REDUCTION PRINCIPLE AND ASYMPTOTIC PHASE FOR CENTER MANIFOLDS OF PARABOLIC SYSTEMS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. We prove the reduction principle and study other attractivity properties of the center and center-unstable manifolds in the vicinity of a steady-state solution for quasilinear systems of parabolic partial differential equations with fully nonlinear boundary conditions on bounded or exterior domains.

1. Introduction

In his illuminating short paper [13], K. Palmer proved a fundamental lemma saying that any given solution of an ODE can be tracked by a solution on the center manifold as long as the given solution stays in a small ball around an equilibrium. Magically, this simple assertion implies the existence of asymptotic phase as well as an important Pliss Reduction Principle [14] saying that every compact invariant set in a small ball centered at the equilibrium is a graph over an invariant set for the reduced ODE on the center manifold. Moreover, if the latter invariant set is (asymptotically) stable for the flow on the center manifold then the former invariant set is (asymptotically) stable for the full flow.

The objective of this paper is to give a generalization of these ODE results for a fairly broad class of parabolic partial differential equations. Specifically, in the current paper we continue the work began in [9, 10], and prove Palmer's Fundamental Lemma, the Pliss reduction principle, the existence of asymptotic phase, and some other properties of the center and center-unstable manifolds in the vicinity of a steady-state solution for quasilinear systems of parabolic partial differential equations with fully nonlinear boundary conditions on bounded or exterior domains.

We consider the equations

$$\begin{aligned} \partial_t u(t) + A(u(t))u(t) &= F(u(t)), \quad \text{on } \Omega, \ \ t > 0, \\ B_j(u(t)) &= 0, \quad \text{on } \partial\Omega, \ \ t \geq 0, \ \ j = 1, \cdots, m, \\ u(0) &= u_0, \quad \text{on } \Omega, \end{aligned} \tag{1.1}$$

on a (possibly unbounded) domain Ω in $\mathbb{R}^n$ with compact boundary $\partial\Omega$, where the solution $u(t,x)$ takes values in $\mathbb{C}^N$. The main part of the differential equation is given by a linear differential operator $A(u)$ of order $2m$ (with $m \in \mathbb{N}$) whose

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matrix–valued coefficients depend on the derivatives of u up to order $2m-1$, and F is a general nonlinear reaction term acting on the derivatives of u up to order $2m-1$. Therefore the differential equation is quasilinear. Our analysis focusses on the fully nonlinear boundary conditions

$$[B_j(u)](x) := b(x, u(x), \nabla u(x), \cdots, \nabla^{m_j} u(x)) = 0, \quad x \in \partial\Omega, \; j = 1, \cdots, m,$$

for the partial derivatives of u up to order $m_j \leq 2m-1$. We assume mild local regularity of the coefficients and that the linearization at a given steady state u_* is normally elliptic and satisfies the Lopatinskii–Shapiro condition. For illustration, we give a simple example where $N=1$ and $m=2$ (see e.g. [3] or [9, §6] for the system case $N>1$). In the case of the quasilinear heat equation with a nonlinear Neumann boundary condition

$$\begin{aligned} \partial_t u(t) - a(u(t))\Delta u(t) &= f(u(t)), \quad \text{on } \Omega, \; t > 0, \\ b(\nabla u(t)) &= 0, \quad \text{on } \partial\Omega, \; t \geq 0, \\ u(0) &= u_0, \quad \text{on } \Omega, \end{aligned}$$

we have to require that $a, f \in C^1(\mathbb{R})$, $b \in C^2(\mathbb{R})$ are real, and that there is a steady state $u_* \in W^2_p(\Omega)$ with $a(u_*) \geq \delta > 0$ and $|b'(u_*) \cdot \nu| \geq \delta > 0$ for the outer unit normal ν. Fully nonlinear boundary conditions appear naturally in the treatment of free boundary problems, see e.g. [7] or [15]. The equations (1.1) are a model case for such problems.

Our fairly general setup is explained in the next section. The local existence of solutions and the existence and properties of invariant manifolds for (1.1) have been studied in [9, 10], where we have also discussed related literature. To make the present paper readable independently of [9, 10], we quote several results from these papers in the next section. In particular, Theorem 2.7 taken from [10] says that the center manifold is locally exponentially attractive with a tracking solution if there is no unstable spectrum and the flow on the center manifold is stable. Moreover, here and in our theorems in Section 3 we assume that the center (or the center–unstable) manifold is finite dimensional. Versions of Theorem 2.7 have been shown for simpler boundary conditions in more abstract settings, see [11, §9.3] and also [4, 12, 16, 17, 18, 19]. We note that in [12] a version of Palmer's lemma for elliptic problems on infinite cylinders has been proved.

In Section 3, we establish the analogues for (1.1) of Palmer's lemma in Lemmas 3.1 and 3.4. The Pliss reduction principle is proved in Theorem 3.6 in the setting of Theorem 2.7. Moreover, in Theorem 3.2 we show that any solution of (1.1) on $\mathbb{R}_+$ that stays in a small ball centered at the equilibrium u_* already converges exponentially to a solution on the center–unstable manifold. Similarly, if the flow is stable on the center manifold, then any solution starting near the equilibrium either leaves a certain neighborhood or it belongs to the center–stable manifold and converges exponentially to a solution on the center manifold, see Theorem 3.5. In these two results, we allow for unstable spectrum in contrast to Theorem 2.7 of [10]. A stronger version of Theorem 3.5 was shown in the recent paper [16] in an abstract setting, but only for problems with linear boundary conditions and assuming that the center manifold consists of equilibria only.

Notation. We set $D_k = -i\partial_k = -i\partial/\partial x_k$ and use the multi index notation. The k–tensor of the partial derivatives of order k is denoted by ∇^k, and we let $\underline{\nabla}^k u = (u, \nabla u, \cdots, \nabla^k u)$. For an operator A on a Banach space we write $\operatorname{dom}(A)$, $\ker(A)$, ran(A), $\sigma(A)$, and $\rho(A)$ for its domain, kernel, range, spectrum, and resolvent set, respectively. $\mathcal{B}(X,Y)$ is the space of bounded linear operators between two Banach spaces X and Y, and $\mathcal{B}(X) := \mathcal{B}(X,X)$. A ball in X with the radius r and center at u will be denoted by $B_X(u,r)$. For an open set $U \subset \mathbb{R}^n$ with (sufficiently regular) boundary ∂U or for a Banach space U, $C^k(U)$ are the spaces of k–times continuously differentiable functions on U. We write $BC^k(\overline{U})$ for the space of $u \in C^k(U)$ such that u and its derivatives up to order k are bounded and have continuous extensions to ∂U. This space is endowed with the supnorm. For unbounded U, $C_0^k(\overline{U})$ consists of $u \in BC^k(\overline{U})$ such that u and its derivatives up to order k vanish at infinity. Similar spaces are used on ∂U. By $W_p^k(U)$ we denote the Sobolev spaces, see e.g. [1, Def.3.1], and by $W_p^s(U)$ the Slobodetskii spaces, see [1, Thm.7.48] or [20, Rem.4.4.1.2]. Finally, $J \subset \mathbb{R}$ is a closed interval with nonempty interior, c is a generic constant, and $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ is a generic nondecreasing function with $\varepsilon(r) \to 0$ as $r \to 0$.

2. Setting and preliminaries

In this section we recall the setting and results from [9, 10] needed in the sequel. Let $\Omega \subset \mathbb{R}^n$ be an open connected set with a compact boundary $\partial\Omega$ of class C^{2m} and outer unit normal $\nu(x)$, where $m \in \mathbb{N}$ is given by (2.4) below. Throughout this paper, we fix a finite exponent p with

$$p > n + 2m. \tag{2.1}$$

Let $E = \mathbb{C}^N$ with $\mathcal{B}(E) = \mathbb{C}^{N\times N}$ for some fixed $N \in \mathbb{N}$. We put

$$X_0 = L_p(\Omega;\mathbb{C}^N), \quad X_1 = W_p^{2m}(\Omega;\mathbb{C}^N), \quad X_{1-1/p} = W_p^{2m(1-1/p)}(\Omega;\mathbb{C}^N),$$

and denote the norms of these spaces by $|\cdot|_0$, $|\cdot|_1$, and $|\cdot|_{1-1/p}$, respectively. (We warn the reader that in [9, 10] the latter norm was denoted by $|\cdot|_p$.) We set

$$Y_0 = L_p(\partial\Omega;\mathbb{C}^N), \quad Y_{j1} = W_p^{2m\kappa_j}(\partial\Omega;\mathbb{C}^N), \quad Y_{j,1-1/p} = W_p^{2m\kappa_j - 2m/p}(\partial\Omega;\mathbb{C}^N),$$
$$Y_1 = Y_{11} \times \cdots \times Y_{m1}, \quad Y_{1-1/p} = Y_{1,1-1/p} \times \cdots \times Y_{m,1-1/p}$$

for $j \in \{1,\cdots,m\}$, $m_j \in \{0,\cdots,2m-1\}$ given by (2.4), and the numbers

$$\kappa_j = 1 - \frac{m_j}{2m} - \frac{1}{2mp}\,. \tag{2.2}$$

Here the Sobolev–Slobodetskii spaces on $\partial\Omega$ are defined via local charts, see Theorem 7.53 in [1] or Definition 3.6.1 in [20]. We observe that

$$X_1 \hookrightarrow X_{1-1/p} \hookrightarrow X_0, \quad Y_{j1} \hookrightarrow Y_{j,1-1/p} \hookrightarrow Y_0,$$

and also that

$$X_{1-1/p} \hookrightarrow C_0^{2m-1}(\overline{\Omega};\mathbb{C}^N), \quad \text{and} \quad Y_{j,1-1/p} \hookrightarrow C^{2m-1-m_j}(\partial\Omega;\mathbb{C}^N) \tag{2.3}$$

by (2.1), (2.2), and standard properties of Sobolev spaces, cf. [20, §4.6.1]. Our basic equations (1.1) involve the operators given by

$$\begin{aligned}
[A(u)v](x) &= \sum_{|\alpha|=2m} a_\alpha(x,u(x),\nabla u(x),\cdots,\nabla^{2m-1}u(x))\, D^\alpha v(x), \quad x \in \Omega,\\
[F(u)](x) &= f(x,u(x),\nabla u(x),\cdots,\nabla^{2m-1}u(x))), \quad x \in \Omega,\\
[B_j(u)](x) &= b_j(x,(\gamma u)(x),(\gamma\nabla u)(x),\cdots,(\gamma\nabla^{m_j}u)(x)), \quad x \in \partial\Omega,
\end{aligned} \tag{2.4}$$

for $j \in \{1, \cdots, m\}$ and functions $u \in X_{1-1/p}$ and $v \in X_1$, where γ is the spatial trace operator and the integers $m \in \mathbb{N}$ and $m_j \in \{0, \cdots, 2m-1\}$ are fixed. In view of (2.3), only continuous functions will be inserted into the nonlinearites. Thus we will omit γ in $B_j(u)$ and in similar expressions. We set $B = (B_1, \cdots, B_m)$. We assume throughout that the coefficients in (2.4) satisfy

(R) $a_\alpha \in C^1(E \times E^n \times \cdots \times E^{(n^{2m-1})}; BC(\overline{\Omega}; \mathcal{B}(E)))$ for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 2m$,
$a_\alpha(x, 0) \longrightarrow a_\alpha(\infty)$ in $\mathcal{B}(E)$ as $x \to \infty$, if Ω is unbounded,
$f \in C^1(E \times E^n \times \cdots \times E^{(n^{2m-1})}; BC(\overline{\Omega}; E))$,
$b_j \in C^{2m+1-m_j}(\partial\Omega \times E \times E^n \times \cdots \times E^{(n^{m_j})}; E)$ for $j \in \{1, \cdots, m\}$.

We will need one more degree of smoothness of the coefficients as recorded in the following hypothesis:

(RR) $a_\alpha \in C^2(E \times E^n \times \cdots \times E^{(n^{2m-1})}; BC(\overline{\Omega}; \mathcal{B}(E)))$ for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 2m$,
$f \in C^2(E \times E^n \times \cdots \times E^{(n^{2m-1})}; BC(\overline{\Omega}; E))$,
$b_j \in C^{2m+2-m_j}(\partial\Omega \times E \times E^n \times \cdots \times E^{(n^{m_j})}; E)$ for $j \in \{1, \cdots, m\}$.

For each $k \in \mathbb{N}_0$, we fix an order of the multi indices $\beta \in \mathbb{N}_0^n$ with $|\beta| = k$. We order the n^k components of a k–tensor in the same way, thus using β as the label for the component corresponding to $\beta \in \mathbb{N}_0^n$ with $|\beta| = k$. For a function w depending on $z \in E^{(n^k)}$, we denote by $\partial_\beta w$ its partial derivative with respect to β-th argument. It is not difficult to see that

$$A \in C^1(X_{1-1/p}; \mathcal{B}(X_1, X_0)) \quad \text{and} \quad F \in C^1(X_{1-1/p}; X_0) \tag{2.5}$$

with the locally bounded derivatives

$$[F'(u)v](x) = \sum_{k=0}^{2m-1} \sum_{|\beta|=k} i^k \, (\partial_\beta f)(x, u(x), \nabla u(x), \cdots, \nabla^{2m-1} u(x)) \, D^\beta v(x),$$

$$\begin{aligned}[A'(u)w]v(x) &= A'(u)[v, w](x) \\ &= \sum_{|\alpha|=2m} \sum_{k=0}^{2m-1} \sum_{|\beta|=k} (\partial_\beta a_\alpha)(x, u(x), \cdots, \nabla^{2m-1} u(x)) \, [\partial^\beta v(x), D^\alpha w(x)]\end{aligned}$$

for $x \in \Omega$, $u, v \in X_{1-1/p}$, and $w \in X_1$, see formula (25) of [9] and the text before it. (Observe that $(\partial_\beta a_\alpha)(x, z) : E^2 \to E$ is bilinear.) We further have

$$B_j \in C^1(X_{1-1/p}; Y_{j,1-1/p}) \cap C^1(X_1; Y_{j1}), \qquad j \in \{1, \cdots, m\}, \tag{2.6}$$

with the locally bounded derivatives

$$[B_j'(u)v](x) = \sum_{k=0}^{m_j} \sum_{|\beta|=k} i^k \, (\partial_\beta b_j)(x, u(x), \nabla u(x), \cdots, \nabla^{m_j} u(x)) \, D^\beta v(x),$$

where $x \in \partial\Omega$ and $u, v \in X_{1-1/p}$, resp. $u, v \in X_1$. The continuous differentiability of $B_j : X_{1-1/p} \to Y_{j,1-1/p}$ was shown in Corollary 12 of [9], and $B_j \in C^1(X_1; Y_{1,1-1/p})$ can be proved by the arguments used in step (4) and (5) of the proof of Proposition 10 of [9], see in particular inequality (69) in [9]. We set $B'(u) = (B_1'(u), \cdots, B_m'(u))$.

The symbols of the principal parts of the linear differential operators are the matrix–valued functions given by

$$\mathcal{A}_\#(x, z, \xi) = \sum_{|\alpha|=2m} a_\alpha(x, z) \, \xi^\alpha, \quad \mathcal{B}_{j\#}(x, z, \xi) = \sum_{|\beta|=m_j} i^{m_j} \, (\partial_\beta b_j)(x, z) \, \xi^\beta$$

for $x \in \overline{\Omega}$, $z \in E \times \cdots \times E^{(n^{2m-1})}$ and $\xi \in \mathbb{R}^n$, resp. $x \in \partial\Omega$, $z \in E \times \cdots \times E^{(n^{m_j})}$ and $\xi \in \mathbb{R}^n$. We further set $\mathcal{A}_\#(\infty, \xi) = \sum_{|\alpha|=2m} a_\alpha(\infty)\, \xi^\alpha$ if Ω is unbounded. We introduce the *normal ellipticity* and the *Lopatinskii–Shapiro condition* for $A(u_0)$ and $B'(u_0)$ at a function $u_0 \in X_{1-1/p}$ as follows:

(E) $\sigma(\mathcal{A}_\#(x, \underline{\nabla}^{2m-1} u_0(x), \xi)) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\} =: \mathbb{C}_+$ and (if Ω is unbounded) $\sigma(\mathcal{A}_\#(\infty, \xi)) \subset \mathbb{C}_+$, for $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n$ with $|\xi| = 1$.

(LS) Let $x \in \partial\Omega$, $\xi \in \mathbb{R}^n$, and $\lambda \in \overline{\mathbb{C}_+}$ with $\xi \perp \nu(x)$ and $(\lambda, \xi) \neq (0, 0)$. The function $\varphi = 0$ is the only solution in $C_0(\mathbb{R}_+; \mathbb{C}^N)$ of the ODE system

$$\lambda\varphi(y) + \mathcal{A}_\#(x, \underline{\nabla}^{2m-1} u_0(x), \xi + i\nu(x)\partial_y)\varphi(y) = 0, \quad y > 0,$$
$$\mathcal{B}_{j\#}(x, \underline{\nabla}^{m_j} u_0(x), \xi + i\nu(x)\partial_y)\varphi(0) = 0, \quad j \in \{1, \cdots, m\}.$$

We refer to [3], [5], [6], and the references therein for more information concerning these conditions. We can now state our basic hypothesis.

Hypothesis 2.1. *Condition (R) holds, and (E), (LS) hold at a steady state $u_* \in X_1$ of (1.1), i.e., $A(u_*)u_* = F(u_*)$ on Ω, $B(u_*) = 0$ on $\partial\Omega$.*

For the investigation of (1.1), we need several spaces of functions on $J \times \Omega$ and $J \times \partial\Omega$, where $J \subset \mathbb{R}$ is a closed interval with a nonempty interior. The base space and solution space of (1.1) are

$$\mathbb{E}_0(J) = L_p(J; L_p(\Omega; \mathbb{C}^N)) = L_p(J; X_0),$$
$$\mathbb{E}_1(J) = W_p^1(J; L_p(\Omega; \mathbb{C}^N)) \cap L_p(J; W_p^{2m}(\Omega; \mathbb{C}^N)) = W_p^1(J; X_0) \cap L_p(J; X_1),$$

respectively, equipped with the natural norms. We need the crucial embeddings

$$\mathbb{E}_1(J) \hookrightarrow BC(J; X_{1-1/p}) \hookrightarrow BC(J; C_0^{2m-1}(\overline{\Omega}; \mathbb{C}^N)), \tag{2.7}$$

see Theorem III.4.10.2 of [2] for the first and (2.3) for the second embedding. We note that the norm of the first embedding is uniform for intervals J of length greater than a fixed $\ell > 0$. Observe that (2.7) implies that the trace operator γ_0 at time $t = 0$ is continuous from $\mathbb{E}_1(J)$ to $X_{1-1/p}$ if $0 \in J$. The boundary data of our linearized equations will be contained in the spaces

$$\mathbb{F}_j(J) = W_p^{\kappa_j}(J; L_p(\partial\Omega; \mathbb{C}^N)) \cap L_p(J; W_p^{2m\kappa_j}(\partial\Omega; \mathbb{C}^N))$$
$$= W_p^{\kappa_j}(J; Y_0) \cap L_p(J; Y_{j1}), \qquad j \in \{1, \cdots, m\},$$

endowed with their natural norms, where $\mathbb{F}(J) := \mathbb{F}_1(J) \times \cdots \times \mathbb{F}_m(J)$. We have

$$\mathbb{F}_j(J) \hookrightarrow BC(J; Y_{j,1-1/p}) \hookrightarrow BC(J \times \partial\Omega) \quad \text{and} \quad \gamma_0 \in \mathcal{B}(\mathbb{F}_j(J), Y_{j,1-1/p}) \tag{2.8}$$

if $0 \in J$, see [6, §3].

For $\alpha, \beta \in \mathbb{R}$, we set $e_\alpha(t) = e^{\alpha t}$ for $t \in \mathbb{R}$ and define the function $e_{\alpha,\beta}$ by setting $e_{\alpha,\beta}(t) = e_\alpha(t)$ for $t \leq 0$ and $e_{\alpha,\beta}(t) = e_\beta(t)$ for $t \geq 0$. Then we introduce the weighted spaces

$$\mathbb{E}_k(\mathbb{R}_\pm, \alpha) = \{v : e_\alpha v \in \mathbb{E}_k(\mathbb{R}_\pm)\}, \qquad \mathbb{F}(\mathbb{R}_\pm, \alpha) = \{v : e_\alpha v \in \mathbb{F}(\mathbb{R}_\pm)\},$$
$$\mathbb{E}_k(\alpha, \beta) = \{v : e_{\alpha,\beta} v \in \mathbb{E}_k(\mathbb{R})\}, \qquad \mathbb{F}(\alpha, \beta) = \{v : e_{\alpha,\beta} v \in \mathbb{F}(\mathbb{R})\}, \tag{2.9}$$

where $k = 0, 1$, endowed with the canonical norms $\|v\|_{\mathbb{E}_0(\mathbb{R}_+,\alpha)} = \|e_\alpha v\|_{\mathbb{E}_0(\mathbb{R}_+)}$ etc. We also use the analogous norms on compact intervals J.

We assume that Hypothesis 2.1 holds. Due to (2.5) and (2.6), we can linearize the problem (1.1) at the steady state $u_* \in X_1$ obtaining the operators defined by

$$A_* = A(u_*) + A'(u_*)u_* - F'(u_*) \in \mathcal{B}(X_1, X_0),$$

$$B_{j*} = B_j'(u_*) \in \mathcal{B}(X_{1-1/p}, Y_{j,1-1/p}) \cap \mathcal{B}(X_1, Y_{j1}).$$

We set $B_* = (B_{1*}, \cdots, B_{m*})$. We further define the nonlinear maps

$$\begin{aligned} &G \in C^1(X_1; X_0) \quad \text{and} \quad H_j \in C^1(X_{1-1/p}; Y_{j,1-1/p}) \cap C^1(X_1; Y_{j1}) \\ &\text{with } G(0) = H_j(0) = 0 \quad \text{and} \quad G'(0) = H_j'(0) = 0 \end{aligned} \tag{2.10}$$

for $j \in \{1, \cdots, m\}$ by setting

$$\begin{aligned} G(v) &= \big(A(u_*)v - A(u_* + v)v\big) - \big(A(u_* + v)u_* - A(u_*)u_* - [A'(u_*)u_*]v\big) \\ &\quad + \big(F(u_* + v) - F(u_*) - F'(u_*)v\big), \\ H_j(v) &= B_j'(u_*)v - B_j(u_* + v), \end{aligned}$$

for $v \in X_1$, resp. $v \in X_{1-1/p}$. Again, we put $H(v) = (H_1(v), \cdots, H_m(v))$. The corresponding Nemytskii operators are denoted by

$$\mathbb{G}(v)(t) = G(v(t)), \qquad \mathbb{H}_j(v)(t) = H_j(v(t)), \qquad \mathbb{H}(v)(t) = H(v(t))$$

for $v \in \mathbb{E}_1^{\text{loc}}(J)$ (which is the space of $v : J \to X_0$ such that $v \in \mathbb{E}_1([a, b])$ for all intervals $[a, b] \subset J$).

Theorem 14 of [9] shows that (1.1) generates a local semiflow on the solution manifold

$$\mathcal{M} = \{u_0 \in X_{1-1/p} : B(u_0) = 0\}.$$

In particular, a function u_0 is the initial value of the (unique) solution $u \in \mathbb{E}_1([0, T])$ of (1.1) for some $T > 0$ if and only if $u_0 \in \mathcal{M}$. Setting $v = u - u_*$ and $v_0 = u_0 - u_*$, we further see that $u_0 \in \mathcal{M}$ if and only if $v_0 \in X_{1-1/p}$ and $B_* v_0 = H(v_0)$ and that $u \in \mathbb{E}_1([0, T])$ solves (1.1) if and only if $v \in \mathbb{E}_1([0, T])$ satisfies

$$\begin{aligned} \partial_t v(t) + A_* v(t) &= G(v(t)) \qquad \text{on } \Omega, \quad \text{a.e. } t > 0, \\ B_{j*} v(t) &= H_j(v(t)) \qquad \text{on } \partial\Omega, \quad t \geq 0, \quad j \in \{1, \cdots, m\}, \\ v(0) &= v_0, \qquad \text{on } \Omega. \end{aligned} \tag{2.11}$$

Remark 2.2. Let Hypothesis 2.1 hold. Theorem 14(a) of [9] and (2.7) then imply the following facts: For each given $T > 0$, there is a radius $\rho = \rho(T) > 0$ such that for every $u_0 = u_* + v_0 \in \mathcal{M}$ with $|v_0|_{1-1/p} \leq \rho$ there exists a unique solution $u = u_* + v$ of (1.1) on $[0, T]$, and $|w(t)|_{1-1/p} \leq c\|v\|_{\mathbb{E}_1([0,T])} \leq c|v_0|_{1-1/p}$ for all $t \in [0, T]$ with constants $c = c(T)$ independent of u_0 in this ball. Moreover, Proposition 15 of [9] and the Sobolev embedding imply that

$$|v(t)|_1 \leq c|v_0|_{1-1/p} \qquad \text{for all } \ t \in [T_0, T], \qquad \Diamond$$

where $T > T_0 > 0$ and the constant may depend on T_0.

We now recall some results from [9] regarding the solvability of the inhomogeneous linear problem

$$\begin{aligned} \partial_t v(t) + A_* v(t) &= g(t) \qquad \text{on } \Omega, \quad \text{a.e. } t \in J, \\ B_* v(t) &= h(t) \qquad \text{on } \partial\Omega, \quad t \in J, \\ v(0) &= v_0, \qquad \text{on } \Omega, \end{aligned} \tag{2.12}$$

in weighted function spaces on the unbounded interval $J \in \{\mathbb{R}_+, \mathbb{R}_-, \mathbb{R}\}$. We assume that Hypothesis 2.1 holds. (Actually, when dealing only with (2.12) we do not have to assume that $u_* \in X_1$ is a steady state of (1.1).) We recall from Theorem 2.1 of [6] that on a bounded interval $J = [a, b]$ the boundary value problem obtained by combining the first two lines of (2.12) with the initial condition $v(a) = v_0$ has a unique solution $v \in \mathbb{E}_1([a,b])$ if and only if $g \in \mathbb{E}_0([a,b])$, $h \in \mathbb{F}([a,b])$, $v_0 \in X_{1-1/p}$, and $B_* v_0 = h(a)$. A solution $v \in \mathbb{E}_1^{\rm loc}(J)$ of (2.12) on J will be denoted by $v = S(v_0, g, h)$, where $J \subset \mathbb{R}$ is any closed interval containing 0. We stress that this notation incorporates the compatibility condition $B_* v_0 = h(0)$ because of the second line in (2.12) and the embeddings (2.7) and (2.8). Moreover, the solution $S(v_0, g, h)$ is unique if $J = \mathbb{R}_+$, but uniqueness may fail on $J = \mathbb{R}_-$.

We define the operator $A_0 = A_*|_{\ker B_*}$ with the domain

$$\mathrm{dom}(A_0) = \{u \in X_1 : B_{j*} u = 0, j = 1, \dots, m\}.$$

It is known that $-A_0$ generates an analytic semigroup on X_0 which we denote by $T(\cdot)$. We need the extrapolation space X_{-1} of A_0 defined as the completion of X_0 with respect to the norm $|u_0|_{-1} = |(\mu + A_0)^{-1} u_0|_0$ for some fixed $\mu \in \rho(-A_0)$. There exists an extension A_{-1} of A_0 to X_{-1} which generates the analytic semigroup $T_{-1}(\cdot)$ extending $T(\cdot)$ to X_{-1}. We further employ the map

$$\Pi = (\mu + A_{-1}) \mathcal{N}_1 \in \mathcal{B}(Y_1, X_1) \tag{2.13}$$

where $\mathcal{N}_1 \in \mathcal{B}(Y_1, X_1)$ is the solution operator, $\mathcal{N}_1 : \varphi \mapsto u$, of the elliptic boundary value problem $(\mu + A_*)u = 0$ on Ω, $B_* u = \varphi$ on $\partial\Omega$, see Proposition 5 of [9]. This proposition also gives a right inverse

$$\mathcal{N}_{1-1/p} \in \mathcal{B}(Y_{1-1/p}, X_{1-1/p}) \tag{2.14}$$

of B_*. Due to Proposition 6 of [9], the solution $v \in \mathbb{E}_1^{\rm loc}(J)$ of the problem (2.12) is given by the variation of constants formula

$$v(t) = T(t-\tau)v(\tau) + \int_\tau^t [T(t-s)g(s) + T_{-1}(t-s)\Pi h(s)]\, ds \tag{2.15}$$

for all $t \geq \tau$ in J.

In order to treat solutions of (2.11) or (2.12) on the intervals $J = \mathbb{R}_\pm$, we assume that the (rescaled) semigroup $\left\{e^{\delta t} T(t)\right\}_{t \geq 0}$ is *hyperbolic* for $\delta \in [\delta_1, \delta_2]$ for some segment $[\delta_1, \delta_2] \subset \mathbb{R}$ (i.e., $\sigma(-A_0 + \delta) \cap i\mathbb{R} = \emptyset$). Let P be the (stable) spectral projection for $-A_0 + \delta$ corresponding to the part of $\sigma(-A_0 + \delta)$ in the open left halfplane, and set $Q = I - P$. Then $T(t)$ is invertible on QX_0 with the inverse $T_Q(-t)Q$, and $\|e^{t\delta} T(t) P\|$, $\|e^{-t\delta} T_Q(-t) Q\| \leq c e^{-\epsilon t}$ for $t \geq 0$ and some $\epsilon > 0$. If $e_\delta T(\cdot)$ is hyperbolic on X_0, then $e_\delta T_{-1}(\cdot)$ is hyperbolic on X_{-1} with projections P_{-1} and $Q_{-1} = I - P_{-1}$ being the extensions of P and Q, respectively. Moreover, Q_{-1} maps X_{-1} into $\mathrm{dom}(A_0)$, and P leaves invariant $X_{1-1/p}$, X_1, and $\mathrm{dom}(A_0)$. The projections commute with the semigroup and its generator as well as with their extrapolations. (See [9, §2] for these facts and related references.)

When needed, we assume that $T(\cdot)$ has an *exponential trichotomy*, i.e., there is a splitting

$$\begin{aligned} &\sigma(-A_0) = \sigma_s \cup \sigma_c \cup \sigma_u \qquad \text{with} \\ &\max \mathrm{Re}\, \sigma_s < -\omega_s < -\underline{\omega}_c < \min \mathrm{Re}\, \sigma_c \leq 0 \leq \max \mathrm{Re}\, \sigma_c < \overline{\omega}_c < \omega_u < \min \mathrm{Re}\, \sigma_u. \end{aligned} \tag{2.16}$$

(If Ω is bounded, $\sigma(-A_0)$ is discrete and thus (2.16) automatically holds with $\sigma_u \subset i\mathbb{R}$ and arbitrarily small $\underline{\omega}_c = \overline{\omega}_c$.) We take numbers $\alpha \in [\underline{\omega}_c, \omega_s]$ and $\beta \in [\overline{\omega}_c, \omega_u]$ and denote by P_k the spectral projections for $-A_0$ corresponding to σ_k, $k = s, c, u$. We set $P_{\rm cs} = P_{\rm s} + P_{\rm c}$, $P_{\rm cu} = P_{\rm c} + P_{\rm u}$, and $P_{\rm su} = P_{\rm s} + P_{\rm u}$. Then the rescaled semigroups $e_\alpha T(\cdot)$ and $e_{-\beta} T(\cdot)$ are hyperbolic on X_0 with stable projections $P_{\rm s}$ and $P_{\rm cs}$, respectively. The restriction of $T(t)$ to $P_k X_0$ yields a group denoted by $T_k(t)$, $t \in \mathbb{R}$, where $k = c, u, cu$.

When needed, we impose the following assumption which is weaker than (2.16): There exist positive numbers $\omega_s, \omega_u, \omega_{cu}, \omega_{cs} > 0$ such that at least one of the following assertions holds:

$$\sigma(-A_0) = \sigma_s \cup \sigma_{cu} \quad \text{with} \quad \max \operatorname{Re} \sigma_s < -\omega_s < -\omega_{cu} < \min \operatorname{Re} \sigma_{cu}\,, \tag{2.17}$$

$$\sigma(-A_0) = \sigma_{cs} \cup \sigma_u \quad \text{with} \quad \max \operatorname{Re} \sigma_{cs} < \omega_{cs} < \omega_u < \min \operatorname{Re} \sigma_u\,. \tag{2.18}$$

We continue to use notation P_k for the spectral projections for $-A_0$ corresponding to the sets σ_k, $k \in \{s, cs, cu, u\}$. Using standard facts, we see that $P_{\rm u} X_0 \subset P_{\rm cu} X_0 \subset \operatorname{dom}(A_0)$ and that on $P_{\rm cu} X_0$ the norms in X_0, $X_{1-1/p}$ and X_1 are equivalent. Finally, we recall the notation $X^0_{1-1/p} = \{z_0 \in X_{1-1/p} : B_* z_0 = 0\}$ for the tangent space at u_* to the nonlinear phase space $\mathcal{M} = \{u_0 \in X_{1-1/p} : B(u_0) = 0\}$ for (1.1), and that $\mathcal{P} = I - \mathcal{N}_{1-1/p} B_*$ projects $X_{1-1/p}$ onto $X^0_{1-1/p}$, see (2.14) and remarks preceding Theorem 14 in [9]. In the theorems stated below, the invariant manifolds are graphs over the corresponding spectral subspaces of $X^0_{1-1/p}$, where it is important to note that $P_{\rm cu} X_0 \subset \operatorname{dom}(A_0) \subset X^0_{1-1/p}$.

We now recall the main result Theorem 4.2 of the paper [10] where one constructs a *local center manifold* $\mathcal{M}_{\rm c}$ and shows some of its basic properties. In particular, $\mathcal{M}_{\rm c}$ is a C^1–manifold in $X_{1-1/p}$ being tangent to $P_{\rm c} X_0$ at u_*. It is given by functions $u_* + v(0)$ for initial values $v(0)$ of solutions $v \in \mathbb{E}_1(\alpha, -\beta)$ to a modified version of (2.11) on $J = \mathbb{R}$ which involves a nonlocal cutoff. This fact is not needed below, but it is described in detail in [10].

We assume that the spectrum of $-A_0$ has the trichotomy decomposition described in (2.16), and recall that this assumption automatically holds if the spatial domain Ω is bounded.

Theorem 2.3. *Assume that Hypothesis 2.1 and* (2.16) *hold. Take any* $\alpha \in (\underline{\omega}_c, \omega_s)$ *and* $\beta \in (\overline{\omega}_c, \omega_u)$. *There exists a radius* $\rho > 0$ *and maps* $\phi_c \in C^1(P_{\rm c} X_0; P_{\rm su} X_{1-1/p})$ *and* $\Phi_c \in C^1(P_{\rm c} X_0; \mathbb{E}_1(\alpha, -\beta))$ *with bounded derivatives such that the following assertions hold.*

(a) We have $\phi_c(0) = 0$ *and* $\phi_c'(0) = 0$. *The local center manifold is given by*

$$\begin{aligned} \mathcal{M}_c &:= \big\{u_0 = u_* + z_0 + \phi_c(z_0) : z_0 \in P_{\rm c} X_0\big\} \cap B_{X_{1-1/p}}(u_*, \rho) \\ &= \big\{u_0 = u_* + v(0) : v = \Phi_c(P_c(u_0 - u_*))\big\} \cap B_{X_{1-1/p}}(u_*, \rho). \end{aligned} \tag{2.19}$$

If $u_0 \in \mathcal{M}_c$, *then the function* v *from* (2.19) *is given by* $v = P_{\rm c} v + \phi_c(P_{\rm c} v)$. *It further solves the equation* (2.11) *(at least) for* $t \in [-3, 3]$, *so that* $\mathcal{M}_{\rm c} \subset \mathcal{M}$. *The dimension of* $\mathcal{M}_{\rm c}$ *is equal to* $\dim P_{\rm c} X_0$.

(b) Let $u_0 \in \mathcal{M}_{\rm c}$ *and* v *be given by* (2.19). *If the forward solution* u *of* (1.1) *exists and stays in* $B_{X_p}(u_*, \rho)$ *on* $[0, t_1]$ *for some* $t_1 > 0$, *then* $u(t) = v(t) + u_* \in \mathcal{M}_c$ *for* $0 \le t \le t_1$. *If the function* $\hat{u} = v + u_*$ *stays in* $B_{X_p}(u_*, \rho)$ *on* $[t_0, 0]$ *for some* $t_0 < 0$, *then* $\hat{u}(t) \in \mathcal{M}_{\rm c}$ *and* $\hat{u}$ *solves* (1.1) *for* $t_0 \le t \le 0$.

(c) Assume that $v(t) + u_* \in \mathcal{M}_{\rm c}$ *for all* $t \in (a, b)$ *and some* $a < 0 < b$. *The then function* $y = P_{\rm c} v$ *satisfies the equations*

$$\begin{aligned} &\dot{y}(t) = -A_0 P_{\rm c} y(t) + P_{\rm c} \Pi H(y(t) + \phi_c(y(t))) + P_{\rm c} G(y(t) + \phi_c(y(t))), \\ &y(0) = P_{\rm c}(u_0 - u_*), \end{aligned} \tag{2.20}$$

on $P_{\mathrm{c}}X_0$ *for* $t \in (a,b)$. *Moreover,* $v \in C((a,b);X_1)$ *and*

$$B_*\phi_c(P_{\mathrm{c}}v_0) = B_*v_0 = H(v_0), \tag{2.21}$$

$$P_{\mathrm{su}}(A_*v_0 - G(v_0)) = \phi_c'(P_{\mathrm{c}}v_0)P_{\mathrm{c}}(A_*v_0 - G(v_0)). \tag{2.22}$$

(d) If $u \in \mathbb{E}_1^{\mathrm{loc}}(\mathbb{R})$ *solves* (1.1) *on* $\mathbb{R}$ *with* $|u(t) - u_*|_{1-1/p} < \rho$ *for all* $t \in \mathbb{R}$, *then* $u(t) \in \mathcal{M}_{\mathrm{c}}$ *for all* $t \in \mathbb{R}$.

(e) In addition, assume that (RR) holds. Then there is a $\rho_0 > 0$ *such that the map* $\phi_c : P_{\mathrm{c}}X_0 \cap B_{X_{1-1/p}}(0,\rho_0) \to P_{\mathrm{su}}X_1$ *is Lipschitz.*

In addition to Theorem 2.3, one constructs a *local center–stable manifold* $\mathcal{M}_{\mathrm{cs}}$ assuming (2.18) (see Theorem 5.1 of [10]), and a *local center–unstable manifold* $\mathcal{M}_{\mathrm{cu}}$ assuming (2.17) (see Theorem 5.2 of [10]). These manifolds are of class C^1 in $X_{1-1/p}$, and are tangent to $P_{\mathrm{cs}}X^0_{1-1/p}$, resp. to $P_{\mathrm{cu}}X_0$, at u_*. They are described in the following two theorems. Recall that $\mathcal{P} = I - \mathcal{N}_{1-1/p}B_*$. In Theorem 2.4 the map ϑ_{cs} takes care of the compatibility conditions.

Theorem 2.4. *Assume Hypothesis 2.1 and* (2.18). *Take any* $\beta \in (\omega_{cs},\omega_u)$. *Then there exists a radius* $\rho > 0$ *and maps* $\phi_{cs} \in C^1(P_{\mathrm{cs}}X^0_{1-1/p};P_{\mathrm{u}}X_0)$, $\vartheta_{cs} \in C^1(P_{\mathrm{cs}}X^0_{1-1/p};P_{\mathrm{cs}}X_{1-1/p})$ *and* $\Phi_{cs} \in C^1(P_{\mathrm{cs}}X^0_{1-1/p};\mathbb{E}_1(\mathbb{R}_+,-\beta))$ *having bounded derivatives such that the following assertions hold.*

(a) We have $\phi_{cs}(0) = \vartheta_{cs}(0) = 0$ *and* $\phi'_{cs}(0) = \vartheta'_{cs}(0) = 0$. *The local center–stable manifold is given by*

$$\begin{aligned}\mathcal{M}_{cs} &:= \big\{u_0 = u_* + z_0 + \vartheta_{cs}(z_0) + \phi_{cs}(z_0) : z_0 \in P_{\mathrm{cs}}X^0_{1-1/p}\big\} \cap B_{X_{1-1/p}}(u_*,\rho)\\ &= \big\{u_0 = u_* + v(0) : v = \Phi_{cs}(P_{\mathrm{cs}}\mathcal{P}(u_0 - u_*))\big\} \cap B_{X_{1-1/p}}(u_*,\rho).\end{aligned} \tag{2.23}$$

If $u_0 \in \mathcal{M}_{cs}$, *then the function* v *from* (2.23) *solves the equation* (2.11) *(at least) for* $t \in [0,4]$. *Thus,* $\mathcal{M}_{\mathrm{cs}} \subset \mathcal{M}$.

(b) Let $u_0 \in \mathcal{M}_{cs}$ *and* v *be given by* (2.23). *Assume that a forward or a backward solution* u *of* (1.1) *exists and stays in* $B_{X_p}(u_*,\rho)$ *on* $[0,t_0]$ *or on* $[-t_0,0]$ *for some* $t_0 > 0$. *Set* $v(t) = u(t) - u_*$ *for* $-t_0 \le t \le 0$ *in the second case. We then have*[1]

$$u(t) = u_* + v(t) = u_* + P_{\mathrm{cs}}\mathcal{P}v(t) + \phi_{cs}(P_{\mathrm{cs}}\mathcal{P}v(t)) + \vartheta_{cs}(P_{\mathrm{cs}}\mathcal{P}v(t)) \in \mathcal{M}_{\mathrm{cs}} \tag{2.24}$$

for $0 \le t \le t_0$ *or* $-t_0 \le t \le 0$, *respectively.*

(c) It holds $\mathcal{M}_{\mathrm{cs}} \cap \mathcal{M}_{\mathrm{u}} = \{u_*\}$.

Theorem 2.5. *Assume Hypothesis 2.1 and* (2.17). *Let* $\alpha \in (\omega_{cu},\omega_s)$. *Then there exists a radius* $\rho > 0$ *and maps* $\phi_{cu} \in C^1(P_{\mathrm{cu}}X_0;P_{\mathrm{s}}X_{1-1/p})$ *and* $\Phi_{cu} \in C^1(P_{\mathrm{cu}}X_0;\mathbb{E}_1(\mathbb{R}_-,\alpha))$ *with bounded derivatives such that the following assertions hold.*

(a) We have $\phi_{cu}(0)=0$ *and* $\phi'_{cu}(0)=0$. *The center-unstable manifold is given by*

$$\begin{aligned}\mathcal{M}_{cu} &:= \big\{u_0 = u_* + z_0 + \phi_{cu}(z_0) : z_0 \in P_{\mathrm{cu}}X_0\big\} \cap B_{X_{1-1/p}}(u_*,\rho)\\ &= \{u_0 = u_* + v(0) : v = \Phi_{cu}(P_{\mathrm{cu}}(u_0 - u_*))\} \cap B_{X_{1-1/p}}(u_*,\rho).\end{aligned} \tag{2.25}$$

If $u_0 \in \mathcal{M}_{\mathrm{cu}}$, *then the function* v *from* (2.25) *solves the equation* (2.11) *(at least) for* $t \in [-4,0]$. *Thus,* $\mathcal{M}_{\mathrm{cu}} \subset \mathcal{M}$. *The dimension of* $\mathcal{M}_{\mathrm{cu}}$ *is equal to* $\dim P_{\mathrm{cu}}X_0$.

(b) Let $u_0 \in \mathcal{M}_{\mathrm{cu}}$ *and* v *be given by* (2.25). *If the function* $\hat{u} = u_* + v$ *stays in* $B_{X_p}(u_*,\rho)$ *on* $[t_0,0]$ *for some* $t_0 < 0$, *then* $\hat{u}(t) = u_* + v(t) \in \mathcal{M}_{\mathrm{cu}}$ *and* $\hat{u}$ *solves* (1.1) *for* $t_0 \le t \le 0$. *If the forward solution* u *of* (1.1) *exists and stays in* $B_{X_p}(u_*,\rho)$ *on*

[1] We corrected in formula (2.24) a misprint found in [10].

$[0, t_1]$ for some $t_1 > 0$, then $u(t) \in \mathcal{M}_{\rm cu}$ and we set $v(t) = u(t) - u_*$, for $0 \le t \le t_1$. We futher have $v(t) = P_{\rm cu}v(t) + \phi_{cu}(P_{\rm cu}v(t))$ for $t \in [t_0, 0]$, resp. $t \in [0, t_1]$.

(c) It holds $\mathcal{M}_{\rm cu} \cap \mathcal{M}_{\rm s} = \{u_*\}$.

(d) Assume, in addition, that (RR) holds. Then there is a $\rho_0 > 0$ such that the map $\phi_{cu} : P_{\rm cu}X_0 \cap B_{X_{1-1/p}}(0, \rho_0) \to P_{\rm s}X_1$ is Lipschitz.

Moreover, the local stable and unstable manifolds $\mathcal{M}_{\rm s}$ and $\mathcal{M}_u$ for (1.1) were constructed in Theorem 4.1 of [10]. These manifolds have analogous properties and are used to prove further properties of the center manifold summarized in the following corollary (see Corollary 5.3 [10]), where we assume throughout that radii $\rho > 0$ in the above theorems are the same.

Corollary 2.6. *Assume that Hypothesis 2.1 and* (2.16) *hold. We then have $\mathcal{M}_{\rm c} \cap \mathcal{M}_{\rm cs} \cap \mathcal{M}_{\rm cu}$, $\mathcal{M}_{\rm c} \cap \mathcal{M}_{\rm s} = \{u_*\}$, and $\mathcal{M}_{\rm c} \cap \mathcal{M}_{\rm u} = \{u_*\}$.*

Finally, we cite a result from [10] that describes the stability of the steady state u_* of (1.1) and the attractivity of $\mathcal{M}_{\rm c}$. As in Theorem 2.3, one assumes that Hypothesis 2.1 and (2.16) hold. In parabolic problems, the center and center–unstable manifolds are finite dimensional in many cases; e.g., if the spatial domain Ω is bounded. Moreover, there are important applications where $\mathcal{M}_{\rm c}$ consists of equilibria only, see e.g. [8], [15], [16]. Thus it is quite possible that one can check the stability of u_* with respect to the semiflow on $\mathcal{M}_{\rm c}$ generated by (1.1) without knowing a priori that u_* is stable with respect to the full semiflow of (1.1) on $\mathcal{M}$.

Theorem 2.7 below states, see Theorem 6.1 of [10], that u_* is stable on $\mathcal{M}$ under the following conditions: $s(-A_0) \le 0$, u_* is stable on $\mathcal{M}_{\rm cu} = \mathcal{M}_{\rm c}$, $P_{\rm cu} = P_{\rm c}$ has finite rank, and the additional regularity assumption (RR) holds. Here $s(-A_0) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(-A_0)\}$. In fact, one has a stronger result saying that each solution starting sufficiently close to u_* converges exponentially to a solution on $\mathcal{M}_{\rm c}$. Here one can assume that $s(-A_0) \le 0$ without loss of generality since $-A_0$ has no spectrum in the open right halfplane provided u_* is stable and $P_{\rm cu}$ has finite rank, due to Theorem 4.1 of [10].

Theorem 2.7. *Assume that the spectrum of $-A_0$ admits a splitting $\sigma(-A_0) = \sigma_s \cup \sigma_c$ corresponding to the spectral projections $P_{\rm s}$ and $P_{\rm c}$ such that $P_{\rm c}$ has finite rank, $\sigma_c \subset i\mathbb{R}$, and there is a number α with $\max \operatorname{Re}\sigma_s < -\alpha < 0$.*

Suppose that for each $r > 0$ there is a $\rho > 0$ such that for $u_0 \in \mathcal{M}_{\rm c}$ with $|P_{\rm c}(u_0 - u_*)|_0 < \rho$ the solution u of (1.1) *exists and $u(t) \in \mathcal{M}_{\rm c} \cap B_{X_{1-1/p}}(u_*, r)$ for all $t \ge 0$. Then there is a $\overline{\rho} > 0$ such that for every $u_0 = u_* + v_0 \in \mathcal{M}$ with $|v_0|_{1-1/p} \le \overline{\rho}$ the solution $u = u_* + v$ of* (1.1) *exists on $\mathbb{R}_+$ and there is a solution $\overline{u}$ of* (1.1) *on $\mathbb{R}_+$ such that $\overline{u}(t) \in \mathcal{M}_{\rm c}$ for all $t \ge 0$ and*

$$|u(t) - \overline{u}(t)|_1 \le ce^{-\alpha t}\, |P_{\rm s}v_0 - \phi_c(P_{\rm c}v_0)|_{1-1/p}$$

for all $t \ge 1$ and a constant c independent of u_0 and t. Moreover, u_* is stable for (1.1)*, i.e.: For each $r > 0$ there exists a $\rho' > 0$ such that for every $u_0 \in \mathcal{M} \cap B_{X_{1-1/p}}(u_*, \rho')$ the solution u of* (1.1) *exists on $\mathbb{R}_+$ and $u(t) \in B_{X_{1-1/p}}(u_*, r)$ for all $t \ge 0$.*

3. The Palmer Fundamental Lemma and Pliss Reduction Principle

Let u be a solution of (1.1), u_* be the equilibrium, and let $v = u - u_*$, $v_0 = u(0) - u_*$. Let $|v_0|_{1-1/p} \le \rho$ where $\rho > 0$ is sufficiently small. Throughout, we assume condition (RR) so that ϕ_c and ϕ_{cu} are Lipschitz into X_1, see Theorem 2.3(e) and Theorem 2.5(d). We first establish a version of the Palmer Fundamental Lemma in our PDE context.

Lemma 3.1. [The Fundamental Lemma] *Assume that Hypothesis 2.1, condition (RR), and* (2.17) *hold. Then there exist* $C, C', C''r, > 0$ *and* $\alpha \in (0, \omega_s)$ *such that if* v *is a solution of* (2.11) *satisfying* $|v(t)|_{1-1/p} \le r$ *for all* $0 \le t \le T$ *with some* $T > 1$*, then there exists a solution* z *of* (2.11) *on* $[0,T]$ *such that* $u_* + z(t) \in \mathcal{M}_{\rm c}$ *for all* $t \in [0,T]$*,* $P_{\rm cu} z(T) = P_{\rm cu} v(T)$*, and*

$$|v(t) - z(t)|_1 \le C e^{-\alpha t} |P_{\rm s} v_0 - \phi_{cu}(P_{\rm cu} v_0)|_{1-1/p} \tag{3.1}$$

holds for all $1 \le t \le T$*. Given* $T_0 > 1$*, the constants are uniform for* $T \ge T_0$*. For every* $t \in [0,T]$*, we further have*

$$|v(t) - z(t)|_{1-1/p} \le C' e^{-\alpha t} |P_{\rm s} v_0 - \phi_{cu}(P_{\rm cu} v_0)|_{1-1/p} \le C'' e^{-\alpha t} |v_0|_{1-1/p}. \tag{3.2}$$

Here, equation (3.1) holds for $t \ge 1$ (or for $t \ge a$ for any $a > 0$) because one can control the X_1-norm of the solution only for strictly positive t, see Proposition 15 of [9] and Remark 2.2 above.

Proof. Part 1. We assume that $T \ge 3$. For a general $T_0 > 0$ the proof is similar. Let v be a solution of (2.11) on $[0,T]$ such that $|v(t)|_{1-1/p} \le r$ for all $t \in [0,T]$, where a sufficiently small $r > 0$ is to be chosen below. Due to (2.17) there are constants $\delta \in (\omega_{cu}, \omega_s)$ and $N \ge 1$ such that $\|e^{-tA_0 P_{\rm cu}} P_{\rm cu}\|_{\mathcal{B}(X_0)} \le N e^{-\delta t}$ for all $t \le 0$. Using Theorem 2.5, we find a radius $\rho_{cu} > 0$ such that the restriction $\phi_{cu} : P_{\rm cu} X_0 \cap \overline{B}_{X_0}(0, \rho_{cu}) \to X_1$ is Lipschitz with Lipschitz constant ℓ, such that $u_* + \xi + \phi_{cu}(\xi) \in \mathcal{M}_{\rm cu}$ for all $\xi \in P_{\rm cu} X_0 \cap \overline{B}_{X_0}(0, \rho_{cu})$, and such that there is a constant $c_0 > 0$ with $|P_{\rm cu} z(t)|_0 \le c_0 |P_{\rm cu} z(1)|_0$ for all $0 \le t \le 1$ and any solution z of (2.11) with $u_* + z(t) \in \mathcal{M}_{\rm c}$ for $t \in [0,1]$. We set

$$\varepsilon_1(R) = \max_{x \in X_1, |x|_1 \le R} \{ \|G'(x)\|_{\mathcal{B}(X_1, X_0)}, \|H'(x)\|_{\mathcal{B}(X_1, Y_1)} \}. \tag{3.3}$$

Because of (2.10), we can fix a (small) number $R > 0$ such that

$$d := N \varepsilon_1(R)(1 + \|P_{\rm cu} \Pi\|_{\mathcal{B}(Y_1, X_0)})(1 + \ell) < \omega_s - \delta, \tag{3.4}$$

$$(1 + c_0) R \, \|P_{\rm cu}\|_{\mathcal{B}(X_1, X_0)} \le \rho_{cu}\,. \tag{3.5}$$

To choose $r > 0$, we note that Remark 2.2 (with $T = 1/2$) implies the inequality

$$|v(t)|_1 \le c |v(t - 1/2)|_{1-1/p} \le cr \quad \text{for all} \quad t \in [1/2, T].$$

Here and below the constants do not depend on v, T, t, R or r, and we let r be less than the radius indicated by Remark 2.2. We can now take small $r > 0$ such that

$$\begin{gathered} |v(t)|_1 \le R \qquad \text{for all} \quad 1/2 \le t \le T, \\ r(1 + \ell) \|P_{\rm cu}\|_{\mathcal{B}(X_{1-1/p}, X_1)} \le R/2 \ \text{ and } \ r \, \|P_{\rm cu}\|_{\mathcal{B}(X_{1-1/p}, X_0)} \le \rho_{cu}. \end{gathered} \tag{3.6}$$

Part 2. We define

$$w = P_{\rm s} v - \phi_{cu}(P_{\rm cu} v) \quad \text{and} \quad \varphi = v - w = P_{\rm cu} v + \phi_{cu}(P_{\rm cu} v)$$

on $[0,T]$. The function w belongs to $P_{\rm s} X_{1-1/p}$ and, due to the last inequality in (3.6), the function φ belongs to $\mathcal{M}_{\rm cu}$. Note that $P_{\rm cu} \varphi = P_{\rm cu} v$ and that

$$(A_* + \mu) x = (A_0 + \mu)(x - \mathcal{N}_1 B_* x) = (A_{-1} + \mu) x - \Pi B_* x$$

for every $x \in X_1$, see (2.13). One proves the analogues of formulas (2.21) and (2.22) for ϕ_{cu} as in the proof of Theorem 4.2 of [10] for the map ϕ_c. ¿From these identities we infer

$$\begin{aligned} B_* w &= B_* v - B_* \phi_{cu}(P_{\rm cu} v) = H(v) - H(\varphi) =: h, \qquad (3.7)\\ \dot w &= P_{\rm s}(-A_* v + G(v)) - \phi_{cu}'(P_{\rm cu} v) P_{\rm cu}(G(v) - A_* v)\\ &\quad - \phi_{cu}'(P_{\rm cu}\varphi) P_{\rm cu}(A_*\varphi - G(\varphi)) + P_{\rm s}(A_*\varphi - G(\varphi))\\ &= -P_{\rm s} A_* w + P_{\rm s}(G(v) - G(\varphi)) + \phi_{cu}'(P_{\rm cu} v) P_{\rm cu}(A_* w + G(\varphi) - G(v))\\ &= -A_{-1} P_{\rm s} w + P_{\rm s}\Pi h + P_{\rm s}(G(v) - G(\varphi)) - \phi_{cu}'(P_{\rm cu} v) P_{\rm cu}(\Pi h + G(v) - G(\varphi))\\ &=: -A_{-1} P_{\rm s} w + P_{\rm s}\Pi h + P_{\rm s} g. \qquad (3.8) \end{aligned}$$

In the penultimate line we used that $P_{\rm cu} A_{-1} w = A_{-1} P_{cu} w = 0$. Applying the variation of constant formula in X_{-1}, we therefore obtain

$$w(t) = T(t-\tau) P_{\rm s} w(\tau) + \int_\tau^t T_{-1}(t-\sigma) P_{\rm s}(g(\sigma) + \Pi h(\sigma))\, d\sigma \qquad (3.9)$$

for $0 \le \tau < t \le T$. Let $\alpha \in [0, \omega_s)$. Proposition 10(II) in [9] implies that

$$\max\Big\{ \|g\|_{\mathbb{E}_0([\tau,t],\alpha)},\, \|h\|_{\mathbb{F}([\tau,t],\alpha)} \Big\} \le \varepsilon(r) \|w\|_{\mathbb{E}_1([\tau,t],\alpha)}. \qquad (3.10)$$

Arguing as in the proof of Proposition 8 of [9] (see inequality (43) there), we can then estimate

$$\begin{aligned} e^{-\alpha\tau}\|w\|_{\mathbb{E}_1([\tau,t],\alpha)} &\le c\big(|w(\tau)|_{1-1/p} + e^{-\alpha\tau}\|g\|_{\mathbb{E}_0([\tau,t],\alpha)} + e^{-\alpha\tau}\|h\|_{\mathbb{F}([\tau,t],\alpha)}\big)\\ &\le c\big(|w(\tau)|_{1-1/p} + e^{-\alpha\tau}\varepsilon(r)\|w\|_{\mathbb{E}_1([\tau,t],\alpha)}\big). \end{aligned}$$

We note that the constants do not depend on τ if $0 \le \tau \le t - 1/4$, say. Making $r > 0$ sufficiently small and using (2.7), we arrive at

$$\begin{aligned} e^{-\alpha\tau}\|w\|_{\mathbb{E}_1([\tau,t],\alpha)} &\le c|w(\tau)|_{1-1/p},\\ |w(t)|_{1-1/p} &\le c e^{-\alpha t}\|w\|_{\mathbb{E}_1([\tau,t],\alpha)} \le c e^{-\alpha(t-\tau)}|w(\tau)|_{1-1/p} \end{aligned} \qquad (3.11)$$

for all $0 \le \tau < t \le T$. Again the constants are uniform for $\tau \in [0, t - 1/4]$.

Part 3. Since $|P_{\rm cu} v(T)|_0 \le \rho_{cu}$ by (3.6), there exists the backward solution $z = P_{\rm cu} z + \phi_{cu}(P_{\rm cu} z)$ of (2.11) such that $u_* + z$ belongs to $\mathcal{M}_{\rm cu}$ and $P_{\rm cu} z(T) = P_{\rm cu} v(T)$. Due to Theorem 2.5, $z(t)$ exists at least for $t \in [T-3, T]$ and $\|z\|_{\mathbb{E}_1([T-3,T])} \le c|P_{\rm cu} z(T)|_0 \le cr$. Thus, $|z(T-3)|_{1-1/p} \le cr$ by (2.7) and so Proposition 5 of [9] yields $|z(t)|_1 \le cr \le R$ for all $t \in [T-2, T]$, after decreasing $r > 0$ if needed. Let $t_0 \in [1/2, T-2]$ be the minimal time such that $z(t)$ with $u_* + z$ on $\mathcal{M}_{\rm cu}$ exists and the inequality $|z(t)|_1 \le R$ holds for all $t_0 \le t \le T$. We set

$$y = P_{\rm cu}(v - z). \qquad (3.12)$$

Denoting

$$g_1 = G(v) - G(z) \quad \text{and} \quad h_1 = B_*(v - z) = H(v) - H(z),$$

we obtain

$$\begin{aligned} y' &= P_{\rm cu}\big(-A_*(v-z) + g_1\big)\\ &= -P_{\rm cu}\big((A_0 + \mu)(v - z - \mathcal{N}_1 h_1) - \mu(v - z)\big) + P_{\rm cu} g_1\\ &= -A_0 P_{\rm cu} y + P_{\rm cu}(g_1 + \Pi h_1). \qquad (3.13) \end{aligned}$$

This equation yields

$$y(t) = -\int_t^T e^{-(t-\tau)A_0 P_{\rm cu}} P_{\rm cu}(g_1(\tau) + \Pi h_1(\tau))\, d\tau,$$

$$|y(t)|_0 \le N(1 + \|P_{\rm cu}\Pi\|_{\mathcal{B}(Y_1,X_0)}) \int_t^T e^{-\delta(t-\tau)} \varepsilon_1(R) |v(\tau) - z(\tau)|_1\, d\tau \tag{3.14}$$

for all $t \in [t_0, T]$. We observe that

$$v - z = w + \phi_{cu}(P_{\rm cu}v) - \phi_{cu}(P_{\rm cu}z) + y \tag{3.15}$$

holds on $[t_0, T]$. Putting $d_0 = d(1+\ell)^{-1}$ and recalling (3.4) and (3.5), we further estimate

$$e^{\delta t}|y(t)|_0 \le d \int_t^T e^{\delta\tau}|y(\tau)|_0\, d\tau + d_0 \int_t^T e^{\delta\tau}|w(\tau)|_1\, d\tau.$$

We then deduce from a Gronwall-type inequality that

$$\begin{aligned} e^{\delta t}|y(t)|_0 &\le d_0 \int_t^T e^{\delta\tau}|w(\tau)|_1\, d\tau + dd_0 \int_t^T e^{d(\tau-t)} \int_\tau^T e^{\delta\sigma}|w(\sigma)|_1\, d\sigma\, d\tau \\ &= d_0 \int_t^T e^{\delta\tau}|w(\tau)|_1\, d\tau + dd_0 \int_t^T e^{\delta\sigma}|w(\sigma)|_1 \int_t^\sigma e^{d(\tau-t)}\, d\tau\, d\sigma \\ &= d_0 \int_t^T e^{d(\sigma-t)} e^{\delta\sigma}|w(\sigma)|_1\, d\sigma. \end{aligned}$$

Since we have chosen d and δ as in (3.4), we can take $\alpha \in (d+\delta, \omega_s)$. So, Hölder's inequality and (3.11) imply

$$\begin{aligned} |y(t)|_0 &\le d_0 \int_t^T e^{(d+\delta)(\sigma-t)}|w(\sigma)|_1\, d\sigma \\ &\le \frac{d_0 e^{-\alpha t}}{(d+\delta-\alpha)^{\frac{1}{p'}}} \|w\|_{\mathbb{E}_1([t,T],\alpha)} \le c|w(t)|_{1-1/p} \end{aligned} \tag{3.16}$$

for all $t \in [t_0, T]$, where the constant c is uniform for $t \le T - 1/4$, say. If $t \in [T-1/4, T]$, we can estimate in (3.16) the $\mathbb{E}_1$ norm on $[t, T]$ by that on $[t-1/4, T]$ and obtain

$$|y(t)|_0 \le c|w(t-1/4)|_{1-1/p} \tag{3.17}$$

with a uniform constant. In view of (3.12), we have $z = P_{\rm cu}(v-y) + \phi_{cu}(P_{\rm cu}(v-y))$. Thus, $|P_{\rm cu}(v-y)|_0 = |P_{\rm cu}z|_0 \le \|P_{\rm cu}\|_{\mathcal{B}(X_1,X_0)}R \le \rho_{cu}$ due (3.5). Using $|v(t_0)|_{1-1/p} \le r$, (3.6), (3.16), and (3.11) with $\tau = 0$, we then deduce

$$\begin{aligned} |z(t_0)|_1 &\le (1+\ell)|P_{\rm cu}(v-y)|_1 \le (1+\ell)\|P_{\rm cu}\|_{\mathcal{B}(X_{1-1/p},X_1)}\, |v(t_0)|_{1-1/p} + c|y(t_0)|_0 \\ &\le R/2 + c|w(t_0)|_{1-1/p} \le R/2 + c|v_0|_{1-1/p} \le R/2 + cr < R, \end{aligned}$$

provided $r > 0$ is small enough. It follows that $t_0 = 1/2$. Since $u_* + z(1/2) \in \mathcal{M}_{\rm cu}$ we can extend $u_* + z$ on $\mathcal{M}_{\rm cu}$ to the time interval $[0, T]$ due to Theorem 2.5(a). The estimates (3.16), (3.17) and (3.11) now imply that the inequalities

$$\begin{aligned} |P_{\rm cu}(v(t) - z(t))|_0 &= |y(t)|_0 \le ce^{-\alpha t}|w(0)|_{1-1/p} = ce^{-\alpha t}|P_{\rm s}v_0 - \phi_{cu}(P_{\rm cu}v_0)|_{1-\frac{1}{p}}, \\ |P_{\rm s}(v(t) - z(t))|_{1-1/p} &= |w(t) + \phi_{cu}(P_{\rm cu}v(t)) - \phi_{cu}(P_{\rm cu}z(t))|_{1-1/p} \\ &\le |w(t)|_{1-1/p} + c|y(t)|_0 \le ce^{-\alpha t}|P_{\rm s}v_0 - \phi_{cu}(P_{\rm cu}v_0)|_{1-1/p} \end{aligned}$$

hold on $[1/2, T]$. These relations and Theorem A.1 of [10] yield (3.1).

Part 4. To establish (3.2), it remains to show that

$$|v(t) - z(t)|_{1-1/p} \le c|P_{\rm s}v_0 - \phi_{cu}(P_{\rm cu}v_0)|_{1-1/p} \qquad \text{for all } \ t \in [0,1]. \tag{3.18}$$

We first note that (3.15) also holds on $[0,T]$ and that $|P_{\rm cu}z(t)|_0 \le \rho_{cu}$ for $0 \le t \le 1$ due to (3.5) and the text before this inequality. Let g and h be given by (3.8) and (3.7). Theorem 2.1 of [6] gives a $\psi \in \mathbb{E}_1([0,1])$ such that

$$\begin{aligned} \partial_t\psi(t) + A_*\psi(t) &= g(t) \qquad \text{on } \Omega, \quad \text{a.e. } t > 0,\\ B_*\psi(t) &= h(t) \qquad \text{on } \partial\Omega, \quad t \ge 0,\\ \psi(0) &= w(0), \qquad \text{on } \Omega. \end{aligned}$$

Using also (3.10) with $\alpha = 0$, we further obtain

$$\|\psi\|_{\mathbb{E}_1([0,1])} \le c(|w(0)|_{1-\frac{1}{p}} + \|g\|_{\mathbb{E}_0([0,1])} + \|h\|_{\mathbb{F}([0,1])}) \le c|w(0)|_{1-\frac{1}{p}} + \varepsilon(r)\|w\|_{\mathbb{E}_1([0,1])}.$$

In view of (2.15) and (3.9), we have $w = P_{\rm s}\psi$ and thus

$$\begin{aligned} \|w\|_{\mathbb{E}_1([0,1])} &\le c\|\psi\|_{\mathbb{E}_1([0,1])} \le c|w(0)|_{1-\frac{1}{p}} + \varepsilon(r)\|w\|_{\mathbb{E}_1([0,1])},\\ \|w\|_{\mathbb{E}_1([0,1])} &\le c|w(0)|_{1-1/p} = c|P_{\rm s}v_0 + \phi_{cu}(P_{\rm cu}v_0)|_p, \end{aligned} \tag{3.19}$$

possibly after decreasing $r > 0$. Equation (3.15) and (2.7) now yield

$$\begin{aligned} |v(t) - z(t)|_{1-1/p} &\le c(|w(t)|_{1-1/p} + |y(t)|_0) \le c(\|w\|_{\mathbb{E}_1([0,1])} + |y(t)|_0)\\ &\le c(|P_{\rm s}v_0 + \phi_{cu}(P_{\rm cu}v_0)|_{1-1/p} + |y(t)|_0) \end{aligned} \tag{3.20}$$

for all $t \in [0,1]$. To control $y(t)$, we use again (3.13). As in the proof of Proposition 10 of [9] (letting there $v = 0$ in steps (1), (4) and (5)), one can show that

$$\|g_1\|_{\mathbb{E}_0([t,1])} + \|h_1\|_{L^p([t,1];Y_1)} \le c\|v - z\|_{L^p([t,1];X_1)}$$

where the constant does not depend on $t \in [0,1]$. Combined with (3.15), (3.19) and (3.1), these facts yield

$$\begin{aligned} y(t) &= e^{-(t-1)A_0P_{\rm cu}}y(1) - \int_t^1 e^{-(t-\tau)A_0P_{\rm cu}}P_{\rm cu}(g_1(\tau) + \Pi h_1(\tau))\,d\tau,\\ |y(t)|_0 &\le c(|y(1)|_0 + \|v - z\|_{L^p([t,1];X_1)}) \le c(|y(1)|_0 + \|w\|_{L^p([t,1];X_1)} + \|y\|_{\mathbb{E}_0([t,1])})\\ &\le c(|P_{\rm s}v_0 + \phi_{cu}(P_{\rm cu}v_0)|_{1-1/p} + \|y\|_{\mathbb{E}_0([t,1])}) \end{aligned}$$

where we could employ that ϕ_{cu} is Lipschitz with values in X_1 since $|P_{\rm cu}v|_0$ and $|P_{\rm cu}z|_0$ are less than ρ_{cu} on $[0,1]$. It follows

$$|y(t)|_0^p \le c|P_{\rm s}v_0 + \phi_{cu}(P_{\rm cu}v_0)|_{1-1/p}^p + c\int_t^1 |y(\tau)|_0^p\,d\tau$$

for all $t \in [0,1]$, so that $|y(t)|_0^p \le c|P_{\rm s}v_0 + \phi_{cu}(P_{\rm cu}v_0)|_{1-1/p}^p$ by Gronwall's inequality. In view of (3.20), we have established (3.18). □

Our first theorem says that the center–unstable manifold attracts solutions which stay in small ball around u_* for all $t \ge 0$ and that there exists a tracking solution $u_* + \overline{z}$ on $\mathcal{M}_{\rm cu}$.

Theorem 3.2. [Asymptotic Phase] *Assume that Hypothesis 2.1, condition (RR), (2.17), and* $\dim P_{\rm cu}X_0 < \infty$ *hold. Then there exists constants* $r, c > 0$ *and* $\alpha \in (0, \omega_s)$ *such that : If a solution* v *of* (2.11) *exists and satisfies* $|v(t)|_{1-1/p} \le r$ *for all* $t \ge 0$*, then there is a solution* $\overline{z}$ *of* (2.11) *such that* $u_* + \overline{z}(t) \in \mathcal{M}_{\rm c}$ *and*

$$|v(t) - \overline{z}(t)|_1 \le ce^{-\alpha t}|P_{\rm s}v_0 - \phi_{cu}(P_{\rm cu}v_0)|_{1-1/p} \qquad \text{for all } \ t \ge 1, \tag{3.21}$$

$$|v(t) - \overline{z}(t)|_{1-1/p} \le ce^{-\alpha t}|P_{\rm s}v_0 - \phi_{cu}(P_{\rm cu}v_0)|_{1-1/p} \qquad \text{for all } \ t \ge 0. \tag{3.22}$$

Proof. We choose $r > 0$ so small that Lemma 3.1 can be applied to v. Lemma 3.1 gives solutions z_n with $u_* + z_n$ on $\mathcal{M}_{\rm cu}$ tracking v on $[0, n]$ for every $n \in \mathbb{N}$ with $n \ge 2$. Lemma 3.1 also yields

$$|P_{\rm cu}z_n(0)|_0 \le |P_{\rm cu}v(0)|_0 + |P_{\rm cu}(z_n(0) - v(0))|_0 \le cr$$

for all $n \in \mathbb{N}$. Hence, there exists a subsequence $n_j \to \infty$ so that $P_{\rm cu}z_{n_j}(0) \to \zeta \in P_{\rm cu}X_0$ as $j \to \infty$. Let $\overline{z}$ be the solution of (2.11) on $[-2, 2]$ such that $P_{\rm cu}\overline{z}(0) = \zeta$ and $u_* + \overline{z}(t) \in \mathcal{M}_{\rm cu}$ for $t \in [-2, 2]$, decreasing $r > 0$ if needed to apply Theorem 2.5 and Remark 2.2. We also have $|\overline{z}(t)|_{1-1/p} \le c|\overline{z}(0)|_0 \le |\zeta|_0 \le c_1 r$ for all $t \in [0, 2]$ and some constant $c_1 > 0$. Let b denote the supremum of $t_1 > 1$ such that $\overline{z}(t)$ exists on $[0, t_1]$ and stays in the ball $\overline{B}_{1-1/p}(0, (1 + c_1 + C'')r)$, where C'' is given by Lemma 3.1. We thus have $b \ge 2$. If we take a sufficiently small $r > 0$, Theorem 2.5(b) shows that $u_* + \overline{z}(t) \in \mathcal{M}_{\rm cu}$ for $t \in [0, b)$.

As in Theorem 4.2 of [10] one can prove that the functions $P_{\rm cu}z_n$ and $P_{\rm cu}\overline{z}$ satisfy an ordinary differential equation analogous to (2.20). Since the initial data $P_{\rm cu}z_{n_j}(0)$ converge to $P_{\rm cu}\overline{z}(0)$, we thus obtain $P_{\rm cu}z_{n_j}(t) \to P_{\rm cu}\overline{z}(t)$ in X_0 as $j \to \infty$ for $t \in [0, b)$ and hence

$$z_{n_j}(t) = P_{\rm cu}z_{n_j}(t) + \phi_{cu}(P_{\rm cu}z_{n_j}(t)) \to \overline{z}(t)$$

in $X_1 \hookrightarrow X_{1-1/p}$, using also the Lipschitz property of ϕ_{cu} stated in Theorem 2.5(d) and decreasing $r > 0$ if necessary. Lemma 3.1 now implies that

$$\begin{aligned} |\overline{z}(t)|_{1-1/p} &\le \limsup_{j\to\infty} |z_{n_j}(t) - v(t)|_{1-1/p} + |v(t)|_{1-1/p} \\ &\le C''e^{-\alpha t}|v_0|_{1-1/p} + r \le (C'' + 1)r < (C'' + 1 + c_1)r. \end{aligned} \tag{3.23}$$

for all $t \in [0, b)$. As a result, $b = \infty$ and $\overline{z}$ exists for all $t \ge 0$ and stays in $\overline{B}_{1-1/p}(0, (C'' + 1 + c_1)r)$. If $r > 0$ is chosen small enough, then Lemma 3.3 below shows that $u_* + \overline{z}(t) \in \mathcal{M}_{\rm cs}$ for all $t \ge 0$. Hence, $u_* + \overline{z}$ is contained in $\mathcal{M}_{\rm c}$ by Corollary 2.6 since $u_* + \overline{z}(t) \in \mathcal{M}_{\rm cu}$ for all $t \ge 0$. The convergence properties follow from Lemma 3.1 as in (3.23). □

Lemma 3.3. *Assume that Hypothesis 2.1 and* (2.18) *hold. If* $u_* + v_0 \in \mathcal{M}_{\rm cs}$*, then*

$$v_0 = P_{\rm cs}v_0 + \phi_{cs}(P_{\rm cs}\,\mathcal{P}\,v_0), \tag{3.24}$$

where $\mathcal{P} = I - \mathcal{N}_{1-1/p}B_*$. *Moreover, there exists a* $\overline{\rho} > 0$ *such that if* v *is a solution of* (2.11) *on* $\mathbb{R}_+$ *staying in* $B(u_*, \overline{\rho})$*, then* $u_* + v(t) \in \mathcal{M}_{\rm cs}$ *for all* $t \ge 0$.

Proof. Theorem 2.4(b) shows that

$$v_0 = P_{\rm cs}\,\mathcal{P}\,v_0 + \vartheta_{cs}(P_{\rm cs}\,\mathcal{P}\,v_0) + \phi_{cs}(P_{\rm cs}\,\mathcal{P}\,v_0)$$

if $u_* + v_0 \in \mathcal{M}_{\rm cs}$. Then $P_{\rm cs}v_0 = P_{\rm cs}\,\mathcal{P}\,v_0 + \vartheta_{cs}(P_{\rm cs}\,\mathcal{P}\,v_0)$, and the first assertion follows. In the framework of Theorem 2.4, the second assertion can be shown as Theorem 2.3(d) (see the proof of Theorem 4.2(e) in [10]). □

Given a solution $u_* + v$ near u_*, in Lemma 3.1 we have constructed a tracking solution on $\mathcal{M}_{\rm cu}$ for finite time intervals. In the next lemma, we construct such a solution on the center manifold in the case of trichotomy if also $u_* + v(0) \in \mathcal{M}_{\rm cs}$.

Lemma 3.4. *Assume that Hypothesis 2.1, condition (RR), and* (2.16) *hold. Then there exist* $C, r > 0$ *such that: If there is a solution* v *of* (2.11) *with* $u_* + v$ *on* $\mathcal{M}_{\rm cs}$ *staying in* $B_{1-1/p}(0,r)$ *on* $[0,T]$ *for some* $T > 1$*, then there is a solution* z *of* (2.11) *on* $[0,T]$ *such that* $u_* + z(t) \in \mathcal{M}_{\rm c}$ *for all* $t \in [0,T]$*,* $P_{\rm cu}z(T) = P_{\rm cu}v(T)$*, and*

$$|v(t) - z(t)|_1 \le Ce^{-\alpha t}|P_{\rm s}v_0 - \phi_{cu}(P_{\rm cu}v_0)|_{1-1/p} \qquad \text{for all } \ t \ge 1,$$
$$|v(t) - z(t)|_{1-1/p} \le Ce^{-\alpha t}|P_{\rm s}v_0 - \phi_{cu}(P_{\rm cu}v_0)|_{1-1/p} \qquad \text{for all } \ t \ge 0,$$

Given $T_0 > 1$*, the constants are uniform for* $T \ge T_0$*.*

Proof. We assume that $T \ge 3$. For a general $T_0 > 0$ the proof is similar. Formula (3.24) says that $v = P_{\rm cs}v + \phi_{cs}(P_{\rm cs}\mathcal{P}v)$. We set $w = P_{\rm s}v - \phi_{cu}(P_{\rm cu}v)$. Since $|P_{\rm c}v(T)|_{1-1/p} \le cr$, for sufficiently small $r > 0$ Theorem 2.5 gives a solution z of (2.11) on $[T-3,T]$ such that $P_{\rm c}z(T) = P_{\rm c}v(T)$ and $u_* + z$ belongs to $\mathcal{M}_{\rm c}$. Moreover, $|z(T)|_1 \le c|P_{\rm c}z(T)|_0 \le cr$. Here and below the constants do not depend on v, T, t, r and the number $R > 0$ introduced later. We have $\mathcal{M}_{\rm c} = \mathcal{M}_{\rm cs} \cap \mathcal{M}_{\rm cu}$ due to Corollary 2.6, and thus

$$z = P_{\rm c}z + \phi_c(P_{\rm c}z) = P_{\rm cs}z + \phi_{cs}(P_{\rm cs}\mathcal{P}z) = P_{\rm cu}z + \phi_{cu}(P_{\rm cu}z).$$

As a result, $P_{\rm s}z = \phi_{cu}(P_{\rm cu}z)$ and $P_{\rm u}z = \phi_{cs}(P_{\rm cs}\mathcal{P}z)$. Hence,

$$\begin{aligned} v - z = w + \phi_{cu}(P_{\rm cu}v) - \phi_{cu}(P_{\rm cu}z) + P_{\rm c}(v-z) \\ + \phi_{cs}(P_{\rm cs}\mathcal{P}v) - \phi_{cs}(P_{\rm cs}\mathcal{P}z). \end{aligned} \tag{3.25}$$

We set $y = P_{\rm c}(v-z)$. Given a small $R > 0$ to be determined later, let $t_0 \in [1/2, T)$ be the minimal time such that the solution $z(t)$ of (2.11) with $u_* + z$ on $\mathcal{M}_{\rm c}$ exists and the inequality $|z(t)|_1 \le R$ holds for all $t_0 \le t \le T$. As in part 3 of the proof of Lemma 3.1, we obtain that $1/2 \le t_0 \le T-2$ exists if $r > 0$ is chosen small enough. We further note that Remark 2.2 shows that $|v(t)|_1 \le c|v(t-1/2)|_{1-1/p} \le cr \le R$ where we decrease $r > 0$ if needed.

Theorems 2.4 and 2.5 imply that the maps $\phi_{cs} : P_{\rm cs}X^0_{1-1/p} \to P_{\rm u}X_0$ and $\phi_{cu} : P_{\rm cu}X_0 \to P_{\rm s}X_{1-1/p}$ are Lipschitz with a constant $\varepsilon(R)$ on balls of radius R in the respective domain spaces. Moreover, $\phi_{cu} : P_{\rm cu}X_0 \to P_{\rm s}X_1$ is Lipschitz on this ball due to (RR) and Theorem 2.5(d) (after possibly decreasing R). We thus deduce

$$\begin{aligned} |v(t) - z(t)|_{1-1/p} \le |w(t)|_{1-1/p} + \varepsilon(cR)|P_{\rm cu}(v(t) - z(t))|_0 \\ + c\varepsilon(cR)|P_{\rm cs}\mathcal{P}(v(t) - z(t))|_{1-1/p} + c|y(t)|_0 \end{aligned}$$

from (3.25) and $X_1 \hookrightarrow X_{1-1/p} \hookrightarrow X_0$. Decreasing $R > 0$ if needed, we obtain

$$|v(t) - z(t)|_{1-1/p} \le c(|w(t)|_{1-1/p} + |y(t)|_0) \tag{3.26}$$

for $t_0 \le t \le T$. Proceeding similarly, inequality (3.26) then leads to

$$\begin{aligned} |v(t) - z(t)|_1 &\le |w(t)|_1 + c|P_{\rm cu}(v(t) - z(t))|_0 + c|P_{\rm cs}\mathcal{P}(v(t) - z(t))|_{1-1/p} + c|y(t)|_0 \\ &\le |w(t)|_1 + c(|w(t)|_{1-1/p} + |y(t)|_0) \le c(|w(t)|_1 + |y(t)|_0), \end{aligned} \tag{3.27}$$

for all $t_0 \le t \le T$.

Let $\delta \in (\underline{\omega_c}, \omega_s)$. As in the proof of (3.13) and (3.14) in Lemma 3.1, we infer

$$\begin{aligned} y'(t) &= P_{\rm c}\big(-A_*(v(t) - z(t)) + G(v(t)) - G(z(t))\big) \\ &= -A_0P_{\rm c}y(t) + P_{\rm c}\big(\Pi(H(v(t)) - H(z(t))) + G(v(t)) - G(z(t))\big), \\ y(t) &= -\int_t^T e^{-(t-\tau)A_0P_{\rm c}}P_{\rm c}\big(\Pi(H(v(\tau)) - H(z(\tau))) + G(v(\tau)) - G(z(\tau))\big)\,d\tau, \end{aligned}$$

$$|y(t)|_0 \le c\varepsilon_1(R)\int_t^T e^{-\delta(t-\tau)}|v(\tau)-z(\tau)|_1\,d\tau.$$

Inequality (3.27) then implies

$$e^{\delta t}|y(t)|_0 \le c\varepsilon_1(R)\int_t^T e^{\delta\tau}|y(\tau)|_0\,d\tau + c\varepsilon(R)\int_t^T e^{\delta t}|w(\tau)|_1\,d\tau.$$

Arguing as in the proof of (3.16) in Lemma 3.1 (with $d = d_0 = c\varepsilon(R)$ being small), we conclude that

$$|y(t)|_0 \le d\int_t^T e^{(d+\delta)(\sigma-t)}|w(\sigma)|_1\,d\sigma.$$

We now fix a sufficiently small $R > 0$ such that $d + \delta < \omega_s$, where $d = c\varepsilon(R)$. Let $\alpha \in (d+\delta, \omega_s)$. If we take a sufficiently small $r > 0$, we can apply estimates (3.11) and (3.19) from Lemma 3.1. Using first Hölder's inequality, we thus derive

$$|y(t)|_0 \le ce^{-\alpha t}\|w\|_{\mathbb{E}_1([t,T],\alpha)} \le c|w(t)|_{1-1/p} \tag{3.28}$$

for all $t_0 \le t \le T - 1/4$. As in (3.17), we also obtain

$$|y(t)|_0 \le c|w(t-1/4)|_{1-1/p} \tag{3.29}$$

for all $t \in [T-1/4, T]$. Observe that

$$z = P_{\mathrm{c}}z + \phi_c(P_{\mathrm{c}}z) = P_{\mathrm{c}}(v-y) + \phi_c(P_{\mathrm{c}}(v-y)). \tag{3.30}$$

Remark 2.2 further yields

$$|v(t)|_1 \le c|v(t-1/2)|_{1-1/p} \le cr \le R/2 \tag{3.31}$$

for all $t \in [t_0, T]$ and a sufficiently small $r > 0$. Using (3.30), (3.31) and (3.28), we infer

$$|z(t_0)|_1 \le c\big(|v(t_0)|_1 + |y(t_0)|_0\big) \le R/2 + c|w(t_0)|_{1-1/p}\,.$$

Since $|w(t_0)|_{1-1/p} \le c|v(t_0)|_{1-1/p}$, we finally conclude that $|z(t_0)|_1 \le R/2 + cr < R$, decreasing $r > 0$ again, if needed. Thus, $t_0 = 1/2$. Now the estimates (3.26), (3.28), (3.29) and (3.11) imply

$$\begin{aligned}|v(t)-z(t)|_{1-1/p} &\le c\big(|w(t)|_{1-1/p} + |y(t)|_0\big)\\ &\le ce^{-\alpha t}|w(0)|_{1-1/p} = ce^{-\alpha t}|P_{\mathrm{s}}v_0 - \phi_{cu}(P_{\mathrm{cu}}v_0)|_{1-1/p}\end{aligned}$$

for all $t \in [1/2, T]$. We can extend this estimate to $t \in [0, T]$ as in part 4) of the proof of Lemma 3.1. Finally, decreasing $r > 0$ if necessary, one can use the estimate (A.2) in [10] applied to $u_* + v$ and $u_* + z$, obtaining the inequality

$$|v(t)-z(t)|_1 \le c|v(t-1/2) - z(t-1/2)|_{1-1/p}$$

and completing the proof. □

Our second theorem extends the stability Theorem 2.7 to the case of unstable spectrum. It says that the center manifold locally attracts the center–stable manifold with a tracking solution if the flow on $\mathcal{M}_{\mathrm{c}}$ is stable.

Theorem 3.5. *Assume that Hypothesis 2.1, condition (RR),* $\dim P_{\mathrm{c}}X_0 < \infty$, *and the trichotomy condition* (2.16) *hold. Suppose that* u_* *is stable for the flow on* $\mathcal{M}_{\mathrm{c}}$. *Then there exist sufficiently small* $r > 0$ *and* $\rho > 0$ *such that for each solution* v *of* (2.11) *with* $|v_0|_{1-1/p} \le \rho$ *either*

(a) *there exists* $t > 0$ *such that* $|v(t)|_{1-1/p} > r$, *or*
(b) $u_* + v_0 \in \mathcal{M}_{\mathrm{cs}}$.

Moreover, in case (b) the solution v of (2.11) with $u_* + v(t) \in \mathcal{M}_{\rm cs}$ exists for all $t \ge 0$, satisfies $|v(t)|_{1-1/p} \le r$ for all $t \ge 0$, and there exists a solution $\overline{z}$ of (2.11) on $\mathbb{R}_+$ with $u_* + \overline{z}$ on $\mathcal{M}_{\rm c}$ such that

$$|v(t) - z(t)|_1 \le Ce^{-\alpha t}|P_{\rm s}v_0 - \phi_{cu}(P_{\rm cu}v_0)|_{1-1/p} \qquad \text{for all } \ t \ge 1, \tag{3.32}$$

$$|v(t) - z(t)|_{1-1/p} \le Ce^{-\alpha t}|P_{\rm s}v_0 - \phi_{cu}(P_{\rm cu}v_0)|_{1-1/p} \qquad \text{for all } \ t \ge 0. \tag{3.33}$$

Proof. Let $r > 0$ be the radius determined in Lemma 3.4. We choose a small $\rho \in (0, r)$ to be fixed later. Let v_0 with $u_* + v_0 \in \mathcal{M}_{\rm cs}$ satisfy $|v_0|_{1-1/p} \le \rho$. Denote by T the supremum of all $t > 0$ such that $v(t)$ exists and satisfies $|v(\tau)|_{1-1/p} < r$ for all $0 \le \tau \le t$ and $|v(T)|_{1-1/p} = r$. Remark 2.2 implies that $T > 1$ for sufficiently small $\rho > 0$. By Lemma 3.4, there exists a solution z_T of (2.11) on $[0, T]$ such that $P_{\rm c}z_T(T) = P_{\rm c}v(T)$, $u_* + z_T(t) \in \mathcal{M}_{\rm c}$ for $0 \le t \le T$, and

$$\begin{aligned} |z_T(t) - v(t)|_1 &\le ce^{-\alpha t}|P_{\rm s}v_0 - \phi_{cu}(P_{\rm cu}v_0)|_{1-1/p}, \qquad t \in [1, T], \\ |z_T(t) - v(t)|_{1-1/p} &\le ce^{-\alpha t}|P_{\rm s}v_0 - \phi_{cu}(P_{\rm cu}v_0)|_{1-1/p} \le c\rho, \qquad t \in [0, T]. \end{aligned} \tag{3.34}$$

Here and below, c does not depend on T and ρ. Using this estimate and Remark 2.2, it follows that

$$|z_T(0)|_{1-1/p} \le |z_T(0) - v(0)|_{1-1/p} + |v(0)|_{1-1/p} \le c\rho + c\rho = c\rho. \tag{3.35}$$

Since u_* is stable for the flow on $\mathcal{M}_{\rm c}$, we can choose ρ so small that

$$|z_T(T)|_{1-1/p} = \mathrm{dist}_{X_{1-1/p}}(u_* + z_T(T), u_*) \le r/2.$$

We then obtain

$$|v(T)|_{1-1/p} \le |v(T) - z_T(T)|_{1-1/p} + |z_T(T)|_{1-1/p} \le c\rho + r/2 < r$$

provided $\rho > 0$ is sufficiently small. This strict inequality is a contradiction if T is finite, and hence $T = \infty$. As a consequence, (3.35) holds for all $T > 1$. Since $P_{\rm c}X_0$ is finite dimensional, there thus exists a sequence of $T_n \to \infty$ such that $P_{\rm s}z_{T_n}(0)$ converge as $n \to \infty$ to some $\zeta \in P_{\rm c}X_0$ with $|\zeta|_0 \le c\rho$. Using Theorem 2.3(a), we find a solution $\overline{z}$ of (2.11) on some time interval $[0, t_0)$ such that $P_{\rm c}\overline{z}(0) = \zeta$ and $u_* + \overline{z}$ belongs to $\mathcal{M}_{\rm c}$. Since $|\overline{z}(0)|_{1-1/p} \le c|\zeta|_0 \le c\rho$, the stability of u_* on $\mathcal{M}_{\rm c}$ implies that $\overline{z}(t)$ exists and $|\overline{z}(t)|_{1-1/p} \le r$ for all $t \ge 0$, if $\rho > 0$ is sufficiently small. As in the proof of Theorem 3.2 we then deduce that $z_{T_n}(t)$ converges to $\overline{z}(t)$ in X_1 as $n \to \infty$. Thus, the required estimate (3.32) follows from (3.34). □

In our last theorem, we extend the stability Theorem 2.7 from the set $K = \{u_*\}$ to larger invariant sets K.

Theorem 3.6. [Reduction Principle] *Assume the conditions of Theorem 2.7. There exists small numbers $\rho, \rho_0 > 0$ such that if $K \subset B_{X_{1-1/p}}(u_*, \rho)$ is a backward and forward globally invariant set for (1.1), then the following assertions hold:*

(a) *$K \subset \mathcal{M}_{\rm c}$ and there exists a set $K_0 \subset B_{P_{\rm c}X_0}(0, \rho_0)$ such that*

$$K = \{u_* + w_0 + \phi_c(w_0) : \, w_0 \in K_0\} \tag{3.36}$$

and K_0 is forward and backward invariant with respect to the flow induced by the ODE (2.20).

(b) *If K_0 is stable, resp. asymptotically stable, for the flow induced by the ODE (2.20), then K is stable, resp. asymptotically stable, for the flow of (1.1).*

Proof. Let $r > 0$ be the radius determined in Lemma 3.1 and Theorem 3.2. Take $\rho > 0$ smaller than the radius described in Theorem 2.3 such that

$$\rho < r/2 \quad \text{and} \quad \rho_0 := \rho \|P_\mathrm{c}\|_{\mathcal{B}(X_{1-1/p},X_0)} < r/2 \tag{3.37}$$

hold. Let $K \subset B_{X_{1-1/p}}(u_*, \rho)$ be a backward and forward globally invariant set for (1.1); that is, for each $u_* + v_0 \in K$ the solution v of (2.11) with $v(0) = v_0$ exists for all $t \in \mathbb{R}$ and $u_* + v(t) \in K$ for all $t \in \mathbb{R}$.

(a) The inclusion $K \subset \mathcal{M}_\mathrm{c}$ follows from Theorem 2.3(d) by our choice of ρ since K is invariant. We define $K_0 = \{P_\mathrm{c}(u_0 - u_*) : u_0 \in K\}$. Then $K_0 \subset B_{P_\mathrm{c}X_0}(0, \rho_0)$. For $y_0 \in K_0$, the function $v_0 = y_0 + \phi_c(y_0)$ satisfies $u_0 = u_* + v_0 \in K$. The solution v of (2.11) with the initial datum $v(0) = v_0$ thus exists for all $t \in \mathbb{R}$ and satisfies $u_* + v(t) \in K \subset B_{X_{1-1/p}}(u_*, \rho)$. By Theorem 2.3 (c), the function $y = P_\mathrm{c}v$ solves the ODE (2.20) for all $t \in \mathbb{R}$ and thus K_0 is invariant for the flow induced by (2.20).

(b) First, we claim that the following assertions hold provided the numbers $\overline{r}_0, \overline{r} > 0$ are chosen small enough:

$$\text{if } \operatorname{dist}_{P_\mathrm{c}X_0}(y, K_0) \le \overline{r}_0 \quad \text{then } |y|_{1-1/p} \le r, \tag{3.38}$$

$$\text{if } \operatorname{dist}_{X_{1-1/p}}(u_* + v, K) \le \overline{r} \quad \text{then } |v|_{1-1/p} \le r, \tag{3.39}$$

$$\text{if } \operatorname{dist}_{X_{1-1/p}}(u_* + v, K) \le \overline{r} \quad \text{then} \quad |P_\mathrm{s}v - \phi_c(P_\mathrm{c}v)|_{1-1/p} \le c \operatorname{dist}_{X_{1-1/p}}(u_* + v, K). \tag{3.40}$$

To show (3.38), we recall that $K_0 \subset B_{P_\mathrm{c}X_0}(0, \rho_0)$ with $\rho_0 = \rho\|P_\mathrm{c}\|_{\mathcal{B}(X_{1-1/p},X_0)}$. Thus, choosing $w_0 \in K_0$ appropriately and using (3.37), we have

$$\begin{aligned} |y|_{1-1/p} &\le |y - w_0|_{1-1/p} + |w_0|_{1-1/p} \\ &\le 2 \operatorname{dist}_{X_{1-1/p}}(y, K_0) + \rho\|P_\mathrm{c}\|_{\mathcal{B}(X_{1-1/p},X_0)} \\ &\le 2\overline{r}_0 + \rho\|P_\mathrm{c}\|_{\mathcal{B}(X_{1-1/p},X_0)} \le r/2 + r/2 = r, \end{aligned}$$

provided $\overline{r}_0 > 0$ is small enough. The proof of (3.39) is analogous. To show (3.40), let us pick a $w \in K$ such that

$$|u_* + v - w|_{1-1/p} \le 2 \operatorname{dist}_{X_{1-1/p}}(u_* + v, K).$$

We note that $w = u_* + w_0 + \phi_c(w_0)$ for some $w_0 \in K_0$ due to (3.36) and recall that $w_0 \in B_{P_\mathrm{c}X_0}(0, \rho_0)$ and $v \in B_{X_{1-1/p}}(0, r)$ by (3.39). Using the Lipschitz property of ϕ_c on small balls stated in Theorem 2.3, we then obtain

$$\begin{aligned} |P_\mathrm{s}v - \phi_c(P_\mathrm{c}v)|_{1-1/p} &\le |P_\mathrm{s}v - \phi_c(w_0)|_{1-1/p} + |\phi_c(w_0) - \phi_c(P_\mathrm{c}v)|_{1-1/p} \\ &\le |P_\mathrm{s}v - P_\mathrm{s}(w - u_*)|_{1-1/p} + c|P_\mathrm{c}(w - u_*) - P_\mathrm{c}v|_{1-1/p} \\ &\le c|u_* + v - w|_{1-1/p} \le c \operatorname{dist}_{X_{1-1/p}}(u_* + v, K). \end{aligned}$$

Next, let us assume that K_0 is stable, that is, that for each $\overline{r}_0 > 0$ there is a $\overline{\rho}_0 > 0$ such that if $\operatorname{dist}_{P_\mathrm{c}X_0}(y(0), K_0) \le \overline{\rho}_0$ then $\operatorname{dist}_{P_\mathrm{c}X_0}(y(t), K_0) \le \overline{r}_0$ for all $t \ge 0$ for the solution y in $P_\mathrm{c}X_0$ of the ODE (2.20). Here we choose $\overline{r}_0 > 0$ such that (3.38) holds, but possibly $\overline{r}_0$ will be further decreased below. To prove that K is stable, let $\overline{r} > 0$ be given where we may assume that $\overline{r} < r$ and that $\overline{r}$ is so small that (3.39) and (3.40) hold. We have have to find $\overline{\rho} > 0$ such that if $\operatorname{dist}_{X_{1-1/p}}(u_* + v_0, K) \le \overline{\rho}$ then the solution v of (2.11) with the initial data $v(0) = v_0$ exists for all $t \ge 0$ and satisfies $\operatorname{dist}_{X_{1-1/p}}(u_* + v(t), K) \le \overline{r}$ for all $t \ge 0$. Let us fix a $\overline{\rho} < \overline{r}$ to be determined later.

Since $\operatorname{dist}_{X_{1-1/p}}(u_* + v_0, K) \le \overline{\rho} < \overline{r}$, either the solution $v(t)$ of (2.11) is defined for all $t \ge 0$ and satisfies $\operatorname{dist}_{X_{1-1/p}}(u_* + v(t), K) < \overline{r}$ for all $t \ge 0$, or there is a number T such that $\operatorname{dist}_{X_{1-1/p}}(u_* + v(t), K) < \overline{r}$ for all $0 \le t < T$ with equality for $t = T$. (We can again assume that T is larger than 1 due to Remark 2.2.) Suppose that the second option holds. By (3.39), we have $|v(t)|_{1-1/p} \le r$ for all $0 \le t \le T$. Lemma 3.1 thus yields a solution z of (2.11) on $[0, T]$ with $u_* + z(t) \in \mathcal{M}_{\mathrm{c}}$. (We recall that $\mathcal{M}_{\mathrm{c}} = \mathcal{M}_{\mathrm{cu}}$ due to the setting assumed in the theorem.) Moreover, z satisfies $P_{\mathrm{c}} z(T) = P_{\mathrm{c}} v(T)$ and the estimate

$$|v(t) - z(t)|_{1-1/p} \le ce^{-\alpha t} |P_{\mathrm{s}} v_0 - \phi_c(P_{\mathrm{c}} v_0)|_{1-1/p} \tag{3.41}$$

for all $0 \le t \le T$.

We pause to remark the inequality

$$\operatorname{dist}_{P_{\mathrm{c}} X_0}(P_{\mathrm{c}} v(0), K_0) \le c \operatorname{dist}_{X_{1-1/p}}(u_* + v(0), K), \tag{3.42}$$

proved as follows: Pick a $w_0 \in K_0$ such that $w = u_* + w_0 + \phi_c(w_0) \in K$ satisfies the inequality $|u_* + v(0) - w|_{1-1/p} \le 2 \operatorname{dist}_{X_{1-1/p}}(u_* + v(0), K)$. We then establish the claim (3.42) by computing

$$\begin{aligned}
\operatorname{dist}_{P_{\mathrm{c}} X_0}(P_{\mathrm{c}} v(0), K_0) &\le |P_{\mathrm{c}} v(0) - w_0|_{P_{\mathrm{c}} X_0} = |P_{\mathrm{c}}(v(0) - w_0 - \phi_c(w_0))|_{P_{\mathrm{c}} X_0} \\
&\le \|P_{\mathrm{c}}\|_{\mathcal{B}(X_{1-1/p}, X_0)} |v(0) - w_0 - \phi_c(w_0)|_{1-1/p} \\
&= \|P_{\mathrm{c}}\|_{\mathcal{B}(X_{1-1/p}, X_0)} |u_* + v(0) - w|_{1-1/p} \le c \operatorname{dist}_{X_{1-1/p}}(u_* + v(0), K).
\end{aligned}$$

Using (3.41), (3.42), and (3.40) with v replaced by v_0, we obtain

$$\begin{aligned}
\operatorname{dist}_{P_{\mathrm{c}} X_0}(P_{\mathrm{c}} z(0), K_0) &\le |P_{\mathrm{c}} z(0) - P_{\mathrm{c}} v(0)|_0 + \operatorname{dist}_{P_{\mathrm{c}} X_0}(P_{\mathrm{c}} v(0), K_0) \\
&\le c|P_{\mathrm{s}} v_0 - \phi_c(P_{\mathrm{c}} v_0)|_{1-1/p} + c \operatorname{dist}_{X_{1-1/p}}(u_* + v(0), K) \\
&\le c \operatorname{dist}_{X_{1-1/p}}(u_* + v_0, K) \le c\overline{\rho}.
\end{aligned} \tag{3.43}$$

By Theorem 2.3(c), $y(t) = P_{\mathrm{c}} z(t)$ satisfies the ODE (2.20). If $\overline{\rho} > 0$ is chosen sufficiently small, then (3.43) yields

$$\operatorname{dist}_{P_{\mathrm{c}} X_0}(y(0), K_0) \le c\overline{\rho} \le \overline{\rho}_0,$$

where $\overline{\rho}_0 > 0$ was chosen above depending on $\overline{r}_0$. Since K_0 is stable, it follows that

$$\operatorname{dist}_{P_{\mathrm{c}} X_0}(y(t), K_0) \le \overline{r}_0 \ \text{ for all } 0 \le t \le T \tag{3.44}$$

(and thus $|y(t)|_{1-1/p} \le r$ by (3.38)). Using $u_* + z(t) \in \mathcal{M}_{\mathrm{c}}$ and (3.36), and also the Lipschitz property of ϕ_c, we estimate

$$\begin{aligned}
&\operatorname{dist}_{X_{1-1/p}}(u_* + z(t), K) \\
&\qquad = \inf_{w_0 \in K_0} \big| \big(u_* + P_{\mathrm{c}} z(t) + \phi_c(P_{\mathrm{c}} z(t))\big) - \big(u_* + w_0 + \phi_c(w_0)\big) \big|_{1-1/p} \\
&\qquad \le \inf_{w_0 \in K_0} \big(c|P_{\mathrm{c}} z(t) - w_0|_0 + c|P_{\mathrm{c}} z(t) - w_0|_0 \big) \\
&\qquad \le c \operatorname{dist}_{P_{\mathrm{c}} X_0}(P_{\mathrm{c}} z(t), K_0)
\end{aligned} \tag{3.45}$$

for all $0 \le t \le T$. By means of (3.45), (3.41), (3.40), and (3.44), we deduce

$$\begin{aligned}
\operatorname{dist}_{X_{1-1/p}}(u_* + v(t), K) &\le \operatorname{dist}_{X_{1-1/p}}(u_* + z(t), K) + |v(t) - z(t)|_{1-1/p} \\
&\le c \operatorname{dist}_{P_{\mathrm{c}} X_0}(P_{\mathrm{c}} z(t), K_0) + ce^{-\alpha t} |P_{\mathrm{s}} v_0 - \phi_c(P_{\mathrm{c}} v_0)|_{1-1/p}) \qquad (3.46) \\
&\le c \operatorname{dist}_{P_{\mathrm{c}} X_0}(y(t), K_0) + c \operatorname{dist}_{X_{1-1/p}}(u_* + v_0, K) \\
&\le c\overline{r}_0 + c\overline{\rho} < \overline{r}/2 + \overline{r}/2 = \overline{r}
\end{aligned}$$

for all $0 \le t \le T$, provided that $\overline{r}_0 > 0$ and $\overline{\rho} > 0$ are sufficiently small. This strict inequality is a contradiction that proves $T = \infty$. In particular, the inequality $\mathrm{dist}_{X_{1-1/p}}(u_* + v(t), K) < \overline{r}$ holds for all $t \ge 0$ and thus K is stable.

To prove the asymptotic stability of K assuming that K_0 is asymptotically stable, we apply Theorem 3.2 to the solution $v(t)$ that has been just constructed for all $t \ge 0$. That is, we take the solution $\overline{z}$ with $u_* + \overline{z}(t) \in \mathcal{M}_c$ that tracks the solution v as described in (3.22). Replacing z by $\overline{z}$ in (3.43) and (3.46), we obtain

$$\mathrm{dist}_{P_c X_0}(P_c \overline{z}(0), K_0) \le c\, \mathrm{dist}_{X_{1-1/p}}(u_* + v_0, K), \tag{3.47}$$

$$\begin{aligned} \mathrm{dist}_{X_{1-1/p}}(u_* + v(t), K) &\le c\, \mathrm{dist}_{P_c X_0}(P_c \overline{z}(t), K_0) \\ &\quad + c e^{-\alpha t} |P_s v_0 - \phi_c(P_c v_0)|_{1-1/p}\big), \end{aligned} \tag{3.48}$$

for all $t \ge 0$. Set $\overline{y}(t) = P_c \overline{z}(t)$. Due to the asymptotic stability of K_0 and (3.47), if $\mathrm{dist}_{X_{1-1/p}}(u_* + v_0, K)$ is sufficiently small, then one has $\mathrm{dist}_{P_c X_0}(\overline{y}(t), K_0) \to 0$ as $t \to \infty$. Finally, due to (3.48), we conclude that $\mathrm{dist}_{X_{1-1/p}}(u_* + v(t), K) \to 0$ as $t \to \infty$. □

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