

# $L^p$ -spectrum of degenerate hypoelliptic Ornstein-Uhlenbeck operators

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## Abstract

We describe the spectrum of degenerate hypoelliptic Ornstein-Uhlenbeck operators  $\mathcal{A} = \sum_{i,j=1}^n q_{ij} D_{ij} + \sum_{i,j=1}^n b_{ij} x_j D_i$  in  $L^p(\mathbb{R}^n)$ ,  $1 \leq p < +\infty$ , and in  $C_0(\mathbb{R}^n)$ . We show that the spectrum of  $\mathcal{A}$  is the sum of  $(-\infty, 0]$  and the spectrum of the drift term. Our result gives a complete picture of the spectral properties of Ornstein-Uhlenbeck operators in  $L^p$  spaces.

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## 1 Introduction

The aim of the present paper is the full description of the spectrum of possibly degenerate Ornstein-Uhlenbeck operators

$$\mathcal{A} = \sum_{i,j=1}^n q_{ij} D_{ij} + \sum_{i,j=1}^n b_{ij} x_j D_i = \text{Tr}(QD^2) + \langle Bx, D \rangle, \quad x \in \mathbb{R}^n, \quad (1.1)$$

in  $L^p(\mathbb{R}^n)$ ,  $1 \leq p < +\infty$ , and in  $C_0(\mathbb{R}^n)$ . Here  $Q = (q_{ij})$  is a real, constant, symmetric and positive semidefinite matrix and  $B = (b_{ij})$  is a nonzero real matrix. The semidefiniteness of the matrix  $Q$  is responsible for the possible degeneracy of  $\mathcal{A}$ . Throughout we assume that  $\mathcal{A}$  is hypoelliptic, which can be stated as follows: the symmetric matrices

$$Q_t = \int_0^t e^{sB} Q e^{sB^T} ds$$

have nonzero determinant for some (equivalently, for all)  $t > 0$ . In the literature one can find several equivalent conditions for hypoellipticity. In particular, on p. 148 of [10] it is pointed out that the hypoellipticity of  $\mathcal{A}$  is equivalent to the property

$$\ker(Q) \text{ does not contain nontrivial subspaces which are invariant for } B^T, \quad (1.2)$$

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see also [13, Appendix]. Here nontrivial means different from  $\{0\}$ .

The hypoellipticity assumption implies that the associated Markov semigroup  $(T(t))_{t \geq 0}$  has the following explicit representation formula due to Kolmogorov [12]

$$(T(t)f)(x) = \frac{1}{(4\pi)^{n/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^n} e^{-\langle Q_t^{-1}y, y \rangle/4} f(e^{tB}x - y) dy, \quad x \in \mathbb{R}^n, t > 0. \quad (1.3)$$

The parabolic equation  $u_t = \mathcal{A}u$ , known as Kolmogorov equation, is solved by the function  $u(t, x) = T(t)f(x)$  for a large class of initial data  $f$ . In recent years, both the semigroup  $(T(t))_{t \geq 0}$  and its generator  $\mathcal{A}$  have extensively been studied. Several applications in physics and finance for the operator  $\mathcal{A}$  and its evolutionary counterpart  $\mathcal{A} - D_t$  can be found in the survey [21]. They were also used in the context of rotating fluids, see e.g. [9]. These operators were also the leading example for an intensive research on elliptic and parabolic problems with unbounded coefficients, see e.g. [14].

In the analytical study of  $\mathcal{A}$ , even in the nondegenerate case the classical  $L^p$  and Schauder estimates do not apply because of the unboundedness of the first order coefficients. Regularity properties in spaces of continuous functions were proved in [4] in the nondegenerate case and in [15] in the degenerate case. Schauder estimates can then be deduced by means of interpolation techniques. Moreover,  $L^p$  estimates were established in [20] and in [19] in the nondegenerate case, by a semigroup approach, and in [1] in the degenerate case.

The underlying stochastic process admits an invariant measure  $\mu$  if and only if all eigenvalues of the drift matrix  $B$  have negative real parts. This means that  $\mu$  is a probability measure satisfying

$$\int_{\mathbb{R}^n} (T(t)f)(x) d\mu(x) = \int_{\mathbb{R}^n} f(x) d\mu(x)$$

for every  $t \geq 0$  and continuous and bounded function  $f$  on  $\mathbb{R}^n$ . The invariant measure is unique and absolutely continuous with respect to the Lebesgue measure having the (Gaussian) density

$$\rho(x) = \frac{1}{(4\pi)^{n/2}(\det Q_\infty)^{1/2}} e^{-\langle Q_\infty^{-1}x, x \rangle/4} \quad \text{with} \quad Q_\infty = \int_0^\infty e^{sB} Q e^{sB^T} ds,$$

see [5, Chapter II. 6].

The semigroup  $(T(t))_{t \geq 0}$  and its generator  $\mathcal{A}$  have widely been investigated in the weighted spaces  $L^p(\mathbb{R}^n, d\mu)$ , if  $\sigma(B) \subset \mathbb{C}_-$ . Here the unboundedness of the coefficients of  $\mathcal{A}$  is balanced by the exponential decay of the density  $\rho$  which leads to a much better behavior in several respects. For instance, the generator has compact resolvent in  $L^p(\mathbb{R}^n, d\mu)$  if  $p \in (1, \infty)$ , which is not true in the unweighted spaces  $L^p$ . The domain of the generator in  $L^p(\mathbb{R}^n, d\mu)$  was computed in [16] for  $p = 2$  and in [20] for  $p \in (1, \infty)$  in the nondegenerate case. See also [2, 3] for the analogous problem on an infinite-dimensional Hilbert space  $E$  instead of  $\mathbb{R}^n$ . In the degenerate case a sharp inclusion for the domain was shown in [8] for  $p = 2$ , whereas the case  $p \neq 2$  is still an open problem, indicating that the general picture of Ornstein-Uhlenbeck operators is still not complete.

In [18] the spectrum of  $\mathcal{A}$  in  $L^p(\mathbb{R}^n, d\mu)$  was completely described also in the degenerate case, provided that  $\sigma(B) \subset \mathbb{C}_-$ . The situation is much different in the spaces  $L^p = L^p(\mathbb{R}^n)$  with respect to the Lebesgue measure, e.g., since  $\mathcal{A}$  does not have a compact resolvent here. For some choices of  $B$  the spectrum of  $\mathcal{A}$  was computed in  $L^p$  in [17]. This paper is the starting point of our investigation.

The operator  $\mathcal{A}$  can be seen as the sum of the diffusion term  $\sum_{i,j=1}^n q_{ij} D_{ij}$  and of the drift term  $\mathcal{L} = \sum_{i,j=1}^n b_{ij} x_j D_i$ . The spectral properties of the drift term are fully understood by [17]. There it was proved that the spectrum of the realization  $\mathcal{L}_p$  of  $\mathcal{L}$  in  $L^p$  is the line  $-\text{tr}(B)/p + i\mathbb{R}$  unless  $B$  is similar to a diagonal matrix with purely imaginary eigenvalues. In this last case the spectrum of  $\mathcal{L}_p$  can be either  $i\mathbb{R}$  or a discrete, explicitly given subgroup  $G$  of  $i\mathbb{R}$ , see Theorem 2.2 and Proposition 2.3.

In [17] it is further shown that the boundary spectrum of the realization  $\mathcal{A}_p$  of  $\mathcal{A}$  in  $L^p$  contains the spectrum of  $\mathcal{L}_p$  without further assumptions on the matrices  $Q$  and  $B \neq 0$ . Here  $\mathcal{A}_p$  is defined as the generator of  $(T(t))_{t \geq 0}$  in  $L^p$ , see Proposition 2.4. The spectrum of  $\mathcal{A}_p$  has been computed in [17] if  $\sigma(B)$  is contained in the left or in the right open half-plane. In this case,  $\sigma(\mathcal{A}_p)$  is equal to  $\{\mu \in \mathbb{C} : \text{Re } \mu \leq -\text{tr}(B)/p\}$ . So it depends on  $p$  and is far from being discrete. In addition, and this is the main step in [17], if all the eigenvalues of  $B$  have positive real parts, then the open half-plane  $\{\mu \in \mathbb{C} \mid \text{Re } \mu < -\text{tr}(B)/p\}$  consists of eigenvalues.

In this paper we complete the picture computing the spectrum of  $\mathcal{A}_p$  without any further restriction on  $Q = Q^T \geq 0$  and  $B \neq 0$ , apart from hypoellipticity. We prove that  $\sigma(\mathcal{A}_p)$  is given as the sum of the spectra of its diffusion part (i.e.,  $(-\infty, 0]$ ) and of the drift term  $\mathcal{L}_p$ .

**Theorem 1.1.** *Let (1.2) be true and  $p \in [1, \infty]$ . Then the spectrum of  $\mathcal{A}_p$  is given by*

$$\sigma(\mathcal{A}_p) = (-\infty, 0] + \sigma(\mathcal{L}_p).$$

*In particular, we have either  $\sigma(\mathcal{A}_p) = (-\infty, 0] + G$  or  $\sigma(\mathcal{A}_p) = \{\mu \in \mathbb{C} \mid \text{Re } \mu \leq -\text{tr}(B)/p\}$ , according to  $\sigma(\mathcal{L}_p)$  being a discrete subgroup  $G = \frac{2\pi i}{\tau} \mathbb{Z}$  of  $i\mathbb{R}$  or the whole line  $-\text{tr}(B)/p + i\mathbb{R}$ . Moreover, the semigroup  $(T(t))_{t \geq 0}$  satisfies the weak spectral mapping theorem*

$$\sigma(T(t)) = \overline{\exp(t\sigma(\mathcal{A}_p))}, \quad t \geq 0.$$

*We even have  $\sigma(T(t)) \setminus \{0\} = e^{t\sigma(\mathcal{A}_p)}$  except for the case that  $\sigma(\mathcal{L}_p) = G = \frac{2\pi i}{\tau} \mathbb{Z}$  and  $t/\tau$  is irrational.*

We note that for  $p = 2$  the spectral mapping theorem was proved for perturbed Ornstein-Uhlenbeck operators with  $Q = I$  and  $B = 2I$  by completely different methods in [6].

If  $B = B^T$  and  $QB = BQ$ , by separation of variables one can transform the Ornstein-Uhlenbeck operator into the form  $\mathcal{A} = \Delta + \sum_{i=1}^n b_i x_i D_i$ . Here the problem can be reduced to one dimensional problems, see [17, Theorem 5.1]. However, this is far from being the general case. We also stress that  $\mathcal{A}_2$  does not possess eigenvalues if  $B$  has an eigenvalue with negative real part or if  $B$  is skew-symmetric and  $Q = I$ , as we will see in Section 3. So we have to proceed in a different way than in [17] or [18], where eigenfunctions played a crucial role.

Instead, we start by reducing  $\mathcal{A}$  to a canonical form with an upper quasi triangular drift matrix whose diagonal is formed by  $1 \times 1$  and  $2 \times 2$  blocks containing the real and complex conjugate eigenvalues of  $B$ , respectively. The transformation is made through a linear change of variables that leaves the spectrum unchanged.

The second step consists in a scaling procedure leading to a new operator  $\mathcal{C}$  in the limit which is the sum of an Ornstein-Uhlenbeck operator in one or two variables and a drift operator acting in the remaining ones. The scaling and the limit allow us to get rid of the upper off-diagonal blocks of the drift matrix of  $\mathcal{A}$  and to separate the variables. We can recover the spectrum of  $\mathcal{A}_p$  from that of the limit operator  $\mathcal{C}$ .

The main part of the proof is thus devoted to the investigation of the spectrum of  $\mathcal{C}$ . Here we can assume that  $B$  has an eigenvalue with nonnegative real part, since the other case is already covered by the main result in [17]. The above splitting then reduces the problem to Ornstein-Uhlenbeck operators in  $\mathbb{R}$  or in  $\mathbb{R}^2$  where  $B$  has one nonnegative eigenvalue or two complex conjugate eigenvalues with nonnegative real parts. We further have to treat eigenvalues in  $i\mathbb{R}$  and with positive real part separately. The detailed study of these four cases is mainly based on the construction of approximate eigenfunctions.

The paper is structured as follows. In Section 2 we recall the known generator properties of the drift operator  $\mathcal{L} = \sum_{i,j=1}^n b_{ij}x_j D_i$  and its spectrum, as computed in [17]. We provide further details in the case where the generated group is periodic. We also collect the known properties on  $\mathcal{A}$ . Most of the results are contained in [17], where it is assumed that  $Q$  is positive definite. However, we explain why they continue to hold with minor modifications in the degenerate hypoelliptic setting. Corollary 2.7 and Proposition 2.8 establish the inclusion  $\sigma(\mathcal{A}_p) \subseteq (-\infty, 0] + \sigma(\mathcal{L}_p)$  by means of general spectral theory of semigroups. In Section 3 we show that there are no eigenvalues of  $\mathcal{A}_2$  in many cases. Finally, Section 4 is devoted to the proof of Theorem 1.1. Here also the spectral mapping theorem follows mainly from general theory, whereas the proof of the other inclusion  $(-\infty, 0] + \sigma(\mathcal{L}_p) \subseteq \sigma(\mathcal{A}_p)$  requires a sophisticated analysis of the four cases indicated above.

**Warning:** Throughout the whole paper, we write  $L^\infty$  for  $C_0(\mathbb{R}^n)$ , which is the space of continuous functions on  $\mathbb{R}^n$  vanishing at infinity, endowed with the supremum norm.

**Notation.**  $L^p$  stands for  $L^p(\mathbb{R}^n)$  if  $p \in [1, \infty)$  and  $C_c^\infty$  for  $C_c^\infty(\mathbb{R}^n)$ .

The spectrum and the resolvent set of a linear operator  $\mathcal{B}$  are denoted by  $\sigma(\mathcal{B})$  and  $\rho(\mathcal{B})$ , respectively. The *spectral bound* of  $\mathcal{B}$  is defined by  $s(\mathcal{B}) = \sup\{\operatorname{Re} \mu \mid \mu \in \sigma(\mathcal{B})\}$  and the *boundary spectrum* is  $\sigma(\mathcal{B}) \cap \{\mu \in \mathbb{C} \mid \operatorname{Re} \mu = s(\mathcal{B})\}$ . The *approximate point spectrum*  $\sigma_{ap}(\mathcal{B})$  of  $\mathcal{B}$  is the subset of  $\sigma(\mathcal{B})$  of all complex numbers  $\mu$  for which there is a sequence  $(v_n)$  in its domain  $D(\mathcal{B})$  such that  $\|v_n\| = 1$  and  $\|\mathcal{B}v_n - \mu v_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . The sequence  $(v_n)$  is called an *approximate eigenvector* relative to the *approximate eigenvalue*  $\mu$ . The topological boundary of the spectrum of  $\mathcal{B}$  is always contained in  $\sigma_{ap}(\mathcal{B})$  (see [7, Proposition IV.1.10]).

We write  $\mathcal{B}_p$  to indicate a realization of a (differential) operator  $\mathcal{B}$  in  $L^p$ , that is when  $\mathcal{B}$  is provided with a specific domain in  $L^p$ . However, we sometimes omit the suffix  $p$  in the proofs, to shorten the notation.

If  $B$  is a matrix,  $B^T$  denotes its transpose. We set  $\mathbb{C}_+ = \{\mu \in \mathbb{C} \mid \operatorname{Re} \mu > 0\}$  and  $\mathbb{C}_- = \{\mu \in \mathbb{C} \mid \operatorname{Re} \mu < 0\}$ . When  $p = \infty$ , by  $1/p$  we mean 0.

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## 2 Basic and known results

We collect background material from [17] and prove auxiliary results concerning the drift term and the Ornstein-Uhlenbeck operator.

## 2.1 Properties of $\mathcal{L}$

Let  $B = (b_{ij}) \neq 0$  be a real  $n \times n$  matrix and consider the drift operator

$$\mathcal{L} = \sum_{i,j=1}^n b_{ij} x_j D_i$$

defined on its maximal domain

$$D(\mathcal{L}_p) = \{u \in L^p \mid \mathcal{L}u \in L^p\}$$

in  $L^p$  for  $1 \leq p \leq \infty$ , where  $\mathcal{L}u$  is understood in the sense of distributions. We write  $\mathcal{L}_p$  for  $(\mathcal{L}, D(\mathcal{L}_p))$  and recall the following results, whose proofs can be found in [17, Section 2].

**Proposition 2.1.** *Let  $1 \leq p \leq \infty$ . The operator  $\mathcal{L}_p$  generates the  $C_0$ -group  $(S(t))_{t \in \mathbb{R}}$  in  $L^p$  defined by*

$$(S(t)f)(x) = f(e^{tB}x), \quad (2.1)$$

the space  $C_c^\infty$  is a core of  $\mathcal{L}_p$ , and we have

$$\|S(t)f\|_p = e^{-\frac{t}{p} \operatorname{tr}(B)} \|f\|_p \quad (2.2)$$

for  $f \in L^p$  and  $t \in \mathbb{R}$ .

We next describe the spectrum of  $\mathcal{L}_p$  distinguishing several cases.

**Theorem 2.2.** *Let  $1 \leq p \leq \infty$ .*

- (a) *Let  $\operatorname{tr}(B) \neq 0$ . Then  $\sigma(\mathcal{L}_p) = -\operatorname{tr}(B)/p + i\mathbb{R}$ .*
- (b) *Let  $\operatorname{tr}(B) = 0$  and  $B$  be not similar to a diagonal matrix with purely imaginary eigenvalues. Then  $\sigma(\mathcal{L}_p) = i\mathbb{R}$ .*
- (c) *Let  $B$  be similar to a diagonal matrix with nonzero eigenvalues  $\pm i\sigma_1, \pm i\sigma_2, \dots, \pm i\sigma_k$  in  $i\mathbb{R}$  and possibly 0, where  $\sigma_r \sigma_s^{-1} \notin \mathbb{Q}$  for some  $r, s \in \{1, \dots, k\}$ . Then  $\sigma(\mathcal{L}_p) = i\mathbb{R}$ .*
- (d) *Let  $B$  be similar to a diagonal matrix with nonzero eigenvalues  $\pm i\sigma_1, \pm i\sigma_2, \dots, \pm i\sigma_k$  in  $i\mathbb{R}$  and possibly 0, where  $\sigma_r \sigma_s^{-1} \in \mathbb{Q}$  for all  $r, s \in \{1, \dots, k\}$ . Then  $(S(t))_{t \in \mathbb{R}}$  is periodic and  $\sigma(\mathcal{L}_p)$  is the discrete subgroup  $G = \{i(n_1\sigma_1 + \dots + n_k\sigma_k) \mid (n_1, \dots, n_k) \in \mathbb{Z}^k\}$ .*

In the sequel we need more information about case (d) above in which  $(S(t))_{t \in \mathbb{R}}$  is periodic.

**Proposition 2.3.** *Let  $B$  be similar to a diagonal matrix with nonzero eigenvalues  $\pm i\sigma_1, \pm i\sigma_2, \dots, \pm i\sigma_k$  in  $i\mathbb{R}$  and possibly 0, with  $2k \leq n$ . Assume that for every  $j \in \{2, \dots, k\}$  we have  $\sigma_j = \frac{p_j}{q_j} \sigma_1$  for some coprime integers  $p_j$  and  $q_j$ . Then  $(S(t))$  is periodic with period  $\tau = 2\pi N \sigma_1^{-1}$ , where  $N$  is the least common multiple of  $q_2, \dots, q_k$ . Moreover, the set  $G$  from Theorem 2.2 is given by  $G = \frac{\sigma_1}{N} i\mathbb{Z} = \frac{2\pi}{\tau} i\mathbb{Z}$ .*

*Proof.* We denote a point in  $\mathbb{R}^n$  by  $x = (x_1, y_1, \dots, x_k, y_k, w_{2k+1}, \dots, w_n)$  and set  $z_j = (x_j, y_j)$ . Possibly after a change of variables we obtain

$$S(t)f(x) = f(e^{it\sigma_1} z_1, \dots, e^{it\sigma_k} z_k, w_{2k+1}, \dots, w_n), \quad (2.3)$$

see Theorem 2.6 of [17]. If  $0 \notin \sigma(B)$ , the components  $w_j$  are not present. Formula (2.3) yields  $S(\tau)f = f$ .

We prove that the set  $G$  defined in Theorem 2.2 (d) coincides with  $\frac{\sigma_1}{N}i\mathbb{Z}$ . The inclusion  $\subseteq$  easily follows from the form of the numbers  $\sigma_j$  and the definition of  $N$ . To show the other inclusion, we first observe that the greatest common divisor of  $N, Np_2/q_2, \dots, Np_k/q_k$  is equal to 1. Indeed, otherwise there would exist a prime number  $p$  dividing  $N, \dots, Np_k/q_k$ . Let  $\alpha \in \mathbb{N}$  be the greatest exponent for which  $p^\alpha$  divides  $N$ . Then  $p^\alpha$  occurs in the prime factorization of some  $q_j$ . Since  $p_j$  and  $q_j$  are coprime,  $p$  cannot divide  $Np_j/q_j$ , and this is a contradiction. As a result, each integer  $m$  can be written as

$$m = m_1N + m_2 \frac{Np_2}{q_2} + \dots + m_k \frac{Np_k}{q_k},$$

for suitable  $m_j \in \mathbb{Z}$ . This is equivalent to saying that the element  $\frac{\sigma_1}{N}m$  can be written as  $m_1\sigma_1 + \dots + m_k\sigma_k$  and concludes the proof.  $\square$

## 2.2 Properties of $\mathcal{A}$

We turn our attention to the Ornstein-Uhlenbeck operator defined in (1.1) and to the associated semigroup  $(T(t))_{t \geq 0}$  given by (1.3). We always assume the hypoellipticity condition (1.2) and  $1 \leq p \leq \infty$ . We do not need the full description of the domain of the generator, but only the fact that smooth functions with compact support are a core. We point out, however, that the domain has been described in [20, Section 4] and in [19] in the nondegenerate case and in [1] in the degenerate one.

**Proposition 2.4.** *The semigroup  $(T(t))_{t \geq 0}$  is strongly continuous on  $L^p$ ,  $1 \leq p \leq \infty$ , and satisfies the estimate*

$$\|T(t)\| \leq e^{-\frac{t}{p} \operatorname{tr}(B)} \quad (2.4)$$

for  $t \geq 0$ . Moreover,  $C_c^\infty$  is a core for the generator  $\mathcal{A}_p$ .

*Proof.* If the diffusion matrix  $Q$  is positive definite, the stated properties and a partial description of the domain of the generator have been proved in Section 3 of [17]. However, the same proofs hold in the degenerate hypoelliptic case. We only sketch them and refer to [17] for more details. To show (2.4), we write  $T(t)f = S(t)(g_t * f)$  where

$$g_t(y) = \frac{1}{(4\pi)^{n/2}(\det Q_t)^{1/2}} e^{-\langle Q_t^{-1}y, y \rangle/4}$$

and  $S(t)$  is defined in (2.1). The estimate (2.4) then follows from (2.2), Young's inequality for convolutions, and  $\|g_t\|_1 = 1$ . Since  $T(t)f \rightarrow f$  in  $L^p$  as  $t \rightarrow 0^+$  for  $f \in C_c^\infty$ , by density (2.4) implies the strong continuity of  $(T(t))_{t \geq 0}$  for every  $1 \leq p \leq \infty$ .

Let  $\mathcal{A}_p$  and  $\mathcal{S}(\mathbb{R}^n)$  denote the generator of  $(T(t))_{t \geq 0}$  in  $L^p$  and the Schwartz class, respectively. One easily checks that  $\mathcal{S}(\mathbb{R}^n) \subseteq D(\mathcal{A}_p)$  and  $\mathcal{A}_p f = \mathcal{A}f$  for every  $f \in \mathcal{S}(\mathbb{R}^n)$ . Moreover,  $\mathcal{S}(\mathbb{R}^n)$  is dense in  $L^p$  and invariant for  $(T(t))_{t \geq 0}$  by (1.3). Therefore it is a core of  $\mathcal{A}_p$ . By a truncation argument we conclude that  $C_c^\infty$  is a core for  $\mathcal{A}_p$ .  $\square$

We recall Theorem 3.3 and Corollary 3.5 of [17].

**Proposition 2.5.** *The boundary spectrum of  $\mathcal{A}_p$  contains the spectrum of the drift  $\mathcal{L}_p$ .*

**Corollary 2.6.** *The growth bound of  $(T(t))_{t \geq 0}$  in  $L^p$  is  $\omega_p = -\operatorname{tr}(B)/p$ .*

Standard semigroup theory then yields first inclusions of the spectra.

**Corollary 2.7.** *The spectrum of  $\mathcal{A}_p$  belongs to the half-plane  $\{\mu \in \mathbb{C} \mid \operatorname{Re} \mu \leq -\operatorname{tr}(B)/p\}$ , and that of  $T(t)$  to the closed ball  $\overline{B}(0, e^{-\frac{t}{p}\operatorname{tr}(B)})$ .*

If  $\sigma(\mathcal{L}_p)$  is the whole line  $-\operatorname{tr}(B)/p + i\mathbb{R}$ , the half-plane  $\{\mu \in \mathbb{C} \mid \operatorname{Re} \mu \leq -\operatorname{tr}(B)/p\}$  coincides with the sum  $(-\infty, 0] + \sigma(\mathcal{L}_p)$ . This is not the case if  $\sigma(\mathcal{L}_p)$  is a discrete subgroup of  $i\mathbb{R}$ , which occurs when the group generated by the drift part is periodic (see Theorem 2.2 (d)). However, also in this case we have the inclusion  $\sigma(\mathcal{A}_p) \subseteq (-\infty, 0] + \sigma(\mathcal{L}_p)$  as proved in the next proposition.

**Proposition 2.8.** *Let  $B$  be similar to a diagonal matrix with nonzero eigenvalues  $\pm i\sigma_1, \dots, \pm i\sigma_k$  in  $i\mathbb{R}$  and possibly 0. Assume that the quotient  $\sigma_r \sigma_s^{-1}$  is rational for all  $r$  and  $s$ . Then  $\sigma(\mathcal{A}_p) \subseteq (-\infty, 0] + \sigma(\mathcal{L}_p)$ .*

*Proof.* Let  $\tau > 0$  be the period of  $(S(t))_{t \in \mathbb{R}}$ , see Proposition 2.3. Also  $e^{tB^T}$  is  $\tau$ -periodic and in particular  $e^{\tau B^T} = I$ . By the representation formula (1.3) we have

$$(T(\tau)f)(x) = \frac{1}{(4\pi)^{n/2}(\det Q_\tau)^{1/2}} \int_{\mathbb{R}^n} e^{-\langle Q_\tau^{-1}y, y \rangle/4} f(x-y) dy,$$

showing that  $T(\tau) = T_\tau(1)$  where  $(T_\tau(t))_{t \geq 0}$  is the semigroup generated by the diffusion operator  $A_\tau = \operatorname{Tr}(Q_\tau D^2)$ , whose spectrum is  $(-\infty, 0]$ . Take  $\mu = a + ib \in \sigma(\mathcal{A}_p)$ . The spectral inclusion Theorem IV.3.6 of [7] yields that  $e^{\tau(a+ib)}$  belongs to  $\sigma(T(\tau)) = \sigma(T_\tau(1))$ . Since  $(T_\tau(t))_{t \geq 0}$  is analytic, from Corollary IV.3.12 of [7] we infer the identity  $\sigma(T_\tau(1)) \setminus \{0\} = e^{\sigma(A_\tau)} = (0, 1]$ . It follows that  $a \leq 0$  and  $\tau b = 2m\pi$  for some  $m \in \mathbb{Z}$  and hence  $ib \in \frac{2\pi}{\tau}i\mathbb{Z} = G$ , using also Proposition 2.3. Therefore  $\sigma(\mathcal{A}_p)$  is contained in  $(-\infty, 0] + G$ .  $\square$

The spectrum of the Ornstein-Uhlenbeck operators has been computed in [17, Section 4] if either  $\sigma(B) \subset \mathbb{C}_-$  or  $\sigma(B) \subset \mathbb{C}_+$ . The proofs in this paper are written only in the uniformly elliptic case where  $Q$  is positive definite, but in the introduction of [17] it is pointed out that they also work only assuming the hypoellipticity condition (1.2).

To explain why this condition suffices, we recall that the spectrum of  $\mathcal{A}_p$  is determined in [17] at first under the assumption  $\sigma(B) \subset \mathbb{C}_+$  by exhibiting explicit eigenfunctions for the eigenvalues  $\mu < -\operatorname{tr}(B)/p$ . These are computed using the matrix

$$\tilde{Q}_\infty = \int_0^\infty e^{-sB} Q e^{-sB^T} ds.$$

The above integral converges since the matrix semigroup  $(e^{-sB})_{s \geq 0}$  is exponentially stable. Moreover  $\tilde{Q}_\infty$  is nondegenerate under condition (1.2). Since  $\tilde{Q}_\infty$ , and not  $Q$ , enters all calculations, all results still hold in the hypoelliptic setting provided that  $\sigma(B) \subset \mathbb{C}_+$ , including the extreme cases  $p = 1, \infty$ .

The case  $\sigma(B) \subset \mathbb{C}_-$  follows from the preceding one by a simple duality argument, which we describe now. The formal adjoint of  $\mathcal{A}$  is given by

$$\mathcal{A}^* = \sum_{i,j=1}^n q_{ij} D_{ij} - \sum_{i,j=1}^n b_{ij} x_j D_i - \operatorname{tr}(B).$$

Let  $\mathcal{A}_p^*$  be the realization of  $\mathcal{A}^*$  in  $L^{p'}$ ,  $1/p + 1/p' = 1$ , as the generator of the semigroup (1.3) with  $-B$  replacing  $B$  (also in the definition of  $Q_t$ ) multiplied by the exponential factor

$e^{-t \operatorname{tr}(B)}$ . Notice that the spectrum of the drift matrix is now contained in  $\mathbb{C}_+$ . Therefore, for every  $\mu \in \mathbb{C}$  with  $\operatorname{Re} \mu < \operatorname{tr}(B)/p' - \operatorname{tr}(B) = -\operatorname{tr}(B)/p$  the operator  $\mu - \mathcal{A}_p^*$  is not injective. Let  $(\mathcal{A}_p', D(\mathcal{A}_p'))$  denote the adjoint of  $\mathcal{A}_p$  in  $L^{p'}$ . Recalling that  $C_c^\infty$  is a core for  $\mathcal{A}_p^*$  and  $\mathcal{A}_p$ , it is easily seen that  $D(\mathcal{A}_p^*) \subseteq D(\mathcal{A}_p')$  and  $\mathcal{A}_p' f = \mathcal{A}_p^* f$  for every  $f \in D(\mathcal{A}_p^*)$ . Since  $\mu - \mathcal{A}_p^*$  is not injective, it follows that  $\mu - \mathcal{A}_p$  is not surjective and hence  $\mu \in \sigma(\mathcal{A}_p)$ . The other inclusion is provided by Corollary 2.7. Note that this works in the extreme cases  $p = 1, \infty$  as well.

We state the results discussed above.

**Theorem 2.9.** *Let  $1 \leq p \leq \infty$  and (1.2) be true. If either  $\sigma(B) \subset \mathbb{C}_-$  or  $\sigma(B) \subset \mathbb{C}_+$ , then  $\sigma(\mathcal{A}_p) = \{\mu \in \mathbb{C} \mid \operatorname{Re} \mu \leq -\operatorname{tr}(B)/p\}$ . In the latter case, every  $\mu$  with  $\operatorname{Re} \mu < -\operatorname{tr}(B)/p$  is an eigenvalue.*

### 3 Preliminary considerations

In contrast to [17] we cannot use eigenvalues in the proof of our main result. To show this we rule out eigenvalues of  $\mathcal{A}$  if the spectrum of  $B$  intersects  $\mathbb{C}_-$  or if  $B$  is skew-symmetric and  $Q = I$ , where we assume that  $p = 2$ .

First, we assume that some eigenvalue of  $B$  has a negative real part. Suppose that  $\mu \in \mathbb{C}$  with  $\operatorname{Re} \mu < -\frac{1}{2} \operatorname{tr}(B)$  was an eigenvalue of  $\mathcal{A}_2$  with eigenfunction  $f \in L^2 \setminus \{0\}$ . The spectral mapping theorem for the point spectrum shows that  $T(t)f = e^{\mu t} f$  for every  $t \geq 0$ , see Theorem IV.3.7 and Corollary IV.3.8 of [7]. Denoting by  $\hat{f}$  the Fourier transform of  $f$ , the representation formula (1.3) implies that the equation  $T(t)f = e^{\mu t} f$  is equivalent to

$$\hat{f}(e^{-tB^T} \xi) = e^{(\mu + \operatorname{tr}(B))t} e^{|Q_t^{1/2} e^{-tB^T} \xi|^2} \hat{f}(\xi), \quad (3.1)$$

where  $t \geq 0$ , see Section 4 of [17]. We compute

$$\begin{aligned} |Q_t^{1/2} e^{-tB^T} \xi|^2 &= \langle Q_t e^{-tB^T} \xi, e^{-tB^T} \xi \rangle = \int_0^t \langle e^{sB} Q e^{sB^T} e^{-tB^T} \xi, e^{-tB^T} \xi \rangle ds \\ &= \int_0^t |Q^{1/2} e^{(s-t)B^T} \xi|^2 ds = \int_0^t |Q^{1/2} e^{-sB^T} \xi|^2 ds \end{aligned} \quad (3.2)$$

for  $\xi \in \mathbb{R}^n$ . Take  $\lambda \in \sigma(B) = \sigma(B^T)$  with  $\operatorname{Re} \lambda < 0$ . Let  $P$  be the spectral projection of  $B^T$  corresponding to  $\lambda$ . Fix  $\varepsilon > 0$  with  $\operatorname{Re} \lambda + \varepsilon < 0$ . Then there exists a constant  $M > 0$  such that  $\|e^{sB^T} P\| \leq M e^{(\operatorname{Re} \lambda + \varepsilon)s}$  for every  $s \geq 0$ . Observe that also  $-B$  satisfies (1.2), so that there is a constant  $\nu > 0$  with

$$\int_0^1 |Q^{1/2} e^{-sB^T} \xi|^2 ds = \left\langle \int_0^1 e^{-sB} Q e^{-sB^T} \xi ds, \xi \right\rangle \geq \nu |\xi|^2.$$

Let  $t \in [m, m+1)$  for some  $m \in \mathbb{N}_0$ . Inserting  $P$  in (3.2), it follows

$$\begin{aligned} |Q_t^{1/2} e^{-tB^T} \xi|^2 &\geq \int_0^m |Q^{1/2} e^{-sB^T} \xi|^2 ds = \sum_{k=0}^{m-1} \int_0^1 |Q^{1/2} e^{-rB^T} e^{-kB^T} \xi|^2 dr \\ &\geq \sum_{k=0}^{m-1} \nu \|P\|^{-2} |P e^{-kB^T} \xi|^2 \geq \sum_{k=0}^{m-1} \nu (M \|P\|)^{-2} e^{-2(\operatorname{Re} \lambda + \varepsilon)k} |P \xi|^2 \\ &\geq c e^{-2(\operatorname{Re} \lambda + \varepsilon)t} |P \xi|^2 \end{aligned}$$

for some constant  $c > 0$ . Integrating (3.1) on  $\mathbb{R}^n$ , we derive

$$\begin{aligned} \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi &= e^{2t(\operatorname{Re}\mu + \frac{1}{2}\operatorname{tr}(B))} \int_{\mathbb{R}^n} e^{2|Q_t^{1/2} e^{-tB^T} \xi|^2} |\hat{f}(\xi)|^2 d\xi \\ &\geq \exp(2c\alpha^2 e^{-2(\operatorname{Re}\lambda + \varepsilon)t}) e^{2t(\operatorname{Re}\mu + \frac{1}{2}\operatorname{tr}(B))} \int_{\{|P\xi| \geq \alpha\}} |\hat{f}(\xi)|^2 d\xi, \end{aligned}$$

for every  $t \geq 0$  and  $\alpha > 0$ . Letting  $t \rightarrow +\infty$ , the right hand side blows up unless  $\int_{\{|P\xi| \geq \alpha\}} |\hat{f}(\xi)|^2 d\xi = 0$ . Since  $\alpha > 0$  is arbitrary and the set  $\{P\xi = 0\}$  has measure 0, this would imply  $\hat{f} = 0$  and thus  $f = 0$  in  $L^2$ , which is a contradiction.

Second, we assume that  $B = -B^T$  and  $Q = I$ . Recalling that  $\operatorname{tr}(B) = 0$ , we now suppose there was an eigenvalue  $\mu$  of  $\mathcal{A}_2$  with  $\operatorname{Re}\mu < 0$ . Arguing as before, we rewrite (3.1) as

$$e^{-|Q_t^{1/2} e^{-tB^T} \xi|^2} \hat{f}(e^{-tB^T} \xi) = e^{\mu t} \hat{f}(\xi), \quad t \geq 0,$$

and then integrate over  $\mathbb{R}^n$  to obtain

$$\int_{\mathbb{R}^n} e^{-2|Q_t^{1/2} \xi|^2} |\hat{f}(\xi)|^2 d\xi = e^{2\operatorname{Re}\mu t} \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi.$$

Observing

$$|Q_t^{1/2} \xi|^2 = \langle Q_t \xi, \xi \rangle = \int_0^t \langle e^{sB} e^{sB^T} \xi, \xi \rangle ds = \int_0^t \langle e^{s(B+B^T)} \xi, \xi \rangle ds = t|\xi|^2,$$

we derive

$$\begin{aligned} \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi &= \int_{\mathbb{R}^n} e^{-2t(|\xi|^2 + \operatorname{Re}\mu)} |\hat{f}(\xi)|^2 d\xi \\ &= \int_{\{|\xi|^2 > -\operatorname{Re}\mu\}} e^{-2t(|\xi|^2 + \operatorname{Re}\mu)} |\hat{f}(\xi)|^2 d\xi + \int_{\{|\xi|^2 < -\operatorname{Re}\mu\}} e^{-2t(|\xi|^2 + \operatorname{Re}\mu)} |\hat{f}(\xi)|^2 d\xi. \end{aligned}$$

The first integral in the last line tends to 0 as  $t \rightarrow +\infty$  by dominated convergence. The second integral tends to  $+\infty$  by monotone convergence, if  $\int_{\{|\xi|^2 < -\operatorname{Re}\mu\}} |\hat{f}(\xi)|^2 d\xi > 0$ . Therefore we have either  $\|\hat{f}\|_2 = +\infty$  or  $\|\hat{f}\|_2 = 0$ , and we get a contradiction in any case.

By duality one deduces from the above examples that, if  $\sigma(B)$  intersects both  $\mathbb{C}_-$  and  $\mathbb{C}_+$ , a point  $\lambda$  can be in the spectrum of  $\mathcal{A}_2$  even though  $\lambda - \mathcal{A}_2$  is injective and has dense range. Approximate eigenvalues will thus play a central role.

In order to describe the spectrum of  $\mathcal{A}$ , we will reduce the drift matrix  $B$  to a quasi triangular upper matrix. This is done as follows. If  $M$  is an invertible real  $n \times n$  matrix, we define the change of variables

$$\Phi_M : L^p \rightarrow L^p, \quad (\Phi_M u)(y) = u(M^{-1}y). \quad (3.3)$$

Setting  $v = \Phi_M u$ , one easily calculates that  $\mathcal{A}u(x) = \mathcal{A}_0 v(Mx)$  for  $x \in \mathbb{R}^n$ , where

$$\mathcal{A}_0 v = \operatorname{Tr}(Q_0 D^2 v) + \langle B_0 y, Dv \rangle$$

with  $y = Mx$ ,  $Q_0 = MQM^T$ , and  $B_0 = MBM^{-1}$ . We conclude

$$\mathcal{A} = \Phi_M^{-1} \mathcal{A}_0 \Phi_M \quad \text{with} \quad D(\mathcal{A}_p) = \Phi_M^{-1} D(\mathcal{A}_{0,p}).$$

We observe that the new operator  $\mathcal{A}_0$  is still hypoelliptic, see (1.2), and that the spectrum is invariant under this transformation.

Applying Schur's theorem for real matrices (see e.g. Theorem 2.3.4 in [11]), we can now choose a real orthogonal matrix  $M$  such that  $MBM^{-1} = T$  with

$$T = \begin{pmatrix} B_1 & * & * & \cdots & * \\ 0 & B_2 & * & \cdots & * \\ 0 & 0 & B_3 & * & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_l \end{pmatrix} \quad (3.4)$$

where each  $B_j$  is a real  $1 \times 1$  matrix with a real eigenvalue of  $B$ , or a real  $2 \times 2$  matrix with a pair of nonreal complex conjugate eigenvalues  $\alpha_j \pm i\beta_j$ . The diagonal blocks  $B_j$  may be arranged in any prescribed order. By  $*$  we denote an arbitrary block.

## 4 The spectrum of $\mathcal{A}_p$

The spectrum of the Ornstein-Uhlenbeck operators  $\mathcal{A}_p$  depends on the spectrum of the drift operator  $\mathcal{L}_p$  which in turn is determined by  $B$ . If  $B$  is similar to a diagonal matrix with nonzero eigenvalues  $\pm i\sigma_1, \pm i\sigma_2, \dots, \pm i\sigma_k$  in  $i\mathbb{R}$  and possibly 0 and if all ratios  $\sigma_r\sigma_s^{-1}$  belong to  $\mathbb{Q}$ , then Theorem 2.2 shows that  $\sigma(\mathcal{L}_p)$  is a discrete subgroup  $G = \frac{2\pi i}{\tau}\mathbb{Z}$  of  $i\mathbb{R}$ , where  $S(\tau) = I$ . In this case we prove that  $\sigma(\mathcal{A}_p) = (-\infty, 0] + G$ . In all the remaining cases, the spectrum of  $\mathcal{L}_p$  is the vertical line  $-\text{tr}(B)/p + i\mathbb{R}$  and we show that  $\sigma(\mathcal{A}_p)$  is the half-plane  $\{\mu \in \mathbb{C} \mid \text{Re } \mu \leq -\text{tr}(B)/p\}$ . These results, which are the main achievement of the paper, are stated in Theorem 1.1, which we rewrite below for convenience.

**Theorem 4.1.** *Let (1.2) be true and  $p \in [1, \infty]$ . Then the spectrum of  $\mathcal{A}_p$  is given by*

$$\sigma(\mathcal{A}_p) = (-\infty, 0] + \sigma(\mathcal{L}_p). \quad (4.1)$$

*In particular, we have either  $\sigma(\mathcal{A}_p) = (-\infty, 0] + G$  or  $\sigma(\mathcal{A}_p) = \{\mu \in \mathbb{C} \mid \text{Re } \mu \leq -\text{tr}(B)/p\}$ , according to  $\sigma(\mathcal{L}_p)$  being the discrete subgroup  $G = \frac{2\pi i}{\tau}\mathbb{Z}$  of  $i\mathbb{R}$  or the whole line  $-\text{tr}(B)/p + i\mathbb{R}$ . Moreover, the semigroup  $(T(t))_{t \geq 0}$  satisfies the weak spectral mapping theorem*

$$\sigma(T(t)) = \overline{\exp(t\sigma(\mathcal{A}_p))}, \quad t \geq 0. \quad (4.2)$$

*We even have  $\sigma(T(t)) \setminus \{0\} = e^{t\sigma(\mathcal{A}_p)}$  except for the case that  $\sigma(\mathcal{L}_p) = G = \frac{2\pi i}{\tau}\mathbb{Z}$  and  $t/\tau$  is irrational.*

*Proof.* Theorem 2.9 shows the equality (4.1) if  $\sigma(B) \subset \mathbb{C}_-$  or  $\sigma(B) \subset \mathbb{C}_+$ . Moreover, by Corollary 2.7 and Proposition 2.8 the inclusion  $\sigma(\mathcal{A}_p) \subseteq (-\infty, 0] + \sigma(\mathcal{L}_p)$  always holds. Therefore we only have to prove the other inclusion  $(-\infty, 0] + \sigma(\mathcal{L}_p) \subseteq \sigma(\mathcal{A}_p)$  in two remaining cases: one eigenvalue of  $B$  has a positive real part or one eigenvalue of  $B$  lies on the imaginary axis. Note that these cases may overlap and that the first one includes situations covered by Theorem 2.9. The inclusion  $(-\infty, 0] + \sigma(\mathcal{L}_p) \subseteq \sigma(\mathcal{A}_p)$  is established in these two cases in the following two subsections.

To prove the (weak) spectral mapping theorem, we take (4.1) for granted. Let  $t > 0$ . The spectral inclusion Theorem IV.3.6 of [7] and Corollary 2.7 show that

$$e^{t\sigma(\mathcal{A}_p)} \subseteq \sigma(T(t)) \setminus \{0\} \subseteq \{\mu \in \mathbb{C} \mid 0 < |\mu| \leq e^{-t \text{tr}(B)/p}\} =: B_t.$$

We thus even obtain  $\sigma(T(t)) \setminus \{0\} = e^{t\sigma(\mathcal{A}_p)}$  if  $\sigma(\mathcal{A}_p) = \{\mu \in \mathbb{C} \mid \operatorname{Re} \mu \leq -\operatorname{tr}(B)/p\}$ . In the other case Proposition 2.3 yields  $\sigma(\mathcal{L}_p) = G = \frac{2\pi i}{\tau} \mathbb{Z}$ . Now  $\operatorname{tr}(B) = 0$ . By (4.1) we can now write

$$e^{t\sigma(\mathcal{A}_p)} = \{e^{ta} e^{2\pi i m t / \tau} \mid a \leq 0, m \in \mathbb{Z}\} = (0, 1] \cdot S_t,$$

with  $S_t = \{e^{2\pi i m t / \tau} \mid m \in \mathbb{Z}\}$ . There are two subcases.

First, let  $t/\tau$  be irrational. Then the set  $S_t$  is dense in the unit circle and it follows that  $\overline{\exp(t\sigma(\mathcal{A}_p))}$  is equal to  $\overline{B_0}$ ; i.e., (4.2) is true.

Second, let  $t/\tau = j/k$  for coprime  $j, k \in \mathbb{N}$ . Then  $S_t$  coincides with the set  $\Gamma_k$  of  $k$ th unit roots so that  $e^{t\sigma(\mathcal{A}_p)} = (0, 1] \cdot \Gamma_k$ . On the other hand, we have  $S(t)^k = S(j\tau) = I$ . As in the proof of Proposition 2.8, we deduce that  $T(t)^k = T(kt) = T_{kt}(1)$  for the analytic semigroup  $(T_{kt}(s))_{s \geq 0}$  generated by  $\operatorname{Tr}(Q_{kt} D^2)$ . The spectrum of  $T_{kt}(1)$  is thus equal to  $[0, 1]$  and hence  $\sigma(T(t)) \setminus \{0\} = (0, 1] \cdot \Gamma_k$  as required.  $\square$

#### 4.1 The case $\sigma(B) \cap \mathbb{C}_+ \neq \emptyset$

We show the remaining inclusion in the proof of Theorem 4.1 in the first case.

**Proposition 4.2.** *Let  $\sigma(B) \cap \mathbb{C}_+ \neq \emptyset$ . Then  $(-\infty, 0] + \sigma(\mathcal{L}_p) \subseteq \sigma(\mathcal{A}_p)$ .*

In the proof we use degenerate Ornstein-Uhlenbeck operators depending on different sets of variables, as we explain now. We let  $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^m$  and write a point  $z \in \mathbb{R}^n$  accordingly as  $z = (x, y)$ . Let  $B_1$  and  $B_2$  be real  $k \times k$  and  $m \times m$  matrices, respectively, and  $Q_2$  a real, symmetric and positive semidefinite  $m \times m$  matrix. We consider the operators

$$\mathcal{L}^{(1)} = \langle B_1 x, D_x \rangle \quad \text{and} \quad \mathcal{A}^{(2)} = \operatorname{Tr}(Q_2 D^2) + \langle B_2 y, D_y \rangle. \quad (4.3)$$

Here  $\mathcal{L}^{(1)}$  is a drift operator on  $L^p(\mathbb{R}^k)$  and  $\mathcal{A}^{(2)}$  is an Ornstein-Uhlenbeck operator on  $L^p(\mathbb{R}^m)$ , which is assumed to be hypoelliptic (recall that  $L^\infty$  means  $C_0$ ). Let  $(S_1(t))_{t \geq 0}$  and  $(T_2(t))_{t \geq 0}$  be the generated semigroups. Then  $(S_1(t) \otimes T_2(t))_{t \geq 0}$  acting on  $L^p(\mathbb{R}^k) \otimes L^p(\mathbb{R}^m)$  can be extended to a  $C_0$ -semigroup on  $L^p(\mathbb{R}^n)$ , whose generator is the closure  $\mathcal{C}_p$  of  $\mathcal{C} = \mathcal{L}^{(1)} + \mathcal{A}^{(2)}$  initially defined on  $D(\mathcal{L}_p^{(1)}) \otimes D(\mathcal{A}_p^{(2)})$ . Since  $C_c^\infty(\mathbb{R}^k)$  and  $C_c^\infty(\mathbb{R}^m)$  are cores for  $\mathcal{L}^{(1)}$  and  $\mathcal{A}^{(2)}$ , respectively, it follows that  $C_c^\infty(\mathbb{R}^n)$  is a core for  $\mathcal{C}_p$ .

*Proof of Proposition 4.2.* Let  $\lambda = \alpha + i\beta \in \sigma(B)$  with  $\alpha > 0$ . As explained at the end of Section 3, using a change of variables we can assume that our operator is given by

$$\mathcal{A} = \operatorname{Tr}(Q_0 D^2) + \langle T x, D \rangle,$$

where  $T$  is in the quasi triangular form (3.4), its last block  $B_l$  corresponds to  $\lambda$ , and  $Q_0$  is the transformed diffusion matrix. We distinguish between the cases  $\beta = 0$  and  $\beta \neq 0$ . (Below we tacitly assume that  $T \neq B_l$  since the easier case  $T = B_l$  can be treated analogously.)

**Case 1.  $\beta = 0$ .** Denote the nonreal eigenvalues of  $B$  by  $\{\alpha_1 \pm i\beta_1, \dots, \alpha_k \pm i\beta_k\}$  with  $0 \leq 2k < n$  and the real ones by  $\{\lambda_{2k+1}, \dots, \lambda_n\}$  with  $\lambda_n = \alpha > 0$ . We write a point in  $\mathbb{R}^n$  as  $x = (x_1, y_1, \dots, x_k, y_k, w_{2k+1}, \dots, w_n)$ . We use a scaling argument in which the variables  $z_j = (x_j, y_j)$  relative to conjugated eigenvalues are coupled and which leaves the last variable unscaled. Let  $D_{z_j} = (D_{x_j}, D_{y_j})$  for  $j = 1, \dots, k$ . The scaling operator is defined by

$$I_r u(x) = u \left( \frac{z_1}{r^{\gamma_1}}, \frac{z_2}{r^{\gamma_2}}, \dots, \frac{z_k}{r^{\gamma_k}}, \frac{w_{2k+1}}{r^{\gamma_{2k+1}}}, \dots, \frac{w_{n-1}}{r^{\gamma_{n-1}}}, w_n \right)$$

for  $r > 0$  and with  $\gamma_1 = 1$  and  $\gamma_i > \gamma_j > 0$  for  $i < j$ . Observe that  $\|I_r^{-1}\| = \|I_r\|^{-1}$  on  $L^p$ . Let  $u \in C_c^\infty$ . Computing  $I_r^{-1}\mathcal{A}I_r u$ , one finds that

$$\lim_{r \rightarrow +\infty} I_r^{-1}\mathcal{A}I_r u = \mathcal{C}u \quad \text{in } L^p,$$

for the limit operator

$$\mathcal{C}u = \nu D_{w_n}^2 u + \lambda_n w_n D_{w_n} u + \sum_{j=1}^k \langle B_j z_j, D_{z_j} u \rangle + \sum_{j=2k+1}^{n-1} \lambda_j w_j D_{w_j} u.$$

The constant  $\nu$  is the component  $\langle Q_0 e_n, e_n \rangle$  of  $Q_0$  where  $e_n = (0, \dots, 0, 1)$ . It is positive, which can be explained as follows. The last row vector in the matrix  $T$  is  $\lambda_n e_n$ . This means that the transpose of  $T$  maps  $e_n$  to  $\lambda_n e_n$ . Let  $X$  be the one-dimensional subspace spanned by  $e_n$ . It is invariant for the transpose of  $T$ . Since  $\mathcal{A}$  is hypoelliptic,  $X$  is not contained in the kernel of  $Q_0$ . It follows  $Q_0 e_n \neq 0$  and hence  $\nu = |Q_0^{1/2} e_n|^2 > 0$ .

Note that we can write  $\mathcal{C} = \mathcal{L}^{\text{lim}} + \mathcal{A}^{\text{lim}}$  with

$$\mathcal{A}^{\text{lim}} = \nu D_n^2 + \lambda_n w_n D_n, \quad \mathcal{L}^{\text{lim}} = \sum_{j=1}^k \langle B_j z_j, D_{z_j} \rangle + \sum_{j=2k+1}^{n-1} \lambda_j w_j D_{w_j}.$$

We endow  $\mathcal{C}$  with the domain described before the proof and call it  $\mathcal{C}_p$ . We first establish a crucial spectral property of  $\mathcal{C}_p$ .

**Claim.** *Every  $\mu$  with  $\text{Re } \mu < -\text{tr}(B)/p$  is an approximate eigenvalue for  $\mathcal{C}_p$ .*

Since  $\lambda_n > 0$ , every  $\lambda$  with  $\text{Re } \lambda < -\lambda_n/p$  is an eigenvalue of the one dimensional operator  $\mathcal{A}_p^{\text{lim}}$  by Theorem 2.9. Theorem 2.2 shows that  $\mathcal{L}_p^{\text{lim}}$  possesses the approximate eigenvalue  $-c/p$ , where

$$c = 2 \sum_{j=1}^k \alpha_j + \sum_{i=2k+1}^{n-1} \lambda_i = \text{tr}(B) - \lambda_n.$$

Now, fix  $\mu \in \mathbb{C}$  with  $\text{Re } \mu < -\text{tr}(B)/p$  and set  $\lambda = \mu + c/p$ . Note that  $\text{Re } \lambda < -\lambda_n/p$ . Choose an eigenfunction  $u_1 = u_1(w_n)$  of  $\mathcal{A}_p^{\text{lim}}$  for  $\lambda$  with  $\|u_1\|_{L^p(\mathbb{R})} = 1$ . Given  $\varepsilon > 0$ , there is a function  $u_2 = u_2(z_1, \dots, z_k, w_{2k+1}, \dots, w_{n-1})$  in  $D(\mathcal{L}_p^{\text{lim}})$  with  $\|u_2\|_{L^p(\mathbb{R}^{n-1})} = 1$  and  $\|\mathcal{L}^{\text{lim}} u_2 + \frac{c}{p} u_2\|_{L^p(\mathbb{R}^{n-1})} \leq \varepsilon$ . The function  $u = u_1 u_2$  thus belongs to  $D(\mathcal{C}_p)$ , has norm one in  $L^p$  and satisfies

$$\mathcal{C}u - \mu u = (\mathcal{A}^{\text{lim}} u_1 - \lambda u_1) u_2 + (\mathcal{L}^{\text{lim}} u_2 + \frac{c}{p} u_2) u_1 = (\mathcal{L}^{\text{lim}} u_2 + \frac{c}{p} u_2) u_1, \quad (4.4)$$

which yields  $\|\mathcal{C}u - \mu u\|_p \leq \varepsilon$ . So the claim is proved.

Take  $\lambda_0 \in \rho(\mathcal{A}_p)$ . By similarity,  $\lambda_0$  belongs to  $\rho(I_r^{-1}\mathcal{A}I_r)$  with resolvent  $R(\lambda_0, I_r^{-1}\mathcal{A}_p I_r) = I_r^{-1}R(\lambda_0, \mathcal{A}_p)I_r$  for all  $r > 0$ . It follows  $\|R(\lambda_0, I_r^{-1}\mathcal{A}_p I_r)\| \leq \|R(\lambda_0, \mathcal{A}_p)\| =: C$  and

$$\|u\|_p = \|R(\lambda_0, I_r^{-1}\mathcal{A}_p I_r)(\lambda_0 - I_r^{-1}\mathcal{A}I_r)u\|_p \leq C \|(\lambda_0 - I_r^{-1}\mathcal{A}I_r)u\|_p \quad (4.5)$$

for  $u \in C_c^\infty$ . Letting  $r \rightarrow +\infty$ , we infer  $\|u\|_p \leq C \|(\lambda_0 - \mathcal{C})u\|_p$ . Since  $C_c^\infty$  is a core for  $\mathcal{C}_p$ , this shows that  $\lambda_0$  cannot be an approximate eigenvalue of  $\mathcal{C}_p$ , and hence  $\text{Re } \lambda_0 \geq -\text{tr}(B)/p$  by the claim. This means that  $(-\infty, 0] + \sigma(\mathcal{L}_p)$  is contained in  $\sigma(\mathcal{A}_p)$ .

**Case 2.  $\beta \neq 0$ .** We rearrange the blocks in (3.4) such that the first blocks contain the real eigenvalues  $\lambda_1, \dots, \lambda_k$  with  $0 \leq k < n$  and the other blocks contain the complex ones

$\alpha_{k+1} \pm i\beta_{k+1}, \dots, \alpha_{k+m} \pm i\beta_{k+m}$  for  $n = k + 2m$ , where  $\alpha_{k+m} = \alpha > 0$  and  $\beta_{k+m} = \beta \neq 0$ . As a consequence, a point of  $\mathbb{R}^n$  is denoted by  $x = (w_1, w_2, \dots, w_k, z_{k+1}, \dots, z_{k+m})$  with  $z_j = (x_j, y_j)$ . The scaling operator is now defined by

$$J_r u(x) = u\left(\frac{w_1}{r^{\gamma_1}}, \frac{w_2}{r^{\gamma_2}}, \dots, \frac{w_k}{r^{\gamma_k}}, \frac{z_{k+1}}{r^{\gamma_{k+1}}}, \dots, \frac{z_{k+m-1}}{r^{\gamma_{k+m-1}}}, z_{k+m}\right) \quad (4.6)$$

with  $\gamma_1 = 1$  and  $\gamma_i > \gamma_j > 0$  for  $i < j$ . For every  $u \in C_c^\infty$  we have

$$\lim_{r \rightarrow +\infty} J_r^{-1} \mathcal{A} J_r u = \mathcal{C}u \quad \text{in } L^p,$$

where the limit operator is given by

$$\begin{aligned} \mathcal{C}u &= \text{Tr}(Q_0^\dagger D_{k+m}^2 u) + \langle B_{k+m} z_{k+m}, D_{z_{k+m}} u \rangle \\ &+ \sum_{j=1}^k \lambda_j w_j D_{w_j} u + \sum_{j=1}^{m-1} \langle B_{k+j} z_{k+j}, D_{z_{k+j}} u \rangle. \end{aligned} \quad (4.7)$$

Here  $Q_0^\dagger$  is the lower right  $2 \times 2$  submatrix of  $Q_0$ .

As before we introduce  $\mathcal{C}_p$  and claim that the open half-plane  $\{\mu \in \mathbb{C} \mid \text{Re } \mu < -\text{tr}(B)/p\}$  is contained in its approximate spectrum.

To prove the claim, we split  $\mathcal{C}$  as the sum  $\mathcal{A}^{\text{lim}} + \mathcal{L}^{\text{lim}}$  for the Ornstein-Uhlenbeck operator

$$\mathcal{A}^{\text{lim}} = \text{Tr}(Q_0^\dagger D_{k+m}^2) + \langle B_{k+m} z_{k+m}, D_{z_{k+m}} \rangle$$

in the last two variables and the drift operator

$$\mathcal{L}^{\text{lim}} = \sum_{j=1}^k \lambda_j w_j D_{w_j} + \sum_{j=1}^{m-1} \langle B_{k+j} z_{k+j}, D_{z_{k+j}} \rangle$$

acting in the remaining variables.

We show that  $\mathcal{A}^{\text{lim}}$  is hypoelliptic by verifying (1.2). Let  $Y$  be a real subspace of  $\mathbb{R}^2$  which is invariant for  $B_{k+m}^T$ . Suppose that  $\dim Y = 1$ . Then there would exist a real eigenvalue for  $B_{k+m}^T$ , but this is not the case as  $\sigma(B_{k+m}^T) = \{\alpha \pm i\beta\}$ . We thus have either  $Y = \{0\}$  or  $Y = \mathbb{R}^2$ . Suppose that  $\mathbb{R}^2 \subseteq \ker(Q_0^\dagger)$ . In this case the real subspace of  $\mathbb{R}^n$  spanned by  $e_{n-1} = (0, \dots, 0, 1, 0)$  and  $e_n = (0, \dots, 0, 1)$  would be invariant for the transpose of the drift matrix  $T$  and it would be contained in  $\ker(Q_0)$ . This contradicts the hypoellipticity of  $\mathcal{A}$ .

Since  $\alpha_{k+m} > 0$ , by Theorem 2.9 every  $\lambda$  with  $\text{Re } \lambda < -2\alpha_{k+m}/p$  is an eigenvalue of  $\mathcal{A}_p^{\text{lim}}$ . Moreover, Theorem 2.2 yields that  $\sigma_{ap}(\mathcal{L}_p^{\text{lim}})$  contains the number  $-c/p$  with

$$c = \sum_{j=1}^k \lambda_j + \sum_{j=1}^{m-1} 2\alpha_{j+k}.$$

Take  $\mu \in \mathbb{C}$  with  $\text{Re } \mu < -\text{tr}(B)/p$  and set  $\lambda = \mu + c/p$ . Take an eigenfunction  $u_1 = u_1(x_{k+m}, y_{k+m})$  of  $\mathcal{A}_p^{\text{lim}}$  for  $\lambda$  with  $\|u_1\|_{L^p(\mathbb{R}^2)} = 1$ . Given  $\varepsilon > 0$ , we have a function  $u_2 = u_2(w_1, \dots, w_k, z_{k+1}, \dots, z_{k+m-1})$  in  $D(\mathcal{L}_p^{\text{lim}})$  satisfying  $\|u_2\|_{L^p(\mathbb{R}^{n-2})} = 1$  and  $\|\mathcal{L}_p^{\text{lim}} u_2 + \frac{c}{p} u_2\|_{L^p(\mathbb{R}^{n-2})} \leq \varepsilon$ . As in (4.4),  $u = u_1 u_2$  is an approximate eigenfunction for  $\mathcal{C}_p$  with approximate eigenvalue  $\mu$ . We can then proceed as at the end of Case 1, see (4.5).  $\square$

## 4.2 The case $\sigma(B) \cap i\mathbb{R} \neq \emptyset$

To deal with imaginary eigenvalues of  $B$ , we need a second type of transformation. We introduce an isometry  $S : L^p \rightarrow L^p$  by

$$Su(x) = e^{is\phi(x)}u(x), \quad (4.8)$$

where  $s \in \mathbb{R}$  and the function  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$  is chosen below. For  $u \in C_c^\infty$ , say, the operator  $\mathcal{A}$  given by (1.1) is transformed into

$$S^{-1}\mathcal{A}Su(x) = \mathcal{A}u - s^2\langle QD\phi, D\phi \rangle u + 2is\langle QD\phi, Du \rangle + is\operatorname{Tr}(QD^2\phi)u + is\langle Bx, D\phi \rangle u. \quad (4.9)$$

Let  $\lambda \in \sigma(B) \cap i\mathbb{R}$ . Then we have either  $\lambda = 0$  or  $\lambda = i\beta \neq 0$ . The next two propositions show the spectral inclusion needed for Theorem 4.1 separately for these two cases.

**Proposition 4.3.** *Let  $0 \in \sigma(B)$ . Then  $(-\infty, 0] + \sigma(\mathcal{L}_p) \subseteq \sigma(\mathcal{A}_p)$ .*

*Proof.* In the proof we write  $\mathcal{A}$  for  $\mathcal{A}_p$  and similarly for the other operators involved. Observe that the kernel of  $B^T$  is a nontrivial subspace which is invariant for  $B^T$ . Condition (1.2) thus yields a vector  $\xi \in \ker(B^T)$  with  $Q\xi \neq 0$ . Then  $\langle Q\xi, \xi \rangle = |Q^{1/2}\xi|^2 \neq 0$ . We set  $\phi(x) = \xi \cdot x$  for  $x \in \mathbb{R}^n$ . Equation (4.9) then becomes

$$S^{-1}\mathcal{A}Su(x) = \mathcal{A}u - s^2\langle Q\xi, \xi \rangle u + 2is\langle Q\xi, Du \rangle + is\langle x, B^T\xi \rangle u = \tilde{\mathcal{A}}u - s^2\langle Q\xi, \xi \rangle u \quad (4.10)$$

where we have defined  $\tilde{\mathcal{A}}u = \mathcal{A}u + 2is\langle Q\xi, Du \rangle$  and used  $B^T\xi = 0$ . Let  $k \in \mathbb{N}$  and the isometry  $V_k : L^p \rightarrow L^p$  be given by

$$V_k u(x) = k^{-n/p}u(k^{-1}x). \quad (4.11)$$

For  $u \in C_c^\infty$ , we compute

$$V_k^{-1}\tilde{\mathcal{A}}V_k u = k^{-2}\operatorname{Tr}(QD^2u) + \langle Bx, Du \rangle + k^{-1}2is\langle Q\xi, Du \rangle \rightarrow \mathcal{L}u = \langle Bx, Du \rangle,$$

as  $k \rightarrow +\infty$ . Set  $\tilde{\mathcal{A}}_k = V_k^{-1}\tilde{\mathcal{A}}V_k$ . Then  $\rho(\tilde{\mathcal{A}}_k) = \rho(\tilde{\mathcal{A}})$  by similarity, where we omit the subscript  $p$ . We want to show the inclusion  $\sigma(\mathcal{L}) \subseteq \sigma(\tilde{\mathcal{A}})$ , for which we need the next fact.

**Claim.** Let  $\lambda \in \rho(\tilde{\mathcal{A}}) \cap \rho(\mathcal{L})$  and  $f \in L^p$ . We then obtain

$$R(\lambda, \tilde{\mathcal{A}}_k)f \rightarrow R(\lambda, \mathcal{L})f \quad \text{in } L^p \quad \text{as } k \rightarrow \infty. \quad (4.12)$$

Since  $C_c^\infty$  is a core of  $(\mathcal{L}, D(\mathcal{L}_p))$  by Proposition 2.1, it suffices to prove the convergence on the dense subspace  $(\lambda - \mathcal{L})C_c^\infty$ . Let  $f = \lambda u - \mathcal{L}u$  for some  $u \in C_c^\infty$ . Using the identity

$$R(\lambda, \tilde{\mathcal{A}}_k)f - R(\lambda, \mathcal{L})f = R(\lambda, \tilde{\mathcal{A}}_k)(\mathcal{L} - \tilde{\mathcal{A}}_k)R(\lambda, \mathcal{L})f$$

we deduce

$$\|R(\lambda, \tilde{\mathcal{A}}_k)f - R(\lambda, \mathcal{L})f\|_p \leq \|R(\lambda, \tilde{\mathcal{A}}_k)\| \|\mathcal{L}u - \tilde{\mathcal{A}}_k u\|_p \leq \|R(\lambda, \tilde{\mathcal{A}})\| \|\mathcal{L}u - \tilde{\mathcal{A}}_k u\|_p$$

and the claim follows.

Now, let  $\lambda_0 \in \sigma(\mathcal{L})$ . Suppose that  $\lambda_0 \in \rho(\tilde{\mathcal{A}})$ . Then there exists a radius  $r > 0$  such that  $\lambda \in \rho(\tilde{\mathcal{A}}) = \rho(\tilde{\mathcal{A}}_k)$  whenever  $|\lambda - \lambda_0| < r$ . Take  $\lambda$  with  $|\lambda - \lambda_0| < r$  and  $\operatorname{Re} \lambda > \operatorname{Re} \lambda_0$ . Then  $\lambda$  also belongs to  $\rho(\mathcal{L})$  by Theorem 2.2. The formula (4.12) thus yields

$$\|R(\lambda, \mathcal{L})f\|_p \leq \liminf_{k \rightarrow \infty} \|R(\lambda, \tilde{\mathcal{A}}_k)f\|_p \leq \|R(\lambda, \tilde{\mathcal{A}})\| \|f\|_p.$$

for every  $f \in L^p$ . In the limit  $\lambda \rightarrow \lambda_0$  the left-hand side blows up, whereas the right-hand side remains bounded. By this contradiction,  $\lambda_0$  belongs to  $\sigma(\tilde{\mathcal{A}})$  and consequently  $\lambda_0 - s^2\langle Q\xi, \xi \rangle$  to  $\sigma(\mathcal{A})$ , see (4.10). As  $s \in \mathbb{R}$  is arbitrary, we conclude that  $\sigma(\mathcal{L}) + (-\infty, 0] \subseteq \sigma(\mathcal{A})$ .  $\square$

We next treat the remaining case.

**Proposition 4.4.** *Let  $\beta \neq 0$  and  $i\beta \in \sigma(B)$ , then  $(-\infty, 0] + \sigma(\mathcal{L}_p) \subseteq \sigma(\mathcal{A}_p)$ .*

In order to show this proposition, we proceed as in Case 2 of the proof of Proposition 4.2 obtaining the same limit operator  $\mathcal{C} = \mathcal{A}^{\text{lim}} + \mathcal{L}^{\text{lim}}$ , see (4.7). But now we cannot use Theorem 2.9 to determine the spectrum of  $\mathcal{A}^{\text{lim}}$ , since the  $2 \times 2$  drift matrix of  $\mathcal{A}^{\text{lim}}$  has the purely imaginary eigenvalues  $\pm i\beta$ . Instead we directly compute the spectrum of  $\mathcal{A}^{\text{lim}}$ . We start with a first-order operator that will appear in a scaling limit.

**Lemma 4.5.** *Let  $b, s, \mu_1, \mu_2 \in \mathbb{R}$  and set  $\mathcal{T}_\infty u = bx_2 D_1 u - bx_1 D_2 u - s^2(\mu_1 \frac{x_1^2}{|x|^2} + \mu_2 \frac{x_2^2}{|x|^2})u$ . Let  $\mathcal{T}_{\infty,p}$  be the realization of  $\mathcal{T}_\infty$  in  $L^p(\mathbb{R}^2)$  endowed with domain  $D(\mathcal{T}_\infty) = \{u \in L^p(\mathbb{R}^2) \mid \mathcal{T}_\infty u \in L^p(\mathbb{R}^2)\}$ , where  $\mathcal{T}_\infty u$  is understood in the sense of distributions. Then, for every  $m \in \mathbb{Z}$ , the number  $imb - s^2(\mu_1 + \mu_2)/2$  is an eigenvalue of  $\mathcal{T}_{\infty,p}$  possessing an eigenfunction  $u$  in  $C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ .*

*Proof.* In polar coordinates  $(\rho, \theta)$ , our operator is expressed by

$$\mathcal{T}_\infty u = -b\partial_\theta u - s^2(\mu_1 \cos^2 \theta + \mu_2 \sin^2 \theta)u.$$

Let  $\varphi \in C_c^\infty(0, \infty)$  and  $m \in \mathbb{Z}$ . Set  $u(x) = \varphi(|x|)e^{im\theta}e^{-s^2(\mu_1 - \mu_2)\sin(2\theta)/(4b)}$  for  $x \in \mathbb{R}^2$ . Then  $u$  belongs to  $C_c^\infty(\mathbb{R}^2 \setminus \{0\})$  and straightforward computations show that  $\mathcal{T}_\infty u = \lambda u$  with  $\lambda = imb - s^2(\mu_1 + \mu_2)/2$ .  $\square$

**Lemma 4.6.** *Let  $\mathcal{A}^\diamond$  be a hypoelliptic Ornstein-Uhlenbeck operator on  $\mathbb{R}^2$  whose drift matrix  $B^\diamond$  has the eigenvalues  $\pm i\beta$  for  $\beta \in \mathbb{R} \setminus \{0\}$ . Then  $(-\infty, 0] + i\beta\mathbb{Z} = \sigma_{ap}(\mathcal{A}_p^\diamond) = \sigma(\mathcal{A}_p^\diamond)$ .*

*Proof.* We divide the proof in four steps.

1) *Put  $\mathcal{A}^\diamond$  in a canonical form.* Let  $\mu_1$  and  $\mu_2$  be the two nonnegative eigenvalues of the diffusion matrix  $Q^\diamond$  of  $\mathcal{A}^\diamond$ . There is an invertible matrix  $M_1 \in \mathbb{R}^{2 \times 2}$  such that

$$M_1 B^\diamond M_1^{-1} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}.$$

Then take an orthogonal  $U_2 \in \mathbb{R}^{2 \times 2}$  such that  $U_2(M_1 Q^\diamond M_1^T)U_2^T = D$  for the diagonal matrix  $D$  with diagonal elements  $\mu_1$  and  $\mu_2$ . Since  $U_2$  is  $2 \times 2$  orthogonal, we obtain

$$U_2 \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} U_2^T = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} =: B^\circ,$$

where  $b = \pm\beta$ . The change of variables (3.3) with  $M = U_2 M_1$  thus yields

$$\mathcal{A}^\circ = \Phi_M \mathcal{A}^\diamond \Phi_M^{-1} = \mu_1 D_{11} u + \mu_2 D_{22} u + bx_2 D_1 u - bx_1 D_2 u.$$

with  $D(\mathcal{A}_p^\circ) = \Phi_M D(\mathcal{A}_p^\diamond)$ .

We observe that there are two possible cases: either  $\mu_1$  and  $\mu_2$  are both positive, or one of them is positive and the other one zero. In the first case  $\mathcal{A}^\circ$  is a nondegenerate Ornstein-Uhlenbeck operator, in the second one it is a degenerate hypoelliptic operator.

2) *Scale  $\mathcal{A}^\circ$  by the isometry (4.8).* We now set  $\phi(x) = |x|$  for all  $x = (x_1, x_2) \in \mathbb{R}^2$  in (4.8). Observe that  $D\phi(x) = \frac{1}{|x|}x$  and  $\langle B^\circ x, D\phi \rangle = 0$ . For  $u \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$  and  $s \in \mathbb{R}$  the

formula (4.9) thus yields

$$\begin{aligned} S^{-1}\mathcal{A}^\circ Su(x) &= \mathcal{A}^\circ u - s^2 \left( \mu_1 \frac{x_1^2}{|x|^2} + \mu_2 \frac{x_2^2}{|x|^2} \right) u + 2is \left( \mu_1 \frac{x_1}{|x|} D_1 u + \mu_2 \frac{x_2}{|x|} D_2 u \right) \\ &\quad + is \left( \mu_1 \frac{x_2^2}{|x|^3} + \mu_2 \frac{x_1^2}{|x|^3} \right) u =: \mathcal{T}u. \end{aligned}$$

By similarity, it is enough to treat  $\mathcal{T}$ .

3) Show that  $i\beta\mathbb{Z} + (-\infty, 0] \subseteq \sigma(\mathcal{T}_p) = \sigma(\mathcal{A}_p^\diamond)$ . We scale the operator  $\mathcal{T}$  through the isometries  $V_k$  in (4.11), obtaining

$$\begin{aligned} V_k^{-1}\mathcal{T}V_k u &= \frac{1}{k^2}\mu_1 D_{11}u + \frac{1}{k^2}\mu_2 D_{22}u + bx_2 D_1 u - bx_1 D_2 u - s^2 \left( \mu_1 \frac{x_1^2}{|x|^2} + \mu_2 \frac{x_2^2}{|x|^2} \right) u \\ &\quad + \frac{1}{k} 2is \left( \mu_1 \frac{x_1}{|x|} D_1 u + \mu_2 \frac{x_2}{|x|} D_2 u \right) + \frac{1}{k} is \left( \mu_1 \frac{x_2^2}{|x|^3} + \mu_2 \frac{x_1^2}{|x|^3} \right) u. \end{aligned}$$

Set  $\mathcal{T}_k = V_k^{-1}\mathcal{T}V_k$ . With the limit operator  $\mathcal{T}_\infty$  from Lemma 4.5, it follows

$$\mathcal{T}_k u \longrightarrow \mathcal{T}_\infty u = bx_2 D_1 u - bx_1 D_2 u - s^2 \left( \mu_1 \frac{x_1^2}{|x|^2} + \mu_2 \frac{x_2^2}{|x|^2} \right) u$$

in  $L^p$  as  $k \rightarrow \infty$ , for every  $u \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ .

We now argue as in the proof of Proposition 4.2. Take  $\lambda_0 \in \rho(\mathcal{T}_p)$ . By similarity, we have  $\lambda_0 \in \rho(\mathcal{T}_k)$  and  $\|R(\lambda_0, \mathcal{T}_k)\| \leq C$  for every  $k \in \mathbb{N}$ . For  $u \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$  we derive  $\|u\|_p \leq C \|(\lambda_0 - \mathcal{T}_k)u\|_p$ , and thus  $\|u\|_p \leq C \|(\lambda_0 - \mathcal{T}_\infty)u\|_p$  letting  $k \rightarrow +\infty$ . Recalling that  $b = \pm\beta$ , Lemma 4.5 implies that  $\lambda_0 \notin i\beta\mathbb{Z} - s^2(\mu_1 + \mu_2)/2$  for all  $s \in \mathbb{R}$ . We have this shown the inclusion  $i\beta\mathbb{Z} + (-\infty, 0] \subseteq \sigma(\mathcal{T}_p) = \sigma(\mathcal{A}_p^\diamond)$ .

4) Compute the spectrum of  $\mathcal{A}_p^\diamond$ . Theorem 2.2 (d) shows that  $i\beta\mathbb{Z}$  is the spectrum of the drift operator  $\mathcal{L}^\diamond = \langle B^\diamond x, D \rangle$ . From Proposition 2.8 we deduce  $\sigma(\mathcal{A}_p^\diamond) \subseteq (-\infty, 0] + i\beta\mathbb{Z}$  and hence  $\sigma(\mathcal{A}_p^\diamond) = i\beta\mathbb{Z} + (-\infty, 0]$  by step 3). In particular,  $\sigma(\mathcal{A}_p^\diamond)$  coincides with its topological boundary so that  $\sigma(\mathcal{A}_p^\diamond) = \sigma_{ap}(\mathcal{A}_p^\diamond)$ .  $\square$

*Proof of Proposition 4.4.* As already pointed out, we proceed as in Case 2 of the proof of Proposition 4.2. The operator  $\mathcal{A}$  thus has the form  $\mathcal{A} = \text{Tr}(Q_0 D^2) + \langle Tx, D \rangle$  with  $T$  as in (3.4). As after (4.6), the functions  $J_r^{-1}\mathcal{A}J_r u$  tend to  $\mathcal{C}u$  in  $L^p$  as  $r \rightarrow +\infty$  for  $u \in C_c^\infty$ , where  $\mathcal{C} = \mathcal{A}^{\text{lim}} + \mathcal{L}^{\text{lim}}$  is defined in (4.7) and it is split into the same operators  $\mathcal{A}^{\text{lim}}$  and  $\mathcal{L}^{\text{lim}}$ . In particular,  $\mathcal{A}^{\text{lim}}$  is hypoelliptic. Lemma 4.6 yields the spectral identity  $\sigma(\mathcal{A}_p^{\text{lim}}) = \sigma_{ap}(\mathcal{A}_p^{\text{lim}}) = (-\infty, 0] + i\beta\mathbb{Z}$ .

We next want to show the equality  $\sigma(\mathcal{L}_p^{\text{lim}}) + i\beta\mathbb{Z} = \sigma(\mathcal{L}_p)$ . Let  $\bar{B}$  denote the coefficient matrix of  $\mathcal{L}^{\text{lim}}$ . Observe that it is diagonalizable since it has  $n-2$  eigenvalues (counted with multiplicities) and that  $\sigma(B) = \sigma(\bar{B}) \cup \{\pm i\beta\}$ . Hence, case (b) of Theorem 2.2 does not occur for  $\mathcal{L}^{\text{lim}}$ . Let  $\mathcal{L}^{\text{lim}}$  fall under cases (a) or (c) of Theorem 2.2 (so that  $\mathcal{L}$  cannot fall under case (d)). Theorem 2.2 then leads to

$$\sigma(\mathcal{L}_p^{\text{lim}}) = -\text{tr}(\bar{B})/p + i\mathbb{R} = -\text{tr}(B)/p + i\mathbb{R}.$$

The asserted equality thus follows from Theorem 2.2. In case (d), Theorem 2.2 and Proposition 2.3 yield

$$\sigma(\mathcal{L}_p^{\text{lim}}) = \{i(n_1\sigma_1 + \cdots + n_{m-1}\sigma_{m-1}) \mid (n_1, \dots, n_{m-1}) \in \mathbb{Z}^{m-1}\}.$$

If  $\beta/\sigma_1$  is rational, we infer from these results

$$\begin{aligned}\sigma(\mathcal{L}_p) &= \{i(n_1\sigma_1 + \cdots + n_m\beta) \mid (n_1, \dots, n_m) \in \mathbb{Z}^m\} \\ &= \{i(n_1\sigma_1 + \cdots + n_{m-1}\sigma_{m-1}) \mid (n_1, \dots, n_{m-1}) \in \mathbb{Z}^{m-1}\} + i\beta\mathbb{Z} \\ &= \sigma(\mathcal{L}^{\text{lim}}) + i\beta\mathbb{Z}.\end{aligned}$$

Otherwise, it follows  $\sigma(\mathcal{L}_p) = i\mathbb{R} = \sigma(\mathcal{L}^{\text{lim}}) + i\beta\mathbb{R}$  as well.

Observe that also  $\sigma(\mathcal{L}_p^{\text{lim}}) = \sigma_{ap}(\mathcal{L}_p^{\text{lim}})$ . Take  $\lambda_1 \in \sigma(\mathcal{A}_p^{\text{lim}})$  and  $\lambda_2 \in \sigma(\mathcal{L}_p^{\text{lim}})$ . As in (4.4), we check that  $\mu = \lambda_1 + \lambda_2$  is an approximate eigenvalue for  $\mathcal{C}_p$ ; i.e.,  $(-\infty, 0] + i\beta\mathbb{Z} + \sigma(\mathcal{L}^{\text{lim}}) = (-\infty, 0] + \sigma(\mathcal{L}_p)$  is contained in  $\sigma_{ap}(\mathcal{C}_p)$ . Arguing as in (4.5), we finally see that  $\sigma_{ap}(\mathcal{C}_p)$  is a subset of  $\sigma(\mathcal{A}_p)$ .  $\square$

Propositions 4.2, 4.3 and 4.4 conclude the proof of Theorem 4.1.

## References

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