POLYNOMIAL INTERNAL AND EXTERNAL STABILITY OF WELL-POSED LINEAR SYSTEMS

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Abstract. We introduce polynomial stabilizability and detectability of well-posed systems in the sense that a feedback produces a polynomial stable C_0–semigroup. Using these concepts, the polynomial stability of the given C_0–semigroup governing the state equation can be characterized via polynomial bounds on the transfer function. We further give sufficient conditions for polynomial stabilizability and detectability in terms of decompositions into a polynomial stable and an observable part. Our approach relies on recent a characterization of polynomial stable C_0–semigroups on a Hilbert space by resolvent estimates.

1. Introduction

Weakly damped or weakly coupled linear wave type equations often have polynomially decaying classical solutions without being exponentially stable, see e.g. [1], [2], [3], [4], [7], [14], [15], [16], [17], [22], and the references therein. In these contributions various methods have been used, partly based on resolvent estimates. Recently this spectral theory has been completed for the case of bounded semigroups $T(\cdot)$ in a Hilbert space with generator A. Here one can now characterize the 'polynomial stability' $\|T(t)(I-A)^{-1}\| \leq ct^{-1/\alpha}$, $t \geq 1$, of $T(\cdot)$ by the polynomial bound $\|R(i\tau, A)\| \leq c|\tau|^{\alpha}$, $|\tau| \geq 1$, on the resolvent of A. These results are due to Borichev and Tomilov in [6] and to Batty and Duyckaerts in [5], see also [4], [14] and [16] for earlier contributions. We describe this theory in the next section. In a polynomial stable system the spectrum of the generator may approach the imaginary axis as $\operatorname{Im}\lambda \to \pm\infty$. This already indicates that this concept is more subtle than exponential stability. For instance, so far robustness results for polynomial stability are restricted to small regularizing perturbations, see [18].

At least for bounded semigroups in a Hilbert space one has now a solid background which can be used in other areas such as control theory. In the context of observability this was already done in [10] (based on [4] at that time). In this paper we start an investigation of polynomial stabilizability and detectability.

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Stabilizability is one of the basic concepts and topics of linear systems theory. Let the state system be governed by a generator A on the state Hilbert space X, and let Y and U be the observation and the control Hilbert spaces, respectively. For a moment, we simply consider bounded control and observation operators and feedbacks. For a bounded control operator $B : U \longrightarrow X$ we obtain the system

$$x'(t) = Ax(t) + Bu(t), \quad t \geq 0, \qquad x(0) = x_0, \tag{1.1}$$

with the control $u \in L^2_{loc}(\mathbb{R}_+, U)$, the initial state $x_0 \in X$ and the state $x(t) \in X$ at time $t \geq 0$. This system is exponentially stabilizable if one can find a (bounded) feedback $F : X \longrightarrow U$ such that the C_0–semigroup $T_{BF}(\cdot)$ solving the closed–loop system

$$x'(t) = Ax(t) + BFx(t), \quad t \geq 0, \qquad x(0) = x_0, \tag{1.2}$$

is exponentially stable. Observe that $A + BF$ generates $T_{BF}(\cdot)$.

For the dual concept of exponential detectability, one starts with a generator A and a bounded observation operator $C : X \longrightarrow Y$. The output of this system is $y = C(T(\cdot)x_0$. One then looks for a (bounded) feedback $H : Y \longrightarrow X$ such that the C_0–semigroup $T_{HC}(\cdot)$ generated by $A + HC$ becomes exponentially stable.

In our paper we allow for unbounded observation operators C defined on $D(A)$ and control operators B mapping into the larger space $X_{-1} = D(A^*)^*$, where the domains are equipped with the respective graph norm. Here one has to assume that the output map $x_0 \mapsto y$ and the input map $u \mapsto x(t)$ are continuous. Such systems are called *admissible*, see the next section for a precise definition and further information. The monograph [23] investigates these notions in detail. In this framework one can in particular treat boundary control and observation of partial differential equations.

In order to use the full system (A, B, C), one also has to assume the boundedness of the input–output map $u \mapsto y$. This leads to the concept of a well-posed system, which was introduced by G. Weiss and others, see Section 2, the recent survey [24], and e.g. [21], [26], [27]. In well-posed systems, the Laplace transform of the input–output map gives the transfer function of the system, which plays an important role in the present paper. For well-posed systems, it becomes more difficult to determine the generators of the feedback systems, cf. [27]. However, in our arguments we can avoid to use a precise description of these operators. For well-posed systems exponential stabilizability and detectability was discussed in many papers, see e.g. [8], [11], [12], [19], [20], [28], and the references therein.

In this paper we will weaken the exponential stability of the feedback system in the above concepts to polynomial stability. Here the feedback systems are described by equations for the resolvents of the generators of given and the feedback semigroup which are coupled via a perturbation term involving the feedback, see Definitions 3.1 and 3.1. In the study of the resulting concepts of polynomial stabilizability and detectability we pursue two main questions, also treated in the above papers.

We show that a system possesses these properties if it can be decomposed into a polynomial stable and an observable part, see Theorem 4.6 and 4.7. In the exponential case, such results are often called pole-assgnment if the stable part has a finite dimensional complement. Actually one can derive exponential stabilizability from much weaker concepts (optimizability or the finite cost condition), see [8] or [28]. So far it is not clear whether such implications hold for the natural analogues of these concepts to the polynomial setting. Moreover, it is known that optimizability can be characterized by decompositions as above if the resolvent set of the generator contains a strip around $i\mathbb{R}$, see [11] or [20]. In the polynomial setting one here has to fight against the fact that the spectrum may approach the imaginary axis at infinity. So far we only have partial results in this context, not treated below.

The main part of our results is devoted to the relationship between polynomial stability of the given semigroup and polynomial estimates on the transfer function of the system. It is known that A generates an exponentially stable semigroup if (and only if) the system (A, B, C) is exponential stabilizable and detectable and its transfer function is bounded on the right halfplane, see [19] and also [28] for an extension to the concepts of optimizability and estimatibility. (Note that the 'only if' implication is easily shown with 0 feedbacks.) The boundedness of the transfer function is called *external stability*. In Theorem 4.3 we extend these results to our setting, thus requiring polynomial stabilizability and detectability and that the transfer function grows at most polynomially as $|\operatorname{Im}\lambda| \to \infty$. (The latter condition may be called *polynomial external stability.*) If the involved semigroups are bounded, we then obtain polynomial stability of the order one expects, i.e., the sum of the orders in the assumption. The proofs are based on various estimates and manipulations of formulas connecting resolvents, the transfer functions and their variants. We further use the characterization of polynomial stability from [5] and [6].

If the given semigroup is not known to be bounded, then the available theory on polynomial stability does not give the above indicated convergence order. However, in applications one can often check the boundedness of a semigroup by the dissipativity of its generator, possibly for an equivalent norm. Similarly one can characterize well-posed systems with energy dissipation (so called scattering passive systems), see e.g. [21]. Besides the given semigroup, here also the transfer function is contractive which leads to an improvement of our main result for scattering passive systems, see Corollary 4.4. In general, not much is known on the preservation of boundedness under perturbations. In Theorem 5 of the recent paper [18] one finds a result which requires smallness of the perturbations as maps into spaces betweeen $D(A)$ and X. In Proposition 4.5 we show the boundedness in the framework of the present paper. Our approach is based on a characterization of bounded semigroups in terms of L^2–norms of the resolvents of A and A^* due to [9], see Proposition 2.4.

In the next section we discuss the background on polynomial stability and well-posed systems. In Section 3 we introduce polynomial stabilizability and detectability and establish several basic estimates. The last section contains our main results on external polynomial stability and on sufficient criteria for polynomial stabilizability and detectability.

2. Polynomial stability and well-posed systems

We first discuss polynomially stable semigroups. Throughout $T(\cdot)$ denotes a C_0–semigroup on a Banach space X with generator A. There are numbers $\varpi \in \mathbb{R}$ and $M \geq 1$ such that $\|T(t)\| \leq Me^{\varpi t}$ for all $t \geq 0$. The infimum of these numbers ϖ is denoted by $\omega_0(A)$. The semigroup is called *bounded* if $\|T(t)\| \leq M$ for all $t \geq 0$.

We fix some $\omega > \omega_0(A)$. It is well known that then the fractional powers $(\omega - A)^\beta$ exist for $\beta \in \mathbb{R}$. They are bounded operators for $\beta \leq 0$ and closed ones for $\beta > 0$. The domain X_β of $(\omega - A)^\beta$ for $\beta > 0$ is endowed with the norm given by $\|x\|_\beta = \|(\omega - A)^\beta x\|$. The fractional powers satisfy the power law and coincide with usual powers for $\beta \in \mathbb{Z}$. In particular, $(\omega - A)^{-\beta}$ is the inverse of $(\omega - A)^\beta$ for all $\beta \in \mathbb{R}$. We next recall a definition from [4].

Definition 2.1. *A C_0-semigroup $T(\cdot)$ is called* polynomially stable *(of order $\alpha > 0$) if there is a constant $\alpha > 0$ such that*

$$\|T(t)(\omega - A)^{-\alpha}\| \leq ct^{-1} \qquad \textit{for all } \ t \geq 1.$$

(Here and below, we write $c > 0$ for a generic constant.) Note that a larger order α means a weaker convergence property. Due to Proposition 3.1 of [4], a bounded C_0–semigroup $T(\cdot)$ is polynomially stable of order $\alpha > 0$ if and only if

$$\|T(t)(\omega - A)^{-\alpha\gamma}\| \leq c(\gamma)\, t^{-\gamma}, \qquad t \geq 1, \tag{2.1}$$

for all/some $\gamma > 0$. (There is also a partial result for general C_0–semigroups.)

Combined with (2.1), Proposition 3 of [5] yields the following necessary condition for polynomial stability of *bounded* C_0–semigroups. Here we set

$$\mathbb{C}_\pm = \{\lambda \in \mathbb{C} \,\big|\, \operatorname{Re}\lambda \gtrless 0\} \qquad \text{and} \qquad \mathbb{C}_r = r + \mathbb{C}_+ \quad \text{for } \ r \in \mathbb{R}.$$

Proposition 2.2. *Let $T(\cdot)$ be a bounded C_0-semigroup which is polynomially stable of order $\alpha > 0$. Then the spectrum $\sigma(A)$ of A belongs to $\mathbb{C}_-$ and its resolvent is bounded by*

$$\|R(\lambda, A)\| \leq c\,(1 + |\lambda|)^\alpha \qquad \textit{for all } \ \lambda \in \overline{\mathbb{C}_+}. \tag{2.2}$$

Due to Lemma 3.2 in [13], the estimate (2.2) is true if and only if

$$\|R(\lambda, A)(\omega - A)^{-\alpha}\| \leq c \qquad \text{for all } \ \lambda \in \overline{\mathbb{C}_+}. \tag{2.3}$$

If one drops the boundedness assumption, the above result still holds with an epsilon loss in the exponent in the right hand side of (2.2) by Proposition 3.3 of [4] and (2.3). We further note that condition (2.2) implies the inclusion

$$\{\lambda \in \sigma(A) \,\big|\, \operatorname{Re}\lambda \geq -\delta\} \subset \{\lambda \in \mathbb{C}_- \,\big|\, |\operatorname{Im}\lambda| \geq c(-\operatorname{Re}\lambda)^{-1/\alpha}\}$$

for some $c, \delta > 0$, see Proposition 3.7 of [4].

The next result from [6] provides the important converse of the above proposition for bounded semigroups on a Hilbert space, se Theorem 2.4 of [6].

Theorem 2.3. *Let $T(\cdot)$ be a bounded C_0-semigroup on a Hilbert space X such that $\sigma(A) \subset \mathbb{C}_-$ and (2.2) holds for all $\lambda \in i\mathbb{R}$. Then $T(\cdot)$ is polynomially stable of order $\alpha > 0$.*

For general Banach spaces X, in Theorem 5 in [5] this result was shown up to a logarithmic factor in the estimate in semigroup, see also [4], [14] and [16]. The paper [6] gives an example where this logarithmic correction actually occurs. Without assuming its boundedness, the semigroup is still polynomially stable if a holomorphic extension of $R(\lambda, A)(\omega - A)^{-\alpha}$ satisfies (2.3), but here one only obtains the stability order $2\alpha + 1 + \epsilon$ for any $\epsilon > 0$, see Proposition 3.4 of [4].

The proof of Theorem 2.3 is based on the following characterization of the boundedness of C_0–semigroups on Hilbert spaces, see Theorem 2 in [9] and also Lemma 2.1 in [6].

Proposition 2.4. *Let A generate the C_0–semigroup $T(\cdot)$ on the Hilbert space X. The semigroup is bounded if and only if $\mathbb{C}_+ \subset \rho(A)$ and*

$$\sup_{r>0} r \int_{\mathbb{R}} \left(\|R(r + i\tau, A)x\|^2 + \|R(r + i\tau, A^*)x\|^2 \right) d\tau \le c \, \|x\|^2$$

for each $x \in X$.

We now turn our attention to the concept of well-posed systems. From now on, X, U and Y are always Hilbert spaces, A generates the C_0-semigroup $T(\cdot)$ on X and $\omega > \omega_0(A)$. Let X_{-1} be the completion of X with respect the norm given by $\|x\|_{-1} = \|R(\omega, A)x\|$. We sometimes write X_{-1}^A instead of X_{-1} to stress that this *extrapolation space* depends on A. The operator A has a unique extension $A_{-1} \in \mathcal{B}(X, X_{-1})$ which generates a C_0–semigroup given by the continuous extension $T_{-1}(t) \in \mathcal{B}(X_{-1})$ of $T(t)$, $t \ge 0$. We often omit the subscript -1 here. One can define such a space for each linear operator with non–empty resolvent set. Recall that we have set $X_1 = D(A)$.

A bounded linear (obervation) operator $B : U \longrightarrow X_{-1}$ is called *admissible* for A (or the system $(A, B, -)$ is called *admissible*) if the integral

$$\Phi_t u := \int_0^t T(t - s)u(s) \, ds$$

belongs to X for all $u \in L^2(0, t; U)$ and some $t \ge 0$. (The integral is initially defined in X_{-1}.) By Proposition 1.4.2 in [23], this property then holds for all $t \ge 0$ and $\Phi_t \in \mathcal{B}(L^2(0, t; U), X)$. Moreover, these operators are exponentially bounded, see Proposition 4.4.5 in [23].

A bounded linear (control) operator $C : X_1 \longrightarrow Y$ is called *admissible* for A (or the system $(A, -, C)$ is called *admissible*) if the map

$$\Psi_t x := CT(\cdot)x, \qquad x \in X_1,$$

has a bounded extension in $\mathcal{B}(X, L^2(0, t; Y))$ for some $t > 0$. Propositions 4.2.3 and 4.3.3 in [23] show that this fact then holds for all $t \ge 0$ and that the extensions are exponentially bounded. We still denote the extension by Ψ_t. One can extend an admissible observation operator C to the map C_Λ given by

$$C_\Lambda x = \lim_{\lambda \to \infty} C \lambda R(\lambda, A)x$$

with domain $D(C_\Lambda) = \{x \in X \,\big|\, \text{this limit exists in } Y\}$. For each $x \in X$ we have $T(s)x \in D(C_\Lambda)$ for a.e. $s \ge 0$ and $\Psi_t x = C_\Lambda T(\cdot)x$ a.e. on $[0, t]$ for all $t > 0$ by e.g. (5.6) and Proposition 5.3 in [27].

Theorem 4.4.3 of [23] shows that an operator $B \in \mathcal{B}(U, X_{-1})$ is admissible for A if and only if its adjoint $B^* \in \mathcal{B}(D(A^*), U)$ is admissible for A^*. Here we recall that X_{-1} is the dual space of $D(A^*)$, if considered as a Banach space, see e.g. Proposition 2.10.2 in [23].

Let system (A, B, C) be a system with a generator A and admissible control and observation operators B and C. One says that (A, B, C) is *well-posed* if there are bounded linear operators $\mathbb{F}_t : L^2(0, t; U) \longrightarrow L^2(0, t; Y)$ such that

$$\mathbb{F}_{\tau+t}u = \begin{cases} \mathbb{F}_\tau u_1 & \text{on } [0, \tau], \\ \mathbb{F}_t u_2 + \Psi_t \Phi_\tau u_1 & \text{on } [\tau, \tau + t] \end{cases}$$

for all $t, \tau \geq 0$ and $u \in L^2(0, \tau + t; U)$, where $u = u_1$ on $(0, \tau)$ and $u = u_2$ on $(\tau, \tau + t)$, see [26]. Also these (input-output) operators are exponentially bounded by Proposition 4.1 of [26].

One can introduce versions of the maps Ψ_t and $\mathbb{F}_t$ on the time interval $\mathbb{R}_+$ using L^2_{loc} spaces. We denote these extensions by Ψ and $\mathbb{F}$ respectively. For $x_0 \in X$ and $u \in L^2_{loc}(\mathbb{R}_+, U)$ the output of the well-posed system (A, B, C) is then given by $y = \Psi x_0 + \mathbb{F}u$. In [26] it was shown that the Laplace transform $\hat{y}$ of y satisfies

$$\hat{y}(\lambda) = C(\lambda - A)^{-1}x_0 + G(\lambda)\hat{u}(\lambda)$$

for all $\lambda \in \mathbb{C}_\omega$, where $G : \mathbb{C}_\omega \to \mathcal{B}(U, Y)$ is a bounded analytic function. It satisfies $G'(\lambda) = -CR(\lambda, A)^2 B$ and it is thus determined by A, B and C up to an additive constant. (See e.g. Theorem 2.7 in [21].) We call G the *transfer function* of (A, B, C).

Set $Z = D(A) + R(\omega, A_{-1})BU$ and endow it with the norm $\|z\|_Z$ given by the infimum of all $\|x\|_1 + \|R(\omega, A_{-1})Bv\|$ with $z = x + R(\omega, A_{-1})Bv$, $x \in D(A)$ and $v \in U$. Theorem 3.4 and Corollary 3.5 of [21] then yield an extension $\overline{C} \in \mathcal{L}(Z, U)$ of C such that the transfer function is represented as

$$G(\lambda) = \overline{C}R(\lambda, A_{-1})B + D, \qquad \lambda \in \mathbb{C}_\omega, \tag{2.4}$$

for a *feedthrough* operator $D \in \mathcal{L}(U, Y)$. Hence, the operators $\overline{C}R(\lambda, A_{-1})B$ are uniformly bounded on $\overline{\mathbb{C}}_\omega$.

This representation of G is not unique in general since $D(A)$ need not to be dense in Z. Under the additional assumption of *regularity*, one can replace here $\overline{C}$ by C_Λ (possibly for a different D), see Theorem 5.8 in [26] and also Theorem 4.6 in [21] for refinements. We will not use regularity below.

3. Polynomial stabilizability and detectability

In this section we introduce our new concepts and establish their basic properties. We start with the main definitions.

Definition 3.1. *The admissible system* $(A, B, -)$ *is* polynomially stabilizable (of order $\alpha > 0$) *if there exists a generator* A_{BF} *of a polynomially stable* C_0*-semigroup* $T_{BF}(\cdot)$ *on* X *(of order* $\alpha > 0$*) and an admissible observation operator* $F \in \mathcal{L}(D(A_{BF}), U)$ *of* A_{BF} *such that*

$$R(\lambda, A_{BF}) = R(\lambda, A) + R(\lambda, A)BFR(\lambda, A_{BF}) \tag{3.1}$$

for all $\operatorname{Re}\lambda > \max\{\omega_0(A), \omega_0(A_{BF})\}$.

Definition 3.2. *The admissible system* $(A,-,C)$ *is* polynomially detectable (of order $\alpha>0$) *if there exists a generator* A_{HC} *of a polynomially stable* C_0*-semigroup* $T_{HC}(\cdot)$ *(of order* $\alpha>0$*) and an admissible control operator* $H\in\mathcal{L}(Y,X_{-1}^{A_{HC}})$ *of* A_{HC} *such that*

$$R(\lambda,A_{HC})=R(\lambda,A)+R(\lambda,(A_{HC})_{-1})HCR(\lambda,A) \tag{3.2}$$

for all $\operatorname{Re}\lambda>\max\{\omega_0(A),\omega_0(A_{HC})\}$.

Here F, resp. H, plays the role of a feedback. These definitions are inspired the Definition 3.2 in [11] for the exponentially stable case. For this case, in e.g. [28] concepts of exponential stabilizability or detectability were used which are (at least formally) a bit stronger than those in [11], cf. Remark 3.3(b). In our context, one could also include the boundedness of the feedback semigroup $T_{BF}(\cdot)$ or $T_{HC}(\cdot)$ in the above definitions since the theory of polynomial stability works much better in the bounded case, as seen in the previous section. Instead, we make additional boundedness assumptions in some of our results. In applications one can check the boundedness or $T_{BF}(\cdot)$ or $T_{HC}(\cdot)$ by showing that the generators A_{BF} or A_{HC} are dissipative, respectively, where one may use their representation given in the next remark.

Remark 3.3. (a) Let $(A,B,-)$, $(A_{BF},-,F)$, $(A,-,C)$ and $(A_{HC},H,-)$ be admissible. Proposition 4.11 in [12] (with $\beta=\gamma=1$ and $b=c=0$) then shows that the equations (3.1) and (3.2) are equivalent to

$$T_{BF}(t)x=T(t)x+\int_0^t T(t-s)BF_\Lambda T_{BF}(s)x\,ds=T(t)x+\Phi_tF_\Lambda T_{BF}(\cdot)x, \tag{3.3}$$

$$T_{HC}(t)x=T(t)x+\int_0^t T_{HC}(t-s)HC_\Lambda T(s)x\,ds \tag{3.4}$$

for all $t\ge 0$ and $x\in X$, respectively.

(b) Applying $\lambda-A_{-1}$ to (3.1), we see that A_{BF} is restriction of the part $(A_{-1}+BF)|X$ of $A_{-1}+BF$ in X. Similarly, multiplication of (3.2) by $\lambda-A_{HC,-1}$ leads to $A\subset(A_{HC,-1}-HC)|X$. See Proposition 6.6 in [27]. We note that in [28] exponential stabilizability and detectability was in defined in such way that $A_{BF}=(A_{-1}+BF_\Lambda)|X$ and $A_{HC}=(A_{-1}+CH_\Lambda)|X$.

(c) The system $(A,B,-)$ is polynomially stabilizable of order $\alpha>0$ (with feedback F) if and only if $(A^*,-,B^*)$ is polynomially detectable of order $\alpha>0$ (with feedback $H=F^*$). Moreover, the semigroups of the feedback systems are dual to each other.

(d) Let L be a closed operator with $\emptyset\neq\Lambda\subset\rho(L)$ and $\Omega\supset\Lambda$ be connected. If $R(\cdot,L)$ has a holomorphic extension R_λ to Ω, then $\Omega\subset\rho(L)$ and $R_\lambda=R(\lambda,L)$ for every $\lambda\in\Omega$.

Proof of (d). We have $I=(\lambda-L)R(\lambda,L)$ and $x=R(\lambda,L)(\lambda-L)x$ for $x\in D(L)$ and $\lambda\in\Lambda$. The uniqueness of holomorphic extensions yields $I=(\lambda-L_{-1})R_\lambda$ and $x=R_\lambda(\lambda-L)x$ for $\lambda\in\Omega$ and $x\in D(L)$, and thus the assertion. □

In a sequence of lemmas we relate the growth properties of several operators arising in (3.1) or (3.2). We use the spectral bound $s(L)=\sup\{\lambda\mid\lambda\in\sigma(L)\}$ for a closed operator L.

Lemma 3.4. *Let $C \in \mathcal{B}(X_1, Y)$ and $B \in \mathcal{B}(U, X_{-1})$ be admissible observation and control operators for A, respectively and let*

$$\|R(r + i\tau, A)\| \leq c\,|\tau|^\alpha \tag{3.5}$$

for some $r > s(A)$ and $\alpha > 0$ and all $|\tau| \geq 1$. We then obtain the estimates

$$\|CR(r + i\tau, A)\| \leq c\,|\tau|^\alpha \qquad \text{and} \qquad \|R(r + i\tau, A)B\| \leq c\,|\tau|^\alpha$$

for all $|\tau| \geq 1$. Moreover, if (A, B, C) is also well-posed, we have

$$\|\overline{C}R(r + i\tau, A)B\| \leq c\,|\tau|^\alpha$$

for all $|\tau| \geq 1$. Here the constants are uniform for r in bounded intervals.

Proof. Let $\lambda = r + i\tau$ and $\mu = \omega + i\tau$ for $\tau \in \mathbb{R}$ and some $\omega > \max\{0, \omega_0(A)\}$. The resolvent equation yields

$$CR(\lambda, A) = CR(\mu, A) + (\omega - r)CR(\mu, A)R(\lambda, A). \tag{3.6}$$

Let $x \in D(A)$. Since the resolvent is the Laplace transform of $T(\cdot)$, from the admissibility of C and exponential bound of $T(\cdot)$ we deduce

$$\|CR(\mu, A)x\|^2 \leq \Big[\int_0^\infty e^{-\frac{\omega}{2}t} e^{-\frac{\omega}{2}t}\,\|CT(t)x\|\,dt\Big]^2 \leq c\int_0^\infty e^{-\omega t}\,\|CT(t)x\|^2\,dt \tag{3.7}$$
$$\leq c\sum_{n=0}^\infty e^{-\omega n}\,\|CT(\cdot)T(n)x\|^2_{L^2(0,1;Y)} \leq c\sum_{n=0}^\infty e^{-\omega n}\|T(n)x\|^2 \leq c\,\|x\|^2.$$

By density, the formulas (3.5), (3.6) and (3.7) imply

$$\|CR(\lambda, A)\| \leq c + c\,|\tau|^\alpha \leq c\,|\tau|^\alpha$$

for $|\tau| \geq 1$. The second asserted inequality then follows by duality because B^* is an admissible observation operator for A^* and $\|R(\lambda, A)B\| = \|B^*R(\lambda, A^*)\|$. For the final claim, we start from the equation

$$\overline{C}R(\lambda, A)B = \overline{C}R(\mu, A)B + (\omega - r)CR(\mu, A)R(\lambda, A)B$$

for $\lambda = r + i\tau$, $\mu = \omega + i\tau$, $\tau \in \mathbb{R}$ and some $\omega > \max\{0, \omega_0(A)\}$. As noted in the previous section, $\overline{C}R(\mu, A)B : U \to Y$ is uniformly bounded. The third assertion now is a consequence of the two previous ones. □

In the next lemma we deduce resolvent estimates for A from those for A_{BF}.

Lemma 3.5. *Let $B \in \mathcal{L}(U, X_{-1})$ be an admissible control operator for A. Assume that there exist a generator A_{BF} of a C_0-semigroup $T_{BF}(\cdot)$ on X and an admissible observation operator $F \in \mathcal{L}(D(A_{BF}), U)$ of A_{BF} such that* (3.1) *holds. Assume that*

$$\|R(\lambda, A_{BF})\| \leq c\,(1 + |\lambda|^\alpha)$$

for $r < \operatorname{Re}\lambda \leq r + \delta$ and some $r \geq s(A_{BF})$, $\delta > 0$, $\alpha \geq 0$. Suppose that $R(\lambda, A)B$ has a holomorphic extension R^B_λ to $\mathbb{C}_r$ satisfying

$$\|R^B_\lambda\| \leq c\,(1 + |\lambda|^\beta)$$

for $r < \operatorname{Re}\lambda \leq r + \delta$ and some $\beta \geq 0$. Then $R(\cdot, A)$ can be extended to a neighborhood of $\overline{\mathbb{C}_r}$, and we obtain

$$\|R(\lambda, A)\| \leq c\,(1 + |\lambda|^{\alpha+\beta}) \tag{3.8}$$

for $r \le \operatorname{Re}\lambda \le r+\delta$. Moreover, (3.1) *holds on $\overline{\mathbb{C}_r}$. If $r = 0$, then $T(\cdot)$ is polynomially stable with order $2(\alpha+\beta)+1+\eta$ for any $\eta > 0$.*

Proof. By the assumption, (3.1) and Remark 3.3, the resolvent $R(\cdot, A)$ has the extension
$$R(\lambda, A) = R(\lambda, A_{BF}) - R_\lambda^B F R(\lambda, A_{BF})$$
to $\lambda \in \mathbb{C}_r$. Lemma 3.4 and the assumption then imply that
$$\|R(\lambda, A)\| \le c\,(1 + |\lambda|^{\alpha+\beta})$$
for $r < \operatorname{Re}\lambda \le r+\delta$. A standard power series argument allows us to extend this inequality to $\lambda \in \overline{\mathbb{C}_r}$ and to deduce that a neighborhood of $\overline{\mathbb{C}_r}$ belongs to $\rho(A)$. The uniqueness of the holomorphic extension now yields that $R_\lambda^B = R(\lambda, A)B$ on $\overline{\mathbb{C}_r}$ and that (3.1) holds on $\overline{\mathbb{C}_r}$. The last assertion then follows from estimate (3.8) and Propositions 3.4 and 3.6 in [4]. □

The next result is proved as the above lemma.

Lemma 3.6. *Let the operators A, C and H satisfy the assumptions of Definition 3.2 except for the polynomial stability of $T_{HC}(\cdot)$. Assume that*
$$\|R(\lambda, A_{HC})\| \le c\,(1 + |\lambda|^{\alpha})$$
for $r < \operatorname{Re}\lambda \le r+\delta$ and some $r \ge s(A_{HC})$, $\delta > 0$ and $\alpha \ge 0$. Let $CR(\lambda, A)$ have a holomorphic extension R_λ^C to $\mathbb{C}_r$. Suppose that
$$\|R_\lambda^C\| \le c\,(1 + |\lambda|^{\beta})$$
for $r < \operatorname{Re}\lambda \le r+\delta$ and some $\beta > 0$. Then $\rho(A)$ contains a neighborhood of $\overline{\mathbb{C}_r}$, the equality (3.2) *holds on $\overline{\mathbb{C}_r}$, and we obtain*
$$\|R(\lambda, A)\| \le c\,(1 + |\lambda|^{\alpha+\beta})$$
for $r \le \operatorname{Re}\lambda \le r+\delta$. If $r = 0$, then $T(\cdot)$ is polynomially stable with order $2(\alpha+\beta)+1+\eta$ for any $\eta > 0$.

To apply Proposition 2.4, we will need a variant of the above estimates.

Lemma 3.7. *Let A generate a bounded C_0–semigroup and C be an admissible observation operator for A. Then*
$$\sup_{r>0} r \int_{\mathbb{R}} \|CR(r+i\tau, A)x\|^2\, d\tau \le c\,\|x\|^2$$
for all $r > 0$ and $x \in X$.

Proof. Take $r > 0$ and $x \in D(A)$. Since $A - r$ generates the exponentially stable semigroup $(e^{-rt}T(t))_{t\ge 0}$, Plancherel's theorem and the assumption yield
$$\begin{aligned}\|CR(r+i\cdot, A)x\|^2_{L^2(\mathbb{R}_+,Y)} &= \|Ce^{-r\cdot}T(\cdot)x\|^2_{L^2(\mathbb{R}_+,Y)}\\ &= \sum_{n\ge 0}\int_0^1 e^{-2rn}e^{-2rs}\|CT(s)T(n)x\|^2\, ds.\\ &\le c\sum_{n\ge 0} e^{-2rn}\|T(n)x\|^2 \le \frac{c\,\|x\|^2}{1-e^{-2r}} \le \frac{c}{r}\,\|x\|^2.\end{aligned}$$
The assertion follows by density. □

4. Main results

We show that external polynomial stability in the frequency domain, i.e., a polynomial estimate on the transfer function, imply polynomial stability of the state system. We begin with a result involving only the control operator B.

Proposition 4.1. *Let $(A, B, -)$ be admissible and polynomially stabilizable of order $\alpha > 0$. Assume that $R(\lambda, A)B$ has a holomorphic extension to $\mathbb{C}_+$ which is bounded by $c\,(1 + |\lambda|^\beta)$ for $0 < \operatorname{Re}\lambda \le \delta$ and some $\beta \ge 0$, $\delta > 0$. The following assertions hold.*

a) The resolvent $R(\cdot, A)$ can be extended to a neighborhood of $\overline{\mathbb{C}_+}$ and

$$\|R(\lambda, A)\| \le c_\varepsilon\,(1 + |\lambda|^{\alpha+\beta+\varepsilon}) \tag{4.1}$$

for $0 \le \operatorname{Re}\lambda \le \delta$ and every $\varepsilon > 0$. If $T_{BF}(\cdot)$ is bounded, we can choose $\varepsilon = 0$.

b) The semigroup $T(\cdot)$ is polynomially stable. If $T(\cdot)$ is also bounded, then it is polynomially stable of order $\alpha + \beta + \varepsilon$. If in addition $T_{BF}(\cdot)$ is bounded, we can take $\varepsilon = 0$.

Proof. a) Propositions 3.3 and 3.6 in [4] imply that $\sigma(A_{BF}) \subset \mathbb{C}_-$ and

$$\|R(\lambda, A_{BF})\| \le c_\varepsilon(1 + |\lambda|^{\alpha+\varepsilon})$$

holds for $\operatorname{Re}\lambda \ge 0$ and every $\varepsilon > 0$. Using Lemma 3.5, we infer $\sigma(A) \subset \mathbb{C}_-$ and (4.1). If $T_{BF}(\cdot)$ is bounded, we can use Proposition 2.2 instead of the results from [4] and obtain the above estimates with $\varepsilon = 0$.

b) Proposition 3.4 of [4] and (4.1) imply the polynomial stability of $T(\cdot)$. If also $T(\cdot)$ is bounded, it is polynomially stable of order $\alpha + \beta + \varepsilon$ due to Theorem 2.3 and (4.1). □

By duality, the above proposition implies the next one for the observation system $(A, -, C)$.

Proposition 4.2. *Let $(A, -, C)$ be admissible and polynomially detectable of order $\alpha > 0$. Assume that $CR(\cdot, A)$ has a holomorphic extension to $\mathbb{C}_+$ which is bounded by $c\,(1 + |\lambda|^\beta)$ for $0 < \operatorname{Re}\lambda \le \delta$ and some $\beta \ge 0$. The following assertions hold.*

a) The resolvent $R(\cdot, A)$ can be extended to a neighborhood of $\overline{\mathbb{C}_+}$ and estimate (4.1) holds for every $\varepsilon > 0$. If $T_{HC}(\cdot)$ is bounded, we can take $\varepsilon = 0$.

b) The semigroup $T(\cdot)$ is polynomially stable. If $T(\cdot)$ is also bounded, then it is polynomially stable of order $\alpha + \beta + \varepsilon$. If in addition $T_{HC}(\cdot)$ is bounded, we can take $\varepsilon = 0$.

We now can state our main result which uses the full system (A, B, C) and the transfer function G.

Theorem 4.3. *Let (A, B, C) be a well-posed system which is polynomially stabilizable of order $\alpha > 0$ and polynomially detectable of order $\beta > 0$. Assume that G has an holomorphic extension to $\mathbb{C}_+$ which is bounded by $c\,(1 + |\lambda|^\gamma)$ for $0 < \operatorname{Re}\lambda \le \delta$ and some $\gamma \ge 0$ and $\delta > 0$. The following assertions hold.*

a) The extension $\overline{C}$ of C is an admissible observation operator for A_{BF}, $\sigma(A) \subset \mathbb{C}_-$, and

$$\|R(\lambda, A)\| \le c_\varepsilon(1 + |\lambda|^{\alpha+\beta+\gamma+\varepsilon})$$

for $0 < \operatorname{Re}\lambda \le \delta$ *and all* $\varepsilon > 0$*. If* $T_{BF}(\cdot)$ *is bounded, we can take* $\varepsilon = 0$*.*

b) The semigroup $T(\cdot)$ *is polynomially stable. If* $T(\cdot)$ *is bounded, then it is polynomially stable of order* $\alpha+\beta+\gamma+\varepsilon$*. If in addition* $T_{BF}(\cdot)$ *is bounded, we can take* $\varepsilon = 0$*.*

Proof. a) Due to (3.1) and (2.4), we have $D(A_{BF}) \subset Z$ and

$$\begin{aligned} \overline{C}R(\lambda, A_{BF}) &= CR(\lambda, A) + \overline{C}R(\lambda, A)BFR(\lambda, A_{BF}), \\ \overline{C}R(\lambda, A_{BF}) &= CR(\lambda, A) + G(\lambda)FR(\lambda, A_{BF}) - DFR(\lambda, A_{BF}) \end{aligned} \tag{4.2}$$

for $\operatorname{Re}\lambda > \max\{\omega_0(A), \omega_0(A_{BF})\}$. Taking the inverse Laplace transform of this equation, we define

$$\Psi_{BF}x := \mathcal{L}^{-1}(\overline{C}R(\cdot, A_{BF})x) = \Psi x + \mathbb{F}FT_{BF}(\cdot)x - DFT_{BF}(\cdot)x \tag{4.3}$$

for $x \in D(A_{BF})$. By assumption, $\Psi_{BF} : X \longrightarrow L^2_{loc}(\mathbb{R}_+, Y)$ is continuous. For $\tau \ge 0$ and $x \in D(A_{BF})$, the properties of a wellposed system and (3.3) yield

$$\begin{aligned} \Psi_{BF}x(\cdot+\tau) &= \Psi T(\tau)x + \mathbb{F}FT_{BF}(\cdot)T_{BF}(\tau)x + \Psi\Phi_\tau FT_{BF}(\cdot)x \\ &\qquad - DFT_{BF}(\cdot)T_{BF}(\tau)x \\ &= \Psi T_{BF}(\tau)x + \mathbb{F}FT_{BF}(\cdot)T_{BF}(\tau)x - DFT_{BF}(\cdot)T_{BF}(\tau)x \\ &= \Psi_{BF}T_{BF}(\tau)x. \end{aligned}$$

As a result, (Ψ_{BF}, T_{BF}) is an observation system in the sense of [25] or Section 4.3 in [23]. The proof of Theorem 3.3 of [25] and (4.3) thus show that $\Psi_{BF}x = \widetilde{C}T_{BF}(\cdot)x$ for $x \in D(A_{BF})$ and the admissible control operator $\widetilde{C} \in \mathcal{L}(D(A_{BF}), Y)$ for A_{BF} given by

$$\widetilde{C}x = \widehat{\Psi_{BF}}(\lambda)(\lambda - A_{BF})x = \overline{C}R(\lambda, A_{BF})(\lambda - A_{BF})x = \overline{C}x \quad \text{for } x \in D(A_{BF});$$

i.e., $\Psi_{BF}x = \overline{C}T_{BF}(\cdot)x$ for $x \in D(A_{BF})$. Proposition 3.4 of [4] and Lemma 3.4 then yield

$$\|\overline{C}R(\lambda, A_{BF})\| \le c\,(1+|\lambda|^{\alpha+\varepsilon}) \qquad \text{and} \qquad \|FR(\lambda, A_{BF})\| \le c\,(1+|\lambda|^{\alpha+\varepsilon})$$

for $\operatorname{Re}\lambda \ge 0$ and any $\varepsilon > 0$. If $T_{BF}(\cdot)$ is bounded, we can use Proposition 2.2 instead of the results in [4] and derive these estimates with $\varepsilon = 0$. By means of (4.2) and the bound on G, we can now extend $CR(\cdot, A)$ (using the same symbol) to $\mathbb{C}_+$ and obtain

$$\|CR(\lambda, A)\| \le c\,(1+|\lambda|^{\alpha+\gamma+\varepsilon})$$

for $0 < \operatorname{Re}\lambda \le \delta$. Proposition 4.2 then gives

$$\|R(\lambda, A)\| \le c_\varepsilon(1+|\lambda|^{\alpha+\beta+\gamma+\varepsilon})$$

for $0 < \operatorname{Re}\lambda \le \delta$ and all $\varepsilon > 0$, where we can take $\varepsilon = 0$ if $T_{BF}(\cdot)$ is bounded.

b) Proposition 3.4 of [4] and part a) imply the polynomial stability of $T(\cdot)$. If $T(\cdot)$ is bounded, it is polynomially stable of order $\alpha+\beta+\gamma+\varepsilon$ due to Theorem 2.3 and part a), where we can take $\varepsilon = 0$ if $T_{BF}(\cdot)$ is bounded. □

In the above results one obtains the expected stability order of $T(\cdot)$ only if this semigroup is bounded. This property automatically holds in the important case of a *scattering passive* system (A,B,C); i.e., if we have

$$\|y\|^2_{L^2(0,t;Y)} + \|x(t)\|^2 \le \|u\|^2_{L^2(0,t;U)} + \|x_0\|^2$$

for all $u \in L^2(0,t;U)$, $x_0 \in X$ and $t \ge 0$, where $x(t) = T(t)x_0 + \Phi_t u$ is the state and $y = \Psi x_0 + \mathbb{F}u$ is the output of (A,B,C). This class of systems has been characterized and studied in e.g. [21]. In this case $T(t)$ and $G(\lambda)$ are contractions for $t \ge 0$ and $\lambda \in \mathbb{C}_+$ by Proposition 7.2 and Theorem 7.4 of [21].

Corollary 4.4. *Let (A,B,C) be a scattering passive system which is polynomially stabilizable of order $\alpha > 0$ and polynomially detectable of order $\beta > 0$. Then $T(\cdot)$ is polynomially stable of order $\alpha+\beta+\varepsilon$ for each $\varepsilon > 0$. We can take $\varepsilon = 0$ if $T_{BF}(\cdot)$ is bounded.*

Proposition 2.4 yields another suffcient condition for the boundedness of $T(\cdot)$ in the framework of the first two propositions of this section.

Proposition 4.5. *Assume that the assumptions of both Propositions 4.1 and 4.2 hold for some $\alpha > 0$ and for $\beta = 0$. Let $T_{BF}(\cdot)$ and $T_{HC}(\cdot)$ be bounded. Then $T(\cdot)$ is bounded, and hence polynomially stable of order $\alpha > 0$.*

Proof. Definitions 3.1 and 3.2 yield

$$R(r+i\tau,A)x = R(r+i\tau,A_{BF})x - R(r+i\tau,A)BFR(r+i\tau,A_{BF})x, \qquad (4.4)$$

$$R(r+i\tau,A^*)x = R(r+i\tau,A^*_{HC})x - R(r+i\tau,A^*)C^*H^*R(r+i\tau,A^*_{HC})x \qquad (4.5)$$

for all $r > \max\{\omega_0(A),0\}$, $\tau \in \mathbb{R}$ and $x \in X$. We can extend these equations to $r > 0$ using the bounded extensions of $R(\lambda,A)B$ and $R(\lambda,A^*)C^* = (CR(\lambda,A))^*$ which are provided by our assumption. Since $T_{BF}(\cdot)$ and $T_{HC}(\cdot)$ are bounded, Lemma 3.7 implies that the terms on the right hand sides belong to $L^2(\mathbb{R},X)$ as functions in τ, with norms bounded by $cr^{-1/2}\|x\|$. Employing Proposition 2.4, we then deduce the boundedness of $T(\cdot)$ from (4.4) and (4.5). The final assertion now follows from Proposition 4.1. □

We finally present sufficient conditions for polynomial stabilizability and for polynomial detectability by means of a decomposition in to a polynomial stable and an observable part. An admissible system $(A,B,-)$ is called *null controllable in finite time* if for each initial value $x_0 \in X$ there is a time $\tau > 0$ and a control $u \in L^2(0,\tau;U)$ such that $x(\tau) = T(\tau)x_0 + \Phi_\tau u = 0$. We further note that one can extend an operator S to X_{-1} if it commutes with $T(t)$ for all $t \ge 0$ since then $SR(\omega,A) = R(\omega,A)S$.

Theorem 4.6. *Let $(A,B,-)$ be admissible and let $P^2 = P \in \mathcal{B}(X)$ satisfy $T(t)P = T(t)P$ for all $t \ge 0$. Set $X_s = PX$, $X_u = (I-P)X$, $T_s(t) = T(t)P$, $A_u = (I-P)A$ and $B_u = (I-P)B$. Assume that*

(i) *the C_0–semigroup $T_s(\cdot)$ is polynomially stable of order $\alpha > 0$ on X_s and*
(ii) *the system $(A_u,B_u,-)$ is null controllable in finite time on X_u.*

Then the system $(A,B,-)$ is polynomially stabililizable of order $\alpha > 0$.

Proof. First observe that $T_u(\cdot)$ is the C_0–semigroup on X_u generated by A_u and that B_u is admissible for A_u. Due to (ii), for each $x_0 \in X_u$ there is a time $\tau > 0$ and a control $u \in L^2(0,\tau;U)$ such that $x_u(\tau) = T_u(\tau)x_0 + (I-P)\Phi_\tau u = 0$. Extending x_u and u by 0 to (τ,∞), we see that the system $(A_u, B_u, -)$ is optimizable in the sense of Definition 3.1 in [28]. Propositions 3.3 and 3.4 of [28] (or Theorem 2.2 of [8]) then give an operator F_u which satisfies the conditions of Definition 3.1 where $T_{B_uF_u}(\cdot)$ is even exponentially stable, i.e., $\omega_0(A_{B_uF_u}) < 0$. We thus have

$$R(\lambda, A_{B_uF_u}) = R(\lambda, A_u) + R(\lambda, A_u)B_uF_uR(\lambda, A_{B_uF_u}) \tag{4.6}$$

for all $\operatorname{Re}\lambda > \max(\omega_0(A), \omega_0(A_{B_uF_u}))$. We now set

$$F = \begin{pmatrix} 0 \\ F_u \end{pmatrix} \qquad \text{and} \qquad A_{BF} := \begin{pmatrix} A_s & 0 \\ 0 & A_{B_uF_u} \end{pmatrix}.$$

It is then straightforward to check that these operators fulfill the conditions of Definition3.1. □

The next result follows by duality from Theorem 4.6.

Theorem 4.7. *Let $(A,-,C)$ be admissible and let $P^2 = P \in \mathcal{B}(X)$ satisfy $T(t)P = T(t)P$ for all $t \geq 0$. Set $X_s = PX$, $X_u = (I-P)X$, $T_s(t) = T(t)P$, $A_u = (I-P)A$ and $C_u = C(I-P)$. Assume that*

(i) *the C_0–semigroup $T_s(\cdot)$ is polynomially stable of order $\alpha > 0$ on X_s and*
(ii) *the system $(A_u^*, C_u^*, -)$ is null controllable in finite time on X_u.*

Then the system $(A,-,C)$ is polynomially detectable of order $\alpha > 0$.

Remark 4.8. The results of Theorem 4.6 and 4.7 also hold if we replace the condition (ii) by $(ii)'$: The system $(A_u, B_u, -)$ (resp., $(A_u^*, C_u^*, -)$) is polynomially stabilizable of order α.

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