\cdot)y)\|_{L^2([0,\tau],X)} \\
\leq\ \ & \|C_L S(\cdot)x - C_L S(\cdot)y\|_{L^2([0,\tau],Y)} + c(\tau)L_0(r)\,\|S(\cdot)x - S(\cdot)y\|_{L^2([0,\tau],X)} \\
\leq\ \ & \|C_L S(\cdot)x - C_L S(\cdot)y\|_{L^2([0,\tau],Y)} + L_0(r)c_1(\tau)k(R)\|x - y\|_X
\end{aligned}$$

for $x, y \in B(x_0, \rho(r))$ and $t \in [0, \tau]$. Thus, if $L_0(\tilde{r})c_1(\tau)k(R) \leq \kappa/2$ for some $\tilde{r} > 0$, the observability of C and T yields

$$\begin{aligned}
\|C_L\ \ & S(\cdot)x - C_L S(\cdot)y\|_{L^2([0,\tau],Y)} \\
\geq\ \ & \|C_L T(\cdot)x - C_L T(\cdot)y\|_{L^2([0,\tau],Y)} - c_1(\tau)k(R)L_0(\tilde{r})\|x - y\| \geq \tfrac{\kappa}{2}\|x - y\|
\end{aligned}$$

for all $x, y \in X$ with $\|x - x_0\|$, $\|y - x_0\| \leq \rho(\tilde{r})$.

To prove part (b) we proceed in the same way, but we require in addition that $0 < \rho < r_0$ and take $y = x_0$. We thus obtain

$$\|C_L T(\cdot)(x - x_0)\|_{L^2([0,\tau],Y)} \geq \tfrac{\kappa}{2}\|x - x_0\|$$

for all x in a ball around x_0. By linearity, this estimate implies the exact observability of the linear system (2.5). □

5. Applications

In this section we give examples for the main theorems of this paper.

Example 5.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary $\partial\Omega \in C^4$ and let Γ be an open subset of $\partial\Omega$. Consider the damped nonlinear beam equation*

$$\begin{cases} u_{tt} + \Delta^2 u - 2\beta\Delta u_t - f\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = 0, \ x \in \Omega, t > 0, \\ u(t,x) = 0, \ x \in \partial\Omega, \ t \geq 0, \\ \Delta u(t,x) = 0, \ x \in \partial\Omega, \ t \geq 0, \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x), \ x \in \Omega \end{cases} \tag{5.1}$$

with $\beta > 0$ and the output function

$$y(t) = u_t|_\Gamma. \tag{5.2}$$

Equation (5.1) arise in the mathematical study of structural damped nonlinear vibrations of a string or a beam and was considered in [7, 19] *and references therein.*

Let $H = L^2(\Omega)$ and $A\phi = \Delta^2\phi$ with $D(A) = H^4(\Omega) : u = \Delta u = 0$ on $\partial\Omega\}$. It is known that A is a self adjoint, positive, boundedly invertible operator and that

$$H_{\frac{1}{2}} := D(A^{\frac{1}{2}}) = H^2(\Omega) \cap H_0^1(\Omega).$$

Let $H_{-\frac{1}{2}}$ be the dual space of $H_{\frac{1}{2}}$ for the pivot space H.

Set $v = u_t$ and $Z(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$. We can then rewrite the problem (5.1)–(5.2) as the abstract first order ordinary differential equation in the Hilbert space $X = H_{\frac{1}{2}} \times H$

$$\begin{cases} \frac{d}{dt}Z(t) &= \mathcal{A}Z(t) + F(Z(t)), \ Z(0) = Z_0, \\ y(t) &= \mathcal{C}Z(t). \end{cases} \tag{5.3}$$

Here the linear operator

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset H_{\frac{1}{2}} \times H \to H_{\frac{1}{2}} \times H,$$

is given by

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A & -D \end{pmatrix}, \ \mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}),$$

where the damping operator $D : H_{\frac{1}{2}} \to H$ defined by $D = 2\beta A^{\frac{1}{2}}$ is bounded and positive. Furthermore, we set

$$\mathcal{C} = (0, C) \quad \text{and} \quad C\phi = \phi|_\Gamma \quad \text{for} \ \ \phi \in H_{\frac{1}{2}}$$

and define $F : H_{\frac{1}{2}} \times H \to H_{\frac{1}{2}} \times H$ by

$$F\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} 0 \\ f\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u \end{pmatrix}.$$

For $z \in H_{\frac{1}{2}}$, we have

$$\langle Dz, z\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = \langle 2\beta A^{\frac{1}{2}} z, z\rangle_H = 2\beta\|z\|_{H_0^1(\Omega)} \geq \frac{2\beta}{c}\|z\|_{L^2(\Gamma)},$$

for some $c > 0$ by the trace theorem (see e.g. Theorem 2.5.4 in [14]*). Hence, the assumptions (A1)-(A3) of* [9, Proposition 4.1] *are satisfied, and thus the* *observation operator $\mathcal{C}$ is infinite-time admissible for the semigroup generated by $\mathcal{A}$.*

Assuming $f : [0,\infty) \to \mathbb{R}$ locally Lipschitz and bounded, the mapping F is locally Lipschitz continuous on $H_{\frac{1}{2}} \times H$ and of linear growth. Theorem 3.7 now implies that the Lebesgue extension of $\mathcal{C}$ with respect to the semigroup generated by $\mathcal{A}$ is an admissible observation operator for the problem (5.1)-(5.2).

Example 5.2. *Let Ω be a bounded open subset of $\mathbb{R}^N$ with boundary $\partial\Omega \in C^4$. We consider the following semilinear thermo-elastic system*

$$\begin{cases} w_{tt} + \Delta^2 w + \alpha\Delta\theta = f\left(\int_\Omega |\nabla w|^2 dx\right)\Delta w, \ x \in \Omega, t > 0, \\ \theta_t - \Delta\theta + \sigma\theta - \alpha\Delta w_t = 0, \ x \in \Omega, t > 0, \end{cases} \tag{5.4}$$

with the boundary and initial conditions

$$\begin{cases} \theta(t,x) = w(t,x) = \dfrac{\partial w}{\partial \nu}(t,x) = 0, \ x \in \partial\Omega, \ t \geq 0 \\ w(0,x) = w_0(x), \ w_t(0,x) = w_1(x), \ \theta(0,x) = \theta_1(x), \ \ x \in \Omega \end{cases} \tag{5.5}$$

and the output function

$$y(t,x) = -\nabla\theta(t,x), \quad t \geq 0, \ x \in \Omega. \tag{5.6}$$

Here, the coupling parameter α is positive and the constant σ is non negative. Controllability of corresponding linear system of (5.4)–(5.5) with various boundary conditions and controls are well studied, see [1, 8, 12].

We define the linear operators $A_0 = \Delta^2$ and $A_D = -\Delta$ on $L^2(\Omega) \to L^2(\Omega)$ with the domains

$$D(A_0) = H^4(\Omega) \cap H_0^2(\Omega) \qquad \text{and} \qquad D(A_D) = H^2(\Omega) \cap H_0^1(\Omega).$$

It is well known that A_0 and A_D are selfadjoint positive operators and that

$$D(A_0{}^{\frac{1}{2}}) = H_0^2(\Omega) \quad \text{and} \quad D(A_D^{\frac{1}{2}}) = H_0^1(\Omega).$$

We introduce the Hilbert space $\mathbb{H} := D(A_0{}^{\frac{1}{2}}) \times L^2(\Omega) \times L^2(\Omega)$, equipped with its natural inner product. Set $v = w_t$ and

$$z(t) = \begin{pmatrix} w(t) \\ v(t) \\ \theta(t) \end{pmatrix}, \quad z_0 = \begin{pmatrix} w_0 \\ v_0 \\ \theta_0 \end{pmatrix}.$$

The system (5.4)-(5.5) can be rewritten as an abstract semilinear evolution equation in $\mathbb{H}$ of the form

$$z_t = Az + F(z), \ \ z(0) = z_0 \in \mathbb{H},$$

with the output function

$$y(t) = Cz(t),$$

where A is the linear operator defined by

$$A = \begin{pmatrix} 0 & I & 0 \\ -A_0 & 0 & \alpha A_D \\ 0 & -\alpha A_D & -A_D - \sigma I \end{pmatrix}$$

with domain $D(A) = D(A_0) \times D(A_0{}^{\frac{1}{2}}) \times D(A_D)$, *and the observation operator* $C : D(A) \to Y = 0 \times 0 \times (L^2(\Omega))^N$, $C = (0, 0, -\nabla)$. *Further* $F : \mathbb{H} \to \mathbb{H}$ *is the nonlinear operator given by*

$$F\left(\begin{pmatrix} w \\ v \\ \theta \end{pmatrix}\right) = \begin{pmatrix} 0 \\ f\left(\int_\Omega |\nabla w|^2 dx\right) \Delta w \\ 0 \end{pmatrix}.$$

In Proposition 2.1 of [1], *it was shown that* A *generates a* C_0 *semigroup of contractions on the Hilbert space* $\mathbb{H}$. *Proposition 2.7 of* [1] *also implies that* C *is admissible with respect to* A. *Finally, in Section 3 of* [2] *the pair* (A, C) *was proved exactly observable. If we assume that* $f : [0, +\infty) \to \mathbb{R}$ *is bounded and locally Lipschitz continuous, then* F *is locally Lipschitz on* $\mathbb{H}$ *and satisfies assumption (H). Moreover,* $F(0) = 0$. *Using Theorem 4.4 we deduce that the problem (5.4)–(5.6) is locally exactly observable at* $w_0 = \theta_0 = 0$.

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