Lecture Notes
Spectral Theory

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These lecture notes are based on my course from the summer semester 2015. I kept the numbering and the contents of the results presented in the lectures (except for minor corrections and improvements). Typically, the proofs and calculations in the notes are a bit shorter than those given in the lecture. Moreover, the drawings and many additional, mostly oral remarks from the lectures are omitted here. On the other hand, I have added several lengthy proofs not shown in the lectures (mainly about Sobolev spaces). These are indicated in the text.

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Contents

CHAPTER 1

General spectral theory

General notation. $X \neq \{0\}$, $Y \neq \{0\}$ and $Z \neq \{0\}$ are Banach spaces over $\mathbb{C}$ with norms $\|\cdot\|$ (or $\|\cdot\|_X$ etc.). The space

$$\mathcal{B}(X,Y) = \{T : X \to Y \,\big|\, T \text{ is linear and continuous}\}$$

is endowed with the operator norm $\|T\| = \sup_{\|x\|\leqslant 1}\|Tx\|$, and we abbreviate $\mathcal{B}(X) := \mathcal{B}(X,X)$.

Let $\mathrm{D}(A)$ be a linear subspace of X and $A : \mathrm{D}(A) \to Y$ be linear. Then A, or $(A, \mathrm{D}(A))$, is called *linear operator from X to Y* (and *on X* if $X = Y$) with domain $\mathrm{D}(A)$. We denote by

$$\mathrm{N}(A) = \{x \in \mathrm{D}(A) \,\big|\, Ax = 0\}$$

$$\text{and} \quad \mathrm{R}(A) = \{y \in Y \,\big|\, \exists\, x \in \mathrm{D}(A) \text{ with } y = Ax\}$$

the *kernel* and *range* of A.

1.1. Closed operators

We recall one of the basic examples of an unbounded operator: Let $X = C([0,1])$ be endowed with the supremum norm and let $Af = f'$ with $\mathrm{D}(A) = C^1([0,1])$. Then A is linear, but not bounded. Indeed, the functions $u_n \in \mathrm{D}(A)$ given by $u_n(x) = (1/\sqrt{n})\sin(nx)$ for $n \in \mathbb{N}$ satisfy $\|u_n\|_\infty \to 0$ and

$$\|Au_n\|_\infty \geqslant |u_n'(0)| = \sqrt{n} \to \infty \quad \text{as } n \to \infty.$$

However, if $f_n \in \mathrm{D}(A) = C^1([0,1])$ fulfill $f_n \to f$ and $Af_n = f_n' \to g$ in $C([0,1])$ as $n \to \infty$, then $f \in \mathrm{D}(A)$ and $Af = g$ (see Analysis 1 & 2). This observation leads us to the following basic definition.

Definition 1.1. *Let A be a linear operator from X to Y. The operator A is called* closed *if for all $x_n \in \mathrm{D}(A)$, $n \in \mathbb{N}$, such that there exists $x = \lim_{n\to\infty} x_n$ in X and $y = \lim_{n\to\infty} Ax_n$ in Y, we have $x \in \mathrm{D}(A)$ and $Ax = y$.*

Hence, $\lim_{n\to\infty}(Ax_n) = A(\lim_{n\to\infty} x_n)$ if both (x_n) and (Ax_n) converge and A is closed.

Example 1.2. a) It is clear that every operator $A \in \mathcal{B}(X,Y)$ is closed (with $\mathrm{D}(A) = X$). On $X = C([0,1])$ the operator $Af = f'$ with $\mathrm{D}(A) = C^1([0,1])$ is closed, as seen above.

b) Let $X = C([0,1])$. The operator $Af = f'$ with

$$\mathrm{D}(A) = \{f \in C^1([0,1]) \,\big|\, f(0) = 0\}.$$

is closed in X. Indeed, let $f_n \in \mathrm{D}(A)$ and $f, g \in X$ be such that $f_n \to f$ and $Af_n = f_n' \to g$ in X as $n \to \infty$. Again by Analysis 1 & 2, the function f belongs to $C^1([0,1])$ and $f' = g$. Since $0 = f_n(0) \to f(0)$ as $n \to \infty$, we obtain $f \in \mathrm{D}(A)$. This means that A is closed on X. In the same way we see that $A_1 f = f'$ with

$$\mathrm{D}(A_1) = \{f \in C^1([0,1]) \,|\, f(0) = f'(0) = 0\}$$

is closed in X. There are many more variants.

c) Let $X = C([0,1])$ and $Af = f'$ with

$$\mathrm{D}(A) = C_c^1((0,1]) = \{f \in C^1([0,1]) \,|\, \operatorname{supp} f \subseteq (0,1]\},$$

where the support $\operatorname{supp} f$ of f is the closure of $\{t \in [0,1] \,|\, f(t) \neq 0\}$ in $[0,1]$. This operator is not closed. In fact, consider the maps $f_n \in \mathrm{D}(A)$ given by

$$f_n(t) = \begin{cases} 0, & 0 \leqslant t \leqslant 1/n, \\ (t - 1/n)^2, & 1/n \leqslant t \leqslant 1, \end{cases}$$

for every $n \in \mathbb{N}$. Then, $f_n \to f$ and $f_n' \to f'$ in X as $n \to \infty$, where $f(t) = t^2$. However, $\operatorname{supp} f = [0,1]$ and so $f \notin \mathrm{D}(A)$.

d) Let $X = L^p(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$, and $m : \mathbb{R}^d \to \mathbb{C}$ be measurable. Define $Af = mf$ with

$$\mathrm{D}(A) = \{f \in X \,|\, mf \in X\}.$$

This is the *maximal domain*. Then A is closed. Indeed, let $f_n \to f$ and $Af_n = mf_n \to g$ in X as $n \to \infty$. Then there is a subsequence such that $f_{n_j}(x) \to f(x)$ and $m(x) f_{n_j}(x) \to g(x)$ for a.e. $x \in \mathbb{R}^d$, as $j \to \infty$. Hence, $mf = g$ in $L^p(\mathbb{R}^d)$ and we thus obtain $f \in \mathrm{D}(A)$ and $Af = g$.

e) Let $X = L^1([0,1])$, $Y = \mathbb{C}$, and $Af = f(0)$ with $\mathrm{D}(A) = C([0,1])$. Then A is not closed from X to Y. In fact, look at $f_n \in \mathrm{D}(A)$ given by

$$f_n(t) = \begin{cases} 1 - nt, & 0 \leqslant t \leqslant 1/n, \\ 0, & 1/n \leqslant t \leqslant 1, \end{cases}$$

for $n \in \mathbb{N}$. Then $\|f_n\|_1 = 1/(2n) \to 0$ as $n \to \infty$, but $Af_n = f_n(0) = 1$. $\Diamond$

Definition 1.3. *Let A be a linear operator from X to Y. The* graph *of A is given by*

$$\mathrm{G}(A) = \{(x, Ax) \in X \times Y \,|\, x \in \mathrm{D}(A)\}.$$

The graph norm *of A is defined by $\|x\|_A = \|x\|_X + \|Ax\|_Y$. We write $[\mathrm{D}(A)]$ if we equip $\mathrm{D}(A)$ with $\|\cdot\|_A$.*

Of course, $\|\cdot\|_A$ is equivalent to $\|\cdot\|_X$ if $A \in \mathcal{B}(X,Y)$. We endow $X \times Y$ with the norm $\|(x,y)\|_{X \times Y} = \|x\|_X + \|y\|_Y$. Recall that a seqeunce in $X \times Y$ converges if and only if its components in X and in Y converge.

Lemma 1.4. *Every linear operator A from X to Y satisfies the following assertions.*

(a) $\mathrm{G}(A) \subseteq X \times Y$ *is a linear subspace.*
(b) $[\mathrm{D}(A)]$ *is a normed vector space and* $A \in \mathcal{B}([\mathrm{D}(A)], Y)$.
(c) A *is closed if and only if* $\mathrm{G}(A)$ *is closed in* $X \times Y$ *if and only if* $[\mathrm{D}(A)]$ *is a Banach space.*
(d) Let A be injective and put $\mathrm{D}(A^{-1}) := \mathrm{R}(A)$. *Then, A is closed from X to Y if and only if A^{-1} is closed from Y to X.*

Proof. Parts (a) and (b) follow from the definitions, and assertion (c) implies (d) since

$$\mathrm{G}(A^{-1}) = \{(y, A^{-1}y) \,|\, y \in \mathrm{R}(A)\} = \{(Ax, x) \,|\, x \in \mathrm{D}(A)\}$$

is closed in $Y \times X$ if and only if $\mathrm{G}(A)$ is closed in $X \times Y$. We next show (c).

The operator A is closed if and only if for all $x_n \in \mathrm{D}(A)$, $n \in \mathbb{N}$, and $(x, y) \in X \times Y$ with $(x_n, Ax_n) \to (x, y)$ in $X \times Y$ as $n \to \infty$, we have $x \in \mathrm{D}(A)$ and $Ax = y$; i.e., $(x, y) \in \mathrm{G}(A)$. This property is equivalent to the closedness of $\mathrm{G}(A)$. Since $\|(x, Ax)\|_{X\times Y} = \|x\|_X + \|Ax\|_Y$, a Cauchy sequence or a converging sequence in $\mathrm{G}(A)$ corresponds to a Cauchy or a converging sequence in $[\mathrm{D}(A)]$, respectively. So $[\mathrm{D}(A)]$ is complete if and only if $(\mathrm{G}(A), \|\cdot\|_{X\times Y})$ is complete if and only if $\mathrm{G}(A) \subseteq X \times Y$ is closed. □

The next result is a variant of the open mapping theorem.

Theorem 1.5 (Closed Graph Theorem). *Let X and Y be Banach spaces and A be a closed operator from X to Y. Then A is* bounded *(i.e., $\|Ax\| \leqslant c\,\|x\|$ for some $c \geqslant 0$ and all $x \in \mathrm{D}(A)$) if and only if $\mathrm{D}(A)$ is closed in X. In particular, a closed operator with $\mathrm{D}(A) = X$ belongs to $\mathcal{B}(X, Y)$.*

Proof. "$\Leftarrow$": Let $\mathrm{D}(A)$ be closed in X. Then $\mathrm{D}(A)$ is a Banach space for $\|\cdot\|_X$ and $\|\cdot\|_A$. Since $\|x\|_X \leqslant \|x\|_A$ for all $x \in \mathrm{D}(A)$, a corollary to the open mapping theorem (see e.g. Corollary 4.29 in [**Sc2**]) shows that there is a constant $c > 0$ such that $\|Ax\|_Y \leqslant \|x\|_A \leqslant c\,\|x\|_X$ for all $x \in \mathrm{D}(A)$.

"$\Rightarrow$": Let A be bounded and let $x_n \in \mathrm{D}(A)$ converge to $x \in X$ with respect to $\|\cdot\|_X$. Then $\|Ax_n - Ax_m\|_Y \leqslant c\,\|x_n - x_m\|_X$, and so the sequence $(Ax_n)_n$ is Cauchy in Y. There thus exists $y := \lim_{n\to\infty} Ax_n$ in Y. The closedness of A shows that $x \in \mathrm{D}(A)$; i.e., $\mathrm{D}(A)$ is closed in X. □

We next show that Theorem 1.5 is wrong without completeness and give an example of a non-closed, everywhere defined operator.

Remark 1.6. a) Let T be given by $(Tf)(t) = tf(t)$, $t \in \mathbb{R}$, on $C_c(\mathbb{R})$ with supremum norm. This linear operator is everywhere defined, unbounded and closed. In fact, take $f_n, f, g \in C_c(\mathbb{R})$ such that $f_n(t) \to f(t)$ and $(Tf_n)(t) = tf_n(t) \to g(t)$ uniformly for $t \in \mathbb{R}$ as $n \to \infty$. Then $g(t) = tf(t)$ for all $t \in \mathbb{R}$; i.e., $g = Tf$ and T is closed. Further, pick $\varphi_n \in C_c(\mathbb{R})$ with $\|\varphi_n\|_\infty = 1$ and $\varphi_n(n) = 1$. Since $\|T\varphi_n\|_\infty \geqslant |T\varphi_n(n)| = n$, the operator T is unbounded.

b) Let X be an infinite dimensional Banach space and let $\mathcal{B}$ be an algebraic basis of X, see Theorem III.5.1 in [**La**]. (Hence, for each $x \in X$ there are unique $\alpha_1, \dots, \alpha_n \in \mathbb{C}$, $b_1, \dots, b_n \in \mathcal{B}$ and $n \in \mathbb{N}$ such that $x = \alpha_1 b_1 + \cdots + \alpha_n b_n$.) We may assume that $\|b\| = 1$ for all $b \in \mathcal{B}$. Choose a countable subset $\mathcal{B}_0 = \{b_k \,|\, k \in \mathbb{N}\}$ of $\mathcal{B}$ and set

$$Tb_k = kb_k \text{ for each } b_k \in \mathcal{B}_0, \text{ and } Tb = 0 \text{ for each } b \in \mathcal{B}\backslash\mathcal{B}_0.$$

Then T can be extended to a linear operator on X which is unbounded, since $\|Tb_k\| = k$ and $\|b_k\| = 1$. Thus T is not closed by Theorem 1.5. ◊

Proposition 1.7. *Let A be closed from X to Y, $T \in \mathcal{B}(X, Y)$, and $S \in \mathcal{B}(Z, X)$. Then the following operators are closed.*

(a) $B = A + T$ with $\mathrm{D}(B) = \mathrm{D}(A)$,
(b) $C = AS$ with $\mathrm{D}(C) = \{z \in Z \,|\, Sz \in \mathrm{D}(A)\}$.

PROOF. (a) Let $x_n \in \mathrm{D}(B)$, $n \in \mathbb{N}$, and $x \in X$, $y \in Y$ such that $x_n \to x$ in X and $Bx_n = Ax_n + Tx_n \to y$ in Y as $n \to \infty$. Since T is bounded, there exists $Tx = \lim_{n\to\infty} Tx_n$ and so $Ax_n \to y - Tx$ as $n \to \infty$. The closedness of A then yields $x \in \mathrm{D}(A) = \mathrm{D}(B)$ and $Ax = y - Tx$; i.e., $Bx = Ax + Tx = y$.

(b) Let $z_n \in \mathrm{D}(C)$, $n \in \mathbb{N}$, and $z \in Z$, $y \in Y$ such that $z_n \to z$ in Z and $ASz_n \to y$ in Y as $n \to \infty$. By the boundedness of S, the vectors $x_n := Sz_n$ converge to Sz. Since $Ax_n \to y$ and A is closed, we obtain $Sz \in \mathrm{D}(A)$ and $ASz = y$; i.e., $z \in \mathrm{D}(C)$ and $Cz = y$. □

COROLLARY 1.8. *Let A be linear on X and $\lambda \in \mathbb{C}$. Then the following assertions hold.*

(a) A is closed on X if and only if $\lambda I - A$ is closed on X.
(b) If $\lambda I - A$ is bijective with $(\lambda I - A)^{-1} \in \mathcal{B}(X)$, then A is closed.

PROOF. Assertion (a) is a consequence of Proposition 1.7 since $A = -((\lambda I - A) - \lambda I)$. For the second part, Lemma 1.4 shows that $\lambda I - A$ is closed, and then assertion (a) yields (b). □

The following examples show that closedness can be lost when taking sums or products of closed operators. See the exercises for further related results.

EXAMPLE 1.9. a) Let $E = C_b(\mathbb{R}^2)$ and $A_k = \partial_k$ with

$$\mathrm{D}(A_k) = \{f \in E \,|\, \text{the partial derivative } \partial_k f \text{ exists and belongs to } E\},$$

for $k \in \{1, 2\}$. Set $B = \partial_1 + \partial_2$ on

$$\mathrm{D}(B) := \mathrm{D}(A_1) \cap \mathrm{D}(A_2) = C_b^1(\mathbb{R}^2) = \{f \in C^1(\mathbb{R}^2) \,|\, f, \partial_1 f, \partial_2 f \in E\}.$$

By an exercise, A_1 and A_2 are closed. However, B is not closed.

Indeed, take $\phi_n \in C_b^1(\mathbb{R})$ converging uniformly to some $\phi \in C_b(\mathbb{R})\backslash C^1(\mathbb{R})$. Set $f_n(x, y) = \phi_n(x - y)$ and $f(x, y) = \phi(x - y)$ for $(x, y) \in \mathbb{R}^2$ and $n \in \mathbb{N}$. We then obtain $f \in E$, $f_n \in \mathrm{D}(B)$, $\|f_n - f\|_\infty = \|\phi_n - \phi\|_\infty \to 0$ and $Bf_n = \phi_n' - \phi_n' = 0 \to 0$ as $n \to \infty$, but $f \notin \mathrm{D}(B)$.

b) Let $X = C([0, 1])$, $Af = f'$ with $\mathrm{D}(A) = C^1([0, 1])$ and $m \in C([0, 1])$ such that $m = 0$ on $[0, 1/2]$. Define $T \in \mathcal{B}(X)$ by $Tf = mf$ for all $f \in X$. Then the operator TA with $\mathrm{D}(TA) = \mathrm{D}(A)$ is not closed.

To see this, take functions $f_n \in \mathrm{D}(A)$ such that $f_n = 1$ on $[1/2, 1]$ and $f_n \to f$ in X with $f \notin C^1([0, 1])$. Then, $TAf_n = mf_n' = 0$ converges to 0, but $f \notin \mathrm{D}(A)$. ◊

1.2. The spectrum

DEFINITION 1.10. *Let A be a closed operator on X. The* resolvent set *of A is given by*

$$\rho(A) = \{\lambda \in \mathbb{C} \,|\, \lambda I - A : \mathrm{D}(A) \to X \text{ is bijective}\},$$

and its spectrum *by*

$$\sigma(A) = \mathbb{C}\backslash\rho(A).$$

We further define the point spectrum *of A by*

$$\sigma_p(A) = \{\lambda \in \mathbb{C} \,|\, \exists\, v \in \mathrm{D}(A)\backslash\{0\} \text{ with } \lambda v = Av\} \subseteq \sigma(A),$$

where we call $\lambda \in \sigma_p(A)$ *an* eigenvalue *of* A *and the corresponding* v *an* eigenvector *or* eigenfunction *of* A. *For* $\lambda \in \rho(A)$ *the operator*

$$R(\lambda, A) := (\lambda I - A)^{-1} : X \to X$$

and the set $\{R(\lambda, A) \,|\, \lambda \in \rho(A)\}$ *are called the* resolvent.

Remark 1.11. a) Let A be closed on X and $\lambda \in \rho(A)$. The resolvent $R(\lambda, A)$ has the range $\mathrm{D}(A)$. Corollary 1.8 and Lemma 1.4 further show that $R(\lambda, A)$ is closed and thus it belongs to $\mathcal{B}(X)$ by Theorem 1.5.

b) Let A be a linear operator such that $\lambda I - A : \mathrm{D}(A) \to X$ has a bounded inverse for some $\lambda \in \mathbb{C}$. Then A is closed by Corollary 1.8. In this case, the closedness assumption in Definition 1.10 is redundant. ◊

We set $\mathrm{e}_\lambda(t) = \mathrm{e}^{\lambda t}$ for $\lambda \in \mathbb{C}$, $t \in J$, and any interval $J \subseteq \mathbb{R}$.

Example 1.12. a) Let $X = \mathbb{C}^d$ and $T \in \mathcal{B}(X)$. Then $\sigma(T)$ only consists of the eigenvalues $\lambda_1, \dots, \lambda_m$ of T, where $m \leqslant d$.

b) Let $X = C([0,1])$ and $Au = u'$ with $\mathrm{D}(A) = C^1([0,1])$. Then $\sigma(A) = \sigma_p(A) = \mathbb{C}$. Indeed, e_λ belongs to $\mathrm{D}(A)$ and $A\mathrm{e}_\lambda = \lambda \mathrm{e}_\lambda$ for each $\lambda \in \mathbb{C}$.

c) Let $X = C([0,1])$ and $Au = u'$ with $\mathrm{D}(A) = \{u \in C^1([0,1]) \,|\, u(0) = 0\}$. Then A is closed by Example 1.2. Moreover, $\sigma(A)$ is empty. In fact, let $\lambda \in \mathbb{C}$ and $f \in X$. We then have $u \in \mathrm{D}(A)$ and $(\lambda I - A)u = f$ if and only if $u \in C^1([0,1])$, $u'(t) = \lambda u(t) - f(t)$, $t \in [0,1]$, and $u(0) = 0$, which is equivalent to

$$u(t) = -\int_0^t \mathrm{e}^{\lambda(t-s)} f(s)\,\mathrm{d}s =: (R_\lambda) f(t),$$

for all $0 \leqslant t \leqslant 1$. Hence, $\sigma(A) = \varnothing$ and $R(\lambda, A) = R_\lambda$. ◊

We see in Examples 1.21 and 1.25 that both for unbounded and bounded operators the point spectrum can be empty though the spectrum is void, provided that $\dim X = \infty$.

Let $U \subseteq \mathbb{C}$ be open. The *derivative* of $f : U \to Y$ at $\lambda \in U$ is given by

$$f'(\lambda) = \lim_{\mu \to \lambda} \frac{1}{\mu - \lambda}(f(\mu) - f(\lambda)) \in Y,$$

if the limit exists in Y. In the next theorem we collect the most basic properties of the spectrum and the resolvent of closed operators.

Theorem 1.13. *Let A be a closed operator on X and let $\lambda \in \rho(A)$. Then the following assertions hold.*

(a) $AR(\lambda, A) = \lambda R(\lambda, A) - I$, $AR(\lambda, A)x = R(\lambda, A)Ax$ *for all* $x \in \mathrm{D}(A)$, *and*

$$\frac{1}{\mu - \lambda}(R(\lambda, A) - R(\mu, A)) = R(\lambda, A)R(\mu, A) = R(\mu, A)R(\lambda, A)$$

if $\mu \in \rho(A) \backslash \{\lambda\}$. *The formula in display is called the* resolvent equation.

(b) *The spectrum* $\sigma(A)$ *is closed, where* $B(\lambda, 1/\|R(\lambda,A)\|) \subseteq \rho(A)$ *and*

$$R(\mu, A) = \sum_{n=0}^{\infty} (\lambda - \mu)^n R(\lambda, A)^{n+1} =: R_\mu$$

if $|\lambda-\mu| < 1/\|R(\lambda,A)\| =: r_\lambda$. *This series converges in* $\mathcal{B}(X,[\mathrm{D}(A)])$, *absolutely and uniformly on* $B(\lambda,\delta r_\lambda)$ *for each* $\delta \in (0,1)$. *Moreover,*

$$\|R(\mu,A)\|_{\mathcal{B}(X,[\mathrm{D}(A)])} \leqslant \frac{c(\lambda)}{1-\delta}$$

for all $\mu \in B(\lambda, \delta/\|R(\lambda,A)\|)$ *and a constant* $c(\lambda)$ *given by* (1.1).

(c) *The function* $\rho(A) \to \mathcal{B}(X,[\mathrm{D}(A)])$; $\lambda \mapsto R(\lambda,A)$, *is infinitely often differentiable with*

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}\lambda}\right)^n R(\lambda,A) = (-1)^n\, n!\, R(\lambda,A)^{n+1} \qquad \text{for every} \quad n \in \mathbb{N}.$$

(d) $\|R(\lambda,A)\| \geqslant \frac{1}{d(\lambda,\sigma(A))}$.

Proof. (a) The first assertions follow from

$$x = (\lambda I - A)R(\lambda,A)x = R(\lambda,A)(\lambda I - A)x,$$

where $x \in X$ in the first equality and $x \in \mathrm{D}(A)$ in the second one. For $\mu \in \rho(A)$, we further have

$$(\lambda R(\lambda,A) - AR(\lambda,A))R(\mu,A) = R(\mu,A),$$
$$R(\lambda,A)(\mu R(\mu,A) - AR(\mu,A)) = R(\lambda,A).$$

Subtracting these two equations and also interchanging λ and μ, we deduce the resolvent equation.

(b) Let $|\mu-\lambda| \leqslant \delta/\|R(\lambda,A)\|$ for some $\delta \in (0,1)$ and $x \in X$ with $\|x\| \leqslant 1$. It follows

$$\begin{aligned}
&\|(\lambda-\mu)^n R(\lambda,A)^{n+1}x\|_A \\
&\qquad \leqslant \frac{\delta^n}{\|R(\lambda,A)\|^n}\left(\|AR(\lambda,A)R(\lambda,A)^n x\| + \|R(\lambda,A)^{n+1}x\|\right) \\
&\qquad \leqslant \delta^n\left(\|\lambda R(\lambda,A)\| + 1 + \|R(\lambda,A)\|\right) =: \delta^n c(\lambda). \qquad (1.1)
\end{aligned}$$

Due to this inequality and e.g. Lemma 4.23 in [**Sc2**], the series R_μ converges and can be estimated as asserted. Part (a) yields

$$(\mu I - A)R(\lambda,A) = (\mu-\lambda)R(\lambda,A) + I.$$

Using this fact and that A belongs to $\mathcal{B}([\mathrm{D}(A)],X)$, we infer

$$(\mu I - A)R_\mu = -\sum_{n=0}^{\infty}(\lambda-\mu)^{n+1}R(\lambda,A)^{n+1} + \sum_{n=0}^{\infty}(\lambda-\mu)^n R(\lambda,A)^n = I,$$

and similarly $R_\mu(\mu I - A)x = x$ for all $x \in \mathrm{D}(A)$. Hence, $\mu \in \rho(A)$ and $R_\mu = R(\mu,A)$.

Assertion (c) is a consequence of the power series expansion, as in the scalar case. Part (b) also implies (d). □

Proposition 1.14. *Let* $\Omega \subseteq \mathbb{R}^d$, $m \in C(\Omega)$, $X = C_b(\Omega)$, *and* $Af = mf$ *with* $\mathrm{D}(A) = \{f \in X \,|\, mf \in X\}$. *Then* A *is closed,*

$$\sigma(A) = \overline{m(\Omega)},$$

and $R(\lambda,A)f = \frac{1}{\lambda - m}f$ *for all* $\lambda \in \rho(A)$ *and* $f \in X$.

For every closed (resp., non-empty and compact) subset $S \subseteq \mathbb{C}$ *there is a closed (resp., bounded) operator* B *on a Banach space with* $\sigma(B) = S$.

Proof. The closedness of A can be shown as in Remark 1.6. Let $\lambda \notin \overline{m(\Omega)}$ and $g \in C_b(\Omega)$. The function $f := \frac{1}{\lambda - m} g$ then belongs to $C_b(\Omega)$ and $\lambda f - mf = g$ so that $mf = \lambda f - g \in C_b(\Omega)$. As a result, $f \in \mathrm{D}(A)$ and f is the unique solution in $\mathrm{D}(A)$ of the equation $\lambda f - Af = g$. This means that $\lambda \in \rho(A)$ and $R(\lambda, A)g = \frac{1}{\lambda - m} g$.

In the case that $\lambda = m(z)$ for some $z \in \Omega$, we obtain

$$((\lambda I - A)f)(z) = \lambda f(z) - m(z)f(z) = 0$$

for every $f \in \mathrm{D}(A)$. Consequently, $\lambda I - A$ is not surjective and so $\lambda \in \sigma(A)$. We now conclude that $\sigma(A) = \overline{m(\Omega)}$ since the spectrum is closed.

The final assertion follows from Example 1.12c) if $S = \varnothing$. Otherwise, consider $\Omega = S$. Define A and X as above. Then, $\sigma(A) = S$ and A is bounded if S is compact (where $C_b(S) = C(S)$). □

A similar result is valid in L^p-spaces, see e.g. Example IX.2.6 in [**Co2**].

Example 1.15. Let $X = C_0(\mathbb{R}_+) = \{f \in C(\mathbb{R}_+) \,|\, \lim_{t\to\infty} f(t) = 0\}$ with the supremum norm and $Af = f'$ with

$$\mathrm{D}(A) = C_0^1(\mathbb{R}_+) = \{f \in C^1(\mathbb{R}_+) \,|\, f, f' \in X\}.$$

As in Example 1.2 one sees that A is closed. Moreover, $\sigma(A) = \{\lambda \in \mathbb{C} \,|\, \operatorname{Re}\lambda \leqslant 0\} =: \mathbb{C}_-$ and $\sigma_p(A) = \{\lambda \in \mathbb{C} \,|\, \operatorname{Re}\lambda < 0\}$.

Proof. First note that for $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < 0$, the function e_λ belongs to $\mathrm{D}(A)$ and $A\mathrm{e}_\lambda = \lambda \mathrm{e}_\lambda$. Hence,

$$\{\lambda \in \mathbb{C} \,|\, \operatorname{Re}\lambda < 0\} \subseteq \sigma_p(A) \subseteq \sigma(A),$$

so that $\mathbb{C}_- \subseteq \sigma(A)$ by the closedness of the spectrum.

Next, let $\operatorname{Re}\lambda > 0$ and $f \in X$. We then have $u \in \mathrm{D}(A)$ and $\lambda u - Au = f$ if and only if $u \in X \cap C^1(\mathbb{R}_+)$ and $u'(t) = \lambda u(t) - f(t)$ for all $t \geqslant 0$. This equation is uniquely solved by

$$u(t) = \int_t^\infty \mathrm{e}^{\lambda(t-s)} f(s)\,\mathrm{d}s =: (R_\lambda f)(t), \quad t \geqslant 0.$$

We still have to show that $R_\lambda f \in X$. Let $\varepsilon > 0$. Then there is a number $t_\varepsilon \geqslant 0$ such that $|f(s)| \leqslant \varepsilon$ for all $s \geqslant t_\varepsilon$. We can now estimate

$$|R_\lambda f(t)| \leqslant \int_t^\infty \mathrm{e}^{(\operatorname{Re}\lambda)(t-s)} |f(s)|\,\mathrm{d}s \leqslant \varepsilon \int_0^\infty \mathrm{e}^{-\operatorname{Re}\lambda r}\,\mathrm{d}r = \frac{\varepsilon}{\operatorname{Re}\lambda},$$

for all $t \geqslant t_\varepsilon$, where we substituted $r = s - t$. As a result, $R_\lambda f \in X$ and so $\lambda \in \rho(A)$ with $R_\lambda = R(\lambda, A)$. We have thus shown that $\sigma(A) = \mathbb{C}_-$.

Finally, assume there is a number $\lambda \in \mathrm{i}\mathbb{R}$ and a function $v \in C_0^1(\mathbb{R}_+)$ with $v' = \lambda v$. It follows $v(t) = \mathrm{e}^{\lambda(t-s)} v(s)$ and so $|v(t)| = |v(s)|$ for all $t \geqslant s \geqslant 0$. Letting $t \to \infty$, we infer that $|v(s)| = 0$ for all $s \geqslant 0$; i.e., $v = 0$. Hence, A has no eigenvalues on $\mathrm{i}\mathbb{R}$. □

Complementing Theorem 1.13, we state additional properties of the spectrum if the operator is bounded.

Theorem 1.16. *Let $T \in \mathcal{B}(X)$. Then $\sigma(T)$ is a non-empty compact set. The* spectral radius $r(T) := \max\{|\lambda| \,|\, \lambda \in \sigma(T)\}$ *is given by*

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|T^n\|^{1/n} \leqslant \|T\|,$$

and for $\lambda \in \mathbb{C}$ *with* $|\lambda| > r(T)$ *we have*

$$R(\lambda,T) = \sum_{n=0}^{\infty} \lambda^{-n-1} T^n =: R_\lambda.$$

PROOF. 1) Since $\|T^{n+m}\| \leqslant \|T^n\|\,\|T^m\|$ for all $n,m \in \mathbb{N}$, by an elementary lemma (see Lemma VI.1.4 in [**We**]) there exists the limit

$$\lim_{n\to\infty} \|T^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|T^n\|^{1/n} =: r \leqslant \|T\|.$$

If $|\lambda| > r$, then

$$\limsup_{n\to\infty} \|\lambda^{-n}T^n\|^{1/n} = \frac{1}{|\lambda|} \lim_{n\to\infty} \|T^n\|^{1/n} = \frac{r}{|\lambda|} < 1.$$

Using Lemma 4.23 in [**Sc2**], as in Analysis 1 one now checks that the series R_λ converges in $\mathcal{B}(X)$ for $|\lambda| > r$. Moreover,

$$(\lambda I - T)R_\lambda = \sum_{n=0}^{\infty} \lambda^{-n}T^n - \sum_{n=0}^{\infty} \lambda^{-n-1}T^{n+1} = I,$$

and similarly $R_\lambda(\lambda I - T) = I$. Hence, $\lambda \in \rho(T)$ and $R_\lambda = R(\lambda,T)$. Due to its closedness, the spectrum $\sigma(T) \subseteq \overline{B}(0,r)$ is compact. Therefore, $r(T)$ exists as the maximum of a compact subset of $\mathbb{R}$, and $r(T) \leqslant r$.

2) Take $\Phi \in \mathcal{B}(X)^*$ and define $f_\Phi(\lambda) := \Phi(R(\lambda,T))$ for $\lambda \in D = \mathbb{C}\backslash\overline{B}(0,r(T))$. Note that $f_\Phi : D \to \mathbb{C}$ is complex differentiable and

$$f_\Phi(\lambda) = \sum_{n=0}^{\infty} \lambda^{-n-1}\Phi(T^n) =: S_\lambda \qquad \text{if } \ |\lambda| > r.$$

By e.g. Theorem V.1.11 in [**Co1**], there are unique coeffcients $a_m \in \mathbb{C}$ with

$$f_\Phi(\lambda) = \sum_{m=-\infty}^{\infty} a_m \lambda^m \qquad \text{for } \ \lambda \in D.$$

Hence, the series S_λ converges for all $\lambda \in D$ and so

$$\forall\ \lambda \in D,\ \Phi \in \mathcal{B}(X)^* : \qquad \sup_{n\in\mathbb{N}} |\lambda^{-n-1}\Phi(T^n)| < \infty.$$

A corollary to the uniform boundedness principle (see e.g. Corollary 5.12 in [**Sc2**]) thus yields that

$$c(\lambda) := \sup_{n\in\mathbb{N}} \|\lambda^{-n-1}T^n\| < \infty$$

for each $\lambda \in D$. As a result,

$$\lim_{n\to\infty} \|T^n\|^{1/n} = \lim_{n\to\infty} |\lambda|\,(|\lambda|\,\|\lambda^{-n-1}T^n\|)^{1/n} \leqslant |\lambda| \lim_{n\to\infty} (|\lambda|\,c(\lambda))^{1/n} = |\lambda|$$

for all $|\lambda| > r(T)$. This means that $r = r(T)$.

3) Suppose that $\sigma(T) = \varnothing$. The functions f_Φ from part 2) are then holomorphic on $\mathbb{C}$ for every $\Phi \in \mathcal{B}(X)^*$. Step 1) implies that

$$|f_\Phi(\lambda)| \leqslant \|\Phi\| |\lambda|^{-1} \sum_{n=0}^{\infty} \frac{\|T\|^n}{|\lambda|^n} \leqslant \frac{2\|\Phi\|}{|\lambda|},$$

for all $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant 2\|T\|$. Hence, f_Φ is bounded and thus constant by Liouville's theorem from complex analysis. The above estimate then shows that $\Phi(R(\lambda,T)) = 0$ for all $\lambda \in \mathbb{C}$ and $\Phi \in \mathcal{B}(X^*)$. Employing the Hahn-Banach theorem (see e.g. Corollary 5.10 in [**Sc2**]), we obtain $R(\lambda,T) = 0$, which is impossible since $R(\lambda,T)$ is injective and $X \neq \{0\}$. □

EXAMPLE 1.17. a) We define the Volterra operator V on $X = C([0,1])$ by

$$Vf(t) = \int_0^t f(s)\,\mathrm{d}s$$

for $t \in [0,1]$ and $f \in X$. Then V belongs to $\mathcal{B}(X)$ and

$$|V^n f(t)| \leqslant \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} \|f\|_\infty \,\mathrm{d}s_n \ldots \mathrm{d}s_1 \leqslant \frac{1}{n!}\|f\|_\infty,$$

for all $n \in \mathbb{N}$, $t \in [0,1]$, and $f \in X$. Hence, $\|V^n\| \leqslant 1/n!$. For $f = \mathbb{1}$ we obtain $\|V^n\| \geqslant \|V^n \mathbb{1}\|_\infty = 1/n!$ and so $\|V^n\| = 1/n!$. Theorem 1.16 thus yields

$$r(V) = \lim_{n\to\infty} \left(\frac{1}{n!}\right)^{1/n} = 0 \quad \text{and} \quad \sigma(V) = \{0\}.$$

Observe that $\sigma_p(V) = \varnothing$ since $Vf = 0$ implies that $f = (Vf)' = 0$. Moreover, $\|V\| = 1 > r(V) = 0$.

b) Let left shift L given by $Lx = (x_{n+1})$ on $X \in \{c_0, \ell^p \,|\, 1 \leqslant p \leqslant \infty\}$ has the spectrum $\sigma(L) = \overline{B}(0,1)$. Moreover, $\sigma_p(L) = B(0,1)$ if $X \neq \ell^\infty$ and $\sigma(L) = \overline{B}(0,1)$ if $X = \ell^\infty$.

PROOF. The operator $L \in \mathcal{B}(X)$ has norm 1 (see e.g. Example 2.9 in [**Sc2**]), and so $\sigma(L) \subseteq \overline{B}(0,1)$. Clearly, $L(1,0,\cdots) = 0$. For $|\lambda| < 1$ the sequence $v = (\lambda^n)_{n\in\mathbb{N}}$ belongs to X and satisfies $Lv = (\lambda^{n+1})_n = \lambda v$ so that $B(0,1) \subseteq \sigma_p(L) \subseteq \sigma(L)$. If $|\lambda| = 1$ and $Lx = \lambda x$, then $x_{n+1} = \lambda x_n$ for all $n \in \mathbb{N}$. Hence, $x = (\lambda^{n-1}x_1)_n$ which belongs to X (for $x_1 \neq 0$) if and only if $X = \ell^\infty$. The closedness of the spectrum now implies the assertions. □

For further investigations we need certain subdivisions of the spectrum. We note that in this context also other definitions are used in the literature.

DEFINITION 1.18. *Let A be a linear operator on X. Then we call*

$$\sigma_{ap}(A) = \{\lambda \in \mathbb{C} \,|\, \exists\, x_n \in \mathrm{D}(A) \textit{ with } \|x_n\| = 1 \textit{ for all } n \in \mathbb{N} \textit{ and } \lambda x_n - Ax_n \to 0 \textit{ as } n \to \infty\}$$

the approximate point spectrum *of A and*

$$\sigma_r(A) = \{\lambda \in \mathbb{C} \,|\, (\lambda I - A)\,\mathrm{D}(A) \textit{ is not dense in } X\}$$

the residual spectrum *of A.*

One calls $\lambda \in \sigma_{ap}(A)$ an approximate eigenvalue and the corresponding x_n approximate eigenvectors. If one has $\lambda \in \mathbb{C}$ and $x_n \in \mathrm{D}(A)$ with $\|x_n\| \geqslant \delta > 0$ for all $n \in \mathbb{N}$ and $\lambda x_n - Ax_n \to 0$ as $n \to \infty$, then λ belongs to $\sigma_{ap}(A)$ with approximate eigenvectors $\|x_n\|^{-1}x_n$, since $\|x_n\|^{-1} \leqslant 1/\delta$.

PROPOSITION 1.19. *Let A be closed on X. The following assertions are true (with possibly non-disjoint unions).*

(a) $\sigma_{ap}(A) = \sigma_p(A) \cup \{\lambda \in \mathbb{C} \,|\, (\lambda I - A)\,\mathrm{D}(A) \textit{ is not closed in } X\}$.
(b) $\sigma(A) = \sigma_{ap}(A) \cup \sigma_r(A)$.
(c) $\partial\sigma(A) \subseteq \sigma_{ap}(A)$.

PROOF. 1) Let $\lambda \notin \sigma_{ap}(A)$. Then there is a constant $c > 0$ with $\|(\lambda I - A)x\| \geqslant c\|x\|$ for all $x \in \mathrm{D}(A)$. This lower estimate implies that $\lambda \notin \sigma_p(A)$. Moreover, if $y_n = \lambda x_n - Ax_n \to y$ in X as $n \to \infty$ for some $x_n \in \mathrm{D}(A)$, then the lower estimate shows that (x_n) is Cauchy in X, and so x_n tends to some x in X. Hence, $Ax_n = \lambda x_n - y_n \to \lambda x - y$ and the closedness of A yields $x \in \mathrm{D}(A)$ and $\lambda x - Ax = y$. Consequently, $(\lambda I - A)\,\mathrm{D}(A)$ is closed.

Conversely, if $(\lambda I - A)\,\mathrm{D}(A)$ is closed and $\lambda \notin \sigma_p(A)$, then the inverse $(\lambda I - A)^{-1}$ exists and is closed on its closed domain $(\lambda I - A)\,\mathrm{D}(A)$. The closed graph theorem 1.5 then yields the boundedness of $(\lambda I - A)^{-1}$. Thus,

$$\|x\| = \|(\lambda I - A)^{-1}(\lambda I - A)x\| \leqslant C\|(\lambda I - A)x\|$$

for all $x \in \mathrm{D}(A)$ and a constant $C > 0$. This means that $\lambda \notin \sigma_{ap}(A)$. We thus have shown assertion (a), which implies (b).

2) Let $\lambda \in \partial\sigma(A)$. Then there are points λ_n in $\rho(A)$ with $\lambda_n \to \lambda$ as $n \to \infty$. By Theorem 1.13(d), the norms $\|R(\lambda_n, A)\|$ tend to ∞ as $n \to \infty$, and thus there exist $y_n \in X$ such that $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and $0 \neq a_n := \|R(\lambda_n, A)y_n\| \to \infty$ as $n \to \infty$. Set $x_n = \frac{1}{a_n} R(\lambda_n, A)y_n \in \mathrm{D}(A)$. We then have $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\lambda x_n - Ax_n = (\lambda - \lambda_n)x_n + \frac{1}{a_n} y_n$ converges to 0 as $n \to \infty$. As a result, $\lambda \in \sigma_{ap}(A)$. □

In the next result we determine the spectra of certain operators which (formally) arise as functions $f(A)$ of A, namely the resolvent of A where $f(\mu) = (\lambda - \mu)^{-1}$ for $\lambda \in \rho(A)$ and a affine transformation of A where $f(\mu) = \alpha\mu + \beta$ for $\alpha, \beta \in \mathbb{C}$.

PROPOSITION 1.20. *Let A be closed on X, $\lambda \in \rho(A)$, $\alpha \in \mathbb{C}\backslash\{0\}$ and $\beta \in \mathbb{C}$. Then the following assertions hold.*

(a) $\sigma(R(\lambda, A))\backslash\{0\} = (\lambda - \sigma(A))^{-1} = \{\frac{1}{\lambda - \nu} \,|\, \nu \in \sigma(A)\}$.
(b) $\sigma_j(R(\lambda, A))\backslash\{0\} = (\lambda - \sigma_j(A))^{-1}$ *for* $j \in \{p, ap, r\}$.
(c) *If x is an eigenvector for the eigenvalue $\mu \neq 0$ of $R(\lambda, A)$, then $y = \mu R(\lambda, A)x$ is an eigenvector for the eigenvalue $\nu = \lambda - 1/\mu$ of A. If $y \in \mathrm{D}(A)$ is an eigenvector for the eigenvalue $\nu = \lambda - 1/\mu$ of A with $\mu \in \mathbb{C}\backslash\{0\}$, then $x = \mu^{-1}(\lambda y - Ay)$ is an eigenvector for the eigenvalue μ of $R(\lambda, A)$.*
(d) $r(R(\lambda, A)) = 1/d(\lambda, \sigma(A))$.
(e) *If A is unbounded, then $0 \in \sigma(R(\lambda, A))$.*
(f) $\sigma(\alpha A + \beta I) = \alpha\sigma(A) + \beta$ *and* $\sigma_j(\alpha A + \beta I) = \alpha\sigma_j(A) + \beta$ *for* $j \in \{p, ap, r\}$.

PROOF. For $\mu \in \mathbb{C}\backslash\{0\}$ we have

$$\mu I - R(\lambda, A) = \Big(\big(\lambda - \tfrac{1}{\mu}\big)I - A\Big)\mu R(\lambda, A). \tag{1.2}$$

Observe that the operator $\mu R(\lambda, A) : X \to \mathrm{D}(A)$ is bijective. Hence, the bijectivity of $\mu I - R(\lambda, A)$ is equivalent to that of $(\lambda - \frac{1}{\mu})I - A$. As a result, μ belongs to $\rho(R(\lambda, A))$ if and only if $\lambda - \frac{1}{\mu}$ belongs to $\rho(A)$ if and only if $\mu = (\lambda - \nu)^{-1}$ for some $\nu \in \rho(A)$. We have thus shown (a).

In the same way, one derives assertion (b) for $j = p$, assertion (c) and that $\mu I - R(\lambda, A)$ and $(\lambda - \frac{1}{\mu})I - A$ have the same range. Using also Proposition 1.19, we then deduce (b) for $j = ap$ and $j = r$.

Part (d) is a consequence of (a). In part (e), the inverse $R(\lambda, A)^{-1} = \lambda I - A$ is unbounded so that $0 \in \sigma(R(\lambda, A))$. Similar as (a) and (b), the last assertion follows from the equality

$$\lambda I - (\alpha A + \beta I) = \alpha\big(\tfrac{\lambda-\beta}{\alpha} I - A\big). \qquad \square$$

EXAMPLE 1.21. a) Let $X = L^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, and the translation $T(t)$ be given by $(T(t)f)(s) = f(s+t)$ for $s \in \mathbb{R}$, $f \in X$, and $t \in \mathbb{R}$. Then $\sigma(T(t)) = \partial B(0,1)$ for $t \neq 0$.

PROOF. Recall from Example 4.12 in [**Sc2**] that $T(t)$ is an isometry on X with inverse $(T(t))^{-1} = T(-t)$ for every $t \in \mathbb{R}$. By Theorem 1.16 we have $\sigma(T(t)) \subseteq \overline{B}(0,1)$. Proposition 1.20 further yields $\sigma(T(t))^{-1} = \sigma(T(t)^{-1}) = \sigma(T(-t)) \subseteq \overline{B}(0,1)$ so that $\sigma(T(t)) \subseteq \partial B(0,1)$ for all $t \in \mathbb{R}$. Fix $t \neq 0$. For every $\lambda \in \mathrm{i}\mathbb{R}$, the function e_λ belongs to $C_b(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$ and

$$(T(t)\mathrm{e}_\lambda)(s) = \mathrm{e}^{\lambda(s+t)} = \mathrm{e}^{\lambda t}\mathrm{e}_\lambda(s)$$

for all $s \in \mathbb{R}$. Hence, $\sigma(T(t)) = \sigma_p(T(t)) = \partial B(0,1)$ for $p = \infty$.

If $p \in [1, \infty)$, we use e_λ to construct approximate eigenfunctions. For $n \in \mathbb{N}$ set $f_n = n^{-1/p}\mathbb{1}_{[0,n]}\mathrm{e}_\lambda$. We then have $\|f_n\|_p = n^{-1/p}\|\mathbb{1}_{[0,n]}\|_p = 1$ and (see above)

$$\|T(t)f_n - \mathrm{e}^{\lambda t}f_n\|_p = n^{-1/p}\|\mathrm{e}^{\lambda t}(\mathbb{1}_{[-t,n-t]} - \mathbb{1}_{[0,n]})\|_p = n^{-1/p}|2t|^{-1/p} \to 0,$$

as $n \to \infty$. As a result, $\sigma(T(t)) = \partial B(0,1)$ if $t \neq 0$. $\square$

b) Let $X = C_0(\mathbb{R})$ and $Au = u'$ with $\mathrm{D}(A) = C_0^1(\mathbb{R}) := \{f \in C^1(\mathbb{R}) \,|\, f, f' \in C_0(\mathbb{R})\}$. Then $\sigma(A) = \mathrm{i}\mathbb{R}$ and $\sigma_p(A) = \varnothing$.

PROOF. As in Example 1.15 one sees that $\lambda \in \rho(A)$ if $\operatorname{Re}\lambda \neq 0$ and

$$R(\lambda, A)f(t) = \int_t^\infty \mathrm{e}^{\lambda(t-s)} f(s)\,\mathrm{d}s \qquad \text{if} \quad \operatorname{Re}\lambda > 0,$$

$$R(\lambda, A)f(t) = -\int_{-\infty}^t \mathrm{e}^{\lambda(t-s)} f(s)\,\mathrm{d}s \qquad \text{if} \quad \operatorname{Re}\lambda < 0,$$

for all $t \in \mathbb{R}$ and $f \in X$. Let $\operatorname{Re}\lambda = 0$. Then λ is not an eigenvalue, cf. Example 1.15. Choose $\varphi_n \in C_c^1(\mathbb{R})$ with $\|\varphi_n'\|_\infty \leqslant 1/n$ and $\|\varphi_n\|_\infty = 1$, and set $u_n = \varphi_n \mathrm{e}_\lambda$ for all $n \in \mathbb{N}$. Then, $\|u_n\|_\infty = 1$, $u_n \in \mathrm{D}(A)$ and $Au_n = \varphi_n'\mathrm{e}_\lambda + \varphi_n\mathrm{e}_\lambda' = \varphi_n'\mathrm{e}_\lambda + \lambda u_n$. Since $\|\varphi_n'\mathrm{e}_\lambda\|_\infty \leqslant 1/n$, we obtain $\lambda \in \sigma_{ap}(\mathbb{R})$. $\square$

We now introduce the adjoint of a densely defined linear operator in order to obtain a convenient description of the residual spectrum, for instance.

DEFINITION 1.22. *Let A be a linear operator from X to Y with dense domain. We define its* adjoint *A^* from Y^* to X^* by setting*

$$\mathrm{D}(A^*) = \{y^* \in Y^* \,|\, \exists\, z^* \in X^* \ \forall\, x \in \mathrm{D}(A) : \langle Ax, y^* \rangle = \langle x, z^* \rangle\},$$
$$A^* y^* = z^*.$$

Observe that for all $x \in \mathrm{D}(A)$ and $y^* \in \mathrm{D}(A^*)$ we obtain

$$\langle Ax, y^* \rangle = \langle x, A^* y^* \rangle.$$

We note that the operator $Af = f'$ with $\mathrm{D}(A) = \{f \in C^1([0,1]) \,|\, f(0) = 0\}$ is not densely defined on $X = C([0,1])$ since $\overline{\mathrm{D}(A)} = \{f \in X \,|\, f(0) = 0\}$.

Remark 1.23. Let A be linear from X to Y with $\overline{\mathrm{D}(A)} = X$.

a) Since $\mathrm{D}(A)$ is dense, there is at most one vector $z^* = A^*y^*$ as in Definition 1.22, so that $A^* : \mathrm{D}(A^*) \to X^*$ is a map. It is clear that A^* is linear. If $A \in \mathcal{B}(X,Y)$, then Definition 1.22 coincides with the definition of A^* in §5.4 of [**Sc2**], where $\mathrm{D}(A^*) = Y^*$.

b) The operator A^* is closed from Y^* to X^*.

Proof. Let $y_n^* \in \mathrm{D}(A^*)$, $y^* \in Y^*$, and $z^* \in X^*$ such that $y_n^* \to y^*$ in Y^* and $z_n^* := A^*y_n^* \to z^*$ in X^* as $n \to \infty$. Let $x \in \mathrm{D}(A)$. We derive

$$\langle x, z^*\rangle = \lim_{n\to\infty} \langle x, z_n^*\rangle = \lim_{n\to\infty} \langle Ax, y_n^*\rangle = \langle Ax, y^*\rangle.$$

As a result, $y^* \in \mathrm{D}(A^*)$ and $A^*y^* = z^*$. □

c) If $T \in \mathcal{B}(X,Y)$, then the sum $A + T$ with $\mathrm{D}(A + T) = \mathrm{D}(A)$ has the adjoint $(A + T)^* = A^* + T^*$ with $\mathrm{D}((A + T)^*) = \mathrm{D}(A^*)$.

Proof. Let $x \in \mathrm{D}(A)$ and $y^* \in Y^*$. We obtain

$$\langle (A + T)x, y^*\rangle = \langle Ax, y^*\rangle + \langle x, T^*y^*\rangle.$$

Hence, $y^* \in \mathrm{D}((A + T)^*)$ if and only if $y^* \in \mathrm{D}(A^*)$, and then $(A + T)^*y^* = A^*y^* + T^*y^*$. □

Theorem 1.24. *Let A be a closed operator on X with dense domain. Then the following assertions hold.*

(a) $\sigma_r(A) = \sigma_p(A^*)$.

(b) $\sigma(A) = \sigma(A^*)$ and $R(\lambda, A)^* = R(\lambda, A^*)$ for every $\lambda \in \rho(A)$.

Proof. (a) Due to a corollary of the Hahn-Banach theorem (see e.g. Corollary 5.13 in [**Sc2**]), the set $(\lambda I - A)\,\mathrm{D}(A)$ is not dense in X if and only if there is a vector $y^* \in X^*\backslash\{0\}$ such that $\langle \lambda x - Ax, y^*\rangle = 0$ for every $x \in \mathrm{D}(A)$. This equation is equivalent to $\langle Ax, y^*\rangle = \langle x, \lambda y^*\rangle$, which in turn means that $y^* \in \mathrm{D}(A^*)\backslash\{0\}$ and $A^*y^* = \lambda y^*$; i.e., $\lambda \in \sigma_p(A^*)$.

(b) Let $\lambda \in \rho(A)$. Take $y^* \in \mathrm{D}(A^*)$ and $x \in X$. We compute

$$\begin{aligned}\langle x, R(\lambda, A)^*(\lambda I - A^*)y^*\rangle &= \langle R(\lambda, A)x, (\lambda I - A^*)y^*\rangle \\ &= \langle (\lambda I - A)R(\lambda, A)x, y^*\rangle = \langle x, y^*\rangle,\end{aligned}$$

using Definition 1.22 and that $R(\lambda, A)x$ belongs to $\mathrm{D}(A)$. It follows $R(\lambda, A)^*(\lambda I - A^*)y^* = y^*$ so that $\lambda I - A^*$ is injective. Next, take $x^* \in X^*$. Set $y^* = R(\lambda, A)^*x^*$. For $x \in \mathrm{D}(A)$, we obtain

$$\langle (\lambda I - A)x, y^*\rangle = \langle R(\lambda, A)(\lambda I - A)x, x^*\rangle = \langle x, x^*\rangle.$$

Hence, $y^* \in \mathrm{D}(A^*)$ and $x^* = (\lambda I - A)^*y^* = (\lambda I - A^*)y^*$, where we use Remark 1.23(c). Consequently, the operator $\lambda I - A^*$ is surjective, hence bijective with inverse $R(\lambda, A^*) = R(\lambda, A)^*$.

Conversely, let $\lambda \in \rho(A^*)$. Then λ does not belong to $\sigma_p(A^*) = \sigma_r(A)$, see (a). Take $x \in \mathrm{D}(A)$. Due to a corollary of the Hahn-Banach theorem (see e.g. Corollary 5.10 in [**Sc2**]), there is a functional $y^* \in X^*$ such that $\|y^*\| = 1$ and $\langle x, y^*\rangle = \|x\|$. As above, we calculate

$$\begin{aligned}\|x\| &= \langle x, y^*\rangle = \langle x, (\lambda I - A^*)R(\lambda, A^*)y^*\rangle = \langle (\lambda I - A)x, R(\lambda, A^*)y^*\rangle \\ &\leqslant \|R(\lambda, A^*)\|\,\|\lambda x - Ax\|;\end{aligned}$$

i.e., λ does not belong to $\sigma_{ap}(A)$. Proposition 1.19 thus yields $\lambda \notin \sigma(A)$. □

Example 1.25. Let $X \in \{c_0, \ell^p \,|\, 1 \leqslant p \leqslant \infty\}$. Let $Rx = (0, x_1, x_2, \dots)$ be the right shift on X. Then $\sigma(R) = \overline{B}(0,1)$, $\sigma_p(R) = \emptyset$, $\sigma_r(R) = \overline{B}(0,1)$ for $X = \ell^1$, and $\sigma_r(R) = B(0,1)$ for $X \in \{c_0, \ell^p \,|\, 1 < p < \infty\}$.

Proof. First, let $X \neq \ell^\infty$. From e.g. Example 5.44 of [**Sc2**] we know that $R^* = L$, where the left shift L acts on ℓ^1 if $X = c_0$ and on $\ell^{p'}$ otherwise. Since $\sigma(L) = \overline{B}(0,1)$ by Example 1.17, Theorem 1.24 yields $\sigma(R) = \sigma(R^*) = \sigma(L) = \overline{B}(0,1)$. Similarly, $\sigma_r(R) = \sigma_p(L) = B(0,1)$ if $X = c_0$ or $X = \ell^p$ with $1 < p < \infty$, and $\sigma_r(R) = \sigma_p(L) = \overline{B}(0,1)$ if $X = \ell^1$.

If $X = \ell^\infty$, then $R = L^*$ for L on ℓ^1 so that again $\sigma(R) = \overline{B}(0,1)$.

Clearly, $Rx = 0$ yields $x = 0$. If $\lambda x = Rx = (0, x_1, x_2, \cdots)$ and $\lambda \neq 0$, then $0 = \lambda x_1$ and so $x_1 = 0$. Iteratively one sees that $x = 0$. Hence, R has no eigenvalues. □

Let A be a linear operator from X to Y. Then a linear operator B from X to Y is called A*–bounded* (or if *relatively bounded* with respect to A) if $\mathrm{D}(A) \subseteq \mathrm{D}(B)$ and $B \in \mathcal{B}([\mathrm{D}(A)], Y)$.

Remark 1.26. Let A and B be linear from X to Y with $\mathrm{D}(A) \subseteq \mathrm{D}(B)$.

a) The operator B is A–bounded if and only if there are constants $a, b \geqslant 0$ such that

$$\|Bx\| \leqslant a\,\|Ax\| + b\,\|x\| \tag{1.3}$$

for all $x \in \mathrm{D}(A)$. If $X = Y$ and there exists a point λ in $\rho(A)$, then the A–boundedness of B is also equivalent to the boundedness of $BR(\lambda, A)$. For instance, we then have (1.3) with $a := \|BR(\lambda, A)\|$ and $b := |\lambda|\, a$.

b) Let A be closed and let (1.3) be satisfied with $a < 1$. Then $A + B$ with $\mathrm{D}(A+B) = \mathrm{D}(A)$ is also closed, by an exercise. In view of a), the next result also requires that b in (1.3) is sufficiently small. ◊

Theorem 1.27. *Let A be a closed operator on X and $\lambda \in \rho(A)$. Further, let B be linear on X with $\mathrm{D}(A) \subseteq \mathrm{D}(B)$. Assume that $\|BR(\lambda, A)\| < 1$. Then $A + B$ with $\mathrm{D}(A+B) = \mathrm{D}(A)$ is closed, $\lambda \in \rho(A+B)$,*

$$R(\lambda, A+B) = R(\lambda, A) \sum_{n=0}^{\infty} (BR(\lambda, A))^n = R(\lambda, A)(I - BR(\lambda, A))^{-1},$$

$$\|R(\lambda, A+B)\| \leqslant \frac{\|R(\lambda, A)\|}{1 - \|BR(\lambda, A)\|}\,.$$

Proof. By e.g. Proposition 4.24 of [**Sc2**], the operator $I - BR(\lambda, A)$ has the inverse

$$S_\lambda = \sum_{n=0}^{\infty} (BR(\lambda, A))^n$$

in $\mathcal{B}(X)$. Hence, $\lambda I - A - B = (I - BR(\lambda, A))(\lambda I - A) : \mathrm{D}(A) \to X$ is bijective with the bounded inverse $R(\lambda, A)S_\lambda$. Remark 1.11 thus yields the closedness of $A + B$ on $\mathrm{D}(A)$, and so $\lambda \in \rho(A+B)$. The asserted estimate also follows from Proposition 4.24 of [**Sc2**]. □

The smallness condition in the above theorem is sharp in general: Let $X = \mathbb{C}$, $a \in \mathbb{C} \cong \mathcal{B}(\mathbb{C})$, $a \neq 0$, and $b = a$. Then a is invertible, but $a - a = 0$ is not. Here we have $\lambda = 0$ and $|bR(0, a)| = |\frac{a}{a}| = 1$.

CHAPTER 2

Spectral theory of compact operators

2.1. Compact operators

We first state a few facts from, e.g., Section 1.3 of [**Sc2**]. A non-empty subset $S \subseteq X$ is *compact* if each sequence in S has a subsequence that converges in S. Equivalently, S is compact if every open covering of S has a finite subcovering. We call $S \subseteq X$ *relatively compact* if $\overline{S}$ is compact which means that each sequence in S has a converging subsequence (with limit in $\overline{S}$). Finally, $S \subseteq X$ is relatively compact if and only if it is *totally bounded*; i.e., for each $\varepsilon > 0$ there are finitely many balls in X covering it, where one may chose the centers in S. Recall that compact sets are bounded and closed. The converse is true if and only if X has finite dimension.

DEFINITION 2.1. *A linear map $T : X \to Y$ is called* compact *if $T\overline{B}(0,1)$ is relatively compact in Y. The set of all compact operators is denoted by $\mathcal{B}_0(X,Y)$.*

REMARK 2.2. a) If T is compact, then $T\overline{B}(0,1)$ is bounded and thus T is bounded; i.e., $\mathcal{B}_0(X,Y) \subseteq \mathcal{B}(X,Y)$.

b) Let $T : X \to Y$ be linear. Then the following assertions are equivalent.

(i) T is compact.
(ii) T maps bounded sets of X into relatively compact sets of Y.
(iii) For every bounded sequence $(x_n)_n$ in X there exists a convergent subsequence $(Tx_{n_j})_j$ in Y.

PROOF. $(i) \Rightarrow (ii)$: If T is compact and $B \subseteq X$ is bounded, then $B \subseteq \overline{B}(0,r)$ for some $r \geqslant 0$ and $\overline{TB} \subseteq \overline{T\overline{B}(0,r)} = r\overline{T\overline{B}(0,1)}$, which is compact. Hence, $\overline{TB}$ is compact. The implications $(ii) \Rightarrow (iii) \Rightarrow (i)$ are clear. □

c) The space of *operators of finite rank* is defined by

$$\mathcal{B}_{00}(X,Y) = \{T \in \mathcal{B}(X,Y) \,\big|\, \dim TX < \infty\},$$

see Example 5.16 of [**Sc2**]. For $T \in \mathcal{B}_{00}(X,Y)$, the set $T\overline{B}(0,1)$ is relatively compact, and hence $\mathcal{B}_{00}(X,Y) \subseteq \mathcal{B}_0(X,Y)$.

d) The identity $I : X \to X$ is compact if and only if $\overline{B}(0,1)$ is compact if and only if $\dim X < \infty$. ◊

The next result says that $\mathcal{B}_0(X,Y)$ is a closed two-sided ideal in $\mathcal{B}(X,Y)$.

PROPOSITION 2.3. *The set $\mathcal{B}_0(X,Y)$ is a closed linear subspace of $\mathcal{B}(X,Y)$. Let $T \in \mathcal{B}(X,Y)$ and $S \in \mathcal{B}(Y,Z)$. If one of the operators T or S is compact, then ST is compact.*

PROOF. Let $x_k \in X$, $k \in \mathbb{N}$, satisfy $\sup_{k\in\mathbb{N}}\|x_k\| =: c < \infty$.

1) Let $T, R \in \mathcal{B}_0(X,Y)$ be compact. If $\alpha \in \mathbb{C}$, then αT is also compact. There further exists a converging subsequence $(Tx_{k_j})_j$. Since $(x_{k_j})_j$ is still bounded, there is another converging subsequence $(Rx_{k_{j_l}})_l$. Hence, $((T+R)x_{k_{j_l}})_l$ converges and so $T+R \in \mathcal{B}_0(X,Y)$.

2) Let $T_n \in \mathcal{B}_0(X,Y)$ converge in $\mathcal{B}(X,Y)$ to some $T \in \mathcal{B}(X,Y)$ as $n \to \infty$. Since T_1 is compact, there is a converging subsequence $(T_1 x_{\nu_1(j)})_j$. Because $\|x_{\nu_1(j)}\| \leqslant c$ for all j, there exists a subsubsequence $\nu_2 \subseteq \nu_1$ such that $(T_2 x_{\nu_2(j)})_j$ converges. Note that $(T_1 x_{\nu_2(j)})_j$ still converges. Iteratively, we obtain subsequences $\nu_l \subseteq \nu_{l-1}$ such that $(T_n x_{\nu_l(j)})_j$ converges for all $l \geqslant n$.

Set $u_m = x_{\nu_m(m)}$. Then $(T_n u_m)_m$ converges as $m \to \infty$ for each $n \in \mathbb{N}$. Let $\varepsilon > 0$. Fix an index $N = N_\varepsilon \in \mathbb{N}$ such that $\|T_N - T\| \leqslant \varepsilon$. Let $m \geqslant k \geqslant N$. We then obtain

$$\begin{aligned}\|Tu_m - Tu_k\| &\leqslant \|(T-T_N)u_m\| + \|T_N(u_m-u_k)\| + \|(T_N - T)u_k\| \\ &\leqslant 2\varepsilon c + \|T_N(u_m - u_k)\|.\end{aligned}$$

Therefore (Tu_m) is a Cauchy sequence. So we have shown that T is compact. Hence, $\mathcal{B}_0(X,Y)$ is closed.

3) Let $S \in \mathcal{B}_0(X,Y)$. Since $(Tx_k)_k$ is bounded, there is a converging subsequence $(STx_{k_j})_j$, so that ST is compact.

If $T \in \mathcal{B}_0(X,Y)$, then there is a converging subsequence $(Tx_{k_j})_j$ and thus STx_{k_j} converges. Again, ST is compact. □

Remark 2.4. Strong limits of compact operators may fail to be compact. Consider, e.g., $X = \ell^2$ and $T_n x = (x_1, \dots, x_n, 0, 0, \dots)$ for all $x \in X$ and $n \in \mathbb{N}$. Then $T_n \in \mathcal{B}_{00}(X) \subseteq \mathcal{B}_0(X)$ but $T_n x \to x = Ix$ as $n \to \infty$ for every $x \in X$ and $I \notin \mathcal{B}_0(X)$. ◊

Example 2.5. a) Let $X \in \{C([0,1]), L^p([0,1]) \,|\, 1 \leqslant p \leqslant \infty\}$, $Y = C([0,1])$, and $k \in C([0,1]^2)$. Setting

$$Tf(t) = \int_0^1 k(t,\tau) f(\tau)\,\mathrm{d}\tau,$$

for $f \in X$ and $t \in [0,1]$, we define the integral operator $T : X \to Y$ for the kernel k. We claim that $T \in \mathcal{B}_0(X,Y)$.

Proof. By Analysis 2 or 3, the function Tf is continuous for all $f \in X$ and $T : X \to Y$ is linear. Since $\|Tf\|_\infty \leqslant \|k\|_\infty \|f\|_1 \leqslant \|k\|_\infty \|f\|_p$ (using that $\lambda([0,1]) = 1$), we obtain that $T \in \mathcal{B}(X,Y)$ and thus TB is bounded in Y. where $B := \overline{B}_X(0,1)$. Moreover, for $t, s \in [0,1]$ and $f \in B$ we have

$$\begin{aligned}|Tf(t) - Tf(s)| &\leqslant \int_0^1 |k(t,\tau) - k(s,\tau)|\,|f(\tau)|\,\mathrm{d}\tau \\ &\leqslant \sup_{\tau \in [0,1]} |k(t,\tau) - k(s,\tau)|\,\|f\|_1 \\ &\leqslant \sup_{\tau\in[0,1]} |k(t,\tau) - k(s,\tau)|\,\|f\|_p.\end{aligned}$$

The right hand side tends to 0 as $|t-s| \to 0$ uniformly in $f \in B$, because k is uniformly continuous. Therefore TB is equicontinuous. The Arzela-Ascoli theorem (see e.g. Theorem 1.47 in [**Sc2**]) then implies that TB is relatively compact. Hence, $T \in \mathcal{B}_0(X,Y)$. □

b) Let $X = C([0,1])$ and $Vf(t) = \int_0^t f(s)\,\mathrm{d}s$ for $t \in (0,1)$ and $f \in X$. This defines a bounded operator V on X with norm 1, see Example 1.17. Let $f \in \overline{B}(0,1)$. Then $\|Vf\|_\infty \leqslant 1$ and $\|(Vf)'\|_\infty = \|f\|_\infty \leqslant 1$. The Arzela-Ascoli theorem (see e.g. Corollary 1.48 in [**Sc2**]) then yields the compactness of V.

c) Let $X = L^2(\mathbb{R})$. For $f \in X$, we define

$$Tf(t) = \int_{\mathbb{R}} \mathrm{e}^{-|t-s|} f(s)\,\mathrm{d}s, \qquad t \in \mathbb{R}.$$

Theorem 2.14 of [**Sc2**] yields that $T \in \mathcal{B}(X)$. We claim that T is not compact.

Proof. Take $f_n = \mathbb{1}_{[n,n+1]}$. For $n > m$ in $\mathbb{N}$, we compute $\|f_n\|_2 = 1$ and

$$\begin{aligned}
\|Tf_n - Tf_m\|_2^2 &\geqslant \int_{n+1}^{n+2} \left| \int_n^{n+1} \mathrm{e}^{s-t}\,\mathrm{d}s - \int_m^{m+1} \mathrm{e}^{s-t}\,\mathrm{d}s \right|^2 \mathrm{d}t \\
&= \int_{n+1}^{n+2} \mathrm{e}^{-2t} \big(\mathrm{e}^{n+1} - \mathrm{e}^n - e^{m+1} + \mathrm{e}^m\big)^2\,\mathrm{d}t \\
&\geqslant \tfrac{1}{2}(\mathrm{e}^{-2n-2} - \mathrm{e}^{-2n-4})(\mathrm{e}^{n+1} - 2\mathrm{e}^n)^2 \\
&= \tfrac{1}{2}(\mathrm{e}^{-2} - \mathrm{e}^{-4})(\mathrm{e} - 2)^2 > 0.
\end{aligned}$$

Hence, (Tf_n) has no converging subsequence. □

d) Let $E = L^2(\mathbb{R}^d)$ and $k \in L^2(\mathbb{R}^{2d})$. For $f \in E$, we set

$$Tf(x) = \int_{\mathbb{R}^d} k(x,y) f(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^d.$$

By Example 5.44 in [**Sc2**], this defines an operator $T \in \mathcal{B}(E)$. We claim that T is compact.

Proof. Proposition 4.13 of [**Sc2**] yields $k_n \in C_c(\mathbb{R}^{2d})$ that converge to k in E. Let T_n be the corresponding integral operators in $\mathcal{B}(E)$. There is a closed ball $B_n \subseteq \mathbb{R}^d$ such that $\operatorname{supp} k_n \subseteq B_n \times B_n$. We then have

$$T_n f(x) = \begin{cases} 0, & x \in \mathbb{R}^d \backslash B_n, \\ \int_{B_n} k_n(x,y) f(y)\,\mathrm{d}y, & x \in B_n. \end{cases}$$

for $f \in E$ and $n \in \mathbb{N}$. Let $R_n f = f_{|B_n}$. Fix $n \in N$ and take a bounded sequence (f_k) in E. Arguing as in (a), one finds a subsequence such that $(R_n T_n f_{k_j})_j$ has a limit g_0 in $C(B_n)$. Since B_n has finite measure, this sequence converges also in $L^2(B_n)$. Hence, $T_n f_{k_j}$ tend in E to the 0–extension g of g_0 as $j \in \infty$, and so T_n is compact. Let $f \in \overline{B}_E(0,1)$. Hölder's inequality in the inner integral yields

$$\begin{aligned}
\|Tf - T_n f\|_2^2 &= \int_{\mathbb{R}^d} \left| \int_{R^d} (k(x,y) - k_n(x,y)) f(y)\,\mathrm{d}y \right|^2 \mathrm{d}x \\
&\leqslant \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |k(x,y) - k_n(x,y)|^2\,\mathrm{d}y \int_{\mathbb{R}^d} |f(y)|^2\,\mathrm{d}y\,\mathrm{d}x \\
&\leqslant \|k - k_n\|_2^2
\end{aligned}$$

for all $n \in \mathbb{N}$. The operators T_n thus converge to T in $\mathcal{B}(E)$ so that T is compact by Proposition 2.3. □

Summarizing, integral operators are usually compact if the base space is compact or has finite measure, or if the kernel decays fast enough at infinity.

PROPOSITION 2.6. *Let $T \in \mathcal{B}_0(X,Y)$ and $x_n \xrightarrow{\sigma} x$ in X as $n \to \infty$ (i.e., $\langle x_n, x^* \rangle \to \langle x, x^* \rangle$ as $n \to \infty$ for every $x^* \in X^*$). Then Tx_n converges to Tx in Y as $n \to \infty$.*

PROOF. Let $y^* \in Y^*$. We have $\langle Tx_n - Tx, y^* \rangle = \langle x_n - x, T^* y^* \rangle \to 0$ as $n \to \infty$ for each $y^* \in Y^*$, hence $Tx_n \xrightarrow{\sigma} Tx$ as $n \to \infty$.

Suppose that Tx_n does not converge to Tx. Then there were $\delta > 0$ and a subsequence such that

$$\|Tx_{n_j} - Tx\| \geqslant \delta > 0$$

for all $j \in \mathbb{N}$. By compactness, there is a subsubsequence and a $y \in Y$ such that $Tx_{n_{j_l}} \to y$ as $l \to \infty$. On the other hand, $Tx_{n_{j_l}} \xrightarrow{\sigma} Tx$ and $Tx_{n_{j_l}} \xrightarrow{\sigma} y$. Since weak limits are unique, it follows that $y = Tx$, which is impossible. □

THEOREM 2.7 (Schauder). *An operator $T \in \mathcal{B}(X,Y)$ is compact if and only if its adjoint $T^* \in \mathcal{B}(Y^*, X^*)$ is compact.*

PROOF. 1) Let T be compact and take $y_n^* \in Y^*$ with $\sup_{n\in\mathbb{N}} \|y_n^*\| =: c < \infty$. The set $K := \overline{TB_X(0,1)}$ is a compact metric space for the restriction of the norm of Y. Set $f_n := y_n^*|K \in C(K)$ for each $n \in \mathbb{N}$. Putting $c_1 := \max_{y\in K} \|y\| < \infty$, we obtain

$$\|f_n\|_\infty = \max_{y\in K} |\langle y, y_n^* \rangle| \leqslant cc_1$$

for every $n \in \mathbb{N}$. Moreover, $(f_n)_{n\in\mathbb{N}}$ is equicontinuous since

$$|f_n(y) - f_n(z)| \leqslant \|y_n^*\| \, \|y - z\| \leqslant c \, \|y - z\|$$

for all $n \in \mathbb{N}$ and $y, z \in K$. The Arzela-Ascoli theorem then yields a subsequence $(f_{n_j})_j$ converging in $C(K)$. We thus deduce that

$$\begin{aligned} \|T^* y_{n_j}^* - T^* y_{n_l}^*\|_{X^*} &= \sup_{\|x\|\leqslant 1} |\langle x, T^*(y_{n_j}^* - y_{n_l}^*\rangle| = \sup_{\|x\|\leqslant 1} |\langle Tx, y_{n_j}^* - y_{n_l}^* \rangle| \\ &\leqslant \|f_{n_j} - f_{n_l}\|_{C(K)} \end{aligned}$$

tends to 0 as $j, l \to \infty$. This means that $(T^* y_{n_j}^*)_j$ converges and so T^* is compact.

2) Let T^* be compact. By step 1), the bi-adjoint T^{**} is compact. Let $J_X : X \to X^{**}$ be the canonical isometric embedding. Proposition 5.54 in [**Sc2**] says that $T^{**} J_X = J_Y T$, and hence $J_Y T$ is compact by Proposition 2.3. If (x_n) is bounded in X, we thus obtain a converging subsequence $(J_Y T x_{n_j})_j$ which is Cauchy. Since J_Y is isometric, also $(Tx_{n_j})_j$ is Cauchy and thus converges; i.e., T is compact. □

2.2. The Fredholm alternative

We need some fact from functional analysis. For non-empty sets $M \subseteq X$ and $N_* \subseteq X^*$ we define the *annihilators*

$$\begin{aligned} M^\perp &= \{x^* \in X^* \,|\, \forall y \in M : \langle y, x^* \rangle = 0\}, \\ {}^\perp N_* &= \{x \in X \,|\, \forall y^* \in N_* : \langle x, y^* \rangle = 0\}. \end{aligned}$$

These sets are equal to X^* or X if and only if $M = \{0\}$ or $N_* = \{0\}$, respectively, see e.g. Remark 5.21 in [**Sc2**]. For $T \in \mathcal{B}(X)$ we have

$$\begin{aligned} \mathrm{R}(T)^{\perp} &= \mathrm{N}(T^*), \qquad \overline{\mathrm{R}(T)} = {}^{\perp}\mathrm{N}(T^*), \\ \mathrm{N}(T) &= {}^{\perp}\mathrm{R}(T^*), \qquad \overline{\mathrm{R}(T^*)} \subseteq \mathrm{N}(T)^{\perp}. \end{aligned} \tag{2.1}$$

In particular, $\mathrm{R}(T)$ is dense if and only if T^* is injective; and if $\mathrm{R}(T^*)$ is dense, then T is injective. (See e.g. Proposition 5.46 in [**Sc2**].)

The following theorem extends fundamental results on matrices known from linear algebra.

Theorem 2.8 (Riesz 1918, Schauder 1930). *Let $K \in \mathcal{B}_0(X)$ and $T = I - K$. Then the following assertions hold.*

(a) $\mathrm{R}(T)$ *is closed.*
(b) $\dim \mathrm{N}(T) < \infty$ *and* $\operatorname{codim} \mathrm{R}(T) := \dim X/\mathrm{R}(T) < \infty$.
(c) T is bijective $\Leftrightarrow$ T is surjective $\Leftrightarrow$ T is injective $\Leftrightarrow$ T^* is bijective $\Leftrightarrow$ T^* is surjective $\Leftrightarrow$ T^* is injective.

More precisely, we have

$$\dim \mathrm{N}(T) = \operatorname{codim} \mathrm{R}(T) = \dim \mathrm{N}(T^*) = \operatorname{codim} \mathrm{R}(T^*). \tag{2.2}$$

Corollary 2.9 (Fredholm alternative). *Let $L \in \mathcal{B}_0(X)$, $\lambda \in \mathbb{C}\backslash\{0\}$, and $y \in X$. Then one of the following alternatives holds.*

A) $\lambda x = Lx$ has only the trivial solution $x = 0$.
Then for every $y \in X$ there is a unique solution $x \in X$ of $\lambda x - Lx = y$ given by $x = R(\lambda, L)y$.

B) $\lambda x = Lx$ has an n-dimensional solution space $\mathrm{N}(\lambda I - L)$ for some $n \in \mathbb{N}$.
Then there are n linearly independent solutions $x_1^*, \dots, x_n^* \in X^*$ of $\lambda x^* = L^* x^*$. The equation $\lambda x - Lx = y$ has a solution $x \in X$ if and only if $\langle y, x_k^* \rangle = 0$ for every $k = 1, \dots, n$. Every $z \in X$ satisfying $\lambda z - Lz = y$ is of the form $z = x + x_0$, where $\lambda x - Lx = y$ and $x_0 \in \mathrm{N}(\lambda I - L)$.

Proof of Corollary 2.9. We set $K = \frac{1}{\lambda} L \in \mathcal{B}_0(X)$ and note that $\lambda x - Lx = y$ is equivalent to $(I - K)x = \frac{1}{\lambda} y$. By Theorem 2.8(b), we have either $\dim \mathrm{N}(I - K) = 0$ (case A) or $\dim \mathrm{N}(I - K) = n \in \mathbb{N}$ (case B).

In the first case, $I - K$ is bijective due to Theorem 2.8(c) which yields A).

In the second case, Theorem 2.8(a) shows that $\mathrm{R}(I - K)$ is closed so that $\mathrm{R}(I - K) = {}^{\perp}\mathrm{N}(I - K^*)$ by (2.1). We thus deduce the solvability condition from case B) noting that $\dim \mathrm{N}(I - K^*) = n$ due to (2.2). If $x - Kx = y$ and $z - Kz = y$, then $z - x$ belongs to $\mathrm{N}(I - K)$, as required in case B). □

Example 2.10. Let $X = C([0,1])$ and $Vf(t) = \int_0^t f(s)ds$ for $t \in [0,1]$ and $f \in X$. Then

$$\mathrm{R}(V) = \{g \in C^1([0,1]) \,|\, g(0) = 0\},$$

which is not closed in X. Moreover, $\mathrm{N}(V) = \{0\}$ and V is compact by Examples 1.17 and 2.5, respectively. In this case, $Vf = g$ can not be solved for all $g \in X$. Summing up, the Fredholm alternative fails for $\lambda = 0$. ◊

Proof of Theorem 2.8. **1)** The space $N := \mathrm{N}(T) = T^{-1}(\{0\})$ is closed in X. For $x \in N$ we have $Kx = x \in N$, so that K leaves N invariant and its restriction K_N to N coincides with the identity on N. On the other hand, K_N is still compact so that $\dim N < \infty$ by Remark 2.2d).

2) Since $\dim N < \infty$, there is a closed subspace $C \subseteq X$ such that $N \cap C = \{0\}$ and $N + C = X$; i.e., $X = N \oplus C$, see e.g. Proposition 5.17 in [**Sc2**].

Let $\tilde{T} : C \to \mathrm{R}(T)$ be the restriction of T to C and endow C and $\mathrm{R}(T)$ with the norm of X. (Observe that C is a Banach space, but we do not know yet whether $\mathrm{R}(T)$ is a Banach space.)

If $\tilde{T}x = 0$ for some $x \in C$, then $x \in N$ and so $x = 0$. Let $y \in \mathrm{R}(T)$. Then there is an $x \in X$ such that $Tx = y$. We can write $x = x_0 + x_1$ with $x_0 \in N$ and $x_1 \in C$. Hence, $\tilde{T}x_1 = Tx_1 + Tx_0 = y$, and so $\tilde{T}$ is bijective.

A corollary to the open mapping theorem (see e.g. Corollary 4.31 in [**Sc2**]) now yields that $\mathrm{R}(T) = \mathrm{R}(\tilde{T})$ is closed if and only if $\tilde{T}^{-1} : \mathrm{R}(T) \to C$ is bounded. Assume that $\tilde{T}^{-1}$ was unbounded. Then there would exist elements $\hat{y}_n = \tilde{T}\hat{x}_n$ of $\mathrm{R}(T)$ with $\hat{x}_n \in C$ such that $\hat{y}_n \to 0$ as $n \to \infty$ and $\|\hat{x}_n\| = \|\tilde{T}^{-1}\hat{y}_n\| \geqslant \delta$ for some $\delta > 0$ and all $n \in \mathbb{N}$. We set $x_n = \|\hat{x}_n\|^{-1}\hat{x}_n$ and note that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and that

$$y_n := \tilde{T}x_n = \frac{1}{\|\hat{x}_n\|}\hat{y}_n$$

still converges to 0 as $n \to \infty$. Since K is compact, there exists a subsequence $(x_{n_j})_j$ and a vector $z \in X$ such that $Kx_{n_j} \to z$ as $j \to \infty$. Consequently, $x_{n_j} = y_{n_j} + Kx_{n_j} \to z$ as $j \to \infty$ and so $\|z\| = 1$. Since C is closed, z belongs to C. On the other hand, z is contained in N because of

$$Tz = z - Kz = \lim_{j\to\infty} Kx_{n_j} - K \lim_{j\to\infty} x_{n_j} = 0.$$

Hence, $z \in C \cap N = \{0\}$, which contradicts $\|z\| = 1$. Assertion (a) has been established.

3) Theorem 2.7 shows the compactness of K^* so that $\dim \mathrm{N}(I - K^*) < \infty$ by step 1). Using (2.1) and Proposition 5.23 in [**Sc2**], we further obtain

$$\mathrm{N}(T^*) = \mathrm{R}(T)^\perp \cong (X/\mathrm{R}(T))^*.$$

Since $\mathrm{N}(T^*)$ is finite-dimensional, linear algebra yields that

$$\dim \mathrm{N}(T^*) = \dim(X/\mathrm{R}(T))^* = \dim X/\mathrm{R}(T) = \operatorname{codim} \mathrm{R}(T) \tag{2.3}$$

and so $\operatorname{codim} \mathrm{R}(T) < \infty$; i.e., assertion (b) is true. We next prove in two steps the remaining equalities in (2.2) which then yield the firt part of (c).[1]

4) *Claim A: There is a closed linear subspace $\hat{N}$ with $\dim \hat{N} < \infty$ and a closed linear subspace $\hat{R}$ of X such that*

$$X = \hat{N} \oplus \hat{R}, \quad T\hat{N} \subseteq \hat{N}, \quad T\hat{R} \subseteq \hat{R} \quad \textit{and} \quad T_2 := T_{|\hat{R}} : \hat{R} \to \hat{R} \textit{ is bijective.}$$

Suppose for a moment that Claim A is true. Setting $T_1 := T_{|\hat{N}} \in \mathcal{B}(\hat{N})$, we obtain the following properties.

(i) $\dim \hat{\mathrm{N}}/R(T_1) = \dim \mathrm{N}(T_1)$ (by the dimension formula in $\mathbb{C}^n$).

(ii) $\mathrm{N}(T) = \mathrm{N}(T_1)$. In fact, writing $x = x_1 + x_2$ for $x \in X$, $x_1 \in \hat{N}$ and $x_2 \in \hat{R}$ we deduce that $Tx = 0$ if and only if $T_2x_2 = -T_1x_1 \in \hat{N} \cap \hat{R} = \{0\}$. As T_2 is injective, the latter statement is equivalent to $x_2 = 0$ and $T_1x_1 = 0$. Hence, $x \in \mathrm{N}(T)$ if and only if $x \in \hat{N}$ and $T_1x = 0$; i.e., $\mathrm{N}(T) = \mathrm{N}(T_1)$.

[1]In the lectures (2.2) was not shown. Instead of steps 4) and 5) below, only the (much easier) proof of the first part of (c) was given.

(iii) We define the map

$$\Phi : \hat{N}/\mathrm{R}(T_1) \to X/\mathrm{R}(T); \quad x + \mathrm{R}(T_1) \mapsto x + \mathrm{R}(T),$$

for $x \in \hat{N} \subseteq X$. Because of $\mathrm{R}(T_1) \subseteq \mathrm{R}(T)$, the map Φ is well defined. Of course, it is linear. We want to show that Φ is bijective, which leads to

$$\dim \hat{N}/\mathrm{R}(T_1) = \dim X/\mathrm{R}(T).$$

Proof of (iii). If $\Phi(x + \mathrm{R}(T_1)) = 0$ for some $x \in \hat{N}$, then there is a vector $y \in X$ with $x = Ty$. There are $y_1 \in \hat{N}$ and $y_2 \in \hat{R}$ with $y = y_1 + y_2$. Hence, $T_2 y_2 = x - T_1 y_1$ belongs to $\hat{R} \cap \hat{N} = \{0\}$. Since T_2 is injective, we infer that $y_2 = 0$. Thus, Φ is injective.

Take $x \in X$. There are $x_1 \in \hat{N}$ and $x_2 \in \hat{R} = T\hat{R}$ with $x = x_1 + x_2$. We now conclude that $x - x_1 = x_2 \in \mathrm{R}(T)$ and so

$$\Phi(x_1 + \mathrm{R}(T_1)) = x_1 + \mathrm{R}(T) = x + \mathrm{R}(T).$$

Hence, Φ is bijective. $\Diamond$

The properties (i)–(iii) yield

$$\dim \mathrm{N}(T) = \dim \mathrm{N}(T_1) = \dim \hat{N}/\mathrm{R}(T_1) = \dim X/\mathrm{R}(T) = \operatorname{codim} \mathrm{R}(T). \tag{2.4}$$

Since also K^* is compact by Theorem 2.7, we further obtain

$$\dim \mathrm{N}(T^*) = \operatorname{codim} \mathrm{R}(T^*). \tag{2.5}$$

Combining (2.3)–(2.5), we deduce assertion (c) assuming Claim A.

5) *Proof of Claim A.* We set $N_k = \mathrm{N}(T^k)$ and $R_k = \mathrm{R}(T^k)$ for $k \in \mathbb{N}_0$. Observe that

$$\begin{aligned} \{0\} = N_0 \subseteq N_1 \subseteq N_2 \subseteq \dots, &\qquad X = R_0 \supseteq R_1 \supseteq R_2 \supseteq \dots, \\ TN_k \subseteq N_{k-1} \subseteq N_k, \quad \text{and} &\quad TR_k = R_{k+1} \subseteq R_k \end{aligned} \tag{2.6}$$

for all $k \in \mathbb{N}_0$. We further have

$$T^k = (I - K)^k = I - \sum_{j=1}^{k} \binom{k}{j} (-1)^{j+1} K^j =: I - C_k,$$

where C_k is compact for each $k \in \mathbb{N}$ due to Proposition 2.3. Assertions (a) and (b) now imply that

$$N_k,\ R_k \ \text{ are closed} \quad \text{ and } \quad \dim N_k < \infty \tag{2.7}$$

for every $k \in \mathbb{N}$. We need four more claims to establish Claim A.

Claim 1: There is a minimal $n \in \mathbb{N}_0$ such that $N_n = N_{n+j}$ for all $j \in \mathbb{N}_0$.

Indeed, if it was true that $N_j \subsetneq N_{j+1}$ for all $j \in \mathbb{N}_0$, then Riesz' Lemma (see e.g. Lemma 1.43 in [**Sc2**]) would give $x_j \in N_j$ with $\|x_j\| = 1$ and $d(x_j, N_{j-1}) \geqslant 1/2$ for every $j \in \mathbb{N}_0$. (Here we use that N_{j-1} is closed.) Take $l > k \geqslant 0$. Since $Tx_l + x_k - Tx_k \in N_{l-1}$ by (2.6), we deduce that

$$\|Kx_l - Kx_k\| = \|x_l - (Tx_l + x_k - Tx_k)\| \geqslant 1/2.$$

As a result, $(Kx_k)_k$ has no converging subsequence, which contradicts the compactness of K. So there is a minimal $n \in \mathbb{N}_0$ with $N_n = N_{n+1}$. Let $x \in N_{n+2}$. Then, $Tx \in N_{n+1} = N_n$ so that $x \in N_{n+1}$. This means that $N_{n+1} = N_{n+2}$, and Claim 1 follows by induction.

Claim 2: There is a minimal $m \in \mathbb{N}_0$ such that $R_m = R_{m+j}$ for all $j \in \mathbb{N}_0$.

Indeed, if it was true that $R_{j+1} \subsetneq R_j$ for all $j \in \mathbb{N}_0$, then Riesz' Lemma (see e.g. Lemma 1.44 in [**Sc2**]) would give $x_j \in R_j$ with $\|x_j\| = 1$ and $d(x_j, R_{j+1}) \geqslant 1/2$ for every $j \in \mathbb{N}_0$. (Here we use that R_{j+1} is closed.) Take $l > k \geqslant 0$. Since $Tx_k + x_l - Tx_l \in R_{k+1}$ by (2.6), we deduce that

$$\|Kx_k - Kx_l\| = \|x_k - (Tx_k + x_l - Tx_l)\| \geqslant 1/2.$$

As a result, $(Kx_k)_k$ has no converging subsequence, which contradicts the compactness of K. So there is a minimal $m \in \mathbb{N}_0$ with $R_m = R_{m+1}$. Let $y \in R_{m+1}$. Then there is a vector $x \in X$ such that $y = T^{m+1}x = TT^m x$ and hence $y \in TR_m = TR_{m+1} = R_{m+2}$ by (2.6). This means that $R_{m+1} = R_{m+2}$, and Claim 2 follows by induction.

Claim 3: $N_n \cap R_n = \{0\}$ *and* $N_m + R_m = X$.

Indeed, for the first part, let $x \in N_n \cap R_n$. Then $T^n x = 0$ and there is a vector $y \in X$ such that $T^n y = x$. Hence, $T^{2n} y = 0$ and so $y \in N_{2n} = N_n$ by Claim 1. Consequently, $x = T^n y = 0$.

For the second part, let $x \in X$. Claim 2 yields that $T^m x \in R_m = R_{2m}$, and thus there exists a vector $y \in X$ with $T^m x = T^{2m} y$. Therefore, $x = (x - T^m y) + T^m y$ belongs to $N_m + R_m$.

Claim 4: $n = m$.

Indeed, suppose that $n > m$. Due to Claim 1 and Claim 2, there is an element $x \in N_n \backslash N_m$ and we have $R_n = R_m$. Claim 3 further gives $y \in N_m \subseteq N_n$ and $z \in R_m = R_n$ such that $x = y + z$. Therefore, $z = x - y \in N_n$ so that $z = 0$ by Claim 3. As a result, $x = y \in N_m$, which is impossible.

Second suppose that $n < m$. Due to Claim 1 and Claim 2, we have $N_n = N_m$ and there is an element $x \in R_n \backslash R_m$. Claim 3 further gives $y \in N_m = N_n$ and $z \in R_m \subseteq R_n$ such that $x = y + z$. Therefore, $y = x - z \in R_n$ so that $y = 0$ by Claim 3. As a result, $x = z \in R_m$, which is impossible.

We can now finish the proof of Claim A, setting $\hat{N} := N_n$ and $\hat{R} = R_n$. By (2.7), the spaces N_n and R_n are closed and $\dim N_n < \infty$. From Claims 3 and 4 we then infer that $X = \hat{N} \oplus \hat{R}$. Moreover, (2.6) and Claim 3 yield $T\hat{N} \subseteq \hat{N}$ and $T\hat{R} = \hat{R}$. If $Tx = 0$ for some $x = T^n y \in \hat{R}$ and $y \in X$, then $y \in N_{n+1} = N_n$ by Claim 1. Therefore, $x = 0$ and $T_{|\hat{R}}$ is bijective. □

We next reformulate the Riesz-Schauder theorem in terms of spectral theory. Observe that the Voltera operator in Example 2.10 is compact and has the spectrum $\sigma(V) = \{0\}$ with $\sigma_p(V) = \varnothing$.

Theorem 2.11. *Let* $\dim X = \infty$ *and* $K \in \mathcal{B}(X)$ *be compact. Then the following assertions hold.*

(a) $\sigma(K) = \{0\} \mathbin{\dot\cup} \{\lambda_j \,|\, j \in J\}$, *where* $J \in \{\varnothing, \mathbb{N}, \{1, \dots, n\} \,|\, n \in \mathbb{N}\}$.

(b) $\sigma(K)\backslash\{0\} = \sigma_p(K)\backslash\{0\}$. *For all* $\lambda \in \sigma(K)\backslash\{0\}$ *the range of* $\lambda I - K$ *is closed and*

$$\dim \mathrm{N}(\lambda I - K) = \operatorname{codim} \mathrm{R}(\lambda I - K) < \infty.$$

(c) For all $\varepsilon > 0$ *the set* $\sigma(K)\backslash B(0, \varepsilon)$ *is finite. Hence,* $\lambda_j \to 0$ *as* $j \to \infty$ *if* $J = \mathbb{N}$.

Proof. If $0 \in \rho(K)$, then K would be invertible. Proposition 2.3 now shows that $I = K^{-1}K$ would be compact which contradicts $\dim X = \infty$.

Hence, 0 always belongs to $\sigma(K)$. Observe that thus assertion (a) follows from (c) by taking $\varepsilon = 1/n$ for $n \in \mathbb{N}$.

For $\lambda \in \mathbb{C}\backslash\{0\}$ we have $\lambda I - K = \lambda(I - \frac{1}{\lambda}K)$. Since $\frac{1}{\lambda}K \in \mathcal{B}_0(X)$, Theorem 2.8 implies either $\lambda \in \rho(K)$ or $\lambda \in \sigma_p(K)$ with

$$\dim \mathrm{N}(\lambda I - K) = \dim \mathrm{N}(I - \tfrac{1}{\lambda}K) = \operatorname{codim} \mathrm{R}(I - \tfrac{1}{\lambda}K) = \operatorname{codim} \mathrm{R}(\lambda I - K) < \infty.$$

So we have established (b).

To prove assertion (c), we suppose that for some $\varepsilon_0 > 0$ we have points λ_n in $\sigma(K)\backslash B(0, \varepsilon_0)$ with $\lambda_n \neq \lambda_m$ for all $n \neq m$ in $\mathbb{N}$ and vectors x_n in $X\backslash\{0\}$ with $Kx_n = \lambda_n x_n$. In Linear Algebra it is shown that eigenvectors to different eigenvalues are linearly independent. Hence, the subspaces

$$X_n := \operatorname{lin}\{x_1, \dots, x_n\}$$

satisfy $X_n \subsetneqq X_{n+1}$ for every $n \in \mathbb{N}$. Moreover, $KX_n \subseteq X_n$ and X_n is closed for each $n \in \mathbb{N}$ (since $\dim X_n < \infty$). Riesz' lemma (see e.g. Lemma 1.44 in [**Sc2**]) gives vectors y_n in X_n such that $\|y_n\| = 1$ and $d(y_n, X_{n-1}) \geqslant 1/2$ for each $n \in \mathbb{N}$. There are $\alpha_{n,j} \in \mathbb{C}$ with $y_n = \alpha_{n,1}x_1 + \cdots + \alpha_{n,n}x_n$, and hence

$$\lambda_n y_n - Ky_n = \sum_{j=1}^{n} (\lambda_n - \lambda_j)\alpha_{n,j}x_j = \sum_{j=1}^{n-1} (\lambda_n - \lambda_j)\alpha_{n,j}x_j$$

belongs to X_{n-1}. For $n > m$, the vector $\lambda_n y_n - Ky_n + Ky_m$ is thus contained in X_{n-1} so that

$$\|Ky_n - Ky_m\| = |\lambda_n| \left\| y_n - \frac{1}{\lambda_n}(\lambda_n y_n - Ky_n + Ky_m) \right\| \geqslant \frac{|\lambda_n|}{2} \geqslant \frac{\varepsilon_0}{2}.$$

This fact contradicts the compactness of K. □

Example 2.12. The following bounded operators are not compact since their spectra are not finite or a null sequence.

a) the left and right shifts L and R on c_0 or ℓ^p, $1 \leqslant p \leqslant \infty$, see Examples 1.17 and 1.25;

b) the translation $T(t)f = f(\cdot + t)$ for $t \in \mathbb{R}\backslash\{0\}$, on $L^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, see Example 1.21;

c) multiplication operators $Tf = mf$ on $X = C_b(S)$ for $S \subseteq \mathbb{R}^d$ and a given function $m \in C_b(S)$ if $\overline{m(S)}$ is not finite or a null sequence, see Proposition 1.14. ◊

2.3. The Dirichlet problem and boundary integrals

In this subsection, we discuss a principal application of Theorem 2.8 to partial differential equations. Here we work with real-valued functions for simplicity. We observe that Theorem 2.8 is still true for real scalars by the same proof.

Let $D \subseteq \mathbb{R}^3$ be open, bounded and connected with $\partial D \in C^2$ and outer unit normal ν at ∂D (see part 3) below). Let $\varphi \in C(\partial D)$ be given.

Claim. There is a unique solution u in $C^2(D) \cap C(\overline{D}) := \{u \in C(\overline{D}) \,|\, u_{|D} \in C^2(D)\}$ of the Dirichlet problem

$$\begin{aligned} \Delta u(x) &= 0, \qquad x \in D, \\ u(x) &= \varphi(x), \qquad x \in \partial D. \end{aligned} \tag{2.8}$$

1) Tools from partial differential equations and uniqueness. We first state the strong maximum principle for the Laplacian, see Theorem 2.2.4 in [**Ev**].

(MP) Let $u \in C^2(D) \cap C(\overline{D})$ satisfy $\Delta u = 0$ on D. Then $\max_{\overline{D}} u = \max_{\partial D} u$. If there is a point $x_0 \in D$ such that $u(x_0) = \max_{\overline{D}} u$, then u is constant.

Hence, if $u, v \in C^2(D) \cap C(\overline{D})$ solve (2.8), then $w = u - v \in C^2(D) \cap C(\overline{D})$ satisfies $\Delta w = 0$ on D and $w = 0$ on ∂D. The maximum principle (MP) thus yields that $\max_{\overline{D}} w = 0$. Similarly, the maximum of $-w$ is 0, so that $w = 0$. This means that the problem (2.8) has at most one solution.

Theorem 2.7 of [**PW**] implies the following version of Hopf's lemma.

(HL) Let $u \in C^2(D) \cap C(\overline{D})$ satisfy $\Delta u \geqslant 0$ on D. Assume that there is a point $x_0 \in \partial D$ such that $u(x_0) = \max_{\overline{D}} u$, $\partial_\nu u(x_0)$ exists, and $\partial_\nu u$ is continuous at x_0. Then either u is constant or $\partial_\nu u(x_0) > 0$.

2) The double layer potential. We want to reformulate (2.8) using an integral operator. To this aim, we first consider the "Newton potential" on $\mathbb{R}^3$ given by $\gamma(x) = \frac{1}{4\pi|x|_2}$ for $x \in \mathbb{R}^3\backslash\{0\}$. It satisfies $\Delta\gamma = 0$ on $\mathbb{R}^3\backslash\{0\}$. We define

$$k(x,y) = \frac{\partial}{\partial\nu(y)}\gamma(x-y) = -(\nabla\gamma)(x-y)\cdot\nu(y) = \frac{(x-y)\cdot\nu(y)}{4\pi|x-y|_2^3} \tag{2.9}$$

for all $x \in \mathbb{R}^3$ and $y \in \partial D$ with $x \neq y$, where the dot denotes the Euclidean scalar product in $\mathbb{R}^3$. One introduces the "double layer potential" by setting

$$Sg(x) = \int_{\partial D} k(x,y)g(y)\,\mathrm{d}\sigma(y) \tag{2.10}$$

for all $x \in \mathbb{R}^3\backslash\partial D$ and $g \in C(\partial D)$, where the surface integral is recalled below in part 3). Standard results from Analysis 2 or 3 then imply that $Sg \in C^\infty(\mathbb{R}^3\backslash\partial D)$ and $\Delta Sg = 0$ on $\mathbb{R}^3\backslash\partial D$. For each $\varphi \in C(\partial D)$, one thus obtains the solution $u = Sg_{|D}$ of (2.8) if one can find a map $g \in C(\partial D)$ satisfying

$$\lim_{\substack{x\to z\\ x\in D}} Sg(x) = \varphi(z) \qquad \text{for all } \; z \in \partial D. \tag{2.11}$$

3) The surface integral. A compact boundary $\partial\Omega$ of an open subset $\Omega \subseteq \mathbb{R}^d$ belongs to C^k, $k \in \mathbb{N}$, if there are open subsets $\tilde{U}_j$ and $\tilde{V}_j$ of $\mathbb{R}^d$ and C^k–diffeomorphisms $\Psi_j : \tilde{V}_j \to \tilde{U}_j$, $j \in \{1,\dots,m\}$, such that the functions Ψ_j and Ψ_j^{-1} and their derivatives up to order k have continuous extensions to $\partial\tilde{V}_j$ and $\partial\tilde{U}_j$, respectively, $\partial\Omega \subseteq \tilde{V}_1 \cup \dots \cup \tilde{V}_m$, and Ψ_j maps $V_j := \tilde{V}_j \cap \partial\Omega$ onto $U_j := \tilde{U}_j \cap (\mathbb{R}^{d-1}\times\{0\})$. We set $F_j = \Psi_j^{-1}|_{U_j}$. Below we use these notions for $k = 2$ and $D = \Omega$. We also identify U_j with a subset of $\mathbb{R}^2$ writing $t \in \mathbb{R}^2$ instead of $(t,0) \in \mathbb{R}^3$.

We recall that the surface integral for a (Borel) measurable function $h : \partial D \to \mathbb{R}$ is given by

$$\int_{\partial D} h(y)\,\mathrm{d}\sigma(y) = \sum_{j=1}^m \int_{U_j} \varphi_j(F_j(t))h(F_j(t))\sqrt{\det F'(t)^T F'(t)}dt,$$

if the right hand side exists. Here, $0 \leqslant \varphi_j \in C_c^\infty(\mathbb{R}^3)$ satisfy $\operatorname{supp}\varphi_j \subseteq \tilde V_j$ and $\sum_{j=1}^m \varphi_j = 1$. This definition does not depend on the choice of $\Psi_j : \tilde V_j \to \tilde U_j$ and φ_j. Moreover, $\sigma(B) = \int_{\partial D} \mathbb{1}_B \,\mathrm{d}\sigma$ defines a (finite) measure on the Borel sets of ∂D. In particular, the above integral has the usual properties of integrals. We mostly omit the index $j \in \{1, \dots, m\}$.

Recall that for $y = F(t) \in V$ and $t \in U \subseteq \mathbb{R}^2$, the tangent plane of ∂D at y is spanned by $\partial_1 F(t)$ and $\partial_2 F(t)$, where $t \in U \subseteq \mathbb{R}^2$. Taylor's formula applied to $\Psi^{-1} \in C_b^2(\tilde U)$ at $(t, 0)$ yields that

$$\begin{aligned} x := \Psi^{-1}(s,0) &= y + (\Psi^{-1})'(t,0)\begin{pmatrix} s-t \\ 0 \end{pmatrix} + \mathcal{O}(|s-t|_2^2) \\ &= y + F'(t)(s-t) + \mathcal{O}(|s-t|_2^2). \end{aligned}$$

for $s \in U$. Using that $\nu(y)$ is orthogonal to $\partial_j F(t)$, we deduce that

$$(x-y)\cdot\nu(y) = \nu(y)^T F'(t)(s-t) + \mathcal{O}(|s-t|_2^2) = \mathcal{O}(|s-t|_2^2). \tag{2.12}$$

On the other hand, Ψ^{-1} and Ψ are globally Lipschitz so that

$$c\,|s-t|_2 \leqslant |x-y|_2 \leqslant C\,|s-t|_2 \tag{2.13}$$

for all $x = F(s) \in V$, $y = F(t) \in V$ with $s, t \in U$ and some constants $C, c > 0$. In the following we denote by c various, possibly differing constants.

4) Compactness of a version of S on ∂D. Using (2.9), (2.12) and (2.13), we obtain

$$|k(x,y)| = \frac{|(x-y)\cdot\nu(y)|}{4\pi\,|x-y|_2^3} \leqslant \frac{c}{|x-y|_2} \leqslant \frac{c}{|s-t|_2} \tag{2.14}$$

for all $x = F(s)$ and $y = F(t)$ in V with $x \neq y$. As a result, the integrands

$$\varphi(F(t))k(F(s),F(t))g(F(t))\sqrt{\det F'(t)^T F'(t)}$$

of Sg are bounded by a constant times $|s-t|_2^{-1}\|g\|_\infty$ for all $x = F(s)$ and $y = F(t)$ in ∂D with $x \neq y$. We next set $k(x,x) = 0$ for $x \in \partial D$ and

$$k_n(x,y) = \begin{cases} k(x,y), & |x-y|_2 > 1/n, \\ n^3(4\pi)^{-1}(x-y)\cdot\nu(y), & |x-y|_2 \leqslant 1/n, \end{cases}$$

for $n \in \mathbb{N}$. By means of (2.12) and (2.13), we estimate

$$|k_n(x,y)| \leqslant cn^3|s-t|_2^2 \leqslant c|s-t|_2^{-1} \tag{2.15}$$

if $|x-y|_2 \leqslant 1/n$ because then $|s-t|_2 \leqslant c/n$.

Since k_n is continuous on $\partial D \times \partial D$, we can define an operator $T_n \in \mathcal{B}(C(\partial D))$ by

$$T_n g(x) = \int_{\partial D} k_n(x,y)g(y)\,\mathrm{d}\sigma$$

for $x \in \partial D$ and $g \in C(\partial D)$. As in Example 2.5a), one shows that T_n is compact thanks to the Arzela-Ascoli theorem. For $g \in C(\partial D)$ and $x \in \partial D$, we set $D(x,n) = D \cap B(x, \frac{1}{n})$ and calculate

$$\int_{\partial D} |(k(x,y) - k_n(x,y))g(y)|\,\mathrm{d}\sigma(y) = \int_{D(x,n)} |k(x,y) - k_n(x,y)|\,|g(y)|\,\mathrm{d}\sigma(y)$$

$$
\begin{aligned}
&\leqslant \|g\|_\infty \int_{D(x,n)} (|k(x,y)| + |k_n(x,y)|)\,\mathrm{d}\sigma(y) \\
&\leqslant c\,\|g\|_\infty \sum_{j=1}^{m} \int_{U_j \cap B(s,\frac{c}{n})} \frac{\mathrm{d}t}{|s-t|_2} \\
&\leqslant c\,\|g\|_\infty \int_{B(0,\frac{c}{n})} \frac{\mathrm{d}v}{|v|_2} \\
&\leqslant c\,\|g\|_\infty \int_0^{\frac{c}{n}} \frac{r\mathrm{d}r}{r} \leqslant \frac{c\,\|g\|_\infty}{n}, \qquad (2.16)
\end{aligned}
$$

employing (2.13), (2.14), (2.15), and polar coordinates in $\mathbb{R}^2$.

Hence, for each $x \in \partial D$ the function $y \mapsto k(x,y)g(y)$ is integrable for the surface measure σ on ∂D. So we can define

$$Tg(x) := \int_{\partial D} k(x,y)g(y)\,\mathrm{d}\sigma(y),$$

for $x \in \partial D$ and $g \in C(\partial D)$. By (2.16), the functions $T_n g$ converge uniformly on ∂D to Tg as $n \to \infty$ so that $Tg \in C(\partial D)$. Estimate (2.16) actually implies that the differences $T_n - T$ belong to $\mathcal{B}(C(\partial D))$ and converge to 0 in this space. Hence, T is contained in $\mathcal{B}(C(\partial D))$ and it is compact by Proposition 2.3 since all T_n are compact.

5) Facts from potential theory. In Theorems VIII and IX in Chapter VI of [**Ke**] it is shown that

$$\lim_{\substack{x\to z\\ x\in D}} Sg(x) = Tg(z) - \tfrac{1}{2}g(z) \quad \text{and} \quad \lim_{\substack{x\to z\\ x\in\mathbb{R}^3\setminus D}} Sg(x) = Tg(z) + \tfrac{1}{2}g(z), \quad (2.17)$$

for all $z \in \partial D$ and $g \in C(\partial D)$. We set

$$v(x) = \begin{cases} Sg(x), & x \in \mathbb{R}^3\setminus\overline{D}, \\ g(x), & x \in \partial D. \end{cases} \qquad (2.18)$$

If $v = 0$ on D, then there exists $\partial_\nu v(y) = 0$ for $y \in \partial D$ due to Theorem X in Chapter VI of [**Ke**].

6) Conclusion. Let $\varphi \in (\partial D)$. In view of (2.11) and (2.17), the function $Sg \in C^2(D)$ has an extension $u \in C^2(D) \cap C(\overline{D})$ solving (2.8) provided that $g \in C(\partial D)$ satisfies $\frac{1}{2}g - Tg = -\varphi$.

Since T is compact, thanks to the Fredholm alternative Corollary 2.9 it remains to establish the injectivity of $\frac{1}{2}I - T$. So let $g_0 \in C(\partial D)$ satisfy $\frac{1}{2}g_0 = Tg_0$. By the previous paragraph, the extension of Sg_0 to $\overline{D}$ then solves (2.8) with $\varphi = 0$. This problem has also the trivial solution $u = 0$. The uniqueness of (2.8) thus yields $Sg_0 = 0$ on D.

Define v_0 by (2.18) with g_0 instead of g. Then $\partial_\nu v_0 = 0$ on ∂D due to the result mentioned after (2.18). We are now looking for a contradiction with Hopf's lemma (HL), employing Sg_0 on $\mathbb{R}^3\setminus\overline{D}$. Fix $r_0 > 0$ such that $\overline{D} \subseteq B(0,r_0)$. For $r \geqslant r_0 + 1$, $x \in \partial B(0,r)$ and $y \in \partial D$, from (2.10) and (2.9) we deduce that

$$|k(x,y)| \leqslant \frac{c}{|x-y|_2^2} \leqslant \frac{c}{(r-r_0)^2} \leqslant \frac{c}{r^2},$$

$$|Sg_0(x)| \leqslant \int_{\partial D} \frac{c}{r^2} \|g_0\|_\infty d\sigma \leqslant \frac{c}{r^2}.$$

Suppose that $g_0 \neq 0$. We can thus fix a radius $r \geqslant r_0 + 1$ such that

$$\|g_0\|_\infty > \max_{x \in \partial B(r)} |Sg_0(x)|.$$

In particular, v_0 is not constant on $\overline{B(r)}\backslash D$. Since $\Delta v_0 = 0$ on $B(r)\backslash\overline{D}$, the strong maximum principle (MP) says that v_0 does not attain its maximum on $B(r)\backslash\overline{D}$. Since the maximum exists on $\overline{B(r)}\backslash D$, it must be attained at a point $y_0 \in \partial D$, and hence $v_0(y_0) > v_0(x)$ for all $x \in B(r)\backslash\overline{D}$. Noting that $-\nu(y_0)$ is the outer unit normal of $B(r)\backslash\overline{D}$ at y_0, we infer from (HL) that $\partial_\nu v_0(y_0) < 0$ contradicting $\partial_\nu v_0 = 0$ on ∂D. As a result, $\frac{1}{2}I - T$ is injective and we have established the claim.

2.4. Closed operators with compact resolvent

We now extend the results of Section 2.2 to a class of closed operators introduced in the next definition.

Definition 2.13. *A closed operator A on X has a* compact resolvent *if there exists a $\lambda \in \rho(A)$ such that $R(\lambda, A) \in \mathcal{B}(X)$ is compact.*

Remark 2.14. a) Let A have a compact resolvent $R(\lambda, A)$ and $\mu \in \rho(A)$. The resolvent equation then yields

$$R(\mu, A) = R(\lambda, A) + (\lambda - \mu)R(\lambda, A)R(\mu, A),$$

so that $R(\mu, A)$ is also compact due to Proposition 2.3.

b) Let A be closed on X with $\lambda \in \rho(A)$. Recall that $[\mathrm{D}(A)] = (\mathrm{D}(A), \|\cdot\|_A)$. The following assertions are equivalent.

(i) A has a compact resolvent.
(ii) Each bounded sequence in $[\mathrm{D}(A)]$ has a subsequence that converges in X.
(iii) The inclusion map $J : [\mathrm{D}(A)] \to X$ is compact.

Proof. Let (i) hold. Take $x_n \in \mathrm{D}(A)$ with $\|x_n\|_A \leqslant c$ for $n \in \mathbb{N}$. Set $y_n = \lambda x_n - Ax_n$. Then $\|y_n\| \leqslant (|\lambda| + 1)c$ for every $n \in \mathbb{N}$ so that $x_n = R(\lambda, A)y_n$ has a subsequence which converges in X by (i), and (ii) is true. The implication '(ii) $\Rightarrow$ (iii)' follows from Remark 2.2. Let (iii) be valid. Define $R_\lambda \in \mathcal{B}(X, [\mathrm{D}(A)])$ by $R_\lambda x = R(\lambda, A)x$ for $x \in X$. The operator $R(\lambda, A) = JR_\lambda : X \to X$ is then compact due to Proposition 2.3. $\square$

c) If $T \in \mathcal{B}(X)$ has a compact resolvent, then $\dim X < \infty$. In fact, the identity $I = (\lambda I - T)R(\lambda, T)$ would be compact by Proposition 2.3, if $R(\lambda, T)$ was compact for some $\lambda \in \rho(T)$.

d) Let A be closed and densely defined on X with $\lambda \in \rho(A)$. Then, A has a compact resolvent if and only if A^* has a compact resolvent. Indeed, first note that $\lambda \in \rho(A^*)$ and $R(\lambda, A)^* = R(\lambda, A^*)$ by Theorem 1.24. Theorem 2.7 then yields that the compactness of $R(\lambda, A)$ and $R(\lambda, A)^* = R(\lambda, A^*)$ are equivalent. $\Diamond$

In the next example we use the Hölder space

$$C^\alpha(S) = \Big\{ f \in C_b(S) \,\Big|\, [f]_\alpha = \sup_{x \neq y} \frac{|f(y) - f(x)|}{|x-y|^\alpha} < \infty \Big\}$$

for $\alpha \in (0,1)$ and $S \subseteq \mathbb{R}^d$ which is a Banach space with norm $\|f\|_{C^\alpha} = \|f\|_\infty + [f]_\alpha$. For $\alpha = 1$ the corresponding space is denoted by $C^{1-}(S)$. We have $C^{1-}(S) \hookrightarrow C^\alpha(S) \hookrightarrow C^\beta(S) \hookrightarrow C_b(S)$ if $0 < \beta \leqslant \alpha < 1$ where the embeddings are given by the inclusion map.

Example 2.15. a) Let $U \subseteq \mathbb{R}^d$ be open and bounded, E be a closed subspace of $C(\overline{U})$, and A be closed on E with $\lambda \in \rho(A)$. Assume that $\mathrm{D}(A) \hookrightarrow C^\alpha(\overline{U})$ for some $\alpha \in (0,1)$. A bounded sequence (f_n) in $[\mathrm{D}(A)]$ is thus bounded in $C^\alpha(\overline{U})$. The theorem of Arzela-Ascoli (see e.g. Corollary 1.48 in [**Sc2**]) yields a subsequence that converges in $C(\overline{U})$, and hence in E. By Remark 2.14, A has a compact resolvent in E.

b) Let $X = C([0,1])$, $Au = u'$ and $\mathrm{D}(A) = \{u \in C^1([0,1]) \,|\, u(0) = 0\}$. The spectrum of A is empty by Example 1.12. So part a) implies that A has compact resolvent.

c) Let $X = C([0,1])$, $Au = u'$ and $\mathrm{D}(A) = C^1([0,1])$. The resolvent set of A is empty by Example 1.12. Therefore A has no (compact) resolvent although $[\mathrm{D}(A)]$ is compactly embedded in X by a). $\diamondsuit$

Theorem 2.16. *Let $\dim X = \infty$ and A be a closed operator with compact resolvent. Then the following assertions are true*

(a) $\sigma(A)$ is either empty or $\sigma(A) = \sigma_p(A)$ contains at most countably many eigenvalues λ_j.

(b) If $\sigma(A)$ is infinite, then $|\lambda_j| \to \infty$ as $j \to \infty$.

(c) For all $\lambda_j \in \sigma(A)$, the range of $\lambda_j I - A$ is closed and $\dim N(\lambda_j I - A) = \operatorname{codim} \mathrm{R}(\lambda_j I - A) < \infty$.

Proof. Fix $\mu \in \rho(A)$. By Theorem 2.11, the spectrum $\sigma(R(\mu, A))$ only contains 0 and either no or finitely many eigenvalues μ_j or a nullsequence of eigenvalues μ_j. Moreover, the range of $\mu_j I - R(\mu, A)$ is closed and $\dim N(\mu_j I - R(\mu, A)) = \operatorname{codim} \mathrm{R}(\mu_j I - R(\mu, A)) < \infty$ for all j. Proposition 1.20 yields $(\mu - \sigma(A))^{-1} = \sigma(R(\mu, A)) \backslash \{0\}$ and $(\mu - \sigma_p(A))^{-1} = \sigma_p(R(\mu, A)) \backslash \{0\}$. These facts imply assertions (a) and (b), where $\lambda_j = \mu - \mu_j^{-1}$. Observe that

$$\lambda_j I - A = \mu_j^{-1}(\mu_j I - R(\mu, A))(\mu I - A).$$

Hence, also (c) follows from the results of Theorem 2.11 stated above. $\square$

Example 2.17. a) Let $X = C([0,1])$ and $Au = u'$ with $\mathrm{D}(A) = \{u \in C^1([0,1]) \,|\, u(0) = u(1)\}$. Then A is closed, has a compact resolvent and $\sigma(A) = \sigma_p(A) = 2\pi \mathrm{i} \mathbb{Z}$.

Proof. The closedness is shown as in Example 1.2. Let $f \in X$. A function u belongs to $\mathrm{D}(A)$ and satisfies $u - Au = f$ if and only if $u \in C^1([0,1])$, $u(0) = u(1)$ and $u' = u - f$. These properties are equivalent to

$$u(t) = c\mathrm{e}^t - \int_0^t \mathrm{e}^{t-s} f(s)\,\mathrm{d}s =: R_c f(t), \quad t \in [0,1], \qquad \text{and} \qquad u(0) = u(1),$$

for some $c = c(f)$, which means that

$$c = R_c f(0) = R_c f(1) = c\mathrm{e} - \mathrm{e}\int_0^1 \mathrm{e}^{-s} f(s)\,\mathrm{d}s,$$
$$c = \frac{\mathrm{e}}{\mathrm{e}-1}\int_0^1 \mathrm{e}^{-s} f(s)\,\mathrm{d}s.$$

We thus derive $1 \in \rho(A)$ and

$$R(1,A)f(t) = \frac{\mathrm{e}^{t+1}}{\mathrm{e}-1}\int_0^1 \mathrm{e}^{-s} f(s)\,\mathrm{d}s - \int_0^t \mathrm{e}^{t-s} f(s)\,\mathrm{d}s$$

for $t \in [0,1]$. Due to Example 2.15a), the operator A thus has a compact resolvent and hence $\sigma(A) = \sigma_p(A)$ by Theorem 2.16. Finally, $\lambda \in \mathbb{C}$ belongs to $\sigma_p(A)$ if and only if there is a non-zero $u \in C^1([0,1])$ with $u(0) = u(1)$ and $u' = \lambda u$ which is equivalent to $u = \mathrm{e}_\lambda u(0)$ and $1 = \mathrm{e}_\lambda(0) = \mathrm{e}_\lambda(1) = \mathrm{e}^\lambda$,; i.e., $\lambda \in 2\pi\mathrm{i}\mathbb{Z}$. □

b) Let $X = C([0,1])$ and $Au = u''$ with $\mathrm{D}(A) = \{u \in C^2([0,1]) \,|\, u(0) = u(1) = 0\}$. Then A is closed, has a compact resolvent and $\sigma(A) = \sigma_p(A) = \{-\pi^2 k^2 \,|\, k \in \mathbb{N}\}$.

Proof. We first note that $u(t) = \sin(\pi k t)$ is an eigenfunction for A and the eigenvalue $\lambda = -\pi^2 k^2$, where $k \in \mathbb{N}$. Conversely, if $\lambda \in \sigma_p(A)$, we have a map $u \in C^2([0,1])$ with $u(0) = u(1) = 0$ and $u'' = \lambda u$. There thus exist $a, b, \mu \in \mathbb{C}$ with $\mu^2 = \lambda$ and $u = a\mathrm{e}_\mu + b\mathrm{e}_{-\mu} \neq 0$. The conditions $u(0) = 0$ and $u(1) = 0$ then yield $a + b = 0$ and $a\mathrm{e}^\mu + b\mathrm{e}^{-\mu} = 0$, respectively. Hence, $e^{2\mu} = 1$ and $\mu \neq 0$; i.e., $\mu = \mathrm{i}\pi k$ and $\lambda = -\pi^2 k^2$ for some $k \in \mathbb{Z}\backslash\{0\}$. As in a), it remains to find a point in $\rho(A)$. To this aim, set

$$Rf(t) = \frac{1}{2}\int_0^1 |t-s|\, f(s)\,\mathrm{d}s = \frac{1}{2}\int_0^t (t-s) f(s)\,\mathrm{d}s + \frac{1}{2}\int_t^1 (s-t) f(s)\,\mathrm{d}s,$$

for $t \in [0,1]$ and $f \in X$. Clearly, R is a bounded operator on X and

$$\frac{\mathrm{d}}{\mathrm{d}t} Rf(t) = \frac{1}{2}\int_0^t f(s)\,\mathrm{d}s - \frac{1}{2}\int_t^1 f(s)\,\mathrm{d}s,$$

so that $Rf \in C^2([0,1])$ with $(Rf)'' = f$. Then the function

$$R_0 f(t) = Rf(t) - (1-t)Rf(0) - tRf(1), \qquad t \in [0,1],$$

belongs to $D(A)$ and $AR_0 f = f$. Since A is injective, we have shown that A has the inverse R_0. □

We note that in the above simple examples one could compute the resolvent for all $\lambda \in \rho(A)$, cf. Example 5.9. In this sense the power of Theorem 2.16 is not really needed here.

2.5. Fredholm operators

In this section we study a class of operators which satisfy most of the assertions of the Riesz–Schauder Theorems 2.8 or 2.16. This class turns out to be stable under compact perturbations, and it arises in many applications as we indicate in Example 2.26.

Definition 2.18. *A map* $T \in \mathcal{B}(X,Y)$ *is called a* Fredholm operator *if*
a) its range $\mathrm{R}(T)$ *is closed in* Y*,*
b) $\dim \mathrm{N}(T) < \infty$*.*
c) $\operatorname{codim} \mathrm{R}(T) = \dim Y/\mathrm{R}(T) < \infty$*.*
In this case the index *of* T *is the integer* $\operatorname{ind}(T) = \dim \mathrm{N}(T) - \operatorname{codim} \mathrm{R}(T)$*.*

For a closed operator A in Y we use the above definition with $X = [\mathrm{D}(A)]$. One sometimes calls $\dim \mathrm{N}(T)$ the nullity and $\operatorname{codim} \mathrm{R}(T)$ the defect of T.

Remark 2.19. a) An invertible operator $T \in \mathcal{B}(X,Y)$ is Fredholm. Loosely speaking, Fredholmity means 'invertibility except for finite dimensional spaces', cf. Proposition 2.20 and the exercises.

b) Theorem 2.8 says that the operator $\lambda I - K$ is Fredholm with index 0 if $K \in \mathcal{B}(X)$ is compact and $\lambda \in \mathbb{C}\backslash\{0\}$. The same is true for $\lambda I - A$ if $\lambda \in \mathbb{C}$ and A is closed in X with compact resolvent, by Theorem 2.16.

c) Each integer can occur as an index of a Fredholm operator. For instance, let $T = L^n$ for the left shift L on ℓ^p, $1 \leqslant p \leqslant \infty$, and some $n \in \mathbb{N}$; i.e., $Tx = (x_{n+k})_k$. Because of $\mathrm{R}(T) = \ell^p$ and $\mathrm{N}(T) = \operatorname{lin}\{\mathrm{e}_1, \dots, \mathrm{e}_n\}$, the operator T is Fredholm with index n. Moreover, $S = R^n$ has index $-n$ for the right shift R on ℓ^p and $n \in \mathbb{N}$ since $Sx = (0, \cdots, 0, x_1, x_2, \cdots)$ with n zeros, S is injective, and $\ell^p/\mathrm{R}(S) \cong \operatorname{lin}\{\mathrm{e}_1, \dots, \mathrm{e}_n\}$ (cf. Example 2.20 of [**Sc2**]).

d) If $\dim X = \infty$ then the Fredholm operators are not a linear subspace of $\mathcal{B}(X)$, since e.g. the identity I is Fredholm, but $I - I = 0$ not.

e) One can omit condition a) in Definition 2.18.

Proof. [Kato (1958)] Let $T \in \mathcal{B}(X,Y)$ satisfy $\operatorname{codim} \mathrm{R}(T) = n \in \mathbb{N}_0$. If $n = 0$, then $\mathrm{R}(T) = Y$ and we are done. Hence, take $n \in \mathbb{N}$. The operator $Q : X \to X/\mathrm{N}(T)$; $Qx = x + \mathrm{N}(T) = \hat{x}$, is a surjective contraction with kernel $\mathrm{N}(T)$, see e.g. Proposition 2.19 in [**Sc2**]. By $\hat{T}(x+\mathrm{N}(T)) = \hat{T}\hat{x} := Tx$, we define a bijective operator $\hat{T} \in \mathcal{B}(X/\mathrm{N}(T), \mathrm{R}(T))$ such that $T = \hat{T}Q$, cf. Proposition A.1.3 of [**Co2**]. Further, there are $y_1, \cdots, y_n \in Y$ such that $Y = \mathrm{R}(T) + \operatorname{lin}\{y_1, \cdots, y_n\}$ and $\mathrm{R}(T) \cap \operatorname{lin}\{y_1, \cdots, y_n\} = \{0\}$. On the Banach space $E = (X/\mathrm{N}(T)) \times \mathbb{C}^n$ we define the operator

$$S : E \to Y; \quad S(\hat{x}, (\lambda_1, \cdots, \lambda_n)) = \hat{T}\hat{x} + \sum_{j=1}^{n} \lambda_j y_j = Tx + \sum_{j=1}^{n} \lambda_j y_j.$$

It straightforward to check that S is linear, bounded and bijective. The open mapping theorem (see e.g. Theorem 4.28 in [**Sc2**]) then says that S is an isomorphism. Consequently, $\mathrm{R}(T) = S((X/\mathrm{N}(T)) \times \{0\})$ is closed. $\square$

In context of part e) of the above remark, we stress that for each infinite dimensional Banach space X there are non-closed subspaces Z of X with codimension 1. To see this, take a countable subset $\mathcal{B}_0 = \{b_k \,|\, k \in \mathbb{N}\}$ of an algebraic basis $\mathcal{B}$ of X as in Remark 1.6. Set $\varphi(b_k) = k$ and $\varphi(b) = 0$ for $b \in \mathcal{B}\backslash\mathcal{B}_0$. Then φ extends to an unbounded linear map $\varphi : X \to \mathbb{C}$. Define $\hat{\varphi}$ as in the proof of Remark 2.19e). It is a linear bijection from $X/\mathrm{N}(\varphi)$ to $R(\varphi) = \mathbb{C}$; i.e., $\operatorname{codim} \mathrm{N}(\varphi) = 1$. However, $Z = \mathrm{N}(\varphi)$ is not closed by Proposition III.5.3 of [**Co2**].

The analysis in this section is based on the properties of Fredholm operators established in the next result. (See the exercises for a related characterization of Fredholm operators of any index.)

Proposition 2.20. *Let $T \in \mathcal{B}(X,Y)$ be a Fredholm operator with $\operatorname{ind}(T) = 0$. Then there exists an invertible operator $J \in \mathcal{B}(Y,X)$ and a finite rank operator $K \in \mathcal{B}_{00}(X)$ such that $JT = I_X - K$.*

Proof. By the assumptions, there are closed subspaces X_1 of X and Y_0 of Y such that $X = \mathrm{N}(T) \oplus X_1$, $Y = Y_0 \oplus \mathrm{R}(T)$ and $\dim Y_0 = \operatorname{codim} \mathrm{R}(T) = \dim \mathrm{N}(T)$. (Use e.g. Proposition 5.17 of [**Sc2**].) We set

$$T_1 : X_1 \times Y_0 \to Y; \quad T_1(x_1, y_0) = Tx_1 + y_0.$$

Clearly, T_1 is linear and continuous. If $T_1(x_1, y_0) = 0$ for some (x_1, y_0) in $X_1 \times Y_0$, then $y_0 = -Tx_1 \in \mathrm{R}(T)$ so that $y_0 = 0$. Hence, x_1 belongs to $\mathrm{N}(T) \cap X_1$; i.e., $x_1 = 0$ and T_1 is injective. Let $y \in Y$. Then $y = y_0 + y_1$ for some $y_0 \in Y_0$ and $y_1 = Tx_1$ with $x_1 \in X$. As a result, $y = y_0 + Tx_1 = T_1(x_1, y_0)$ and T_1 is bijective. The inverse $T_1^{-1} : Y \to X_1 \times Y_0$ is then bounded by the open mapping theorem, see e.g. Theorem 4.28 in [**Sc2**]. Observe that $T_1^{-1} y_1 = (x_1, 0)$ if $Tx_1 = y_1$ for some $x_1 \in X_1$.

There exists an isomorphism $S : Y_0 \to \mathrm{N}(T)$ since these spaces have the same finite dimension. Define

$$S_1 : X_1 \times Y_0 \to Y; \quad S_1(x_1, y_0) = x_1 + Sy_0.$$

As above, one checks that S_1 is invertible. We now introduce the invertible operator $J = S_1 T_1^{-1} : Y \to X$ and the map $K := I_X - JT \in \mathcal{B}(X)$. For $x_1 \in X_1$ we compute

$$Kx_1 = x_1 - S_1 T_1^{-1} T x_1 = x_1 - S_1(x_1, 0) = x_1 - x_1 = 0.$$

Since $X = \mathrm{N}(T) \oplus X_1$, we derive $KX \subseteq K\,\mathrm{N}(T)$, and hence $\dim \mathrm{R}(K) \leqslant \dim \mathrm{N}(T) < \infty$ as asserted. □

We can now show an important perturbation result for Fredholmity and the index. The quantitative smallness condition from Theorem 1.27 (on invertibility) is replaced by compactness.

Theorem 2.21. *Let $T \in \mathcal{B}(X,Y)$ be Fredholm and $K \in \mathcal{B}(X,Y)$ be compact. Then $T + K \in \mathcal{B}(X,Y)$ is Fredholm with $\operatorname{ind}(T+K) = \operatorname{ind}(T)$.*

Proof. Set $n = \operatorname{ind}(T) \in \mathbb{Z}$.

1) Let $n = 0$. Proposition 2.20 yields an invertible operator $J \in \mathcal{B}(Y,X)$ and a map $K_1 \in \mathcal{B}_{00}(X)$ such that $JT = I_X - K_1$. The product JK is compact by Proposition 2.3. We thus deduce that $J(T+K) = I_X - (K_1 - JK)$ is Fredholm with index 0 from Theorem 2.8. The invertibility of J easily implies that also $T + K$ is Fredholm with index 0.

2) Let $n > 0$. Set $\tilde{Y} = Y \times \mathbb{C}^n$, and define

$$\tilde{T} : X \to \tilde{Y}; \quad \tilde{T}x = (Tx, 0), \qquad \tilde{K} : X \to \tilde{Y}; \quad \tilde{K}x = (Kx, 0).$$

It is straightforward to check that $\mathrm{R}(\tilde{T})$ is closed, $\mathrm{N}(T) = \mathrm{N}(\tilde{T})$, $\tilde{Y}/\mathrm{R}(\tilde{T}) \cong (Y/\mathrm{R}(T)) \times \{0\}$. In particular, $\operatorname{codim} \mathrm{R}(\tilde{T}) = \operatorname{codim} \mathrm{R}(T) + n$ and so $\tilde{T}$ is Fredholm with index 0. Since $\tilde{K}$ is still compact, by part 1) the operator

$\tilde{T}+\tilde{K}$ is also Fredholm with index 0. Noting that $(\tilde{T}+\tilde{K})x = (Tx+Kx, 0)$, we infer that $T+K$ is Fredholm with index n.

3) Let $n<0$. Set $\hat{X} = X \times \mathbb{C}^{|n|}$, and define

$$\hat{T} : \hat{X} \to Y; \quad \hat{T}(x,\xi) = Tx, \qquad \hat{K} : \hat{X} \to Y; \quad \hat{K}(x,\xi) = Kx.$$

Starting from $\dim \mathrm{N}(\hat{T}) = \dim \mathrm{N}(T) + |n|$ and $\mathrm{R}(\hat{T}) = \mathrm{R}(T)$, one derives the assertion as in part 2). □

To exploit the above result in spectral theory, we need another definition.

DEFINITION 2.22. *For $T \in \mathcal{B}(X)$ we define the* essential spectrum *by*

$$\sigma_{\mathrm{ess}}(T) = \{\lambda \in \mathbb{C} \,|\, \lambda I - T \textit{ is not Fredholm}\}.$$

We also set

$$\sigma^0_{\mathrm{ess}}(T) = \{\lambda \in \mathbb{C} \,|\, \lambda I - T \textit{ is not Fredholm of index 0}\}.$$

For a closed operator A on X we define analogously

$$\sigma_{\mathrm{ess}}(A) = \{\lambda \in \mathbb{C} \,|\, \lambda I - A : [\mathrm{D}(A)] \to X \textit{ is not Fredholm}\},$$
$$\sigma^0_{\mathrm{ess}}(A) = \{\lambda \in \mathbb{C} \,|\, \lambda I - A : [\mathrm{D}(A)] \to X \textit{ is not Fredholm of index 0}\}.$$

Observe that $\sigma_{\mathrm{ess}}(B) \subseteq \sigma^0_{\mathrm{ess}}(B) \subseteq \sigma(B)$ and that $\lambda \in \sigma(B)\backslash\sigma^0_{\mathrm{ess}}(B)$ is an eigenvalue with finite dimensional eigenspace since $\lambda I - B$ is then Fredholm with $\dim \mathrm{N}(B) = \operatorname{codim} \mathrm{R}(B) < \infty$. (Here $B \in \mathcal{B}(X)$ or B is closed on X.)

There are various differing concepts of the essential spectrum in the literature. Typically, they lead to the same *essential spectral radius*

$$r_{\mathrm{ess}}(T) = \sup\{|\lambda| \,|\, \lambda \in \sigma_{\mathrm{ess}}(T)\},$$

if $T \in \mathcal{B}(X)$. The next concept is used below in our perturbation result Theorem 2.25 for $\sigma_{\mathrm{ess}}(A)$.

DEFINITION 2.23. *Let A and B be linear from X to Y with $\mathrm{D}(A) \subseteq \mathrm{D}(B)$. Then B is called A–*compact *(or,* relatively compact with respect to A*) if $B : [\mathrm{D}(A)] \to Y$ is compact.*

Note that an A–compact operator is automatically A–bounded. Relative compactness is further discussed in the excercises.

LEMMA 2.24. *Let A be closed from X to Y and B be A–compact. Then $A+B$ with $\mathrm{D}(A+B) = \mathrm{D}(A)$ is closed and B is relatively compact with respect to $A+B$.*

PROOF. 1) We first check that B is $(A+B)$–compact. Let $(x_n) \subseteq \mathrm{D}(A)$ be bounded for $\|\cdot\|_{A+B}$. In particular, (x_n) is bounded in X.

Suppose that $\alpha_n := \|Ax_n\|$ tends to infinity as $n \to \infty$. Set $\tilde{x}_n = \alpha_n^{-1} x_n$ for $n \in \mathbb{N}$. (We may assume that $Ax_n \neq 0$ for all n.) Then $\tilde{x}_n \to 0$ in X and $(A+B)\tilde{x}_n = \alpha_n^{-1}(A+B)x_n \to 0$ in Y as $n \to \infty$, whereas $\|A\tilde{x}_n\| = 1$ for all n. Since B is A–compact, there is a subsequence such that $B\tilde{x}_{n_j}$ converges to some y in Y as $j \to \infty$. Hence, $A\tilde{x}_{n_j}$ tends to $-y$. The closedness of A then yields $y = 0$, which is impossible since $1 = \|A\tilde{x}_{n_j}\| \to \|y\|$ as $j \to \infty$.

We conclude that there exists a subsequence such that $(Ax_{n_k})_k$ is bounded in Y. Using again the A–compactness of B, we obtain another subsequence $(Bx_{n_{k_l}})_l$ with a limit in Y; i.e., B is $(A+B)$–compact.

2) Let $x_n \in \mathrm{D}(A)$ tend to some x in X and $(A+B)x_n$ to some y in Y as $n \to \infty$. By part 1), there is subsequence and a vector $z \in Y$ such that $Bx_{n_j} \to z$ in Y as $j \to \infty$. As a result, Ax_{n_j} tends to $y - z$. Since A is closed, we infer that $x \in \mathrm{D}(A) = \mathrm{D}(A+B)$ and $Ax = y - z$. This means that $x_{n_j} \to x$ in $[\mathrm{D}(A)]$, and hence $Bx_{n_j} \to Bx = z$ by continuity. It follows $(A+B)x = y$ and so $A+B$ is closed. □

Theorem 2.25. *Let A be closed on X and B be A–compact. Then*

$$\sigma_{\mathrm{ess}}(A+B) = \sigma_{\mathrm{ess}}(A) \qquad \text{and} \qquad \sigma^0_{\mathrm{ess}}(A+B) = \sigma^0_{\mathrm{ess}}(A).$$

Proof. We apply Theorem 2.21 to the operators $\lambda I - A$ and $-B$ from $[\mathrm{D}(A)]$ to X, where $\lambda \in \sigma_{\mathrm{ess}}(A)$ or $\lambda \in \sigma^0_{\mathrm{ess}}(A)$, and to the operators $\lambda I - A - B$ and B from $[\mathrm{D}(A+B)]$ to X, where $\lambda \in \sigma_{\mathrm{ess}}(A+B)$ or $\lambda \in \sigma^0_{\mathrm{ess}}(A+B)$. In the second part we use that B is $(A+B)$–compact by Lemma 2.24. Consequently, $\lambda I - A$ is Fredholm (with index 0) if and only if $\lambda I - A - B$ is Fredholm (with index 0), as asserted. □

We apply the above result to a typical situation arising in partial differential equations. However, we can only give a rough sketch.

Example 2.26. We study the asymptotic stability of stationary solutions to the reaction-convection-diffusion equation

$$\begin{aligned} \partial_t u(t,x) &= d\partial_{xx}u(t,x) + c\partial_x u(t,x) + f(u(t,x)), \qquad t \geqslant 0,\ x \in \mathbb{R}, \\ u(0,x) &= u_0(x), \qquad x \in \mathbb{R}, \end{aligned} \tag{2.19}$$

for diffusion and convection constants $d > 0$ and $c \in \mathbb{R}$, a given initial state $u_0 \in C_{ub}(\mathbb{R}) = \{v \in C_b(\mathbb{R}) \,|\, v \text{ is uniformly continuous}\}$, and a function $f \in C^2(\mathbb{C})$ with $f(\mathbb{R}) \subseteq \mathbb{R}$ describing (auto-)reaction (where we mean real differentiability). We interpret $u(t,x)$ as the density of a species.

It is known that there is a maximal existence time $\bar{t} = \bar{t}(u_0) \in (0,\infty]$ and a unique solution u of (2.19) in the space $C([0,\bar{t}), C_b(\mathbb{R})) \cap C^1((0,\bar{t}), C_b(\mathbb{R})) \cap C((0,\bar{t}), C^2_b(\mathbb{R}))$.[2] Moreover, if $u_0 \geqslant 0$ and $f(0) \geqslant 0$, then $u \geqslant 0$. (See e.g. Proposition 7.3.1 in [**Lu**] or Theorem 3.8 in [**Sc3**].)

Let $u_* \in C^2_b(\mathbb{R},\mathbb{R})$ be a stationary solution of (2.19); i.e., $u(t,x) = u_*(x)$ solves (2.19) which is equivalent to

$$0 = du_*'' + cu_*' + f(u_*) \qquad \text{on } \ \mathbb{R}.$$

One now asks whether such special solutions describe well the behavior of (2.19), at least locally near u_*. One possible answer is the principle of linearized stability. Here, one proceeds similar as for ordinary differential equations. Define in $X = C_b(\mathbb{R})$ the maps A and F by

$$Av = dv'' + cv' \quad \text{with} \quad \mathrm{D}(A) = C^2_b(\mathbb{R}), \qquad F(v) = f \circ v.$$

One can then check that $F \in C^1(X,X)$ with derivative $F'(v) \in \mathcal{B}(X)$ at $v \in X$ given by $F'(v)w = f'(v)w$ for $w \in X$. (One defines differentiablity in Banach spaces analogously as in $\mathbb{R}^n$.) We introduce the linearized operator at u_* by setting

$$A_*v = Av + F'(u_*)v = dv'' + cv' + f'(u_*)v \quad \text{with} \quad \mathrm{D}(A_*) = C^2_b(\mathbb{R}).$$

[2] We note that one needs the uniform continuity of u_0 to obtain the stated continuity of u at $t = 0$.

The principle of linearized stability now says the following. If $s(A_*) := \sup\{\operatorname{Re}\lambda \,|\, \lambda \in \sigma(A)\} < -\delta < 0$, then there are constants $c, r > 0$ such that

$$\forall\, u_0 \in \overline{B}_X(u_*, r) \cap C_{ub}(\mathbb{R}) : \quad \bar{t}(u_0) = \infty \quad \text{and} \quad \|u(t) - u_*\|_\infty \leqslant c\mathrm{e}^{-\delta t}\|u_0 - u_*\|_\infty$$

for all $t \geqslant 0$, where u solves (2.19). (See e.g. Theorem 3.8 in [**Sc3**].) We note that such results fail for certain partial differential equations. Here it works since (2.19) is of 'parabolic type'.

Of course, one now has to compute the sign of the *spectral bound* $s(A_*)$ (or different properties of $\sigma(A_*)$ for more refined versions of the above result). We sketch a partial answer for the important special case that $u_*(s)$ has limits $\xi_\pm$ in $\mathbb{R}$ as $s \to \pm\infty$. Then the limit operators

$$A_\pm = A + F'(\xi_\pm \mathbb{1}) \quad \text{with} \quad \mathrm{D}(A_\pm) = C_b^2(\mathbb{R})$$

have constant coefficients which simplifies the computation of their spectral properties. We now follow the survey article [**Sa**].

Let $\lambda \in \mathbb{C}$. We rewrite $A_\pm u - \lambda u = g$ as a first order system by

$$L(\lambda)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} := \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ \frac{1}{d}(\lambda - f'(\xi_\pm)) & -\frac{c}{d} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{d}g \end{pmatrix},$$

where $(v_1, v_2) \mathrel{\widehat{=}} (u, u')$. We set

$$M_\pm(\lambda) = \begin{pmatrix} 0 & 1 \\ \frac{1}{d}(\lambda - f'(\xi_\pm)) & -\frac{c}{d} \end{pmatrix}.$$

We denote by $X_\pm^u(\lambda)$ the linear span of all (generalized) eigenvectors of $M_\pm(\lambda)$ for eigenvalues μ with $\operatorname{Re}\mu > 0$. Theorems 3.2 and 3.3 and Remark 3.3 in [**Sa**] then yield

$$\begin{aligned}
&\lambda I - A_* \text{ is Fredholm} \iff \lambda \notin \sigma_{\mathrm{ess}}(A_*) \iff \sigma(M_\pm(\lambda)) \cap \mathrm{i}\mathbb{R} = \varnothing,\\
&\operatorname{ind}(\lambda I - A_*) = \dim X_-^u(\lambda) - \dim X_+^u(\lambda),\\
&\lambda \notin \sigma_{\mathrm{ess}}^0(A_*) \iff \sigma(M_\pm(\lambda)) \cap \mathrm{i}\mathbb{R} = \varnothing \quad \text{and} \quad \dim X_-^u(\lambda) = \dim X_+^u(\lambda).
\end{aligned}$$

Note that here non-zero indices naturally occur. The proofs of these results use Theorems 2.21 and 2.25 and properties of the ordinary differential equation governed by the matrices

$$M_\lambda(s) = \begin{pmatrix} 0 & 1 \\ \frac{1}{d}(\lambda - f'(u_*(s))) & -\frac{c}{d} \end{pmatrix}, \qquad s \in \mathbb{R}.$$

One thus has to study the eigenvalues of $M_\pm(\lambda)$ (which is easy) to determine the location of the essential spectrum. It then remains the (difficult) task to locate the eigenvalues of A_*. In particular for one space dimension, corresponding tools are discussed in [**Sa**]. $\Diamond$

CHAPTER 3

Sobolev spaces and weak derivatives

Throughout, $U \subseteq \mathbb{R}^d$ is open and non-empty.

3.1. Basic properties

We are looking for properties of C^1–functions which can be generalized to a theory of derivatives suited to L^p spaces. Looking at the theorem of dominated convergence for instance, one sees that here the basic concepts should not be based on pointwise limits. It turns out that integration by parts is an excellent starting point for such a theory.

Let $f \in C^1(U)$ and $\varphi \in C_c^\infty(U)$. Set $g_0 = \varphi f$ on $\operatorname{supp}\varphi$ and extend it by 0 to a function $g \in C_c^1(\mathbb{R}^d)$. Then $\partial_1 g = (\partial_1\varphi)f + \varphi\partial_1 f$ on U since both sides are trivially equal to 0 on $U\backslash \operatorname{supp}\varphi$. Take a number $a > 0$ such that $\operatorname{supp} g \subseteq (-a,a)^d =: C_d$ and write $x = (x_1, x')$. We then derive

$$\begin{aligned}\int_U (\partial_1 f)\varphi \,\mathrm{d}x &= -\int_U f\partial_1\varphi\,\mathrm{d}x + \int_U \partial_1 g\,\mathrm{d}x\\ &= -\int_U f\partial_1\varphi\,\mathrm{d}x + \int_{C_{d-1}}\int_{-a}^{a} \partial_1 g(x_1,x')\,\mathrm{d}x_1\,\mathrm{d}x'\\ &= -\int_U f\partial_1\varphi\,\mathrm{d}x + \int_{C_{d-1}} (g(a,x') - g(-a,x'))\,\mathrm{d}x'\\ &= -\int_U f\partial_1\varphi\,\mathrm{d}x.\end{aligned}$$

Inductively one shows that

$$\int_U (\partial^\alpha f)\varphi\,\mathrm{d}x = (-1)^{|\alpha|}\int_U f\partial^\alpha\varphi\,\mathrm{d}x, \tag{3.1}$$

for all $f \in C^k(U)$, $\varphi \in C_c^\infty(U)$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leqslant k$, where

$$\partial^\alpha := \partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d} \qquad \text{and} \qquad |\alpha| := \alpha_1 + \cdots + \alpha_d.$$

To imitate (3.1) in a definition, we set

$$L^p_{\mathrm{loc}}(U) = \{f : U \to \mathbb{C} \,\big|\, f \text{ measurable, } f_{|K} \in L^p(K) \text{ for all compact } K \subseteq U\}$$

for $p \in [1,\infty]$. We extend $f \in L^p_{\mathrm{loc}}(U)$ by 0 to a measurable function $f : \mathbb{R}^d \to \mathbb{C}$ without further notice. Convergence in $L^p_{\mathrm{loc}}(U)$ means that the restrictions to K converge in $L^p(K)$ for all compact $K \subseteq U$. Observe that $L^p(U) \subseteq L^p_{\mathrm{loc}}(U) \subseteq L^1_{\mathrm{loc}}(U)$ for all $1 \leqslant p \leqslant \infty$.

Definition 3.1. *Let $f \in L^1_{\mathrm{loc}}(U)$, $\alpha \in \mathbb{N}_0^d$, and $p \in [1,\infty]$. If there is a function $g \in L^1_{\mathrm{loc}}(U)$ such that*

$$\int_U g\varphi\,\mathrm{d}x = (-1)^{|\alpha|}\int_U f\partial^\alpha\varphi\,\mathrm{d}x \tag{3.2}$$

for all $\varphi \in C_c^\infty(U)$*, then* $g =: \partial^\alpha f$ *is called* weak derivative *of* f*. We set*

$$D^{\alpha,p}(U) = \{f \in L^p_{\mathrm{loc}}(U) \,\big|\, \exists\, \partial^\alpha f \in L^p_{\mathrm{loc}}(U)\}.$$

Moreover, one defines the Sobolev spaces *by*

$$W^{k,p}(U) = \{f \in L^p(U) \,\big|\, f \in D^{\alpha,p}(U),\ \partial^\alpha f \in L^p(U) \text{ for all } |\alpha| \leqslant k\}$$

for $k \in \mathbb{N}$ *and endows them with*

$$\|f\|_{k,p} = \begin{cases} \Big(\sum\limits_{0 \leqslant |\alpha| \leqslant k} \|\partial^\alpha f\|_p^p \Big)^{1/p}, & 1 \leqslant p < \infty, \\[2ex] \max\limits_{0 \leqslant |\alpha| \leqslant k} \|\partial^\alpha f\|_\infty, & p = \infty, \end{cases}$$

where $\partial^0 f := f$*. We write* $W^{0,p}(U) = L^p(U)$*,* $\partial^{\mathrm{e}_j} = \partial_j$*,* $\partial = \partial_1$ *if* $d = 1$*, and* $D^{k,p} = \bigcap_{0 \leqslant |\alpha| \leqslant k} D^{\alpha,p}$*.*

As usually, $L^p_{\mathrm{loc}}(U)$, $D^{\alpha,p}(U)$ and $W^{k,p}(U)$ are spaces of equivalence classes modulo the subspace $\mathcal{N} = \{f : U \to \mathbb{C} \,\big|\, f \text{ is measurable, } f = 0 \text{ a.e.}\}$.

Remark 3.2. Let $\alpha, \beta \in \mathbb{N}_0^d$, $p \in [1, \infty]$, and $k \in \mathbb{N}$.

a) We will see in Lemma 3.4 that $\partial^\alpha f$ is uniquely determined for a.e. $x \in U$. Formula (3.1) thus implies that $C^k(U) + \mathcal{N} \subseteq D^{\alpha,p}(U)$ for all $|\alpha| \leqslant k$ and that weak and classical derivatives coincide for $f \in C^k(U)$.

b) $D^{\alpha,p}(U)$ is a vector space and the map $\partial^\alpha : D^{\alpha,p}(U) \to L^p_{\mathrm{loc}}(U)$ is linear.

c) Let $f \in D^{\alpha,p}(U) \cap D^{\alpha+\beta,p}(U)$. Then $\partial^\alpha f$ belongs to $D^{\beta,p}(U)$ and $\partial^\beta \partial^\alpha f = \partial^{\alpha+\beta} f$. Hence, if also $f \in D^{\beta,p}(U)$, then $\partial^\beta f \in D^{\alpha,p}(U)$ and $\partial^\alpha \partial^\beta f = \partial^{\alpha+\beta} f = \partial^\beta \partial^\alpha f$.

Proof. Let $\varphi \in C_c^\infty(U)$ and $f \in (D^{\alpha,p}(U) \cap D^{\alpha+\beta,p}(U)) \subseteq L^p_{\mathrm{loc}}(U)$. Using (3.2) and Schwarz' theorem from Analysis 2, we then compute

$$\begin{aligned} (-1)^{|\beta|} \int_U (\partial^\alpha f)\, \partial^\beta \varphi \,\mathrm{d}x &= (-1)^{|\alpha| + |\beta|} \int_U f\, \partial^\alpha \partial^\beta \varphi \,\mathrm{d}x \\ &= (-1)^{|\alpha+\beta|} \int_U f\, \partial^{\alpha+\beta} \varphi \,\mathrm{d}x \\ &= \int_U (\partial^{\alpha+\beta} f)\, \varphi \,\mathrm{d}x; \end{aligned}$$

i.e., $\partial^\alpha f \in D^{\beta,p}(U)$ and $\partial^\beta \partial^\alpha f = \partial^{\alpha+\beta} f$. □

d) $(W^{k,p}(U), \|\cdot\|_{k,p})$ is a normed vector space. A sequence $(f_n)_n$ converges in $W^{k,p}(U)$ if and only if $(\partial^\alpha f_n)_n$ converges in $L^p(U)$ for each α with $|\alpha| \leqslant k$. Note that $\|f\|_{1,p}^p = \|f\|_p^p + \|\, |\nabla f|_p \,\|_p^p$ for $f \in W^{1,p}(U)$ and $p < \infty$.

e) The map

$$J : W^{k,p}(U) \to L^p(U)^m; \quad f \mapsto (\partial^\alpha f)_{|\alpha| \leqslant k}\,,$$

is a linear isometry, where m is the number of α in $\mathbb{N}_0^d$ with $|\alpha| \leqslant k$ and $L^p(U)^m$ has the norm $\|(f_j)_j\| = |(\|f_j\|_p)_j|_p$. Since the p-norm and the 1-norm on $\mathbb{R}^m$ are equivalent, there are constants $C_k, c_k > 0$ such that

$$c_k \sum_{0\leqslant|\alpha|\leqslant k} \|\partial^\alpha f\|_p \leqslant \|f\|_{k,p} \leqslant C_k \sum_{0\leqslant|\alpha|\leqslant k} \|\partial^\alpha f\|_p$$

for all $f \in W^{k,p}(U)$. $\Diamond$

We start with simple, but instructive one-dimensional examples.

Example 3.3. a) Let $f \in C(\mathbb{R})$ be such that $f_\pm := f_{|\mathbb{R}_\pm}$ belong to $C^1(\mathbb{R}_\pm)$. We then have $f \in D^{1,1}(\mathbb{R})$ with

$$\partial f = \left\{ \begin{array}{ll} f_+' & \text{on } [0,\infty) \\ f_-' & \text{on } (-\infty,0) \end{array} \right\} =: g.$$

For $f(x) = |x|$, we thus obtain $\partial f = \mathbb{1}_{\mathbb{R}_+} - \mathbb{1}_{(-\infty,0)}$, for instance.

Proof. For every $\varphi \in C_c^\infty(\mathbb{R})$, we compute

$$\begin{aligned}\int_{\mathbb{R}} f\varphi' \,\mathrm{d}t &= \int_{-\infty}^0 f_-\varphi' \,\mathrm{d}t + \int_0^\infty f_+\varphi' \,\mathrm{d}t \\ &= -\int_{-\infty}^0 f_-'\varphi \,\mathrm{d}t + f_-\varphi|_{-\infty}^0 - \int_0^\infty f_+'\varphi dt + f_+\varphi'|_0^\infty \\ &= -\int_{\mathbb{R}} g\varphi \,\mathrm{d}t,\end{aligned}$$

since $f_+(0) = f_-(0)$ by the continuity of f. $\Box$

b) The function $f = \mathbb{1}_{\mathbb{R}_+}$ does not belong to $D^{1,1}(\mathbb{R})$.

Proof. Assume there would exist $g = \partial f \in L^1_{\mathrm{loc}}(\mathbb{R})$. Then we would obtain for every $\varphi \in C_c^\infty(\mathbb{R})$ that

$$\int_{\mathbb{R}} g\varphi \,\mathrm{d}t = -\int_{\mathbb{R}} \mathbb{1}_{\mathbb{R}_+}\varphi' \,\mathrm{d}t = -\int_0^\infty \varphi'(t) \,\mathrm{d}t = \varphi(0).$$

Taking φ with $\operatorname{supp}\varphi \subseteq (0,\infty)$, we deduce from Lemma 3.4 below that $g = 0$ on $(0,\infty)$. Similarly, it follows that $g = 0$ on $(-\infty,0)$. Hence, $g = 0$ and so $\varphi(0) = 0$ for all $\varphi \in C_c^\infty(\mathbb{R})$, which is false. $\Box$

c) Set $f(x,y) = \mathbb{1}_{\mathbb{R}_+}(x)$ for $(x,y) \in \mathbb{R}^2$. Then there exists the weak derivatives $\partial_2 f = 0$, and thus $\partial_1\partial_2 f = \partial_2\partial_2 f = 0$. However, the weak derivative $\partial_1 f$ does not exist. (See exercises.) $\Diamond$

So far we have just used the definition of weak derivatives by duality. For further examples and deeper results one needs *mollifiers* which we recall and discuss next.

Fix a function $0 \leqslant \chi \in C^\infty(\mathbb{R}^d)$ with support $\overline{B}(0,1)$ and $\chi > 0$ on $B(0,1)$. For $x \in \mathbb{R}^d$ and $\varepsilon > 0$, we set

$$k(x) = \frac{1}{\|\chi\|_1}\chi(x) \qquad \text{and} \qquad k_\varepsilon(x) = \varepsilon^{-d}k(\tfrac{1}{\varepsilon}x).$$

Note that $0 \leqslant k_\varepsilon \in C^\infty(\mathbb{R}^d)$, $k_\varepsilon(x) > 0$ if and only if $|x|_2 < \varepsilon$, and $\|k_\varepsilon\|_1 = 1$. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\varepsilon > 0$. We now introduce the mollifier G_ε by

$$G_\varepsilon f(x) = (k_\varepsilon * f)(x) = \int_{B(x,\varepsilon)} k_\varepsilon(x-y)f(y) \,\mathrm{d}y = \int_{B(0,\varepsilon)} k_\varepsilon(z)f(x-z) \,\mathrm{d}z, \tag{3.3}$$

for $x \in \mathbb{R}^d$ or $x \in U$, where we have set $f(x) = 0$ for $x \in \mathbb{R}^d \backslash U$. For a subset S of Banach space and $\varepsilon > 0$, we define

$$S_\varepsilon = S + \overline{B}(0, \varepsilon).$$

From e.g. Proposition 4.13 in [**Sc2**] and its proof we recall that

$$G_\varepsilon f \in C^\infty(\mathbb{R}^d), \tag{3.4}$$

$$\operatorname{supp} G_\varepsilon f \subseteq S_\varepsilon \quad \text{for } S := \operatorname{supp} f, \quad S_\varepsilon \text{ is compact if } S \text{ is compact}, \tag{3.5}$$

$$\|G_\varepsilon f\|_{L^p(U)} \leqslant \|G_\varepsilon f\|_{L^p(\mathbb{R}^d)} \leqslant \|f\|_p \quad \text{if } f \in L^p(U) \text{ and } 1 \leqslant p \leqslant \infty, \tag{3.6}$$

$$G_\varepsilon f \to f \text{ in } L^p(U) \quad \text{as } \varepsilon \to 0 \quad \text{if } f \in L^p(U) \text{ and } 1 \leqslant p < \infty \tag{3.7}$$
$$\text{or if } \; p = \infty \text{ and } f \text{ is uniformly continuous.}$$

Lemma 3.4. *Let $K \subseteq U$ be compact. Then there is a function $\psi \in C_c^\infty(U)$ such that $0 \leqslant \psi \leqslant 1$ on U and $\psi = 1$ on K. Let $g \in L^1_{loc}(U)$ satisfy*

$$\int_U g\varphi \, \mathrm{d}x = 0$$

for all $\varphi \in C_c^\infty(U)$. Then $g = 0$ a.e.. In particular, weak derivatives are uniquely defined.

Proof. 1) Let $0 < \delta < \frac{1}{2} \operatorname{dist}(\partial K, \partial U)$. Then $K_{2\delta} = K + \overline{B}(0, 2\delta)$ is compact and $K_{2\delta} \subseteq U$. The function $\psi := G_\delta \mathbb{1}_{K_\delta}$ thus belongs to $C_c^\infty(U)$ by (3.4) and (3.5). Moreover, (3.3) and (3.6) imply that $0 \leqslant \psi(x) \leqslant \|\psi\|_\infty \leqslant \|\mathbb{1}_{K_\delta}\|_\infty = 1$ for all $x \in U$ and

$$\psi(x) = \int_{B(x,\delta)} k_\delta(x-y) \mathbb{1}_{K_\delta}(y) \, \mathrm{d}y = \|k_\delta\|_1 = 1$$

for all $x \in K$. The first claim is shown.

2) Assume that $g \neq 0$ on a Borel set $B \subseteq U$ with $\lambda(B) > 0$. Theorem 2.20 of [**Ru1**] yields a compact set $K \subseteq B \subseteq U$ with $\lambda(K) > 0$. Since $\psi g \in L^1(U)$, the functions $G_\varepsilon(\psi g)$ converge to ψg in $L^1(U)$ as $\varepsilon \to 0$ due to (3.7). Hence, there is a nullset N and a subsequence $\varepsilon_j \to 0$ with $\varepsilon_j \leqslant \delta$ such that $(G_{\varepsilon_j}(\psi g))(x) \to g(x) \neq 0$ as $j \to \infty$ for each $x \in K \backslash N$. For every $x \in K \backslash N$ and $j \in \mathbb{N}$, we also deduce

$$(G_{\varepsilon_j}(\psi g))(x) = \int_U k_{\varepsilon_j}(x-y)\psi(y)\, g(y) \, \mathrm{d}y = 0$$

from the assumption, since the function $y \mapsto k_{\varepsilon_j}(x-y)\psi(y)$ belongs to $C_c^\infty(U)$. This is a contradiction. □

Recall that Hölder's inequality implies that the map

$$L^p(B) \times L^{p'}(B) \to \mathbb{C}; \quad (f,g) \mapsto \int_B fg \, \mathrm{d}x, \tag{3.8}$$

is continuous for all $1 \leqslant p \leqslant \infty$ and Borel sets $B \subseteq \mathbb{R}^d$. The next lemma is the key to many properties of weak derivatives.

Lemma 3.5. *Let $\alpha \in \mathbb{N}_0^d$, $p \in [1, \infty]$, and $\varepsilon > 0$.*

(a) Let $f \in D^{\alpha,p}(U)$ and $p < \infty$. Then the functions $G_\varepsilon f \in C^\infty(U)$ converge to f and $\partial^\alpha(G_\varepsilon f)$ tend to $\partial^\alpha f$ in $L^p_{\mathrm{loc}}(U)$ as $\varepsilon \to 0$. Moreover,

$$\partial^\alpha(G_\varepsilon f)(x) = G_\varepsilon(\partial^\alpha f)(x) \quad \text{if } \; x \in U, \; \varepsilon < d(x, \partial U).$$

For all $\varepsilon_j \to 0$ we obtain a subsequence $\varepsilon_n := \varepsilon_{j_n} \to 0$ such that $G_{\varepsilon_n} f \to f$ and $\partial^\alpha(G_{\varepsilon_n} f) \to \partial^\alpha f$ a.e. on U as $n \to \infty$. If f also belongs to $D^{\beta,p}(U)$ for some $\beta \in \mathbb{N}_0^d$, we can take the same ε_n for all such β.

(b) Let $f, g \in L^p_{\mathrm{loc}}(U)$ and $f_n \in D^{\alpha,p}(U)$ such that $f_n \to f$ and $\partial^\alpha f_n \to g$ in $L^p_{\mathrm{loc}}(U)$ as $n \to \infty$. Then f is contained in $D^{\alpha,p}(U)$ and $\partial^\alpha f = g$. If these limits exist in $L^p(U)$ and for all α with $|\alpha| \leqslant k$, then f is an element of $W^{k,p}(U)$.

PROOF. (a) 1) Let $\varepsilon > 0$ and $x \in U$. If $\varepsilon < d(x, \partial U)$, then the function $y \mapsto \varphi_{\varepsilon,x}(y) := k_\varepsilon(x-y)$ belongs to $C_c^\infty(U)$ since $\operatorname{supp} \varphi_{\varepsilon,x} = \overline{B}(x,\varepsilon)$. Using a corollary to Lebesgue's theorem and (3.2), we can thus deduce

$$\begin{aligned}
\partial^\alpha G_\varepsilon f(x) &= \int_U \partial_x^\alpha k_\varepsilon(x-y) f(y)\,\mathrm{d}y = (-1)^{|\alpha|} \int_U (\partial^\alpha \varphi_{\varepsilon,x})(y) f(y)\,\mathrm{d}y \\
&= \int_U \varphi_{\varepsilon,x}(y)(\partial^\alpha f)(y)\,\mathrm{d}y = (G_\varepsilon \partial^\alpha f)(x).
\end{aligned}$$

2) Choose a compact subset $K \subseteq U$ and fix $\delta > 0$ with $K_\delta \subseteq U$. Take $\varepsilon \in (0, \delta]$. Note that the integrand of $(G_\varepsilon g)(x)$ is then supported in K_δ for every $x \in K$ and $g \in L^1_{\mathrm{loc}}(U)$, see (3.3). Hence, (3.7) and part 1) imply that

$$\mathbb{1}_K \partial^\alpha (G_\varepsilon f) = \mathbb{1}_K G_\varepsilon(\partial^\alpha f) = \mathbb{1}_K G_\varepsilon(\mathbb{1}_{K_\delta} \partial^\alpha f) \longrightarrow \mathbb{1}_K \mathbb{1}_{K_\delta} \partial^\alpha f = \mathbb{1}_K \partial^\alpha f$$

converge in $L^p(K)$ as $\varepsilon \to 0$. So the asserted convergence in $L^p_{\mathrm{loc}}(U)$ is true.

3) For $m \in \mathbb{N}$, we define

$$K_m = \{x \in U \,\big|\, d(x, \partial U) \geqslant \tfrac{1}{m} \text{ and } |x|_2 \leqslant m\}$$

These sets are compact and $\bigcup_{m\in\mathbb{N}} K_m = U$. Let $\varepsilon_j \to 0$. Then, for each $m \in \mathbb{N}$ there is a null set $N_m \subseteq K_m$ and a subsequence $\nu_m \subseteq \nu_{m-1}$ such that $\partial^\alpha G_{\varepsilon_{\nu_m(k)}} f(x)$ tends to $\partial^\alpha f(x)$ and $G_{\varepsilon_{\nu_m(k)}} f(x)$ tends to $f(x)$ for all $x \in K_m \backslash N_m$ as $k \to \infty$. By means of a diagonal sequence, one obtains a subsequence $\varepsilon_n = \varepsilon_{j_n} \to 0$ such that $\partial^\alpha G_{\varepsilon_n} f(x) \to \partial^\alpha f(x)$ and $G_{\varepsilon_n} f(x) \to f(x)$ for $x \in U \backslash (\bigcup_{m\in\mathbb{N}} N_m)$ as $n \to \infty$, where $\bigcup_{m\in\mathbb{N}} N_m$ is a null set. Employing another diagonal sequence, we can achieve this for countably many $\partial^\beta f$ at the same time.

(b) Let $\varphi \in C_c^\infty(U)$ and $S = \operatorname{supp} \varphi$. Since S is compact, we have $f_n \to f$ and $\partial^\alpha f_n \to g$ in $L^p(S)$ as $n \to \infty$. From (3.8) on S we deduce that

$$\begin{aligned}
\int_U f \partial^\alpha \varphi\,\mathrm{d}x &= \lim_{n\to\infty} \int_U f_n \partial^\alpha \varphi\,\mathrm{d}x = (-1)^{|\alpha|} \lim_{n\to\infty} \int_U (\partial^\alpha f_n)\varphi\,\mathrm{d}x \\
&= (-1)^{|\alpha|} \int_U g\varphi\,\mathrm{d}x.
\end{aligned}$$

Hence, $f \in D^{\alpha,p}(U)$ and $\partial^\alpha f = g$. The last assertion then easily follows. $\square$

In the next examples we also argue by approximation, but using a different, more explicit regularisation method.

EXAMPLE 3.6. Let $U = B(0,1)$.

a) Let $d \geqslant 2$, $1 \leqslant p < d$, and $f(x) = \ln|x|_2$ for $x \in U \backslash \{0\}$. Then f belongs to $W^{1,p}(U)$ with

$$\partial_j f(x) = \frac{x_j}{|x|_2^2} =: g_j(x),$$

for $x \neq 0$ and $j \in \{1, \cdots, d\}$. Moreover, $f \in L^q(U) \backslash L^\infty(U)$ for all $q \in [1, \infty)$.

Proof. Using polar coordinates and $|x_j| \leqslant r$, we obtain

$$\|f\|_q^q = c\int_0^1 |\ln r|^q \, r^{d-1}\,\mathrm{d}r < \infty,$$
$$\|g_j\|_p^p \leqslant c\int_0^1 \frac{r^p}{r^{2p}} r^{d-1}\,\mathrm{d}r = c\int_0^1 r^{d-p-1}\,\mathrm{d}r < \infty,$$

since $p < d$. Hence, $f \in L^q(U)$ and $g_j \in L^p(U)$. Define $u_n \in C^\infty(\overline{U}) \subseteq W^{1,p}(U)$ by $u_n(x) = \ln(n^{-2} + |x|_2^2)^{1/2}$ for $n \in \mathbb{N}$. Observe that $\partial_j u_n(x) = (n^{-2} + |x|_2^2)^{-1} x_j$, $u_n(x) \to f(x)$ and $\partial_j u_n(x) \to g_j(x)$ as $n \to \infty$ for all $x \in U\backslash\{0\}$. We have the pointwise bounds

$$|u_n(x)| \leqslant \begin{cases} |f(x)|, & n^{-2} + |x|_2^2 \leqslant 1, \\ \ln\sqrt{2}, & n^{-2} + |x|_2^2 > 1, \end{cases} \qquad |\partial_j u_n(x)| \leqslant g_j(x), \quad x \in U.$$

Lebesgue's theorem thus yields that $u_n \to f$ and $\partial_j u_n \to f_j$ in $L^p(U)$ as $n \to \infty$. The assertion then follows from Lemma 3.5(b). □

(b) Let $p \in [1,\infty)$ and $\beta \in (1-\frac{d}{p}, 1]$. Set $u(x) = |x|_2^\beta$ and $f_j(x) = \beta x_j |x|_2^{\beta-2}$ for $x \in U\backslash\{0\}$ and $j \in \{1,\dots,d\}$. Then $u \in W^{1,p}(U)$ and $\partial_j u = f_j$. (See Example 4.18 in [**Sc2**].) ◊

Proposition 3.7. *For $1 \leqslant p \leqslant \infty$ and $k \in \mathbb{N}$. Then $W^{k,p}(U)$ is a Banach space which is isometrically isomorphic to a closed subspace of a $L^p(U)^m$ for some $m \in \mathbb{N}$. It is separable if $p < \infty$ and reflexive if $1 < p < \infty$. Moreover, $W^{k,2}(U)$ is a Hilbert space endowed with the scalar product*

$$(f|g)_{k,2} = \sum_{|\alpha| \leqslant k} \int_U \partial^\alpha f \,\overline{\partial^\alpha g}\,\mathrm{d}x.$$

Proof. Let $(f_n)_n$ be a Cauchy sequence in $W^{k,p}(U)$. Then $(\partial^\alpha f_n)_n$ is a Cauchy sequence in $L^p(U)$ for every $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leqslant k$ and thus $\partial^\alpha f_n \to g_\alpha$ in $L^p(U)$ for some $g_\alpha \in L^p(U)$ as $n \to \infty$, where we set $f := g_0$. Let $\varphi \in C_c^\infty(U)$ and $|\alpha| \leqslant k$. Since $f_n \in W^{k,p}(U)$, we deduce

$$\int_U f\partial^\alpha\varphi\,\mathrm{d}x = \lim_{n\to\infty}\int_U f_n\partial^\alpha\varphi\,\mathrm{d}x = \lim_{n\to\infty}(-1)^{|\alpha|}\int_U (\partial^\alpha f_n)\varphi\,\mathrm{d}x$$
$$= (-1)^{|\alpha|}\int_U g_\alpha\varphi\,\mathrm{d}x.$$

This means that g_α is the weak derivative $\partial^\alpha f$ so that $f \in W^{k,p}(U)$ and $f_n \to f$ in $W^{k,p}(U)$. Hence, $W^{k,p}(U)$ is a Banach space. We then deduce from Remark 3.2e) that $W^{k,p}(U)$ is isometrically isomorphic to a subspace of $L^p(U)^m$ which is closed by e.g. Remark 2.11 in [**Sc2**]. The remaining claims now follow by isomorphy from known results of functional analysis. □

We next establish product and chain rules. One can derive many variants by modifications of the proofs.

Proposition 3.8. *(a) Let $f, g \in D^{1,1}(U) \cap L^\infty(U)$. Then, fg belongs to $D^{1,1}(U) \cap L^\infty(U)$ and has the derivatives*

$$\partial_j(fg) = (\partial_j f)g + f(\partial_j g), \qquad j \in \{1,\dots,d\}. \tag{3.9}$$

(b) Let $1 \leqslant p \leqslant \infty$, $f \in W^{1,p}(U)$ and $g \in W^{1,p'}(U)$. Then, fg is contained in $W^{1,1}(U)$ and satisfies (3.9).

Proof. 1) Let $f, g \in D^{1,1}(U)$. Set $f_n = G_{\varepsilon_n} f \in C^\infty(U)$ and $g_n = G_{\varepsilon_n} g \in C^\infty(U)$ with $\varepsilon_n \to 0$ as in Lemma 3.5(a). Fix $m \in \mathbb{N}$ and take $\varphi \in C_c^\infty(U)$ and $j \in \{1, \dots, d\}$. Choose an open and bounded set V such that $\operatorname{supp} \varphi \subseteq V \subseteq \overline{V} \subseteq U$. Since $f_n \to f$ and $\partial_j f_n \to \partial_j f$ on $L^1(\overline{V})$ by Lemma 3.5(a), the formulas (3.8) and (3.1) yield

$$\begin{aligned}\int_U f g_m \partial_j \varphi \,\mathrm{d}x &= \lim_{n\to\infty} \int_{\overline{V}} f_n g_m \partial_j \varphi \,\mathrm{d}x \\ &= -\lim_{n\to\infty} \int_{\overline{V}} ((\partial_j f_n) g_m + f_n(\partial_j g_m))\, \varphi \,\mathrm{d}x \\ &= -\int_U ((\partial_j f) g_m + f(\partial_j g_m))\, \varphi \,\mathrm{d}x,\end{aligned}$$

so that $fg_m \in D^{1,1}(U)$ and $\partial_j(fg_m) = (\partial_j f) g_m + f(\partial_j g_m)$.

2) Let $f, g \in D^{1,1}(U) \cap L^\infty(U)$ and g_m as in 1). Note that $g_m \to g$ and $\partial_j g_m \to \partial_j g$ in $L^1_{\mathrm{loc}}(U)$ as $m \to \infty$. Since f is bounded, we obtain

$$\begin{aligned}\int_U f g \partial_j \varphi \,\mathrm{d}x &= \lim_{m\to\infty} \int_U f g_m \partial_j \varphi \,\mathrm{d}x \\ &= \lim_{m\to\infty} -\left[\int_U (\partial_j f) g_m \varphi \,\mathrm{d}x + \int_U f(\partial_j g_m) \varphi \,\mathrm{d}x\right]\end{aligned}$$

using step 1). In the last line, the second integral converges to $\int_U f(\partial_j g)\varphi \,\mathrm{d}x$, again because of $f \in L^\infty(U)$. For the first integral we use that $g_m \to g$ a.e. by Lemma 3.5(a) and that $\|g_m\|_\infty \leqslant \|g\|_\infty$ by (3.6). Lebesgue's theorem (with the majorant $|\partial_j f|\, \|g\|_\infty \|\varphi\|_\infty \mathbb{1}_{\operatorname{supp}\varphi}$) then implies

$$\int_U f g \partial_j \varphi \,\mathrm{d}x = -\int_U ((\partial_j f) g + f(\partial_j g)) \varphi \,\mathrm{d}x.$$

Note that $(\partial_j f) g + f(\partial_j g) \in L^1_{\mathrm{loc}}(U)$ by our assumptions. Part (a) is shown.

3) Let $f \in W^{1,p}(U)$ and $g \in W^{1,p'}(U)$. If $p \in (1, \infty]$ we show (3.9) as in step 2), using (3.8) and that g_m, $\partial_j g_m$ converge in $L^{p'}_{\mathrm{loc}}(U)$ by Lemma 3.5(a). If $p = 1$, we replace the roles of f and g. Hölder's inequality and (3.9) finally yield that $\partial_j(fg)$ is contained in $L^1(U)$. $\square$

Proposition 3.9. *Let $1 \leqslant p \leqslant \infty$, $j \in \{1, \dots, d\}$, and $f \in D^{1,p}(U)$.*

(a) Let f be real-valued and $h \in C^1(\mathbb{R})$ with $h' \in C_b(\mathbb{R})$. Then $h \circ f$ belongs to $D^{1,p}(U)$ with derivatives

$$\partial_j(h \circ f) = (h' \circ f)\partial_j f.$$

(b) Let $V \subseteq \mathbb{R}^d$ be open and $\Phi = (\Phi_1, \cdots, \Phi_d) : V \to U$ be a diffeomorphism such that Φ' and $(\Phi^{-1})'$ are bounded. Then $f \circ \Phi$ belongs to $D^{1,p}(V)$ with derivatives

$$\partial_j(f \circ \Phi) = \sum_{k=1}^d ((\partial_k f) \circ \Phi)\, \partial_j \Phi_k.$$

In both results we can replace $D^{1,p}(U)$ by $W^{1,p}(U)$, where in (a) we then also assume that $h(0) = 0$ if $\lambda(U) = \infty$ and $p < \infty$.

PROOF.[1] By Lemma 3.5(a), there are $f_n \in C^\infty(U)$ such that $f_n \to f$ and $\partial_j f_n \to \partial_j f$ in $L^p_{\mathrm{loc}}(U)$ and a.e. as $n \to \infty$.

(a) The function $h \circ f$ belongs to $L^p_{\mathrm{loc}}(U)$ since

$$|h(f(x))| \leqslant |h(f(x)) - h(0)| + |h(0)| \leqslant \|h'\|_\infty |f(x)| + |h(0)|$$

for all $x \in U$. It is contained in $L^p(U)$ if $f \in L^p(U)$ and if $h(0) = 0$ in the case that $\lambda(U) = \infty$ and $p \neq \infty$. Let $K \subseteq U$ be compact. We obtain that

$$\int_K |h(f_n(x)) - h(f(x))| \,\mathrm{d}x \leqslant \|h'\|_\infty \int_K |f_n(x) - f(x)| \,\mathrm{d}x \longrightarrow 0,$$

$$\int_K |h'(f_n(x))\partial_j f_n(x) - h'(f(x))\partial_j f(x)| \,\mathrm{d}x$$

$$\leqslant \|h'\|_\infty \int_K |\partial_j f_n(x) - \partial_j f(x)| \,\mathrm{d}x + \int_K |h'(f_n(x)) - h'(f(x))| |\partial_j f(x)| \,\mathrm{d}x \to 0$$

as $n \to \infty$ where we also used Lebesgue's theorem and the majorant $2\|h'\|_\infty |\partial_j f|$ in the last integral. Since $h \circ f_n \in C^1(U)$, $\partial_j(h \circ f_n) = (h' \circ f_n)\partial_j f_n$ and $(h' \circ f)\partial_j f$ belongs to $L^p_{\mathrm{loc}}(U)$, Lemma 3.5(b) yields assertion (a). If $f \in W^{1,p}(U)$, then $(h' \circ f)\partial_j f \in L^p(U)$ and so $h \circ f \in W^{1,p}(U)$.

(b) Let $B = \Phi^{-1}(A)$ for an open set $A \subseteq U$ (or the closure of an open set whose boundary has measure 0) and $g \in L^p(A)$. For $p < \infty$, the transformation rule yields

$$\int_B |g(\Phi(x))|^p \,\mathrm{d}x = \int_A |g(y)|^p \,|\det[(\Phi^{-1})'(x)]| \,\mathrm{d}x \leqslant c \,\|f\|^p_{L^p(A)}.$$

An analogous estimate is also true for $p = \infty$. Hence, $f \circ \Phi$ belongs to $L^p_{\mathrm{loc}}(V)$ and $f_n \circ \Phi$ tends to $f \circ \Phi$ in $L^p_{\mathrm{loc}}(V)$. We further have

$$\partial_j(f_n \circ \Phi) = \sum_{k=1}^d ((\partial_k f_n) \circ \Phi)\, \partial_j \Phi_k.$$

The above estimate also implies that $(\partial_k f_n) \circ \Phi$ tends to $(\partial_k f) \circ \Phi$ in $L^p_{\mathrm{loc}}(V)$ as $n \to \infty$. Since $\partial_j \Phi_k$ is uniformly bounded, Lemma 3.5(b) yields that $f \circ \Phi$ in contained in $D^{1,p}(U)$ and that it has the asserted derivative. If $f \in L^p(U)$ we can replace throughout $L^p_{\mathrm{loc}}(V)$ by $L^p(V)$. □

COROLLARY 3.10. *Let $f \in D^{1,1}(U)$ be real-valued. Then the functions f_+, f_- and $|f|$ belong to $D^{1,1}(U)$ with*

$$\partial_j f_\pm = \pm \mathbb{1}_{\{f \gtrless 0\}} \partial_j f \quad \textit{and} \quad \partial_j |f| = \left(\mathbb{1}_{\{f>0\}} - \mathbb{1}_{\{f<0\}}\right) \partial_j f$$

for all $j \in \{1, \ldots, d\}$. Here one can replace $D^{1,1}$ by $W^{1,p}$ for all $1 \leqslant p \leqslant \infty$.

PROOF.[2] We employ the map $h_\varepsilon \in C^1(\mathbb{R})$ given by $h_\varepsilon(t) := \sqrt{t^2 + \varepsilon^2} - \varepsilon \leqslant t$ for $t \geqslant 0$ and $h_\varepsilon(t) := 0$ for $t < 0$, where $\varepsilon > 0$. Observe that $\|h_\varepsilon'\|_\infty = 1$ for $\varepsilon > 0$ and that $h_\varepsilon(t) \to \mathbb{1}_{[0,\infty)}(t)t$ for $t \in \mathbb{R}$ as $\varepsilon \to 0$. Proposition 3.9 shows that $h_\varepsilon \circ f \in D^{1,1}(U)$ and

$$\int_U h_\varepsilon(f)\partial_j \varphi \,\mathrm{d}x = -\int_U h_\varepsilon'(f)(\partial_j f)\varphi \,\mathrm{d}x = -\int_{\{f>0\}} \frac{f}{\sqrt{f^2 + \varepsilon^2}} (\partial_j f)\varphi \,\mathrm{d}x$$

[1] Not shown in the lectures.

[2] Not shown in the lectures.

for each $\varphi \in C_c^\infty(U)$. Using the majorants $\|\partial_j\varphi\|_\infty \mathbb{1}_B|f|$ and $\|\varphi\|_\infty \mathbb{1}_B|\partial_j f|$ with $B = \operatorname{supp}\varphi$, we deduce from Lebesgue's convergence theorem that

$$\int_U f_+\partial_j\varphi\,\mathrm{d}x = -\int_{\{f>0\}} \frac{f}{|f|}(\partial_j f)\varphi\,\mathrm{d}x = -\int_U \mathbb{1}_{\{f>0\}}(\partial_j f)\varphi\,\mathrm{d}x.$$

There thus exists $\partial_j f_+ = \mathbb{1}_{\{f>0\}}\partial_j f \in L^1_{\mathrm{loc}}(U)$. Clearly, $\partial_j f_+ \in L^p(U)$ if $f \in W^{1,p}(U)$. The other claims follow from $f_- = (-f)_+$ and $|f| = f_+ + f_-$. $\square$

We briefly discuss two special cases, namely $d = 1$ and $p = \infty$.

THEOREM 3.11. *Let $J \subseteq \mathbb{R}$ be an open interval, $1 \leqslant p < \infty$, and $f \in L^p_{\mathrm{loc}}(J)$. Then f belongs to $D^{1,p}(J)$ if and only if there is a function $g \in L^p_{\mathrm{loc}}(J)$ and a representative of f which is continuous on $\overline{J}$ and satisfies*

$$f(t) = f(s) + \int_s^t g(\tau)\,\mathrm{d}\tau \tag{3.10}$$

for all $s, t \in \overline{J}$. In this case, $g = \partial f$ a.e..

PROOF. 1) Let $f \in D^{1,p}(J)$. Take the functions $f_n = G_{\varepsilon_n} f \in C^\infty(J)$ from Lemma 3.5(a). Then for a.e. $t \in J$ and for a.e. $t_0 \in J$ we have

$$f(t) - f(t_0) = \lim_{n\to\infty}(f_n(t) - f_n(t_0)) = \lim_{n\to\infty}\int_{t_0}^t f_n'(\tau)\,\mathrm{d}\tau = \int_{t_0}^t \partial f(\tau)\,\mathrm{d}\tau.$$

Fixing one t_0 and noting that $t \mapsto \int_{t_0}^t \partial f(\tau) d\tau$ is continuous, we obtain a continuous representative of f on $\overline{J}$ which fulfills (3.10) for $t \in \overline{J}$, $s = t_0$ and $g = \partial f$. Subtracting two such equations for any given $t, s \in \overline{J}$ and the fixed t_0, we deduce (3.10) with $g = \partial f$ for all $t, s \in \overline{J}$.

2) If (3.10) is satisfied by some $g \in L^p_{\mathrm{loc}}(J)$, take $g_n \in C^\infty(J)$ such that $g_n \to g$ in $L^p_{\mathrm{loc}}(J)$ as $n \to \infty$. For any $s \in J$ and $n \in \mathbb{N}$, the function $f_n(t) := f(s) + \int_s^t g_n(\tau)\,\mathrm{d}\tau$, $t \in J$, belongs to $C^\infty(J)$ with $f_n' = g_n$. For $[a, b] \subseteq J$ with $s \in [a, b]$, we estimate

$$\begin{aligned}
\|f_n - f\|^p_{L^p([a,b])} &= \int_a^b \left|\int_s^t (g_n(\tau) - g(\tau))\,\mathrm{d}\tau\right|^p \mathrm{d}t \\
&\leqslant \int_a^b |t-s|^{p/p'} \left(\int_a^b |g_n(\tau) - g(\tau)|^p\,\mathrm{d}\tau\right)^{p/p} \mathrm{d}t \\
&\leqslant (b-a)^{1+p/p'}\,\|g_n - g\|^p_{L^p([a,b])},
\end{aligned}$$

using (3.10) and Hölder's inequality. Hence, f_n tends to f in $L^p_{\mathrm{loc}}(J)$ as $n \to \infty$. Lemma 3.5(b) then yields $f \in D^{1,p}(J)$ and $\partial f = g$. $\square$

REMARK 3.12. a) Let $J = (a, b)$ for some $a < b$ in $\mathbb{R}$ and $f : J \to \mathbb{C}$. We then have $f \in W^{1,1}(J)$ if and only if f is *absolutely continuous*; i.e., for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all subdivisions $a < \alpha_1 < \beta_1 < \cdots < \alpha_n < \beta_n < b$ with $n \in \mathbb{N}$ and $\sum_{j=1}^n(\beta_j - \alpha_j) \leqslant \delta$ we obtain

$$\sum_{j=1}^n |f(\beta_j) - f(\alpha_j)| \leqslant \varepsilon.$$

In this case, f is differentiable for a.e. $t \in J$ and the pointwise derivative f' is equal a.e. to the weak derivative $\partial f \in L^1(J)$.[3]

Proof. The implication "⇒" is true since (3.10) yields that

$$\sum_{j=1}^{n} |f(\beta_j) - f(\alpha_j)| = \sum_{j=1}^{n} \left| \int_{\alpha_j}^{\beta_j} \partial f(\tau)\,\mathrm{d}\tau \right| \leqslant \int_{\bigcup_{j=1}^{n}(\alpha_j,\beta_j)} |\partial f(\tau)|\,\mathrm{d}\tau =: S,$$

where $S \to 0$ as $\lambda(\bigcup_{j=1}^{n}(\alpha_j,\beta_j)) \to 0$.

The other implication " ⇐" and the last assertion is shown in Theorem 7.20 of [**Ru1**] (combined with our Theorem 3.11). □

b) There is a continuous increasing function $f : [0,1] \to \mathbb{R}$ with $f(0) = 0$ and $f(1) = 1$ such that $f'(t) = 0$ exists for a.e. $t \in [0,1]$. Consequently,

$$1 = f(1) \neq f(0) + \int_0^1 f'(\tau)\,\mathrm{d}\tau = 0,$$

and so f is not absolutely continuous, does not belong to $W^{1,1}(0,1)$ and violates (3.10). (See §7.16 in [**Ru1**].)

c) Let $f \in D^{1,1}(J)$ and ∂f be continuous. Then f is continuously differentiable, since then

$$\frac{f(t) - f(s)}{t - s} = \frac{1}{t - s} \int_s^t \partial f(\tau)\,\mathrm{d}\tau \;\longrightarrow\; \partial f(s)$$

as $t \to s$, thanks to Theorem 3.11. ◊.

Proposition 3.13. *Let U be convex. Then $W^{1,\infty}(U)$ is isomorphic to*

$$C_b^{1-}(U) = \{f \in C_b(U) \,|\, f \text{ is Lipschitz}\},$$

and the norm of $W^{1,\infty}(U)$ is equivalent to

$$\|f\|_{C_b^{1-}} = \|f\|_\infty + [f]_{Lip},$$

where $[f]_{Lip}$ is the Lipschitz constant of f.

Proof.[4] 1) Let $f \in W^{1,\infty}(U)$. Take $\varepsilon_n \to 0$ from Lemma 3.5(a). Let $K \subseteq U$ be compact. For sufficiently large $n \in \mathbb{N}$, Lemma 3.5 and (3.6) yield

$$|\partial_j G_{\varepsilon_n} f(z)| = |G_{\varepsilon_n} \partial_j f(z)| \leqslant \|\partial_j f\|_\infty \leqslant \|f\|_{1,\infty},$$

for all $j \in \{1, \ldots, d\}$ and $z \in K$. Using that $G_{\varepsilon_n} f(x) \to f(x)$ as $n \to \infty$ for all $x \in U \backslash N$ and a null set N, for all $x, y \in U \backslash N$ we thus estimate

$$\begin{aligned}
|f(x) - f(y)| &= \lim_{n\to\infty} |G_{\varepsilon_n} f(x) - G_{\varepsilon_n} f(y)| \\
&= \lim_{n\to\infty} \left| \int_0^1 \nabla G_{\varepsilon_n} f(y + \tau(x - y)) \cdot (x - y)\,\mathrm{d}\tau \right| \\
&\leqslant \|f\|_{1,\infty}\, |x - y|_2 \,.
\end{aligned}$$

Hence, f has a representative with Lipschitz constant $\|f\|_{1,\infty}$.

[3] Note that a Lipschitz continuous function is absolutely continuous and that an absolutely continuous function is uniformly continuous.

[4] Not shown in the lectures.

2) Let $f \in C_b^{1-}(U)$. Take $\varphi \in C_c^\infty(U)$, $j \in \{1, \dots, d\}$, and $\delta > 0$ such that $(\operatorname{supp}\varphi)_\delta \subseteq U$. For $\varepsilon \in (0, \delta]$ the difference quotient $\frac{1}{\varepsilon}(\varphi(x + \varepsilon e_j) - \varphi(x))$ converges uniformly on $\operatorname{supp}\varphi$ as $\varepsilon \to 0$, and hence

$$\begin{aligned}\left|\int_U f \partial_j \varphi \,\mathrm{d}x\right| &= \lim_{\varepsilon\to 0}\left|\int_{\operatorname{supp}\varphi} f(x)\frac{1}{\varepsilon}(\varphi(x+\varepsilon e_j) - \varphi(x))\,\mathrm{d}x\right| \\ &\leqslant \lim_{\varepsilon\to 0}\int_{\operatorname{supp}\varphi}\frac{1}{\varepsilon}|f(y-\varepsilon e_j) - f(y)|\,|\varphi(y)|\,\mathrm{d}y \\ &\leqslant [f]_{\mathrm{Lip}}\|\varphi\|_1.\end{aligned}$$

Since $C_c^\infty(U)$ is dense in $L^1(U)$, the map $\varphi \mapsto -\int_U f \partial_j \varphi \,\mathrm{d}x$ has a continuous linear extension $F_j : L^1(U) \to \mathbb{C}$. There thus exists a function $g_j \in L^\infty(U) = L^1(U)^*$ with $\|g_j\|_\infty = \|F_j\| \leqslant [f]_{\mathrm{Lip}}$ such that

$$-\int_U f \partial_j \varphi \,\mathrm{d}x = F_j(\varphi) = \int_U g_j \varphi \,\mathrm{d}x$$

for all $\varphi \in C_c^\infty(U)$. This means that f has the weak derivative $\partial_j f = g_j \in L^\infty(U)$. As a result, $f \in W^{1,\infty}(U)$ and $\|f\|_{1,\infty} \leqslant \|f\|_\infty + [f]_{\mathrm{Lip}}$. □

In the above proof convexity is only used in part 1). One can extend the result to other classes of domains, see Proposition 3.28. In the spirit of Remark 3.12, we mention Rademacher's theorem (Theorem 5.8.5 in [**Ev**]) which says that a Lipschitz continuous function f is differentiable for a.e. $x \in U$ and that the weak derivative $\partial_j f$ coincides with the pointwise one.

3.2. Density and embedding theorems

In this and the next section we discuss some of the most important theorems on Sobolev spaces.

Definition 3.14. *For $k \in \mathbb{N}$ and $1 \leqslant p < \infty$, the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$ is denoted by $W_0^{k,p}(U)$.*

Theorem 3.15. *Let $k \in \mathbb{N}$ and $p \in [1, \infty)$. We then have*

$$W_0^{k,p}(\mathbb{R}^d) = W^{k,p}(\mathbb{R}^d).$$

Moreover, the set $C^\infty(U) \cap W^{k,p}(U)$ is dense in $W^{k,p}(U)$.

Proof.[5] We prove the theorem only for $k = 1$, the general case can be treated similarly.

1) Let $f \in W^{1,p}(\mathbb{R}^d)$. Take any $\phi \in C^\infty(\mathbb{R})$ such that $0 \leqslant \phi \leqslant 1$, $\phi = 1$ on $[0, 1]$ and $\phi = 0$ on $[2, \infty)$. Set

$$\varphi_n(x) = \phi\left(\tfrac{1}{n}|x|_2\right) \qquad (\text{“cut-off function”})$$

for $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$. We then have $\varphi_n \in C_c^\infty(\mathbb{R}^d)$, $0 \leqslant \varphi_n \leqslant 1$ and $\|\partial_j \varphi_n\|_\infty \leqslant \|\phi'\|_\infty \frac{1}{n}$ for all $n \in \mathbb{N}$, as well as $\varphi_n(x) \to 1$ for all $x \in \mathbb{R}^d$ as $n \to \infty$. Thus $\|\varphi_n f - f\|_p \to 0$ as $n \to \infty$ by Lebesgue's convergence theorem. Further, Proposition 3.8 implies that

$$\begin{aligned}\|\partial_j(\varphi_n f - f)\|_p &= \|(\varphi_n \partial_j f - \partial_j f) + (\partial_j \varphi_n) f\|_p \\ &\leqslant \|\varphi_n \partial_j f - \partial_j f\|_p + \tfrac{1}{n}\|\phi'\|_\infty \|f\|_p\,,\end{aligned}$$

[5] Not shown in the lectures. The first part is Theorem 4.21 in [**Sc2**] for $k = 1$.

and the right hand side tends to 0 as $n \to \infty$ for each $j \in \{1, \ldots, d\}$. Given $\varepsilon > 0$, we can thus fix an index $m \in \mathbb{N}$ such that $\|\varphi_m f - f\|_{1,p} \leqslant \varepsilon$. Due to (3.4) and (3.5), the functions $G_{\frac{1}{n}}(\varphi_m f)$ belong to $C_c^\infty(\mathbb{R}^d)$ for all $n \in \mathbb{N}$. Equation (3.7) and Lemma 3.5 further yield that

$$G_{\frac{1}{n}}(\varphi_m f) \to \varphi_m f \quad \text{and} \quad \partial_j G_{\frac{1}{n}}(\varphi_m f) = G_{\frac{1}{n}} \partial_j(\varphi_m f) \to \partial_j(\varphi_m f)$$

in $L^p(\mathbb{R}^d)$ as $n \to \infty$, for $j \in \{1, \ldots, d\}$. So there is an index $n \in \mathbb{N}$ with

$$\|G_{\frac{1}{n}}(\varphi_m f) - \varphi_m f\|_{1,p} \leqslant \varepsilon,$$

and thus

$$\|G_{\frac{1}{n}}(\varphi_m f) - f\|_{1,p} \leqslant 2\varepsilon.$$

2) For the second assertion, we only have to consider the case $\partial U \neq \emptyset$. Let $f \in W^{1,p}(U)$. Set

$$U_n = \left\{x \in U \,\middle|\, |x|_2 < n \text{ and } d(x, \partial U) > \tfrac{1}{n}\right\}$$

for all $n \in \mathbb{N}$. Then $U_n \subseteq \overline{U_n} \subseteq U_{n+1} \subseteq U$, $\overline{U_n}$ is compact and $\bigcup_{n=1}^\infty U_n = U$. Observe that $U = \bigcup_{n=1}^\infty U_{n+1} \backslash \overline{U_{n-1}}$, where $U_0,\ U_{-1} := \emptyset$. There are functions φ_n in $C_c^\infty(U)$ such that $\operatorname{supp} \varphi_n \subseteq U_{n+1} \backslash \overline{U_{n-1}}$, $\varphi_n \geqslant 0$, and $\sum_{n=1}^\infty \varphi_n(x) = 1$ for all $x \in U$ (see e.g. Theorem 5.6 in [**RR**]).

Fix $\varepsilon > 0$. As in step 1), for each $n \in \mathbb{N}$ there is a number $\delta_n > 0$ such that $g_n := G_{\delta_n}(\varphi_n f) \in C_c^\infty(U)$, $\operatorname{supp} g_n \subseteq (\operatorname{supp} \varphi_n f)_{\delta_n} \subseteq U_{n+1} \backslash \overline{U_{n-1}}$ and $\|g_n - \varphi_n f\|_{1,p} \leqslant 2^{-n}\varepsilon$. Define $g(x) = \sum_{n=1}^\infty g_n(x)$ for all $x \in U$. Observe that on each ball $\overline{B}(x,r) \subseteq U$ this sum is finite. Hence, $g \in C^\infty(U)$. Since $f = \sum_{n=1}^\infty \varphi_n f$, we further have

$$g(x) - f(x) = \sum_{n=1}^\infty (g_n(x) - \varphi_n(x) f(x)),$$

for all $x \in U$ and $n \in \mathbb{N}$. Due to $\|g_n - \varphi_n f\|_{1,p} \leqslant 2^{-n}\varepsilon$, this series converges absolutely in $W^{1,p}(U)$, and

$$\|f - g\|_{1,p} \leqslant \sum_{n=1}^\infty \|g_n - \varphi_n f\|_{1,p} \leqslant \varepsilon. \qquad \square$$

Remark 3.16. a) If U is bounded, then $W_0^{k,p}(U) \neq W^{k,p}(U)$, see Lemma 6.67 in [**RR**].

b) For 'not too bad' ∂U one can replace in $C^\infty(U)$ by $C^\infty(\overline{U})$ in Theorem 3.15, see Theorem 3.27 below.

We now want to study embeddings of Sobolev spaces. We clearly have

$$W^{k,p}(U) \hookrightarrow W^{j,p}(U), \tag{3.11}$$

$$W^{k,p}(U) \hookrightarrow W^{j,q}(U) \qquad \text{if } \lambda(U) < \infty, \tag{3.12}$$

for $k \geqslant j \geqslant 0$, $1 \leqslant q \leqslant p \leqslant \infty$. (Recall that we have set $W^{0,p}(U) = L^p(U)$ for $1 \leqslant p \leqslant \infty$.) The embedding $X \hookrightarrow Y$ means that there is an injective map $J \in \mathcal{B}(X,Y)$. Writing $c = \|J\|$, one obtains $\|f\|_Y \leqslant c\,\|f\|_X$ if one identifies Jf with f.

THEOREM 3.17 (Sobolev, Morrey). *Let $k \in \mathbb{N}$ and $1 \leqslant p < \infty$. We have the following embeddings.*

(a) If $kp < d$, then

$$p^* := \frac{pd}{d-kp} \in (p,\infty) \quad \textit{and} \quad W^{k,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d) \quad \textit{for all} \;\; q \in [p,p^*].$$

(b) If $kp = d$, then

$$W^{k,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d) \qquad \textit{for all} \;\; q \in [p,\infty).$$

(c) If $kp > d$, then there are either $j \in \mathbb{N}_0$ and $\beta \in (0,1)$ such that $k - \frac{d}{p} = j + \beta$ or $k - \frac{d}{p} \in \mathbb{N}$. In the latter case we set $j := k - \frac{d}{p} - 1 \in \mathbb{N}_0$ and take any $\beta \in (0,1)$. Then

$$W^{k,p}(\mathbb{R}^d) \hookrightarrow C_0^{j+\beta}(\mathbb{R}^d)$$

for all $x, y \in \mathbb{R}^d$, $|\alpha| \leqslant j$, a constant $c > 0$, and

$$C_0^{j+\beta}(\mathbb{R}^d) := \{u \in C^j(\mathbb{R}^d) \,\big|\, \partial^\alpha u(x) \to 0 \textit{ as } |x|_2 \to \infty \textit{ and } \partial^\alpha u \textit{ is } \beta\text{–Hölder} \textit{ continuous on } \mathbb{R}^d, \;\; \textit{for all } 0 \leqslant |\alpha| \leqslant j\}.$$

In parts (a) and (b), the embedding J is just the inclusion map, and in part (c) the function Jf is the continuous representative of f.

We rephrase the above results in a slightly modified way using the 'effective regularity index' $k - \frac{d}{p}$ of $W^{k,p}$.

COROLLARY 3.18. *Let $k \in \mathbb{N}$, $j \in \mathbb{N}_0$, and $p \in [1,\infty)$. We have the following embeddings.*

(a) If $q \in [p,\infty)$ and $k - \frac{d}{p} \geqslant j - \frac{d}{q}$, then

$$W^{k,p}(\mathbb{R}^d) \hookrightarrow W^{j,q}(\mathbb{R}^d).$$

(b) If $q \in [p,\infty)$ and $k - \frac{d}{p} = j$, then

$$W^{k,p}(\mathbb{R}^d) \hookrightarrow W^{j,q}(\mathbb{R}^d).$$

(c) If $\beta \in (0,1)$ and $k - \frac{d}{p} = j + \beta$, then

$$W^{k,p}(\mathbb{R}^d) \hookrightarrow C_0^{j+\beta}(\mathbb{R}^d).$$

PROOF OF COROLLARY 3.18. (a) If $(k-j)p = d$, then Theorem 3.17(b) yields that

$$W^{k-j,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d) \tag{3.13}$$

for all $q \in [p,\infty)$. If $(k-j)p > d$, then $W^{k-j,p}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ by (3.11) and $W^{k-j,p}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ by Theorem 3.17(c). Hence the interpolation inequality (3.14) below implies the embedding (3.13) for all $q \in [p,\infty]$ in this case. Let $(k-j)p < d$. By assumption, we have $p \leqslant q \leqslant pd(d-(k-j)p)^{-1}$, so that (3.13) for these q is a consequence of Theorem 3.17(a). Applying the embedding (3.13) to $\partial^\alpha f \in W^{k-j,p}(\mathbb{R}^d)$ for all $|\alpha| \leqslant j$ and $f \in W^{k,p}(\mathbb{R}^d)$, we deduce

$$\|\partial^\alpha f\|_q \leqslant c\,\|\partial^\alpha f\|_{k-j,p} \leqslant c\,\|f\|_{k,p}.$$

So (a) is true. Part (b) follows from (a), and (c) from Theorem 3.17(c). $\square$

EXAMPLE 3.19. Let $d \geqslant 2$, $\alpha \in (0, 1 - \frac{1}{d})$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\varphi \subseteq B(0, 3/4)$ and $\varphi = 1$ on $B(0, 1/2)$. We define

$$f(x) := \begin{cases} \varphi(x)(-\ln|x|_2)^\alpha, & 0 < |x|_2 \leqslant 3/4, \\ 0, & |x|_2 > 3/4 \text{ or } x = 0. \end{cases}$$

Then $f \in W^{1,d}(\mathbb{R}^d) \backslash L^\infty(\mathbb{R}^d)$; i.e., Theorem 3.17(b) is sharp if $d \geqslant 2$.

PROOF. Arguing as in Example 3.3c), one sees that $f \in L^p(\mathbb{R}^d)$ for all $p < \infty$, $f \notin L^\infty(\mathbb{R}^d)$ and that, for all $j \in \{1, \ldots, d\}$, we have

$$\partial_j f(x) = (\partial_j \varphi(x))(-\ln|x|_2)^\alpha - \alpha\varphi(x)(-\ln|x|_2)^{\alpha-1} \frac{x_j}{|x|_2^2}$$

for $0 < |x|_2 < 3/4$ and $\partial_j f(x) = 0$ otherwise. Using polar coordinates, we further estimate

$$\begin{aligned} \left(\int_{\mathbb{R}^d} |\partial_j f|^d \, \mathrm{d}x \right)^{1/d} &\leqslant c \, \|\partial_j \varphi\|_\infty \left(\int_0^{3/4} (\ln r)^{\alpha d} r^{d-1} \, \mathrm{d}r \right)^{1/d} \\ &\qquad + c \, \|\varphi\|_\infty \left(\int_0^{3/4} \frac{(\ln r)^{(\alpha-1)d}}{r^d} r^{d-1} \, \mathrm{d}r \right)^{1/d} \\ &\leqslant c + c \left(\int_0^{3/4} \frac{\mathrm{d}r}{r(\ln r)^{(1-\alpha)d}} \right)^{1/d} < \infty \end{aligned}$$

for some constants $c > 0$, since $(1 - \alpha)d > 1$. □

For the proof of Theorem 3.17 we set $\hat{x}^j = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d) \in \mathbb{R}^{d-1}$ for all $x \in \mathbb{R}^d$, $j \in \{1, \ldots, d\}$ and $d \geqslant 2$. We start with a lemma.

LEMMA 3.20. *Let $d \geqslant 2$ and $f_1, \ldots, f_d \in L^{d-1}(\mathbb{R}^{d-1}) \cap C(\mathbb{R}^{d-1})$. Set $f(x) = f_1(\hat{x}^1) \cdot \ldots \cdot f_d(\hat{x}^d)$ for $x \in \mathbb{R}^d$. We then have $f \in L^1(\mathbb{R}^d)$ and*

$$\|f\|_{L^1(\mathbb{R}^d)} \leqslant \|f_1\|_{L^{d-1}(\mathbb{R}^{d-1})} \cdot \ldots \cdot \|f_d\|_{L^{d-1}(\mathbb{R}^{d-1})}.$$

PROOF.[6] If $d = 2$, then Fubini's theorem shows that

$$\int_{\mathbb{R}^2} |f(x)| \, \mathrm{d}x = \int_{\mathbb{R}} \int_{\mathbb{R}} |f_1(x_2)||f_2(x_1)| \, \mathrm{d}x_1 \, \mathrm{d}x_2 = \|f_1\|_1 \|f_2\|_1,$$

as asserted. Assume that the assertion holds for some $d \in \mathbb{N}$ with $d \geqslant 2$.

Take $f_1, \ldots, f_{d+1} \in L^d(\mathbb{R}^d) \cap C(\mathbb{R}^d)$. Write $y = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $x = (y, x_{d+1}) \in \mathbb{R}^{d+1}$. For a.e. $x_{d+1} \in \mathbb{R}$, the maps $\hat{y}^j \mapsto |f_j(\hat{y}^j, x_{d+1})|^d$ are integrable on $\mathbb{R}^{d-1}$ for each $j \in \{1, \ldots, d\}$ due to Fubini's theorem. Fix such a $x_{d+1} \in \mathbb{R}$ and write

$$\tilde{f}(y, x_{d+1}) := \prod_{j=1}^d f_j(\hat{x}^j).$$

Using Hölder's inequality, we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} |f(y, x_{d+1})| \, \mathrm{d}y &= \int_{\mathbb{R}^d} |\tilde{f}(y, x_{d+1})| \, |f_{d+1}(y)| \, \mathrm{d}y \\ &\leqslant \|f_{d+1}\|_{L^d(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} |\tilde{f}(y, x_{d+1})|^{d'} \, \mathrm{d}y \right)^{1/d'}. \end{aligned}$$

[6] Not shown in the lectures.

We set $g_j(\hat{y}^j) = |f_j(\hat{y}^j, x_{d+1})|^{d'}$ for $j \in \{1, \ldots, d\}$ and $x \in \mathbb{R}^{d+1}$. Since $d'(d-1) = d$, the maps g_j belong to $L^{d-1}(\mathbb{R}^{d-1})$ and the induction hypothesis yields

$$\begin{aligned}\int_{\mathbb{R}^d} |\tilde{f}(y, x_{d+1})|^{d'} \,\mathrm{d}y &= \int_{\mathbb{R}^d} g_1(\hat{y}^1) \cdot \ldots \cdot g_d(\hat{y}^d) \,\mathrm{d}y \leqslant \|g_1\|_{d-1} \cdot \ldots \cdot \|g_d\|_{d-1} \\ &= \prod_{j=1}^d \left(\int_{\mathbb{R}^{d-1}} |f_j(\hat{y}^j, x_{d+1})|^d \,\mathrm{d}y \right)^{\frac{1}{d-1}}.\end{aligned}$$

Integrating over $x_{d+1} \in \mathbb{R}$, we thus arrive at

$$\int_{\mathbb{R}^{d+1}} |f| \,\mathrm{d}x \leqslant \|f_{d+1}\|_d \int_{\mathbb{R}} \prod_{j=1}^d \left(\int_{\mathbb{R}^{d-1}} |f_j(\hat{x}^j)|^d \,\mathrm{d}y \right)^{\frac{1}{d-1} \frac{d-1}{d}} \mathrm{d}x_{d+1}.$$

Applying the d-fold Hölder inequality to the x_{d+1}-integral, we conclude that

$$\begin{aligned}\int_{\mathbb{R}^{d+1}} |f| dx &\leqslant \|f_{d+1}\|_d \prod_{j=1}^d \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^{d-1}} |f_j(\hat{x}^j)|^d \,\mathrm{d}y \right)^{\frac{1}{d} \cdot d} \mathrm{d}x_{d+1} \right)^{\frac{1}{d}} \\ &= \|f_1\|_d \cdot \ldots \cdot \|f_{d+1}\|_d.\end{aligned}$$

□

Recall from Analysis 3 that for $f \in L^p(U) \cap L^q(U)$ and $r \in [p, q]$ with $1 \leqslant p < q \leqslant \infty$, we have

$$\|f\|_r \leqslant \|f\|_p^\theta \, \|f\|_q^{1-\theta} \leqslant \theta \|f\|_p + (1 - \theta)\|f\|_q, \tag{3.14}$$

where $\theta \in [0, 1]$ is given by $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$ and we also used Young's inequality from Analysis 1+2.

Proof of Theorem 3.17.[7] We only prove the case $k = 1$, the rest can be done by induction, see e.g. §5.6.3 in [**Ev**]. Since $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$, in view of (3.14) for assertion (a) it suffices to show

$$W^{1,p}(\mathbb{R}^d) \hookrightarrow L^{p^*}(\mathbb{R}^d).$$

1) Let $f \in C_c^1(\mathbb{R}^d)$. Let first $p = 1 < d$, whence $p^* = \frac{d}{d-1}$. For $x \in \mathbb{R}^d$ and $j \in \{1, \ldots, d\}$, we then obtain

$$\begin{aligned}|f(x)| &= \left| \int_{-\infty}^{x_j} \partial_j f(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_d) \,\mathrm{d}t \right| \leqslant \int_{\mathbb{R}} |\partial_j f(x)| \,\mathrm{d}x_j, \\ |f(x)|^d &\leqslant \prod_{j=1}^d \int_{\mathbb{R}^d} |\partial_j f(x)| \,\mathrm{d}x_j.\end{aligned}$$

Setting $g_j(\hat{x}^j) = (\int_{\mathbb{R}} |\partial_j f(x)| dx_j)^{\frac{1}{d-1}}$, we deduce

$$|f(x)|^{\frac{d}{d-1}} \leqslant \prod_{j=1}^d g_j(\hat{x}^j).$$

After integration over $x \in \mathbb{R}^d$, Lemma 3.20 yields

$$\|f\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)}^{\frac{d}{d-1}} \leqslant \int_{\mathbb{R}^d} g_1(\hat{x}^1) \cdot \ldots \cdot g_d(\hat{x}^d) \,\mathrm{d}x \leqslant \prod_{j=1}^d \|g_j\|_{L^{d-1}(\mathbb{R}^{d-1})}$$

[7] Not shown in the lectures.

$$= \prod_{j=1}^{d} \left(\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} |\partial_j f(x)| \, \mathrm{d}x_j \, \mathrm{d}\hat{x}^j \right)^{\frac{1}{d-1}},$$

$$\|f\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leqslant \prod_{j=1}^{d} \|\partial_j f\|_{L^1(\mathbb{R}^d)}^{\frac{1}{d}} \leqslant \||\nabla f|_1\|_1 \leqslant \|f\|_{1,1}. \tag{3.15}$$

2) Next, let $p \in (1,d)$ and $p^* = \frac{pd}{d-p}$. Set $t_* := \frac{d-1}{d} p^* = \frac{d-1}{d-p} p > 1$. An elementary calculation shows that $(t_* - 1)p' = t_* \frac{d}{d-1} = p^*$. Set

$$g = f|f|^{t-1} = f(f\overline{f})^{\frac{t-1}{2}}$$

for $t > 1$. We compute

$$\begin{aligned} \partial_j g &= \partial_j f |f|^{t-1} + f \frac{t-1}{2} (f\overline{f})^{\frac{t-1}{2}-1} \big((\partial_j f)\overline{f} + f(\partial_j \overline{f}) \big) \\ &= \partial_j f |f|^{t-1} + (t-1) f |f|^{t-3} \operatorname{Re}(f \partial_j \overline{f}), \\ |g| &= |f|^t, \quad |\partial_j g| \leqslant t |\partial_j f| |f|^{t-1}. \end{aligned}$$

Applying (3.15) to g, we estimate

$$\begin{aligned} \|f\|_{\frac{td}{d-1}}^t &= \left(\int_{\mathbb{R}^d} |f|^{t\frac{d}{d-1}} \, \mathrm{d}x \right)^{\frac{d-1}{d}} = \left(\int_{\mathbb{R}^d} |g|^{\frac{d}{d-1}} \, \mathrm{d}x \right)^{\frac{d-1}{d}} \\ &\leqslant \prod_{j=1}^{d} \|\partial_j g\|_1^{\frac{1}{d}} \leqslant \prod_{j=1}^{d} t^{\frac{1}{d}} \left(\int_{\mathbb{R}^d} |\partial_j f| \, |f|^{t-1} \, \mathrm{d}x \right)^{\frac{1}{d}} \\ &\leqslant t \prod_{j=1}^{d} \Big(\int_{\mathbb{R}^d} |\partial_j f|^p \, \mathrm{d}x \Big)^{\frac{1}{dp}} \left(\int_{\mathbb{R}^d} |f|^{(t-1)p'} \, \mathrm{d}x \right)^{\frac{1}{p'd}} \\ &\leqslant t \prod_{j=1}^{d} \||\nabla f|_p\|_p^{\frac{1}{d}} \ \|f\|_{(t-1)p'}^{\frac{t-1}{d}} = t \, \||\nabla f|_p\|_p \ \|f\|_{(t-1)p'}^{t-1}, \end{aligned}$$

where we used Hölder's inequality. For $t = t_*$, we use the properties of t stated above and obtain

$$\|f\|_{p^*} \leqslant p \frac{d-1}{d-p} \||\nabla f|_p\|_p \leqslant p \frac{d-1}{d-p} \|f\|_{1,p}. \tag{3.16}$$

This estimate can be extended to all $f \in W^{1,p}(\mathbb{R}^d)$ by density (see Theorem 3.15). Hence, the inclusion map is the required embedding in (a).

3) Let $p = d$, $f \in C_c^1(\mathbb{R}^d)$, and $t > 1$. Then $p' = \frac{d}{d-1}$, and step 2) yields

$$\|f\|_{t\frac{d}{d-1}} \leqslant t^{\frac{1}{t}} \|f\|_{(t-1)\frac{d}{d-1}}^{1-\frac{1}{t}} \||\nabla f|_p\|_d^{\frac{1}{t}} \leqslant c \left(\|f\|_{(t-1)\frac{d}{d-1}} + \||\nabla f|_p\|_d \right) \tag{3.17}$$

using Young's inequality from Analysis 1+2. For $t = d$, this estimate gives $f \in L^{\frac{d^2}{d-1}}(\mathbb{R}^d)$ and

$$\|f\|_{\frac{d^2}{d-1}} \leqslant c \, \|f\|_{1,d}.$$

Here and below the constants $c > 0$ do not depend on f. For $q \in (d, d\frac{d}{d-1})$, inequality (3.14) further yields

$$\|f\|_q \leqslant c \, (\|f\|_d + \|f\|_{\frac{d^2}{d-1}}) \leqslant c \, \|f\|_{1,d}.$$

Now, we can apply (3.17) with $t = d+1$ and obtain

$$\|f\|_{\frac{d^2+d}{d-1}} \leqslant c\,(\|f\|_{\frac{d^2}{d-1}} + \||\nabla f|_p\|_d) \leqslant c\,\|f\|_{1,d}.$$

As above, we see that $f \in L^q(\mathbb{R}^d)$ for $d \leqslant q \leqslant d\frac{d+1}{d-1}$. We can now iterate this procedure with $t_n = d+n$ and obtain

$$\|f\|_q \leqslant c(q)\|f\|_{1,p}$$

for all $q < \infty$. As above, assertion (b) follows by approximation.

4) Let $p > d$, $f \in C_c^1(\mathbb{R}^d)$, $Q(r) = [-\frac{r}{2}, \frac{r}{2}]^d$ for some $r > 0$, and $x_0 \in Q(r)$. Set $M(r) = r^{-d}\int_{Q(r)} f\,\mathrm{d}x$. We further put $\beta := 1 - \frac{d}{p} \in (0,1)$. Using $|x - x_0|_\infty \leqslant r$ for $x \in Q(r)$, the transformation $y = t(x - x_0)$ and Hölder's inequality, we compute

$$\begin{aligned}
|f(x_0) - M(r)| &= \left| r^{-d}\int_{Q(r)} (f(x_0) - f(x))\,\mathrm{d}x \right| \\
&= r^{-d}\left| \int_{Q(r)} \int_1^0 \frac{\mathrm{d}}{\mathrm{d}t} f(x_0 + t(x - x_0))\,\mathrm{d}t\,\mathrm{d}x \right| \\
&\leqslant r^{-d}\int_{Q(r)} \int_0^1 |\nabla f(x_0 + t(x - x_0)) \cdot (x - x_0)|\,\mathrm{d}t\,\mathrm{d}x \\
&\leqslant r^{1-d}\int_0^1 \int_{Q(r)} |\nabla f(x_0 + t(x - x_0))|_1\,\mathrm{d}x\,\mathrm{d}t \\
&= r^{1-d}\int_0^1 \int_{t(Q(r)-x_0)} |\nabla f(x_0 + y)|_1\,\mathrm{d}y\; t^{-d}\,\mathrm{d}t \\
&\leqslant r^{1-d}\int_0^1 \left[\int_{\mathbb{R}^d} |\nabla f(x_0 + y)|_1^p\,\mathrm{d}y\right]^{\frac{1}{p}} \lambda(t(Q(r) - x_0))^{\frac{1}{p'}} t^{-d}\,\mathrm{d}t \\
&\leqslant c r^{1-d}\||\nabla f|_p\|_p \int_0^1 r^{\frac{d}{p'}} t^{\frac{d}{p'}-d}\,\mathrm{d}t \\
&= C r^{1-\frac{d}{p}}\||\nabla f|_p\|_p
\end{aligned}$$

for constants $C, c > 0$ only depending on d and p, using also that $\frac{d}{p'} - d > -1$ due to $p > d$. A translation then gives

$$\left| f(x_0 + z) - r^{-d}\int_{z+Q(r)} f(y)\,\mathrm{d}y \right| \leqslant C r^\beta \||\nabla f|_p\|_p$$

for all $z \in \mathbb{R}^d$. Taking $x = z$, $x_0 = 0$, $r = 1$, and using Hölder's inequality, we thus obtain

$$\begin{aligned}
|f(x)| &\leqslant \left| f(x) - \int_{x+Q(1)} f\,\mathrm{d}y \right| + \left| \int_{x+Q(1)} f\,\mathrm{d}y \right| \\
&\leqslant C\||\nabla f|_p\|_p + \|f\|_p \leqslant c\,\|f\|_{1,p} \qquad (3.18)
\end{aligned}$$

for all $x \in \mathbb{R}^d$, where c only depends on d and p. Given $x, y \in \mathbb{R}^d$, we find a cube Q of side length $|x - y|_\infty =: r$ such that $x, y \in Q$ and Q is parallel to the axes. Hence,

$$\begin{aligned}|f(x)-f(y)| &\leqslant \Big|f(x)-r^{-d}\int_Q f\,\mathrm{d}y\Big| + \Big|r^{-d}\int_Q f\,\mathrm{d}y - f(y)\Big| \\ &\leqslant 2C\||\nabla f|_p\|_p\,|x-y|_\infty^\beta \leqslant 2C\||\nabla f|_p\|_p\,|x-y|_2^\beta.\end{aligned}$$

If $f \in W^{1,p}(\mathbb{R}^d)$, then there are $f_n \in C_c^1(\mathbb{R}^d)$ converging to f in $W^{1,p}(\mathbb{R}^d)$. By (3.18), f_n is a Cauchy sequence in $C_0(\mathbb{R}^d)$. Hence, f has a representative $\tilde{f} \in C_0(\mathbb{R}^d)$ such that $f_n \to \tilde{f}$ uniformly as $n \to \infty$. So the above estimates imply that

$$\|\tilde{f}\|_\infty + \sup_{x\neq y}\frac{|\tilde{f}(x)-\tilde{f}(y)|}{|x-y|_2^\beta} \leqslant c\,\|f\|_{1,p}.$$

The map $f \mapsto \tilde{f}$ is the required embedding. □

Remark 3.21. Theorem 3.17 remains valied on any open set U instead of $\mathbb{R}^d$ if we replace $W^{k,p}$ by $W_0^{k,p}$. In fact, we obtain the desired estimate for $f \in C_c^\infty(U)$ if we apply Theorem 3.17 to the 0–extension of f. The result for $f \in W_0^{k,p}(U)$ then follows by density. ◊

Corollary 3.22. *Let*[8] *$U \subseteq \mathbb{R}^d$ be open and bounded. We then have* Poincare's inequality

$$\int_U |\nabla u|_p^p\,\mathrm{d}x \geqslant \delta \int_U |u|^p\,\mathrm{d}x \tag{3.19}$$

for some $\delta > 0$ and all $u \in W_0^{1,p}(U)$ and $p \in [1,\infty)$.

Proof. For $p \in [1,d)$, the estimate (3.19) follows from (3.16) since $L^{p^*}(U) \hookrightarrow L^p(U)$. Let $p \in [d,\infty)$. The case $p = d = 1$ is easy to check. For the other cases, fix $r \in (p,\infty)$ and $u \in W_0^{1,p}(U)$. Then (3.14), (3.15) and $W_0^{1,p}(U) \hookrightarrow L^r(U)$ imply

$$\begin{aligned}\|u\|_p &\leqslant c_\varepsilon\|u\|_1 + \varepsilon\|u\|_r \leqslant c_\varepsilon\||\nabla u|_1\|_1 + c\varepsilon\|u\|_{1,p} \\ &\leqslant c_\varepsilon\||\nabla u|_p\|_p + c\varepsilon\|u\|_p + c\varepsilon\||\nabla u|_p\|_p\end{aligned}$$

for all $\varepsilon > 0$ and some constants $c_\varepsilon, c > 0$ independent of u, where c does not depend on $\varepsilon > 0$. Choosing a small ε, we derive (3.19). □

Remark 3.23. We can show (parts of) Theorem 3.17(a) and (c) for the case $p = 2$ using the Fourier transform. We first recall some basic facts. For $f \in L^1(\mathbb{R}^d)$, we define *Fourier transform* by

$$\hat{f}(\xi) = (\mathcal{F}f)(\xi) := \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\,\xi\cdot x} f(x)\,\mathrm{d}x, \quad \xi \in \mathbb{R}^d,$$

where the dot denotes the scalar product in $\mathbb{R}^d$. Plancherel's theorem says that $\mathcal{F}$ can be extended from $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ to an isometric bijection $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$. Moreover, $\mathcal{F}$ is a bijection on the Schwartz space $\mathcal{S}_d$ with inverse given by $(\mathcal{F}^{-1}g)(y) = (\mathcal{F}g)(-y)$. See e.g. Proposition 5.10 and Theorem 5.11 in [**Sc1**]. For Sobolev spaces we have the crucial identity

$$W^{k,2}(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) \mid |\xi|_2^k\,\hat{u} \in L^2(\mathbb{R}^d)\} \tag{3.20}$$

[8]This result was not part of the lectures.

for $k \in \mathbb{N}$, where $|\xi|_2^k$ stands for the function $\xi \mapsto |\xi|_2^k$ (and analogously for similar expressions). Moreover, the norm of $W^{k,2}(\mathbb{R}^d)$ is equivalent to the one given by $\|u\|_2^2 + \|\,|\xi|_2^k\,\widehat{u}\|_2^2 = \|\widehat{u}\|_2^2 + \|\,|\xi|_2^k\,\widehat{u}\|_2^2$ and

$$\mathcal{F}(\partial^\alpha u) = \mathrm{i}\xi^\alpha \widehat{u} \qquad \text{for } \ u \in W^{k,2}(\mathbb{R}^d),\ |\alpha| \leqslant k. \tag{3.21}$$

See e.g. Theorem 5.21 in [**Sc1**]. We further use the Hausdorff–Young inequality

$$\|\mathcal{F}f\|_q \leqslant c\,\|f\|_{q'} \qquad \text{for } \ q \in [2,\infty],\ f \in L^{q'}(\mathbb{R}^d), \tag{3.22}$$

see e.g. Satz V.2.10 in [**We**].

We come to the announced proofs. Let $f \in C_c^\infty(\mathbb{R}^d)$ and $k \in \mathbb{N}$. Then $\widehat{f} \in \mathcal{S}_d \subseteq L^1(\mathbb{R}^d)$.

1) Let $k > d/2$. Using the Fourier inversion formula, (3.22) for $q = \infty$, Hölder's inequality, (3.20) and polar coordinates, we compute

$$\begin{aligned}
\|f\|_\infty = \|\mathcal{F}^{-1}\mathcal{F}f\|_\infty &\leqslant (2\pi)^{-d/2}\,\|(1+|\xi|_2^k)^{-1}(1+|\xi|_2^k)\widehat{f}\|_1 \\
&\leqslant (2\pi)^{-d/2}\,\|(1+|\xi|_2^k)^{-1}\|_2\,\|(1+|\xi|_2^k)\widehat{f}\|_2 \\
&\leqslant c\,\|f\|_{k,2}\left(\int_0^\infty \frac{r^{d-1}}{(1+r^k)^2}\,\mathrm{d}r\right)^{1/2} \leqslant c\,\|f\|_{k,2}
\end{aligned}$$

for constants $c > 0$, since $2k > d$. Next, let $g \in W^{k,2}(\mathbb{R}^d)$. By Theorem 3.15, there are $f_n \in C_c^\infty(\mathbb{R}^d) \subseteq C_0(\mathbb{R}^d)$ which tend to g in $W^{k,2}(\mathbb{R}^d)$ and thus in $L^2(\mathbb{R}^d)$. The above estimate implies that (f_n) is a Cauchy sequence for the sup-norm. The limit of (f_n) in $C_0(\mathbb{R}^d)$ is then a representative of its limit g in $L^2(\mathbb{R}^d)$. In particular, g also satisfies the above estimate and we have shown Theorem 3.17(c) for $p = 2$, $j = 0$ and $\beta = 0$; i.e., $W^{k,2}(\mathbb{R}^d) \hookrightarrow C_0(\mathbb{R}^d)$.

2) Let $k < d/2$ and $2 \leqslant q < 2^* = 2d(d-2k)^{-1}$. The latter is equivalent to

$$\frac{1}{q} > \frac{1}{2} - \frac{k}{d}.$$

To apply Hölder's inequality, we define the number $s \in (2,\infty]$ by

$$\frac{1}{s} = \frac{1}{q'} - \frac{1}{2} = \frac{1}{2} - \frac{1}{q}. \qquad \text{Hence, } \ \frac{1}{s} < \frac{k}{d}.$$

As in part 1), we estimate

$$\begin{aligned}
\|f\|_q &\leqslant c\,\|(1+|\xi|_2^k)^{-1}(1+|\xi|_2^k)\widehat{f}\|_{q'} \leqslant c\,\|(1+|\xi|_2^k)^{-1}\|_s\,\|(1+|\xi|_2^k)\widehat{f}\|_2 \\
&\leqslant c\,\|f\|_{k,2}\left(\int_0^\infty \frac{r^{d-1}}{(1+r^k)^s}\,\mathrm{d}r\right)^{1/s} \leqslant c\,\|f\|_{k,2},
\end{aligned}$$

where we have used that $sk > d$ by the above relations between the exponents. One can again conclude that $W^{k,2}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$, which is Theorem 3.17(a) for $p = 2$ and $q < 2^*$.[9] $\Diamond$

Problem: How to extend the Sobolev embedding to $W^{k,p}(U)$?

This problem can be solved by an *extension operator*; i.e., a map $E \in \mathcal{B}(W^{k,p}(U), W^{k,p}(\mathbb{R}^d))$ satisfying $Eu = u$ on U. We state a rather advanced theorem that provides such an operator for open sets $U \subseteq \mathbb{R}^d$ with a compact *Lipschitz boundary.* This means that each $x \in \partial U$ has a neighborhood V_x in

[9] One can show the limiting case $q = 2^*$ by a refinement of the above argument.

$\mathbb{R}^d$ such that there is a Lipschitz function $f_x : \mathbb{R}^{d-1} \to \mathbb{R}$ such that (possibly after a rotation) we have $U \cap V_x = \{y \in V_x \,|\, y_d < f_x(y_1, \cdots, y_{d-1})\}$. Due to compactness, we can then cover ∂U by finitely many such V_x, cf. §4.9 in [**AF**]. Clearly, compact boundaries of type C^k as defined in Section 2.3 are also compact Lipschitz boundaries. We further need the upper and lower half spaces

$$\mathbb{R}^d_\pm = \{(x', y) \,|\, x' \in \mathbb{R}^d,\ y \gtrless 0\}.$$

THEOREM 3.24 (Stein extension theorem). *Let U have a compact Lipschitz boundary or let $U = \mathbb{R}^d_+$. Let $k \in \mathbb{N}$ and $p \in [1, \infty]$. Then there exists an operator $E_{k,p}$ in $\mathcal{B}(W^{k,p}(U), W^{k,p}(\mathbb{R}^d))$ with $E_{k,p}u = u$ on U. These operators are restrictions of each other. We thus write E_U for all of them.*

An even more general result is proved in Theorem VI.5 of [**St**]. A sketch of the proof is given in §5.25 of [**AF**].

REMARK 3.25. We sketch a proof of the extension theorem for $W^{1,p}(U)$ with $p \in [1, \infty]$ if ∂U is compact and of class C^1.[10] One needs a C^k–boundary to prove the corresponding result on $W^{k,p}(U)$ by the method below.

1) For any open U, $p \in [1, \infty]$ and $k \in N_0$, we define the *0-extension operator* $E_0 : L^p(U) \to L^p(\mathbb{R}^d)$ by $E_0 f(x) = f(x)$ for $x \in U$ and $E_0 f(x) = 0$ for $x \in \mathbb{R}^d \backslash U$ and the *restriction operator* $R_U : W^{k,p}(\mathbb{R}^d) \to W^{k,p}(U)$ by $R_U f = f_{|U}$. Clearly, E_0 is a linear isometry and R_U is a linear contraction. Let $f \in W^{1,p}(\mathbb{R}^d_-) \cap C^1(\overline{\mathbb{R}^d_-})$. We write elements in $\mathbb{R}^d_\pm$ as (y, t). Define

$$E_- f(y, r) = \begin{cases} f(y, t), & (y, t) \in \overline{\mathbb{R}^d_-}, \\ 4f(y, -\frac{t}{2}) - 3f(y, -t), & (y, t) \in \mathbb{R}^d_+. \end{cases}$$

Note that $E_- f$ belongs to $C^1(\mathbb{R}^d)$. One can check that $\|E_- f\|_{W^{1,p}(\mathbb{R}^d)} \leqslant c\,\|f\|_{W^{1,p}(\mathbb{R}^d_-)}$ for a constant $c > 0$.

2) We show that $W^{1,p}(\mathbb{R}^d_-) \cap C^1(\overline{\mathbb{R}^d_-})$ is dense in $W^{1,p}(\mathbb{R}^d_-)$, so that E_- can be extended to an extension operator on $W^{1,p}(\mathbb{R}^d_-)$. In fact, let $f \in W^{1,p}(\mathbb{R}^d_-)$ and $\varepsilon > 0$. Theorem 3.15 yields a function $g \in C^\infty(H_-) \cap W^{1,p}(\mathbb{R}^d_-)$ with $\|f - g\|_{1,p} \leqslant \varepsilon$. Setting $g_n(y, t) = g(y, t - \frac{1}{n})$ for $t \leqslant 0$, $y \in \mathbb{R}^{d-1}$ and $n \in \mathbb{N}$, we define maps g_n in $C^1(\overline{\mathbb{R}^d_-}) \cap W^{1,p}(\mathbb{R}^d_-)$. Observe that

$$\partial^\alpha g_n = R_{\overline{\mathbb{R}^d_-}} S_n E_0 \partial^\alpha g$$

for $0 \leqslant |\alpha| \leqslant 1$, where $S_n \in \mathcal{B}(L^p(\mathbb{R}^d))$ is given by $S_n h(y, t) = h(y, t - \frac{1}{n})$ for $h \in L^p(\mathbb{R}^d)$. One can see that $S_n h \to h$ in $L^p(\mathbb{R}^d)$ as in Example 4.12 of [**Sc2**]. Hence, g_n converges to g in $W^{1,p}(\mathbb{R}^d_-)$ implying the claim.

3) Since $\partial U \in C^1$, there are bounded open sets $U_0, U_1, \ldots, U_m \subseteq \mathbb{R}^d$ such that $U \subseteq U_0 \cup \cdots \cup U_m$, $\overline{U}_0 \subseteq U$ and $\partial U \subseteq U_1 \cup \cdots \cup U_m$, as well as a diffeomorphism $\Psi_j : U_j \to V_j$ such that Ψ'_j and $(\Psi_j^{-1})'$ are bounded and $\Psi_j(U_j \cap U) \subseteq \mathbb{R}^d_-$ and $\Psi_j(U_j \cap \partial U) \subseteq \mathbb{R}^{d-1} \times \{0\}$, for each $j \in \{1, \ldots, m\}$. Theorem 5.6 in [**RR**] provides us with functions $0 \leqslant \varphi_j \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp} \varphi_j \subseteq U_j$ for all $j = 0, 1, \ldots, m$ and $\sum_{j=0}^m \varphi_j(x) = 1$ for all $x \in \overline{U}$.

[10] Not shown in the lectures.

Let $j \in \{1, \ldots, m\}$. Set $S_j g(y) = g(\Psi_j^{-1}(y))$ for $y \in \mathbb{R}^d_- \cap V_j$ and $S_j g(y) = 0$ for $y \in \mathbb{R}^d_- \backslash V_j$, where $g \in W^{1,p}(U_j \cap U)$. For $h \in W^{1,p}(\mathbb{R}^d)$, set $\hat{S}_j h(x) = h(\Psi_j(x))$ for $x \in U_j$ and $\hat{S}_j h(x) = 0$ for $x \in \mathbb{R}^d \backslash U_j$. Take any $\tilde{\varphi}_j \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp} \tilde{\varphi}_j \subseteq U_j$ and $\tilde{\varphi}_j = 1$ on $\operatorname{supp} \varphi_j$ (see Lemma 3.4). Let $f \in W^{1,p}(U)$. We now define

$$E_1 f = E_0 \varphi_0 f + \sum_{j=1}^{m} \tilde{\varphi}_j \hat{S}_j E_- S_j \big(R_{(U_j \cap U)}(\varphi_j f)\big).$$

Using part 2) and Propositions 3.8 and 3.9, we see that E_1 belongs to $\mathcal{B}(W^{1,p}(U), W^{1,p}(\mathbb{R}^d))$. Let $x \in U$. If $x \in U_k$ for some $k \in \{1, \ldots, m\}$, we have $\Psi_k(x) \in \mathbb{R}^d_-$. If $x \notin U_j$, then $\tilde{\varphi}_j(x) = 0$. Thus

$$\begin{aligned} E_1 f(x) &= \varphi_0(x) f(x) + \sum_{\substack{j=1,\ldots,m \\ x \in U_j}} \tilde{\varphi}_j(x)(\varphi_j f)(\Psi_j^{-1}(\Psi_j(x))) \\ &= \sum_{j=0}^{m} \varphi_j(x) f(x) = f(x). \end{aligned}$$

If $x \in U_0 \backslash (U_1 \cup \cdots \cup U_m)$, we also have $E_1 f(x) = \varphi_0 f(x) = f(x)$. $\Diamond$

We can now easily extend and improve the above results.

THEOREM 3.26 (Sobolev–Morrey on U). *Let U have a compact Lipschitz boundary or let $U = \mathbb{R}^d_+$. Theorem 3.17 and Corollary 3.18 then remain true if we replace $\mathbb{R}^d$ by U and $C_0^j(\mathbb{R}^d)$ by*

$$C_0^j(\overline{U}) = \{f \in C^j(U) \,|\, \partial^\alpha f \text{ has a continuous extension to } \partial U,\ \partial^\alpha f(x) \to 0 \text{ as } |x|_2 \to \infty \text{ if } U \text{ is unbounded, for all } 0 \leqslant |\alpha| \leqslant j\}.$$

PROOF. Consider e.g. Theorem 3.17(a). We have the embedding

$$J : W^{k,p}(\mathbb{R}^d) \hookrightarrow L^{p^*}(\mathbb{R}^d)$$

given by the inclusion. Thanks to Theorem 3.24, the map

$$R_U J E_U : W^{k,p}(U) \to L^{p^*}(U)$$

is continuous and injective. The other assertions are proved similarly. $\square$

THEOREM 3.27. *Let U have a compact Lipschitz boundary or let $U = \mathbb{R}^d_+$, $k \in \mathbb{N}$ and $p \in [1, \infty)$. Then*

$$W^{k,p}(U) \cap C^\infty(\overline{U}) \text{ is dense in } W^{k,p}(U),$$

where $C^\infty(\overline{U})$ contains all $f \in C^\infty(U)$ such that f and all its derivatives can continuously be extended to ∂U.

PROOF. Let $f \in W^{k,p}(U)$. Then $E_U f$ belongs to $W^{k,p}(\mathbb{R}^d)$ by Theorem 3.24. Theorem 3.15 yields functions g_n in $C_c^\infty(\mathbb{R}^d)$ that converge to $E_U f$ in $W^{k,p}(\mathbb{R}^d)$. Hence, $R_U g_n$ is contained in $W^{k,p}(U) \cap C^\infty(\overline{U})$ and tends to $f = R_U E_U f$ in $W^{k,p}(U)$ as $n \to \infty$. $\square$

PROPOSITION 3.28. *Let U have a compact C^1–boundary. Then $W^{1,\infty}(U)$ is isomorphic to*

$$C_b^{1-}(U) = \{f \in C_b(U) \,|\, f \text{ is Lipschitz}\},$$

and the norm of $W^{1,\infty}(U)$ *is equivalent to*

$$\|f\|_{C_b^{1-}} = \|f\|_\infty + [f]_{Lip},$$

where $[f]_{Lip}$ *is the Lipschitz constant of* f.

Proof. The extension operator E_1 from Remark 3.25 also is continuous from $C_b^{1-}(U)$ to $C_b^{1-}(\mathbb{R}^d)$. Owing to Proposition 3.13 we have an isomorphism J from $W^{1,\infty}(\mathbb{R}^d)$ to $C_b^{1-}(\mathbb{R}^d)$ given by choosing the continuous representative. Then $R_U J E_1$ is the asserted isomorphism. □

We continue with one of the main compactness results in analysis.

Theorem 3.29 (Rellich-Kondrachov). *Let* $U \subseteq \mathbb{R}^d$ *be bounded and open with a Lipschitz boundary* ∂U, $k \in \mathbb{N}$ *and* $1 \leqslant p < \infty$. *Then the following assertions hold.*

(a) If $kp \leqslant d$ *and* $1 \leqslant q < p^* = \frac{dp}{d-kp} \in (p, \infty]$, *then the inclusion map*

$$J : W^{k,p}(U) \hookrightarrow L^q(U)$$

is compact. (For instance, let $q = p$.*)*

(b) If $k - \frac{d}{p} > j \in \mathbb{N}_0$, *then the embedding*

$$J : W^{k,p}(U) \hookrightarrow C^j(\overline{U})$$

is compact, where Jf *is the continuous representative.*

Recall that a compact embedding $J : Y \hookrightarrow X$ means that any bounded sequence (y_n) in Y has a subsequence such that $(Jy_{n_j})_j$ converges in X.

Proof of Theorem 3.29.[11] We prove the result only for $k = 1$ (and thus $j = 0$), see e.g. Theorem 6.3 of [**AF**] for the other cases. Part (b) follows from the Arzela-Ascoli theorem since Theorem 3.26 gives constants $\beta, c > 0$ such that $|f(x) - f(y)| \leqslant c\,|x-y|^\beta$ and $|f(x)| \leqslant c$ for all $x, y \in U$ and $f \in W^{1,p}(U)$ with $\|f\|_{1,p} \leqslant 1$, where $p > d$.

In the case $p < d$, take $f_n \in W^{1,p}(U)$ with $\|f_n\|_{1,p} \leqslant 1$ for all $n \in \mathbb{N}$. Fix an open bounded set $V \subseteq \mathbb{R}^d$ containing $\overline{U}$. Lemma 3.4 yields a function $\varphi \in C_c^\infty(V)$ which is equal to 1 on $\overline{U}$. Let E_U be given by Theorem 3.24. Set $g_n = \varphi E_U f_n \in W^{1,p}(\mathbb{R}^d)$. These functions have support V and $\|g_n\|_{1,p} \leqslant c\,\|\varphi\|_{1,\infty}\|E_U\| =: M$ for all $n \in \mathbb{N}$. Take $q \in [1, p^*)$ and $\theta \in (0,1]$ with $\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p^*}$. Inequality (3.14) and Theorem 3.17 yield that

$$\begin{aligned}
\|f_n - f_m\|_{L^q(U)} &\leqslant \|g_n - g_m\|_{L^q(V)} \leqslant \|g_n - g_m\|_{L^1(V)}^\theta \|g_n - g_m\|_{L^{p^*}(V)}^{1-\theta} \\
&\leqslant c\,\|g_n - g_m\|_{L^1(V)}^\theta (\|g_n\|_{1,p}^{1-\theta} + \|g_m\|_{1,p}^{1-\theta}) \\
&\leqslant 2cM^{1-\theta}\|g_n - g_m\|_{L^1(V)}^\theta
\end{aligned}$$

for all $n, m \in \mathbb{N}$. So it suffices to construct a subsequence of g_n which converges in $L^1(V)$. For $x \in V$, $n \in \mathbb{N}$ and $\varepsilon > 0$, we compute

$$|g_n(x) - G_\varepsilon g_n(x)| = \left|\int_{\mathbb{R}^d} k_\varepsilon(x-y)(g_n(x) - g_n(y))\,\mathrm{d}y\right|$$

[11] Not shown in the lectures.

$$
\begin{aligned}
&\leqslant \varepsilon^{-d} \int_{B(x,\varepsilon)} k(\tfrac{1}{\varepsilon}(x-y))\, |g_n(x) - g_n(y)| \, \mathrm{d}y \\
&= \int_{B(0,1)} k(z) |g_n(x) - g_n(x - \varepsilon z)| \, \mathrm{d}z \\
&= \int_{B(0,1)} k(z) \left| \int_0^\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} g_n(x - tz) \, \mathrm{d}t \right| \mathrm{d}z \\
&\leqslant \int_{B(0,1)} k(z) \int_0^\varepsilon |\nabla g_n(x - tz) \cdot z| \, \mathrm{d}t \, \mathrm{d}z \\
&\leqslant \int_0^\varepsilon \int_{B(0,1)} k(z)\, |\nabla g_n(x - tz)|_2 \, \mathrm{d}z \, \mathrm{d}t \\
&= \int_0^\varepsilon \int_{B(x,t)} t^{-d} k(\tfrac{1}{t}(x-y))\, |\nabla g_n(y)|_2 \, \mathrm{d}y \, \mathrm{d}t \\
&= \int_0^\varepsilon \|k_t * |\nabla g_n|_2\|_1 \, \mathrm{d}t \leqslant \varepsilon \sup_{0 \leqslant t \leqslant \varepsilon} \|k_t\|_1 \, \||\nabla g_n|_2\|_1 \\
&\leqslant c\varepsilon \, \||\nabla g_n|_p\|_p \leqslant cM\varepsilon,
\end{aligned}
$$

where we have used the transformations $z = \frac{1}{\varepsilon}(x - y)$ and $y = x - tz$, as well as Fubini's theorem, Young's inequality, (3.6) and $L^p(V) \hookrightarrow L^1(V)$. We thus obtain

$$
\|g_n - G_\varepsilon g_n\|_{L^1(V)} \leqslant c\lambda(V) M\varepsilon =: C\varepsilon \tag{3.23}
$$

for all $n \in \mathbb{N}$ and $\varepsilon > 0$. On the other hand, the definition of $G_\varepsilon g_n$ yields

$$
|G_\varepsilon g_n(x)| \leqslant \|k_\varepsilon\|_\infty \, \|g_n\|_{L^1(V)}, \quad \text{and} \quad |\nabla G_\varepsilon g_n(x)| \leqslant \|\nabla k_\varepsilon\|_\infty \, \|g_n\|_{L^1(V)}
$$

for all $x \in V$ and $n \in \mathbb{N}$ and each fixed $\varepsilon > 0$. The Arzela-Ascoli theorem now implies that the set $F_\varepsilon := \{G_\varepsilon g_n \,|\, n \in \mathbb{N}\}$ is relatively compact in $C(\overline{V})$ for each $\varepsilon > 0$, and thus in $L^1(V)$ since $C(\overline{V}) \hookrightarrow L^1(V)$. Let $\delta > 0$ be given and fix $\varepsilon = \frac{\delta}{2C}$. Then there are indeces $n_1, \dots, n_l \in \mathbb{N}$ such that

$$
F_\varepsilon \subseteq \bigcup\nolimits_{j=1}^{l} B_{L^1(V)}(G_\varepsilon g_{n_j}, \tfrac{\delta}{2}) =: \bigcup\nolimits_{j=1}^{l} B_j.
$$

Hence, given $n \in \mathbb{N}$, there is an index n_j such that $G_\varepsilon g_n \in B_j$. The estimates (3.23) and (3.6) then yield

$$
\|g_n - G_\varepsilon g_{n_j}\|_{L^1(V)} \leqslant \|g_n - G_\varepsilon g_n\|_{L^1(V)} + \|G_\varepsilon(g_n - g_{n_j})\|_{L^1(V)} \leqslant C\varepsilon + \delta/2 = \delta.
$$

We have shown that, for each $\delta > 0$, the set $G := \{g_n \,|\, n \in \mathbb{N}\}$ is covered by finitely many open balls B_j of radius $\delta/2$; i.e., G is totally bounded in $L^1(V)$. Thus G contains a subsequence converging in $L^1(V)$ (see e.g. Corollary 1.39 in [**Sc2**]). In the case $p = d$ one simply replaces p^* by any $r \in (q, \infty)$. □

REMARK 3.30. a) Theorem 3.29 is wrong for unbounded domains, in general. In fact, let $k \in \mathbb{N}$, $p \in [1, \infty)$ and define the functions $f_n = f(\cdot - n)$ in $W^{k,p}(\mathbb{R})$ for any function $0 \neq f \in C^\infty(\mathbb{R})$ with $\operatorname{supp} f \subseteq (-1/2, 1/2)$. Then $\|f_n\|_{k,p}$ and $\|f_n - f_m\|_q > 0$ do not depend on $n \neq m$ in $\mathbb{N}$ so that (f_n) is bounded in $W^{k,p}(\mathbb{R})$ and has no subsequence which converges in L^q with $1 \leqslant q < p^*$.

b) The embedding $W^{1,p}(U) \hookrightarrow L^{p^*}(U)$ is never compact, see Example 6.12 in [**AF**]. ◊

We conclude this section with an interpolation estimate for first derivatives. Again there are plenty of variants.

PROPOSITION 3.31. *Let $1 \leqslant p < \infty$. Let either $f \in W_0^{2,p}(U)$ or U have a compact Lipschitz boundary and $f \in W^{2,p}(U)$. Then there are constants $C, \varepsilon_0 > 0$ such that*

$$\Big(\sum_{j=1}^d \|\partial_j f\|_p^p\Big)^{1/p} \leqslant \varepsilon \Big(\sum_{i,j=1}^d \|\partial_{ij} f\|_p^p\Big)^{1/p} + \frac{C}{\varepsilon}\,\|f\|_p,$$

for all $\varepsilon > 0$ if $f \in W_0^{2,p}(U)$ and for all $0 < \varepsilon \leqslant \varepsilon_0$ if $f \in W^{2,p}(U)$.

PROOF.[12] 1) Let $f \in C_c^2(U)$ and extend it to $\mathbb{R}^d$ by 0. Take $j = 1$. Write $x = (t, y) \in \mathbb{R} \times \mathbb{R}^{d-1}$ for $x \in \mathbb{R}^d$. Fix $y \in \mathbb{R}^{d-1}$ and set $g(t) = f(t, y)$ for $t \in \mathbb{R}$. Let $\varepsilon > 0$ and $a, b \in \mathbb{R}$ with $b - a = \varepsilon$. Take any $r \in (a, a + \frac{\varepsilon}{3})$ and $t \in (b - \frac{\varepsilon}{3}, b)$. There there is a number $\overline{s} = \overline{s}(r, t) \in (a, b)$ such that

$$|g'(\overline{s})| = \Big|\frac{g(t) - g(r)}{t - r}\Big| \leqslant \frac{3}{\varepsilon}(|g(t)| + |g(r)|).$$

For every $s \in (a, b)$ we thus obtain

$$|g'(s)| = \Big|g'(\overline{s}) + \int_{\overline{s}}^s g''(\tau)\,\mathrm{d}\tau\Big| \leqslant \frac{3}{\varepsilon}(|g(r)| + |g(t)|) + \int_a^b |g''(\tau)|\,\mathrm{d}\tau.$$

Integrating first over r and then over t, we conclude

$$\begin{aligned}
\frac{\varepsilon}{3}|g'(s)| &\leqslant \frac{3}{\varepsilon}\int_a^{a+\frac{\varepsilon}{3}} |g(r)|\,\mathrm{d}r + |g(t)| + \frac{\varepsilon}{3}\int_a^b |g''(\tau)|\,\mathrm{d}\tau,\\
\frac{\varepsilon^2}{9}|g'(s)| &\leqslant \int_a^{a+\frac{\varepsilon}{3}} |g(r)|\,\mathrm{d}r + \int_{b-\frac{\varepsilon}{3}}^b |g(t)|\,\mathrm{d}t + \frac{\varepsilon^2}{9}\int_a^b |g''(\tau)|\,\mathrm{d}\tau,\\
|g'(s)| &\leqslant \frac{9}{\varepsilon^2}\int_a^b |g(\tau)|\,\mathrm{d}\tau + \int_a^b |g''(\tau)|\,\mathrm{d}\tau\\
&\leqslant \varepsilon^{\frac{1}{p'}}\frac{9}{\varepsilon^2}\Big(\int_a^b |g(\tau)|^p\,\mathrm{d}\tau\Big)^{1/p} + \varepsilon^{\frac{1}{p'}}\Big(\int_a^b |g''(\tau)|^p\,\mathrm{d}\tau\Big)^{1/p}\\
&\leqslant \varepsilon^{\frac{p-1}{p}}2^{\frac{p-1}{p}}\left(\Big(\frac{9}{\varepsilon^2}\Big)^p\int_a^b |g(\tau)|^p\,\mathrm{d}\tau + \int_a^b |g''(\tau)|\,\mathrm{d}\tau\right)^{1/p},
\end{aligned}$$

where we used Hölder's inequality first for the integrals and then in $\mathbb{R}^2$. We take now the p-th power and then integrate over s arriving at

$$\int_a^b |g'(s)|^p\,\mathrm{d}s \leqslant \varepsilon\varepsilon^{p-1}2^{p-1}\left(\frac{9^p}{\varepsilon^{2p}}\int_a^b |g(\tau)|^p\,\mathrm{d}\tau + \int_a^b |g''(\tau)|^p\,\mathrm{d}\tau\right).$$

Now choose $a = a_k = k\varepsilon$ and $b = b_k = (k+1)\varepsilon$ for $k \in \mathbb{Z}$. Summing the integrals on $[k\varepsilon, (k+1)\varepsilon)$ for $k \in \mathbb{Z}$ and then integrating over $y \in \mathbb{R}^{d-1}$, it follows that

$$\begin{aligned}
\int_{\mathbb{R}} |g'(\tau)|^p\,\mathrm{d}\tau &\leqslant \varepsilon^p 2^{p-1}\left(\frac{9^p}{\varepsilon^{2p}}\int_{\mathbb{R}} |g(\tau)|^p\,\mathrm{d}\tau + \int_{\mathbb{R}} |g''(\tau)|^p\,\mathrm{d}\tau\right),\\
\int_U |\partial_1 f|^p\,\mathrm{d}x &\leqslant (2\varepsilon)^p\int_U |\partial_{11} f|^p\,\mathrm{d}x + \frac{36^p}{(2\varepsilon)^p}\int_U |f|^p\,\mathrm{d}x. \qquad (3.24)
\end{aligned}$$

[12] Not shown in the lectures.

2) By approximation, (3.24) can be established for all $f \in W_0^{2,p}(U)$. The same result holds for $\partial_j f$ and $\partial_{jj} f$ with $j \in \{2, \dots, d\}$. We now replace 2ε by ε, sum over j and take the p-th root to arrive at

$$\Big(\sum_{j=1}^d \|\partial_j f\|_p^p\Big)^{1/p} \leqslant \Big(\varepsilon^p \sum_{j=1}^d \|\partial_{jj} f\|_p^p + \frac{36^p}{\varepsilon^p}\|f\|_p^p\Big)^{1/p}$$
$$\leqslant \varepsilon \Big(\sum_{j=1}^d \|\partial_{jj} f\|_p^p\Big)^{1/p} + \frac{36}{\varepsilon}\|f\|_p, \tag{3.25}$$

for all $f \in W_0^{2,p}(U)$, as asserted.

3) If $u \in W^{2,p}(U)$ and U has a Lipschitz boundary, we use the extension operator $E_U \in \mathcal{B}(W^{2,p}(U), W^{2,p}(\mathbb{R}^d))$ from Theorem 3.24 to deduce from (3.25) with $U = \mathbb{R}^d$ that

$$\begin{aligned}
\Big(\sum_{j=1}^d \|\partial_j f\|_{L^p(U)}^p\Big)^{1/p} &\leqslant \Big(\sum_{j=1}^d \|\partial_j E_U f\|_{L^p(\mathbb{R}^d)}^p\Big)^{1/p} \\
&\leqslant \varepsilon \Big(\sum_{j=1}^d \|\partial_{jj} E_U f\|_{L^p(\mathbb{R}^d)}^p\Big)^{1/p} + \frac{36}{\varepsilon}\, \|E_U f\|_{L^p(\mathbb{R}^d)} \\
&\leqslant \varepsilon\, \|E_U f\|_{W^{2,p}(\mathbb{R}^d)} + \frac{36}{\varepsilon}\, \|E_U f\|_{L^p(U)} \\
&\leqslant c\varepsilon\, \|f\|_{W^{2,p}(U)} + \frac{c}{\varepsilon}\, \|f\|_{L^p(U)} \\
&\leqslant c_0\varepsilon \Big(\sum_{i,j=1}^d \|\partial_{ij} f\|_p^p\Big)^{1/p} + c_1\varepsilon\Big(\sum_{j=1}^d \|\partial_j f\|_p^p\Big)^{1/p} + \frac{c}{\varepsilon}\, \|f\|_p
\end{aligned}$$

where we assume that $\varepsilon \in (0, 1]$ and the constants c, c_0, c_1 do not depend on ε or f. Choosing $\varepsilon_1 = \min\{\frac{1}{2c_1}, 1\}$ we arrive at

$$\frac{1}{2}\Big(\sum_{j=1}^d \|\partial_j f\|_p^p\Big)^{1/p} \leqslant c_0\varepsilon\Big(\sum_{i,j=1}^d \|\partial_{ij} f\|_p^p\Big)^{1/p} + \frac{c}{\varepsilon}\|f\|_p$$

if $0 < \varepsilon \leqslant \varepsilon_1$. This inequality implies the assertion for U with a Lipschitz boundary, after replacing ε by $\varepsilon/(2c_0)$ and ε_1 by $\varepsilon_0 = \min\{c_0/c_1, 2c_0\}$. □

Remark 3.32. Arguing as in part 1) of the above proof, one derives Proposition 3.31 on $W^{2,p}(J)$ for every open interval $J \subseteq \mathbb{R}$ with uniform constants. ◇

3.3. Traces

For $p > d$ and Lipschitz boundaries, we have $W^{1,p}(U) \hookrightarrow C(\overline{U})$ by Sobolev's Theorem 3.26 so that the trace map $f \mapsto f_{|\partial U}$ is well defined from $W^{1,p}(U)$ to $C(\partial U)$, say. Theorem 3.11 further implies that $W^{1,1}(a, b) \hookrightarrow C([a, b])$ for $d = 1$. In other cases it is not clear at all how to give a meaning to the mapping $f \mapsto f|(\partial U)$ on $W^{1,p}(U)$ since for reasonable open sets ∂U is a null set. The next result solves this problem. Moreover, it says that $W_0^{1,p}(U)$ is the space of functions in $W^{1,p}(U)$ with trace 0.

THEOREM 3.33 (Trace theorem). *Let $p \in [1, \infty)$ and $U \subseteq \mathbb{R}^d$ be open with a compact boundary ∂U of class C^1. Then the trace map $f \mapsto f_{|\partial U}$ from $W^{1,p}(U) \cap C(\overline{U})$ to $L^p(\partial U, \sigma)$ has a bounded linear extension $\operatorname{tr} : W^{1,p}(U) \to L^p(\partial U, \sigma)$ whose kernel is $W_0^{1,p}(U)$, where σ is the surface measure on ∂U.*

PROOF.[13] 1) Let $u \in C^1(\overline{U})$. By the definition of the surface integral, see Section 2.3 with a slightly different notation, there are finitely many diffeomorphisms $\Psi_j : \widetilde{U}_j \to \widetilde{V}_j$ and $\varphi_j \in C_c^1(\widetilde{U}_j)$ with $0 \leqslant \varphi_j \leqslant 1$ such that $\|u\|^p_{L^p(\partial U, \sigma)}$ is dominated by

$$c \sum_{j=1}^m \int_{V_j} \varphi_j \circ \Psi_j^{-1} \, |u \circ \Psi_j^{-1}|^p \, \mathrm{d}y'$$

where $\widetilde{U}_j$ and $\widetilde{V}_j$ are open subsets of $\mathbb{R}^d$, the sets $\widetilde{U}_j$ cover ∂U, the functions φ_j form a partition of unity subordinated to $\widetilde{U}_j$, $V_j := \{(y', y_d) \in \widetilde{V}_j \mid y_d = 0\}$, $V_{j+} := \{(y', y_d) \in \widetilde{V}_j \mid y_d > 0\}$, $\Psi_j(\widetilde{U}_j \cap \partial U) = V_j$, and $\Psi_j(\widetilde{U}_j \cap U) = V_{j+}$. We set $v = u \circ \Psi_j^{-1}$ and $\psi = \varphi_j \circ \Psi_j^{-1} \in C_c^1(\widetilde{V}_j)$ and drop the indices j below. By means of Fubini's theorem and the fundamental theorem of calculus, we compute

$$\begin{aligned}
\int_V \psi \, |v(y')|^p \, \mathrm{d}y' &= -\int_{V_+} \partial_d(\psi \, |v|^p) \, \mathrm{d}y \\
&= -\int_{V_+} [(\partial_d \psi) \, |v|^p + p\psi |v|^{p-2} \operatorname{Re}(\overline{v} \partial_d v)] \, \mathrm{d}y \\
&\leqslant c \int_{V_+} [|v|^p + |v|^{p-1} \, |\partial_d v|] \, \mathrm{d}y \leqslant c \, \|v\|_p^p + c \, \|v\|_p^{p-1} \, \|\partial_d v\|_p \\
&\leqslant c \, (\|v\|_p^p + \|\partial_d v\|_p^p) \leqslant c \, \|v\|^p_{W^{1,p}(V_+)} \leqslant c \, \|u\|^p_{W^{1,p}(U)}.
\end{aligned}$$

Here we also used Hölder's and Young's inequality, Proposition 3.9 and the transformation rule. As a result, the map $\operatorname{tr} : (C^1(\overline{U}), \|\cdot\|_{1,p}) \to L^p(\partial U, \sigma)$, $\operatorname{tr} u = u|_{\partial U}$, is continuous. Theorem 3.27 allows to extend tr to an operator in $\mathcal{L}(W^{1,p}(U), L^p(\partial U, \sigma))$. If we start with a function $u \in W^{1,p}(U) \cap C(\overline{U})$, then we can construct approximations $u_n \in C^1(\overline{U})$ which converge to u in $W^{1,p}(U)$ and in $C(\overline{U})$, see the proof of Theorem 5.3.3 in [**Ev**]. Hence, $\operatorname{tr} u_n = u_n|_{\partial U}$ tends to $u|_{\partial U}$ uniformly on ∂U and to $\operatorname{tr} u$ in $L^p(\partial U, \sigma)$, so that $\operatorname{tr} u = u|_{\partial U}$.

2a) We next observe that the inclusion $W_0^{1,p}(U) \subseteq \mathrm{N}(\operatorname{tr})$ is a consequence of the continuity of tr since tr vanishes on $C_c^\infty(U)$ and this space is dense in $W_0^{1,p}(U)$ by definition. To prove the converse, we start with the model case that $v \in W^{1,p}(V_+)$ has a compact support in V_+ and $\operatorname{tr} v = 0$. Our density results yield $v_n \in C^1(\overline{V_+})$ converging to v in $W^{1,p}(V_+)$, and hence $\operatorname{tr} v_n = v_n|_V \to 0$ in $L^p(V)$, as $n \to \infty$. Observe that

$$\begin{aligned}
|v_n(y', y_d)| &\leqslant |v_n(y', 0)| + \int_0^{y_d} |\partial_d v_n(y', s)| \, \mathrm{d}s, \\
|v_n(y', y_d)|^p &\leqslant 2 \, |v_n(y', 0)|^p + 2\bigg(\int_0^{y_d} |\partial_d v_n(y', s)| \, \mathrm{d}s \bigg)^p
\end{aligned}$$

[13] Not shown in the lectures.

for $y' \in V$ and $y_d \geqslant 0$. Integrating over y' und employing Hölder's inequality, we obtain

$$\int_V |v_n(y',y_d)|^p \,\mathrm{d}y' \leqslant 2\int_V |v_n(y',0)|^p \,\mathrm{d}y' + 2y_d^{p-1}\int_V\int_0^{y_d} |\partial_d v_n(y',s)|^p \,\mathrm{d}s\,\mathrm{d}y'.$$

We can now let $n \to \infty$ and arrive at

$$\int_V |v(y',y_d)|^p \,\mathrm{d}y' \leqslant 2y_d^{p-1}\int_V\int_0^{y_d} |\partial_d v(y',s)|^p \,\mathrm{d}s\,\mathrm{d}y'. \tag{3.26}$$

We next use a cutoff argument to obtain a support in the interior of V_+. Choose a function $\chi \in C^\infty(\mathbb{R}_+)$ such that $\chi = 0$ on $[0,1]$ and $\chi = 1$ on $[2,\infty)$. Set $\chi_n(s) = \chi(ns)$ for $s \geqslant 0$ and $n \in \mathbb{N}$, and define $w_n = \chi_n v$ on V_+. Note that $w_n \to v$ in $L^p(V_+)$ as $n \to \infty$, $\partial_j w_n = \chi_n \partial_j v$ for $j \in \{1,\dots,d-1\}$ and $\partial_d w_n = \chi_n\partial_d v + n\chi'(n\cdot)v$. Estimate (3.26) further implies

$$\begin{aligned}
&\int_{V_+} |\nabla w_n - \nabla v|_p^p \,\mathrm{d}y\\
&\quad\leqslant c\int_0^{2/n}\int_V |1-\chi_n|^p\,|\nabla v|_p^p \,\mathrm{d}y'\,\mathrm{d}s + cn^p\int_0^{2/n}\int_V |v(y',s)|^p \,\mathrm{d}y'\,\mathrm{d}s\\
&\quad\leqslant c\int_0^{2/n}\int_V |\nabla v|_p^p \,\mathrm{d}y'\,\mathrm{d}s + cn^p\int_0^{2/n} s^{p-1}\int_0^s\int_V |\partial_d v(y',\tau)|^p \,\mathrm{d}y'\,\mathrm{d}\tau\,\mathrm{d}s\\
&\quad\leqslant c\int_0^{2/n}\int_V |\nabla v|_p^p \,\mathrm{d}y'\,\mathrm{d}s + c\int_0^{2/n}\int_V |\partial_d v(y',\tau)|^p \,\mathrm{d}y'\,\mathrm{d}\tau
\end{aligned}$$

for some constants $c > 0$. Because of $v \in W^{1,p}(V_+)$, the above integrals tend to 0 as $n \to \infty$, and so $w_n \to v$ in $W^{1,p}(V_+)$ as $n \to \infty$. Since $w_n = 0$ for $y_d \in (0,1/n]$, we can mollify w_n to obtain a function $\widehat{w}_n \in C_c^\infty(V_+)$ such that $\|\widehat{w}_n - w_n\|_{1,p} \leqslant 1/n$. This means that $\widehat{w}_n \to v$ in $W^{1,p}(V_+)$ as $n \to \infty$.

2b) We come back to $u \in W^{1,p}(U)$ and consider the sets $\widetilde{U}_j$ and $\widetilde{V}_j$ and the functions Ψ_j and φ_j from step 1). Let $v_j = (\varphi_j u)\circ\Psi_j^{-1}$. First, observe that the trace of v_j to the set V_j is given by $(\operatorname{tr}\varphi_j)\circ\Psi_j^{-1}\,(\operatorname{tr}u)\circ\Psi_j^{-1}$ if $u \in C(\overline{U})$, in addition. By continuity one can extend this identity to all $u \in W^{1,p}(U)$. Let $\operatorname{tr}u = 0$. Then we can apply part 2a) to v_j and obtain $\widehat{w}_n^j \in C_c^1(V_{j+})$ converging to v_j in $W^{1,p}(V_{j+})$. The function

$$\widehat{u}_n = \sum_{j=1}^m \widehat{w}_n^j \circ (\Psi_j|U\cap\widetilde{U}_j)$$

thus belongs to $C_c^1(U)$ and converges to u in $W^{1,p}(U)$ as $n \to \infty$. Since $\widehat{u}_n$ has compact support, we can mollify $\widehat{u}_n$ to a function $u_n \in C_c^\infty(U)$ with $\|\widehat{u}_n - u_n\|_{1,p} \leqslant 1/n$. This means that $u_n \to u$ in $W^{1,p}(U)$ as $n \to \infty$, and hence $u \in W_0^{1,p}(U)$. □

Remark 3.34. As in Remark 3.25 we give a much shorter proof in the case $p = 2$ of an even stronger continuity property of tr, but here only in the half space case $\mathbb{R}_+^d = \{(x,y)\,|\,x \in \mathbb{R}^{d-1}, y > 0\}$ with $d \geqslant 2$.

1) Let $u \in C_c^1(\overline{\mathbb{R}_+^d}) = C^1(\overline{\mathbb{R}_+^d})\cap C_c(\overline{\mathbb{R}_+^d})$ and let $\mathcal{F}_x$ be the Fourier transform with respect to $x \in \mathbb{R}^{d-1}$. The map $y \mapsto (\mathcal{F}_x u)(\xi,y)$ then belongs to $C_c^1(\mathbb{R}_+)$

for each $\xi \in \mathbb{R}^{d-1}$. Since we deal with regular functions, we can calculate

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} |\xi|_2 \, |(\mathcal{F}_x u)(\xi,0)|^2 \, \mathrm{d}\xi \\
&\quad = -\int_{\mathbb{R}^{d-1}} |\xi|_2 \int_0^\infty \partial_y \, |(\mathcal{F}_x u)(\xi,y)|^2 \, \mathrm{d}y \, \mathrm{d}\xi \\
&\quad = -2 \operatorname{Re} \int_0^\infty \int_{\mathbb{R}^{d-1}} |\xi|_2 \, \overline{\mathcal{F}_x u} \, \partial_y(\mathcal{F}_x u) \, \mathrm{d}\xi \, \mathrm{d}y \\
&\quad \leqslant 2 \left[\int_0^\infty \int_{\mathbb{R}^{d-1}} |\xi \mathcal{F}_x u|_2^2 \, \mathrm{d}\xi \, \mathrm{d}y \right]^{\frac12} \left[\int_0^\infty \int_{\mathbb{R}^{d-1}} |\partial_y(\mathcal{F}_x u)|^2 \, \mathrm{d}\xi \, \mathrm{d}y \right]^{\frac12} \\
&\quad = 2 \left[\int_0^\infty \int_{\mathbb{R}^{d-1}} |\mathcal{F}_x(\nabla_x u)|_2^2 \, \mathrm{d}\xi \, \mathrm{d}y \right]^{\frac12} \left[\int_0^\infty \int_{\mathbb{R}^{d-1}} |\mathcal{F}_x(\partial_y u)|^2 \, \mathrm{d}\xi \, \mathrm{d}y \right]^{\frac12} \\
&\quad = 2 \left[\int_0^\infty \int_{\mathbb{R}^{d-1}} |\nabla_x u|_2^2 \, \mathrm{d}x \, \mathrm{d}y \right]^{\frac12} \left[\int_0^\infty \int_{\mathbb{R}^{d-1}} |\partial_y u|^2 \, \mathrm{d}x \, \mathrm{d}y \right]^{\frac12} \\
&\quad \leqslant \int_{\mathbb{R}^d_+} |\nabla_x u|_2^2 \, \mathrm{d}z + \int_{\mathbb{R}^d_+} |\partial_y u|_2^2 \, \mathrm{d}z \leqslant \|u\|^2_{W^{1,2}(\mathbb{R}^d_+)} \, ,
\end{aligned}$$

also using Hölder's inequality, (3.21) and Plancherel's theorem. For $s > 0$ we define the *Bessel potential space*

$$H^s(\mathbb{R}^n) = \{ v \in L^2(\mathbb{R}^n) \,\big|\, |\xi|_2^s \, \hat{v} \in L^2(\mathbb{R}^n) \}$$

endowed with the (Hilbertian) norm given by

$$\|v\|^2_{H^s} = \int_{\mathbb{R}^n} (1 + |\xi|_2^{2s}) \, |\hat{v}(\xi)|^2 \, \mathrm{d}\xi.$$

(Recall from (3.20) that $H^k(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n)$ for $k \in \mathbb{N}$.) We have thus shown that the trace map is continuous from $(C_c^1(\overline{\mathbb{R}^d_+}), \|\cdot\|_{1,2})$ to $H^{\frac12}(\mathbb{R}^{d-1})$.

2) Theorem 3.27 implies that $W^{1,2}(\mathbb{R}^d_+) \cap C^1(\overline{\mathbb{R}^d_+})$ is dense in $W^{1,2}(\mathbb{R}^d_+)$. Using cut-off functions as in the first part of Theorem 3.15, one sees the density of $C_c^1(\overline{\mathbb{R}^d_+})$ in $W^{1,2}(\mathbb{R}^d_+) \cap C^1(\overline{\mathbb{R}^d_+})$ for $\|\cdot\|_{1,2}$. Hence, the trace map is continuous from $W^{1,2}(\mathbb{R}^d_+)$ to $H^{\frac12}(\mathbb{R}^{d-1})$.

3) It is possible to show that $\operatorname{tr} : W^{1,2}(\mathbb{R}^d_+) \to H^{\frac12}(\mathbb{R}^{d-1})$ is surjective. One can show analogous boundedness and surjectivity results on domains U with a compact C^1–boundary, and also for $p \in (1,\infty)$ (employing somewhat different spaces of functions on ∂U). See e.g. Theorem 7.39 in [**AF**]. ◊

We can now prove the divergence theorem in Sobolev spaces, which is a crucial tool in many applications to partial differential operators

Theorem 3.35 (Gauß and Green formulas). *Let $U = \mathbb{R}^d$ or $U \subseteq \mathbb{R}^d$ have a compact boundary ∂U of class C^1 with the outer unit normal ν, $p \in [1,\infty]$, $F \in W^{1,p}(U)^d$, and $\varphi \in W^{1,p'}(U)$. We obtain*

$$\int_U \operatorname{div}(F) \varphi \, \mathrm{d}x = -\int_U F \cdot \nabla \varphi \, \mathrm{d}x + \int_{\partial U} \varphi \, \nu \cdot F \, \mathrm{d}\sigma. \tag{3.27}$$

If $u \in W^{2,p}(U)$ and $v \in W^{2,p'}(U)$, we obtain

$$\begin{aligned}\int_U (\Delta u) v \,\mathrm{d}x &= -\int_U \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\partial U} (\partial_\nu u) v \,\mathrm{d}\sigma \\ &= \int_U u \Delta v \,\mathrm{d}x + \int_{\partial U} ((\partial_\nu u) v - u \partial_\nu v) \,\mathrm{d}\sigma.\end{aligned} \tag{3.28}$$

If $U = \mathbb{R}^d$, these formulas hold without the boundary integrals. (We omit tr *in the boundary integrals and set $\partial_\nu u = \sum_{j=1}^d \nu_j \operatorname{tr} \partial_j u$.)*

Proof. We first observe that Green's formulas (3.28) are a straightforward consequence of (3.27) with $F = \nabla u$.

1) Let U be bounded. Gauß' formula (3.27) is shown in analysis courses for $F \in C^1(\overline{U})^d$ and $\varphi \in C^1(\overline{U})$.

a) Let $p \in (1, \infty)$. For $F \in W^{1,p}(U)^d$ and $\varphi \in W^{1,p'}(U)$, Theorem 3.27 provides functions $F_n \in C^1(\overline{U})^d$ and $\varphi_n \in C^1(\overline{U})$ that converge to F and φ in $W^{1,p}(U)^d$ and $W^{1,p'}(U)$ as $n \to \infty$, respectively, since $p, p' < \infty$. Theorem 3.33 then yields that $F_n|_{\partial U} \to \operatorname{tr} F$ in $L^p(U, \sigma)^d$ and $\varphi_n|_{\partial U} \to \operatorname{tr} \varphi$ in $L^{p'}(U, \sigma)$ as $n \to \infty$. Since $\partial_j : W^{1,q}(U) \to L^q(U)$ is continuous for $q \in \{p, p'\}$, we further obtain that the terms with derivatives converge in L^p, respectively in $L^{p'}$. Formula (3.27) now follows by approximation.

b) Next, let $p = 1$ and thus $p' = \infty$. As above, $\operatorname{div} F_n$ and F_n converge to $\operatorname{div} F$ and F in $L^1(U)$ and $L^1(U)^d$, respectively, as well as $F_n|_{\partial U} \to \operatorname{tr} F$ in $L^1(\partial U, \sigma)^d$. By Proposition 3.28, the function φ belongs to $C(\overline{U})$ and one can thus extend it a map $\varphi \in C_c(\mathbb{R}^d)$. Set $\varphi_n = G_{\frac{1}{n}} \varphi \in C_c^\infty(\mathbb{R}^d)$. Properties (3.7) and (3.6) and Lemma 3.4 imply that $\varphi_n \to \varphi$ in $C(\overline{U})$, $\|\nabla \varphi_n\|_\infty \leqslant \|\nabla \varphi\|_\infty$ and $\nabla \varphi_n \to \nabla \varphi$ pointwise a.e., as $n \to \infty$, where we possibly pass to a subsequence also denoted by $(\varphi_n)_n$. We can now take the limit $n \to \infty$ in the first and the third integral of equation (3.27) for F_n and φ_n. For the second integral we also use the estimate

$$\Big| \int_U F_n \cdot \nabla \varphi_n \,\mathrm{d}x - \int_U F \cdot \nabla \varphi \,\mathrm{d}x \Big| \leqslant \|F_n - F\|_1 \, \|\nabla \varphi_n\|_\infty + \int_U |F| \, |\nabla \varphi_n - \nabla \varphi| \,\mathrm{d}x$$

and Lebesgue's theorem for the integral on the right-hand side. The case $p = \infty$ is treated analogously.

2) Let U be unbounded. There is a radius $r > 0$ such that $\partial U \subseteq B(0, r)$. For $k \in \mathbb{N}$ with $k > r$ we define cut-off functions χ_k as the functions φ_n in part 1) of the proof of Theorem 3.15. For $F_k = \chi_k F$ and $\varphi_k = \chi_k \varphi$, the formula (3.27) follows from step 1) replacing U by $U \cap B(0, 2k)$. Observe that $F_k = F$ and $\varphi_k = \varphi$ on ∂U. We compute

$$\begin{aligned}\int_U \operatorname{div}(F_k) \varphi_k \,\mathrm{d}x &= \int_U \chi_k^2 \operatorname{div}(F) \varphi \,\mathrm{d}x + \int_U \chi_k \varphi F \cdot \nabla \chi_k \,\mathrm{d}x, \\ \int_U F_k \cdot \nabla \varphi_k \,\mathrm{d}x &= \int_U \chi_k^2 F \cdot \nabla \varphi \,\mathrm{d}x + \int_U \chi_k \varphi F \cdot \nabla \chi_k \,\mathrm{d}x.\end{aligned}$$

The products $\varphi \operatorname{div}(F)$, φF and $F \cdot \nabla \varphi$ are integrable, $0 \leqslant \chi_k \leqslant 1$, $\chi_k \to \mathbb{1}$ pointwise and $\|\chi_k'\|_\infty \leqslant c/k$. Letting $k \to \infty$, the theorem of dominated convergence implies (3.27) for F and φ. Note that this proof also works for $U = \mathbb{R}^d$, where $\partial U = \varnothing$. □

3.4. Differential operators in L^p spaces

We now use some of above results to study differential operators in L^p spaces and discuss their closedness, spectra and the compactness of their resolvents.

EXAMPLE 3.36. Let $U \subseteq \mathbb{R}^d$ be open, $1 \leqslant p < \infty$, $X = L^p(U)$, and $\alpha \in \mathbb{N}_0^d$. Then the operator $A = \partial^\alpha$ with $\mathrm{D}(A) = \{u \in D^{\alpha,p}(U) \,|\, u, \partial^\alpha u \in X\}$ is closed. Let $J \subseteq \mathbb{R}$ be an open interval and $X = L^p(J)$. Then also $A_k = \partial^k$ with $\mathrm{D}(A_k) = W^{k,p}(J)$ for $k = 1$ or $k = 2$ are closed.

PROOF. Let $u_n \in D(A)$ with $u_n \to u$ and $Au_n = \partial^\alpha u_n \to v$ in X as $n \to \infty$. By Lemma 3.5(b), the function $u \in X$ belongs to $D^{\alpha,p}(U)$ and $\partial^\alpha u = v \in X$. Hence, $u \in \mathrm{D}(A)$ and $Au = v$; i.e., A is closed.

In particular, for $J = U$ the operators A_1 and ∂^2 on $D_2 = \{u \in D^{2,p}(J) \,|\, u, \partial^2 u \in L^p(J)\}$ are closed. Since $\mathrm{D}(A_2) \subseteq D_2$, it remains to show the converse inclusion. Let $u \in D_2$. Lemma 3.5(a) yields functions $u_n \in C^\infty(J)$ such that $u_n \to u$ and $u_n'' \to \partial^2 u$ in $L^p_{\mathrm{loc}}(J)$, as $n \to \infty$. From Proposition 3.31 and Remark 3.32 we obtain a constant $c > 0$ such that for all compact intervals $J' \subseteq J$ and $g \in C^2(J)$ we have

$$\|g'\|_{L^p(J')} \leqslant c\,(\|g\|_{L^p(J')} + \|g''\|_{L^p(J')}).$$

If we insert here $u_n - u_m = g$, we see that u_n' converges in $L^p_{\mathrm{loc}}(J)$ to a function v and hence u belongs to $D^{1,p}(J)$ and $\partial u = v$ by Lemma 3.5(b). We can now conclude that

$$\begin{aligned}\|\partial u\|_{L^p(J')} &= \lim_{n\to\infty} \|\partial u_n\|_{L^p(J')} \leqslant c \lim_{n\to\infty} (\|u_n\|_{L^p(J')} + \|\partial^2 u_n\|_{L^p(J')}) \\ &= c(\|u\|_{L^p(J')} + \|\partial^2 u\|_{L^p(J')}) \leqslant c(\|u\|_{L^p(J)} + \|\partial^2 u\|_{L^p(J)}).\end{aligned}$$

As a result, $\partial u \in X$ and $u \in D(A_2)$. □

We note that in the one-dimensional case ∂^k is closed on $W^{k,p}(J)$ for all $k \in \mathbb{N}$ (by similar proofs based on higher order versions of Proposition 3.31). This property relies on the fact that the domain $W^{k,p}(J)$ does not contain derivatives in other directions. In contrast, ∂_1 on $W^{1,p}((0,1)^2)$ is not closed in $L^p((0,1)^2)$ which can be shown by functions $u_n(x,y) = \varphi_n(y)$ where $\varphi_n \in C^1([0,1])$ converges in $L^p(0,1)$ to a function $\varphi \notin W^{1,p}(0,1)$.

Next, in various settings we study the first derivative in more detail.

EXAMPLE 3.37. Let $1 \leqslant p < \infty$, $X = L^p(\mathbb{R})$ and $A = \partial$ with $\mathrm{D}(A) = W^{1,p}(\mathbb{R})$. Then $\sigma(A) = \mathrm{i}\mathbb{R}$, $\sigma_p(A) = \varnothing$ and

$$(R(\lambda, A)f)(t) = \begin{cases} \int_t^\infty \mathrm{e}^{\lambda(t-s)} f(s)\,\mathrm{d}s, & \operatorname{Re}\lambda > 0, \\ -\int_{-\infty}^t \mathrm{e}^{\lambda(t-s)} f(s)\,\mathrm{d}s, & \operatorname{Re}\lambda < 0, \end{cases}$$

for $t \in \mathbb{R}$ and $f \in X$, cf. Example 1.21 for $X = C_0(\mathbb{R})$.

PROOF. 1) Let $\operatorname{Re}\lambda > 0$ and $\varphi_\lambda = \mathrm{e}_{\operatorname{Re}\lambda} \mathbb{1}_{\mathbb{R}_-}$. By $R_\lambda f(t)$ we denote the integral in the assertion. Since $|R_\lambda(f(t))| \leqslant (\varphi_\lambda * |f|)(t)$ for all $t \in \mathbb{R}$, Young's inequality (see e.g. Theorem 2.14 in [Sc2]) yields that

$$\|R_\lambda f\|_p \leqslant \|\varphi_\lambda\|_1 \, \|f\|_p = \frac{1}{\operatorname{Re}\lambda} \|f\|_p.$$

Let $f_n \in C_c^\infty(\mathbb{R})$ converge to f in $L^p(\mathbb{R})$. We then compute $\frac{\mathrm{d}}{\mathrm{d}t} R_\lambda f_n = \lambda R_\lambda f_n - f_n$, which tends to $\lambda R_\lambda f - f$ in $L^p(\mathbb{R})$ by the above estimate.

Since A is closed by Example 3.36, the function $R_\lambda f$ belongs to $\mathrm{D}(A)$ and $AR_\lambda f = \lambda R_\lambda f - f$; i.e., $\lambda I - A$ is surjective.

2) Let $Au = \mu u$ for some $u \in \mathrm{D}(A)$ and $\mu \in \mathbb{C}$. Theorem 3.11 says that u is continuous so that Remark 3.12c) implies the continuous differentiability of u and so $u' = \mu u$. Consequently, u is equal to a multiple of e_μ. Since $\mathrm{e}_\mu \notin X$, we obtain $u = 0$ and hence $\mu I - A$ is injective. We have shown that $\sigma_p(A) = \varnothing$ and $\lambda \in \rho(A)$ with $R(\lambda, A) = R_\lambda$ if $\operatorname{Re}\lambda > 0$. In the same way, one sees that $\lambda \in \rho(A)$ if $\operatorname{Re}\lambda < 0$, with the asserted resolvent.

3) Let $\lambda \in \mathrm{i}\mathbb{R}$. Take $\varphi_n \in C_c^1(\mathbb{R})$ with $\varphi_n = n^{-1/p}$ on $[-n, n]$, $\varphi_n(t) = 0$ for $|t| \geqslant n+1$ and $\|\varphi_n\|_\infty \leqslant 2n^{-1/p}$ for every $n \in \mathbb{N}$. We then derive $u_n = \varphi_n \mathrm{e}_\lambda \in W^{1,p}(\mathbb{R})$, $Au_n = \lambda u_n + \varphi_n \mathrm{e}_\lambda$,

$$\|u_n\|_p \geqslant \Big(\int_{-n}^{n} |\varphi_n(t)\mathrm{e}_\lambda(t)|^p \,\mathrm{d}t\Big)^{1/p} = n^{-1/p}(2n)^{1/p} = 2^{1/p},$$

$$\|\lambda u_n - Au_n\|_p = \|\varphi_n' \mathrm{e}_\lambda\|_p = \Big(\int_{\{n \leqslant |t| \leqslant n+1\}} |\varphi_n'(t)\mathrm{e}_\lambda(t)|^p \,\mathrm{d}t\Big)^{1/p} \leqslant 2n^{-1/p}2^{1/p},$$

for every $n \in \mathbb{N}$. Consequently, $\lambda \in \sigma_{ap}(A)$ and so $\sigma(A) = \mathrm{i}\mathbb{R}$. □

Example 3.38. Let $1 \leqslant p < \infty$.

a) Let $X = L^p(0,1)$ and $A = \partial$ with $\mathrm{D}(A) = W^{1,p}(0,1)$. Then $\sigma(A) = \sigma_p(A) = \mathbb{C}$ since $\mathrm{e}_\lambda \in \mathrm{D}(A)$ and $A\mathrm{e}_\lambda = \lambda \mathrm{e}_\lambda$ for all $\lambda \in \mathbb{C}$.

b) Let $X = L^p(0,1)$ and $A = \partial$ with $\mathrm{D}(A) = \{u \in W^{1,p}(0,1) \,|\, u(0) = 0\}$ (using the continuous representative of $u \in W^{1,p}(0,1)$ from Theorem 3.11). Then A is closed, $\sigma(A) = \varnothing$, $R(\lambda, A)f(t) = -\int_0^t \mathrm{e}^{\lambda(t-s)} f(s)\,\mathrm{d}s$ for $t \in (0,1)$, $f \in X$ and $\lambda \in \mathbb{C}$, and A has compact resolvent.

Proof. As in Example 3.37 one sees that the above integral defines a bounded inverse of $\lambda I - A$ for all $\lambda \in \mathbb{C}$. In particular, A is closed by Remark 1.11. (Alternatively, take u_n in $\mathrm{D}(A)$ such that $u_n \to u$ and $Au_n \to v$ in X. Example 3.36 yields that $u \in W^{1,p}(0,1)$ and $\partial u = v$. Using Theorem 3.11, we further infer that $0 = u_n(0) \to u(0)$. Hence, $u \in \mathrm{D}(A)$ and $Au = \partial u = v$.) Finally, Remark 2.14 and Theorem 3.29 imply that A has a compact resolvent. □

c) Let $X = L^p(0,\infty)$ and $A = \partial$ with $\mathrm{D}(A) = W^{1,p}(0,\infty)$. Then $\sigma(A) = \{\lambda \in \mathbb{C} \,|\, \operatorname{Re}\lambda \leqslant 0\}$, $\sigma_p(A) = \{\lambda \in \mathbb{C} \,|\, \operatorname{Re}\lambda < 0\}$ and $R(\lambda, A)f(t) = \int_t^\infty \mathrm{e}^{\lambda(t-s)} f(s)\,\mathrm{d}s$ for $t > 0$, $f \in X$ and $\operatorname{Re}\lambda > 0$.

Proof. The operator is closed by Example 3.36. As in Example 3.37, one computes the resolvent for $\operatorname{Re}\lambda > 0$ and checks that $\mathrm{i}\mathbb{R}$ does not contain eigenvalues. If $\operatorname{Re}\lambda < 0$, then e_λ is an eigenfunction. Since $\sigma(A)$ is closed, it is equal to $\{\lambda \in \mathbb{C} \,|\, \operatorname{Re}\lambda < 0\}$. □

d) Let $X = L^p(0,\infty)$ and $A = -\partial$ with $\mathrm{D}(A) = W_0^{1,p}(0,\infty)$. Then A is closed, $\sigma(A) = \{\lambda \in \mathbb{C} \,|\, \operatorname{Re}\lambda \leqslant 0\}$, $\sigma_p(A) = \varnothing$, and $R(\lambda, A)f(t) = \int_0^t \mathrm{e}^{-\lambda(t-s)} f(s)\,\mathrm{d}s$ for $t > 0$, $f \in X$ and $\operatorname{Re}\lambda > 0$.

Proof. If $\operatorname{Re}\lambda > 0$, the formula for the resolvent is verified as in Example 3.37, so that A is closed. The point spectrum is empty since the only possible eigenfunctions e_λ do not fulfill the boundary condition in $\mathrm{D}(A)$.

Let $\operatorname{Re}\lambda < 0$ and $f = \mathbb{1}_{(0,1)}$. If $u \in \mathrm{D}(A)$ satisfies $\lambda u + Au = f$, then $u(0) = 0$ and $\partial u = -\lambda u - f$ is continuous except at $t = 1$. By Remark 3.12c),

u is piecewise C^1 and hence

$$u(t) = \mathrm{e}^{-\lambda t} \int_0^1 \mathrm{e}^{\lambda s}\, \mathrm{d}s, \qquad t \geqslant 1.$$

So u cannot belong to X and thus f is not in the range of $\lambda I - A$; i.e., $\lambda \in \sigma(A)$. The result then follows from the closedness of the spectrum. □

e) Let $X = L^p(0,1)$ and $A = \partial$ with $\mathrm{D}(A) = \{u \in W^{1,p}(0,1) \,|\, u(0) = u(1)\}$. Then A is closed, $\sigma(A) = \sigma_p(A) = 2\pi\mathrm{i}\mathbb{Z}$ and A has a compact resolvent. (These facts can be proved as in Example 2.17 for $X = C([0,1])$, using now Theorem 3.29 for the compactness.) ◊

We now turn our attention to the second derivative and the Laplacian.

Example 3.39. Let $X = L^p(\mathbb{R})$, $1 \leqslant p < \infty$, and $A = \partial^2$ with $\mathrm{D}(A) = W^{2,p}(\mathbb{R})$. Then $\sigma(A) = \mathbb{R}_-$.

Proof. 1) Set $A_1 = \partial$ and $\mathrm{D}(A_1) = W^{1,p}(\mathbb{R})$. Let $\mu \in \mathbb{C}\backslash\mathbb{R}_-$. There exists a number $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$ such that $\mu = \lambda^2$. Observe that

$$(\mu I - A)u = (\lambda I - A_1)(\lambda I + A_1)u \tag{3.29}$$

for all $u \in \mathrm{D}(A)$. Example 3.37 says that $\lambda I \pm A_1$ are invertible. Hence, $\mu I - A$ is injective. Next, for $v \in W^{1,p}(\mathbb{R})$ the function

$$\partial(\lambda I + A_1)^{-1}v = A_1(\lambda I + A_1)^{-1}v = -\lambda(\lambda I + A_1)^{-1}v + v$$

belongs to $W^{1,p}(\mathbb{R})$. This means that $(\lambda I + A_1)^{-1}$ maps $W^{1,p}(\mathbb{R})$ into $\mathrm{D}(A)$. Given $f \in X$, the map $u := (\lambda I + A_1)^{-1}(\lambda I - A_1)^{-1}f$ thus is an element of $\mathrm{D}(A)$ and $\mu u - Au = f$ in view of the factorization (3.29) . We have shown that $\mu \in \rho(A)$ and $R(\mu, A) = (\lambda I + A_1)^{-1}(\lambda I - A_1)^{-1}$.

2) For $\mu \leqslant 0$, we have $\mu = \lambda^2$ for some $\lambda \in i\mathbb{R}$ and (3.29) is still true. The operator $\lambda I - A_1$ is not surjective since its range is not closed by Example 3.37 and Proposition 1.19. Equation (3.29) thus implies that $\mu I - A$ is not surjective. As a result, $\sigma(A) = \mathbb{R}_-$. □

Example 3.40. Let $X = L^p(0,1)$, $1 \leqslant p < \infty$, and $A = \partial^2$ with $\mathrm{D}(A) = W^{2,p}(0,1) \cap W_0^{1,p}(0,1)$. Then A is closed, $\sigma(A) = \sigma_p(A) = \{-\pi^2 k^2 \,|\, k \in \mathbb{N}\}$, and A has a compact resolvent. These facts can be proved as in Example 2.17 for $X = C([0,1])$, using now Theorem 3.29 for the compactness. ◊

Example 3.41. Let $X = L^2(\mathbb{R}^d)$ and $A = \Delta = \partial_{11} + \ldots + \partial_{dd}$ with $\mathrm{D}(A) = W^{2,2}(\mathbb{R}^d)$. Then A is closed and $\sigma(A) = \mathbb{R}_-$.

Proof. We use the Fourier transform $\mathcal{F}$ which is unitary on X, see e.g. Theorem 5.11 in [**Sc1**]. By (3.20) and (3.21), we have

$$\Delta u = \mathcal{F}^{-1}(-|\xi|_2^2 \widehat{u}) \quad \text{for} \quad u \in \mathrm{D}(A) = \{u \in X \,|\, |\xi|_2^2\, \widehat{u} \in X\}.$$

1) Let $\lambda \in C\backslash\mathbb{R}_-$. Set $m_\lambda(\xi) = (\lambda + |\xi|_2^2)^{-1}$ for $\xi \in \mathbb{R}^d$. Observe that $\|m_\lambda\|_\infty \leqslant c_\lambda$ where $c_\lambda = 1/|\operatorname{Im}\lambda|$ if $\operatorname{Re}\lambda \leqslant 0$ and $c_\lambda = 1/|\lambda|$ if $\operatorname{Re}\lambda > 0$. Let $f \in X$. Then $m_\lambda \widehat{f}$ belongs to X so that $R_\lambda f := \mathcal{F}^{-1}(m_\lambda \widehat{f}) \in X$ and

$$\|R_\lambda f\|_2 = \|m_\lambda \widehat{f}\|_2 \leqslant c_\lambda \|\widehat{f}\|_2 = c_\lambda \|f\|_2 \,.$$

Moreover, since $|\xi|_2^2 m_\lambda(\xi)$ is bounded for $\xi \in \mathbb{R}^d$, the function $|\xi|_2^2 \mathcal{F} R_\lambda f = |\xi|_2^2 m_\lambda \widehat{f}$ is an element of X; i.e., $R_\lambda f \in \mathrm{D}(A)$. Similarly we see that

$$(\lambda I - \Delta)R_\lambda f = \mathcal{F}^{-1}(\lambda + |\xi|_2^2)\mathcal{F}\mathcal{F}^{-1} m_\lambda \widehat{f} = f,$$

$$R_\lambda(\lambda I - \Delta)u = \mathcal{F}^{-1} m_\lambda \mathcal{F}\mathcal{F}^{-1}(\lambda + |\xi|_2^2)\widehat{u} = u, \quad u \in \mathrm{D}(A).$$

Thus, $\lambda \in \rho(A)$ and $R(\lambda, A) = R_\lambda$. In particular, A is closed and $\sigma(A) \subseteq \mathbb{R}_-$.

2) Let $\lambda < 0$. Define $m_\lambda(\xi) = (\lambda + |\xi|_2^2)^{-1}$ for $\xi \in \mathbb{R}^d$ with $|\xi| \neq -\lambda$ and $m_\lambda(\xi) = 0$ if $|\xi| = -\lambda$. Then m_λ is unbounded. Suppose that $mg \in L^2(\mathbb{R}^d)$ for all $g \in L^2(\mathbb{R}^d)$. Then the (closed) multiplication operator $M : g \mapsto mg$ in $L^2(\mathbb{R}^d)$ was defined on $L^2(\mathbb{R}^d)$ and thus bounded by the closed graph theorem. Set $B_n = \{\xi \in \mathbb{R}^d \,\big|\, |m_\lambda(\xi)| \geqslant n\}$ for $n \in \mathbb{N}$. Observe that the Lebesgue measure ℓ_n of B_n is contained in $(0, \infty)$. Set $g_n = \ell_n^{-1/2}\, \overline{m_\lambda}\, |m_\lambda|^{-1} \mathbb{1}_{B_n}$. Since $|g_n| = \ell_n^{-1/2}\, \mathbb{1}_{B_n}$ and $|Mg_n| = \ell_n^{-1/2}\, |m_\lambda|\, \mathbb{1}_{B_n}$, we obtain $\|g_n\|_2 = 1$ and $\|Mg_n\|_2 \geqslant n$ for all $n \in \mathbb{N}$. These relations contradict the continuity of M.

So there exists a function $h \in X$ such that $m_\lambda \widehat{h} \notin X$. If there was an element u of $\mathrm{D}(A)$ with $\lambda u - \Delta u = h$, we would obtain as above that $\lambda \widehat{u} + |\xi|_2^2 \widehat{u} = \widehat{h}$ and hence $m_\lambda \widehat{h} = \widehat{u} \in X$ which is impossible. As a result, $\lambda \in \sigma(A)$ and $\sigma(A) = \mathbb{R}_-$. □

Example 3.42. Let $1 < p < \infty$, $X = L^p(\mathbb{R}^d)$, and $A = \Delta$ with $\mathrm{D}(A) = W^{2,p}(\mathbb{R}^d)$. Then A is closed.

Proof. The Calderón–Zygmund estimate says that the graph norm of A is equivalent to $\|\cdot\|_{2,p}$ on $C_c^\infty(\mathbb{R}^d)$, see e.g. Corollary 9.10 in [**GT**]. Let $u \in W^{2,p}(\mathbb{R}^d)$. By Theorem 3.15, there are $u_n \in C_c^\infty(\mathbb{R}^d)$ that converge to u in $W^{2,p}(\mathbb{R}^d)$ as $n \to \infty$, and hence $u_n \to u$ and $\Delta u_n \to \Delta u$ in X. We derive

$$\begin{aligned} \|u\|_{2,p} &= \lim_{n\to\infty} \|u_n\|_{2,p} \leqslant \lim_{n\to\infty} c\,(\|u_n\|_p + \|\Delta u_n\|_p) \\ &= c\,(\|u\|_p + \|\Delta u\|_p) \leqslant c'\,\|u\|_{2,p}\,, \end{aligned}$$

so that $\|\cdot\|_A$ is equivalent to a complete one and thus A is closed. □

Example 3.43 (Dirichlet Laplacian). Let $1 < p < \infty$, $U \subseteq \mathbb{R}^d$ be bounded and open with $\partial U \in C^2$, $X = L^p(U)$, and $A = \Delta$ with $\mathrm{D}(A) = W^{2,p}(U) \cap W_0^{1,p}(U)$. Then A is closed, invertible and has a compact resolvent. In particular, $\sigma(A) = \sigma_p(A)$.

Proof. The closedness of A follows from Theorem 9.14 in [**GT**]. Its bijectivity is shown in Theorem 9.15 of [**GT**]. Remark 2.14, Theorem 3.29 and Theorem 2.16 then imply the other assertions. □

There are variants of Examples 3.42 and 3.43 for $X = L^1(U)$, $X = L^\infty(U)$ and in other sup-norm spaces, see Theorem 5.8 in [**Ta**] as well as Sections 3.1.2 and 3.1.5 in [**Lu**]. In theses cases, the descriptions of the domains are much more complicated and they are not just (subspaces of) Sobolev (or C^2-) spaces. To indicate the difficulties, we note that there is a function $u \notin W^{2,\infty}(B(0,1))$ such that $\Delta u \in L^\infty(B(0,1))$, namely

$$u(x,y) = \begin{cases} (x^2 - y^2)\ln(x^2 + y^2) & \text{for } (x,y) \neq (0,0), \\ 0 & \text{for } (x,y) = (0,0). \end{cases}$$

Then the second derivative

$$\partial_{xx} u(x,y) = 2\ln(x^2 + y^2) + \frac{4x^2}{x^2 + y^2} + \frac{(6x^2 - 2y^2)(x^2 + y^2) - 4x^2(x^2 - y^2)}{(x^2 + y^2)^2}$$

is unbounded around $(0,0)$, but $\Delta u(x,y) = 8\frac{x^2 - y^2}{x^2 + y^2}$ is bounded.

CHAPTER 4

Selfadjoint operators

In this chapter X and Y are Hilbert spaces with scalar product $(\cdot|\cdot)$, unless something else is said.

4.1. Basic properties

Let A be a densely defined linear operator from X to Y. We define the *Hilbert space adjoint* A' as in the Banach space case by

$$\begin{aligned} \mathrm{D}(A') &:= \{y \in Y \,\big|\, \exists\, z \in X \ \forall\, x \in \mathrm{D}(A) : (Ax|y) = (x|z)\}, \\ A'y &:= z. \end{aligned} \tag{4.1}$$

As in Remark 1.23 one sees that $A' : \mathrm{D}(A') \to X$ is a linear map, which is closed from Y to X. Let $T \in \mathcal{B}(X,Y)$. Then $\mathrm{D}(T') = Y$ and T' is given by

$$\forall\ x \in X,\, y \in Y : \qquad \big(x\big|T'y\big) = (Tx|y)\ ,$$

as in (5.9) in [**Sc2**]. We recall from Proposition 5.42 of [**Sc2**] that

$$\|T\| = \|T'\|, \quad T'' = T, \quad (\alpha T + \beta S)' = \overline{\alpha} T' + \overline{\beta} S', \quad (UT)' = T'U'$$

for $T, S \in \mathcal{B}(X,Y)$, $U \in \mathcal{B}(Y,Z)$, a Hilbert space Z, and $\alpha, \beta \in \mathbb{C}$.

Definition 4.1. *A densely defined linear operator A on X is called* selfadjoint *if $A = A'$ (in particular, $\mathrm{D}(A) = \mathrm{D}(A')$ and A must be closed)* skewadjoint *if $A = -A'$, and* normal *if $AA' = A'A$. We say that $T \in \mathcal{B}(X,Y)$ is* unitary *if T is invertible with $T^{-1} = T'$.*

Let $\Phi : X \to X^*$ be the (antilinear) Riesz isomorphism given by $(\Phi(x))(y) = (y|x)$ for all $x, y \in X$. For $\lambda \in \mathbb{C}$ and $X = Y$, we obtain

$$\overline{\lambda} I_X - T' = \Phi^{-1}(\lambda I_{X^*} - T^*)\Phi, \qquad \overline{\lambda} I_X - A' = \Phi^{-1}(\lambda I_{X^*} - A^*)\Phi,$$

with $\mathrm{D}(A') = \Phi^{-1}\,\mathrm{D}(A^*)$, see p.109 of [**Sc2**]. Theorem 1.24 thus implies

$$\begin{gathered} \sigma(A) = \sigma(A^*) = \overline{\sigma}(A'), \qquad \sigma_r(A) = \sigma_p(A^*) = \overline{\sigma}_p(A'), \\ R(\overline{\lambda}, A') = \Phi^{-1} R(\lambda, A^*)\Phi = \Phi^{-1} R(\lambda, A)^*\Phi = R(\lambda, A)' \qquad \text{for} \ \ \lambda \in \rho(A), \end{gathered} \tag{4.2}$$

where the bars mean complex conjugation of each element.

Remark 4.2. a) Selfadjoint, skewadjoint or unitary operators (with $X = Y$) are normal. If $A = A'$ and $\lambda \in \mathbb{C}$, then $\lambda I - A$ is normal.

b) A densely defined linear operator A is skewadjoint if and only if $\mathrm{i}A$ is selfadjoint. (Use that (4.1) yields $(\mathrm{i}A)' = -\mathrm{i}A$.) ◊

Theorem 1.16 says that the spectral radius $r(T)$ is less or equal than $\|T\|$ for each bounded operator T on a Banach space. Already for the matrix $T = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ one has the strict inequality $r(T) = 0 < 1 = \|T\|$. We next show $\|T\| = r(T)$ for normal operators which is the key to their deeper properties.

Proposition 4.3. *Let X and Y be Hilbert spaces. If $T \in \mathcal{B}(X,Y)$, then $\|T'T\| = \|TT'\| = \|T\|^2$. Let $T \in \mathcal{B}(X)$ be normal. We then have $\|T\| = r(T)$, and thus $T = 0$ if $\sigma(T) = \{0\}$.*

Proof. For $x \in X$ we compute

$$\|Tx\|^2 = (Tx|Tx) = \big(T'Tx|x\big) \leqslant \|T'T\|\|x\|^2,$$
$$\|T\|^2 = \sup_{\|x\|\leqslant 1} \|Tx\|^2 \leqslant \|T'T\| \leqslant \|T'\|\|T\| = \|T\|^2;$$

i.e., $\|T\|^2 = \|T'T\|$. We infer that $\|T\|^2 = \|T'\|^2 = \|T''T'\| = \|TT'\|$.

Next, let T be normal. From the first part we then deduce

$$\|T^2\|^2 = \|T^2(T^2)'\| = \|TTT'T'\| = \|TT'TT'\| = \|TT'(TT')'\|$$
$$= \|TT'\|^2 = \|T\|^4,$$

so that $\|T^2\| = \|T\|^2$. Iteratively it follows that $\|T^{2^n}\| = \|T\|^{2^n}$ for all $n \in \mathbb{N}$. Using Theorem 1.16, we conclude

$$r(T) = \lim_{m\to\infty} \|T^m\|^{1/m} = \lim_{n\to\infty} \|T^{2^n}\|^{2^{-n}} = \|T\|. \qquad \square$$

The following concepts turn out to be very useful to compute adjoints, for instance.

Definition 4.4. *Let A and B be linear operators from a Banach space X to a Banach space Y. We say that B* extends *A (and write $A \subseteq B$) if $\mathrm{D}(A) \subseteq \mathrm{D}(B)$ and $Ax = Bx$ for all $x \in \mathrm{D}(A)$.*

Next, let X be a Hilbert space. A linear operator A on X is called symmetric *if we have $(Ax|y) = (x|Ay)$ for all $x, y \in \mathrm{D}(A)$.*

Of course, a selfadjoint operator is symmetric. As we see in Example 4.8, the converse is not true in general.

Remark 4.5. Let A and B be linear operators from a Banach space X to a Banach space Y.

a) The operator B extends A if and only if its graph $\mathrm{G}(B)$ contains $\mathrm{G}(A)$. Let $A \subseteq B$. Then $A = B$ is equivalent to $\mathrm{D}(B) \subseteq \mathrm{D}(A)$.

b) Let $A \subseteq B$, A be surjective and B be injective. Then A and B are equal. Hence, if $X = Y$ and there is a $\lambda \in \mathbb{C}$ such that $\lambda I - A$ is surjective and $\lambda I - B$ is injective, then $A = B$.

Proof. Let $x \in \mathrm{D}(B)$ and set $y = Bx$. The surjectivity of A yields a vector $z \in \mathrm{D}(A) \subseteq \mathrm{D}(B)$ such that $y = Az = Bz$. Since B is injective, we obtain $x = z \in \mathrm{D}(A)$ so that $A = B$. $\square$

c) Let A be densely defined and symmetric on a Hilbert space X. Definition (4.1) implies that $A \subseteq A'$. In particular, A is selfadjoint if and only if $\mathrm{D}(A') \subseteq \mathrm{D}(A)$. $\Diamond$

In the next theorem we give spectral conditions implying that a symmetric operator is selfadjoint. We start with a simple, but crucial lemma.

Lemma 4.6. *Let A be symmetric, $x \in \mathrm{D}(A)$ and $\alpha, \beta \in \mathbb{R}$. Set $\lambda = \alpha + \mathrm{i}\beta$. We have $(Ax|x) \in \mathbb{R}$ and*

$$\|\lambda x - Ax\|^2 = \|\alpha x - Ax\|^2 + |\beta|^2\,\|x\|^2 \geqslant |\beta|^2\,\|x\|^2.$$

If A is also closed, then $\sigma_{ap}(A) \subseteq \mathbb{R}$ and $\|R(\lambda, A)\| \leqslant \frac{1}{|\operatorname{Im}\lambda|}$ for all $\lambda \in \rho(A)\backslash\mathbb{R}$.

PROOF. For $x \in \mathrm{D}(A)$ we have $(Ax|x) = (x|Ax) = \overline{(Ax|x)}$ so that $(Ax|x) = (x|Ax)$ is real. From this fact we deduce that

$$\begin{aligned}\|\lambda x - Ax\|^2 &= (\alpha x - Ax + \mathrm{i}\beta x|\alpha x - Ax + \mathrm{i}\beta x)\\ &= \|\alpha x - Ax\|^2 + 2\,\mathrm{Re}\,(\mathrm{i}\beta x|\alpha x - Ax) + \|\mathrm{i}\beta x\|^2\\ &= \|\alpha x - Ax\|^2 + 2\,\mathrm{Re}(\mathrm{i}\beta\alpha\|x\|^2 - \mathrm{i}\beta\,(x|Ax)\,) + |\beta|^2\,\|x\|^2\\ &= \|\alpha x - Ax\|^2 + |\beta|^2\,\|x\|^2 \geqslant |\beta|^2\,\|x\|^2.\end{aligned}$$

In particular, $\lambda \notin \sigma_{ap}(A)$ if $\mathrm{Im}\,\lambda = \beta \neq 0$.

If $\lambda \in \rho(A)\backslash\mathbb{R}$ and $y \in X$, write $x = R(\lambda, A)y \in \mathrm{D}(A)$. We then calculate

$$\|y\|^2 = \|\lambda x - Ax\|^2 \geqslant |\mathrm{Im}\,\lambda|^2\,\|x\|^2 = |\mathrm{Im}\,\lambda|^2\,\|R(\lambda, A)y\|^2. \qquad \square$$

THEOREM 4.7. *Let X be a Hilbert space and A be densely defined, closed and symmetric.*

(a) Then $\sigma(A)$ is either a subset of $\mathbb{R}$ or $\sigma(A) = \mathbb{C}$ or $\sigma(A) = \{\lambda \in \mathbb{C} \,|\, \mathrm{Im}\,\lambda \geqslant 0\}$ or $\sigma(A) = \{\lambda \in \mathbb{C} \,|\, \mathrm{Im}\,\lambda \leqslant 0\}$.

(b) The following assertions are equivalent.

(i) $A = A'$.

(ii) $\sigma(A) \subseteq \mathbb{R}$.

(iii) $\mathrm{i}I - A'$ and $\mathrm{i}I + A'$ are injective.

(iv) $(\mathrm{i}I - A)\,\mathrm{D}(A)$ and $(\mathrm{i}I + A)\,\mathrm{D}(A)$ are dense.

(c) If $\rho(A) \cap \mathbb{R} \neq \varnothing$, then A is selfadjoint.

(d) Let A be selfadjoint. Then we have

$$\|R(\lambda, A)\| \leqslant \frac{1}{|\mathrm{Im}\,\lambda|} \tag{4.3}$$

for $\lambda \notin \mathbb{R}$. Further, $\sigma(A) = \sigma_{ap}(A)$ is non-empty and A has no selfadjoint extension $B \neq A$.

PROOF. (a) Let $\lambda \in \sigma(A)$ and $\mu \in \rho(A)$. Suppose that $\mathrm{Im}\,\lambda > 0$ and $\mathrm{Im}\,\mu > 0$. The line segment from λ to μ then must contain a point $\gamma \in \partial\sigma(A)$. This point belongs to $\sigma_{ap}(A)$ by Proposition 1.19 and satisfies $\mathrm{Im}\,\gamma > 0$, which contradicts Lemma 4.6 since A is symmetric. Similarly we exclude that $\mathrm{Im}\,\lambda < 0$ and $\mathrm{Im}\,\mu < 0$. Since the spectrum is closed, only the four cases in (a) remain.

(b) Let A be selfadjoint. Lemma 4.6 yields $\sigma_{ap}(A) \subseteq \mathbb{R}$. Due to (4.2) we also have $\sigma_r(A) = \overline{\sigma}_p(A') = \overline{\sigma}_p(A)$. Hence, $\sigma_r(A) = \sigma_p(A) \subseteq \mathbb{R}$. From Proposition 1.19 we thus deduce $\sigma(A) \subseteq \mathbb{R}$; i.e., (i) implies (ii). The implication '(ii)$\Rightarrow$(iii)' is obvious. Equation (4.2) also shows that $\pm\mathrm{i} \in \sigma_p(A')$ if and only if $\mp\mathrm{i} \in \sigma_r(A)$ so that (iii) and (iv) are equivalent. Finally, let (iv) (and thus (iii)) be true. The range of the operator $\mathrm{i}I \pm A$ is closed by Lemma 4.6 and Proposition 1.19. In view of (iv), the map $\mathrm{i}I \pm A$ is then surjective. On the other hand, $\mathrm{i}I \pm A'$ is injective because of (iii), and hence $A = A'$ thanks to Remark 4.5b).

(c) If there is a point λ in $\rho(A) \cap \mathbb{R}$, then $\rho(A)$ contains a ball around λ by its openness. By part (a), the spectrum of A is thus contained in $\mathbb{R}$, and so A is selfadjoint by (b).

(d) Let $A = A'$. If its spectrum was empty, then A is invertible with a selfadjoint inverse. Proposition 1.20 thus yields that $\sigma(A^{-1})$ is equal to $\{0\}$

so that $A^{-1} = 0$ by Proposition 4.3, which is impossible. Hence, $\sigma(A) \neq \varnothing$. Since $\sigma(A) = \sigma(A') \subseteq \mathbb{R}$ by (b), the other parts of assertion (c) follow from Lemma 4.6 and Remark 4.5. $\square$

EXAMPLE 4.8. a) Let $X = L^2(\mathbb{R})$ and $A = \mathrm{i}\partial$ with $\mathrm{D}(A) = W^{1,2}(\mathbb{R})$. Then A is selfadjoint with $\sigma(A) = \mathbb{R}$.

PROOF. For $u, v \in \mathrm{D}(A)$, integrating by parts we deduce

$$(Au|v) = \mathrm{i}\int_{\mathbb{R}} (\partial u)\overline{v}\,\mathrm{d}s = -\mathrm{i}\int_{\mathbb{R}} u\partial\overline{v}\,\mathrm{d}s = \int_{\mathbb{R}} u\overline{\mathrm{i}\partial v}\,\mathrm{d}s = (u|Av)\,,$$

see Theorem 3.35; i.e., A is symmetric. Example 3.37 and Proposition 1.20 further imply that $\sigma(A) = \mathrm{i}\sigma(-\mathrm{i}A) = \mathrm{i}^2\,\mathbb{R} = \mathbb{R}$. Hence, A is selfadjoint. $\square$

b) Let $X = L^2(0,\infty)$ and $A = \mathrm{i}\partial$ on $\mathrm{D}(A) = W^{1,2}(0,\infty)$. Then A is not symmetric.

PROOF. For $u, v \in \mathrm{D}(A)$ with $u(0) = v(0) = 1$, as above an integration by parts implies

$$(Au|v) = \mathrm{i}\int_0^\infty (\partial u)\overline{v}\,\mathrm{d}s = -\mathrm{i}\int_0^\infty u\partial\overline{v}\,\mathrm{d}s - \mathrm{i}\overline{u(0)}v(0) = (u|Av) - \mathrm{i}. \qquad \square$$

c) Let $X = L^2(0,\infty)$ and $A = \mathrm{i}\partial$ on $\mathrm{D}(A) = W_0^{1,2}(0,\infty)$. Then A is symmetric, but not selfadjoint, and $\sigma(A) = \{\lambda \in \mathbb{C} \,|\, \operatorname{Im}\lambda \geqslant 0\}$.

PROOF. Symmetry is shown as in a) using Theorem 3.35 and that now $u, v \in \mathrm{D}(A)$ have trace 0 at $s = 0$. From Example 3.38 and Proposition 1.20 we deduce that $\sigma(A) = -\mathrm{i}\sigma(\mathrm{i}A) = -\mathrm{i}\{\lambda \in \mathbb{C} \,|\, \operatorname{Re}\lambda \leqslant 0\} = \{\lambda \in \mathbb{C} \,|\, \operatorname{Im}\lambda \geqslant 0\}$. Consequently, A is not selfadjoint. $\square$

d) Let $X = L^2(\mathbb{R}^d)$ and $A = \Delta$ with $\mathrm{D}(A) = W^{2,2}(\mathbb{R}^d)$. Then A is selfadjoint with $\sigma(A) = \mathbb{R}_-$.

PROOF. For $u, v \in \mathrm{D}(A)$, Theorem 3.35 yields

$$(Au|v) = \int_{\mathbb{R}^d} (\Delta u)\overline{v}\,\mathrm{d}x = \int_{\mathbb{R}^d} u\Delta\overline{v}\,\mathrm{d}x = (u|Av)\,,$$

so that A is symmetric. In Example 3.41 we have seen that $\sigma(A) = \mathbb{R}_-$, and hence A is self-adjoint. $\square$

e) Let $U \subseteq \mathbb{R}^d$ be open with C^2–boundary and $A = \Delta$ with $\mathrm{D}(A) = W^{2,2}(U) \cap W_0^{1,2}(U)$. Then A is selfadjoint (and has compact resolvent by Example 3.43). In fact, the symmetry of A can be shown as in part d) because the traces of $u, v \in D(A)$ vanish. Then A is selfadjoint since it is invertible by Example 3.43.

f) Let $X = L^2(0,1)$, $A_0 = \partial^2$ with $\mathrm{D}(A_0) = W_0^{2,2}(0,1)$ and A be as in e) with $U = (0,1)$. As in e) we see that A_0 is symmetric. But A_0 is not selfadjoint, since $A_0 \subsetneq A$ and $A = A'$. We further claim that $A_0' = \partial^2$ with $\mathrm{D}(A_0') = W^{2,2}(0,1)$.

PROOF. For $v \in W^{2,2}(0,1)$ and $u \in \mathrm{D}(A)$, we deduce from Theorem 3.35

$$(A_0u|v) = \int_0^1 (\partial^2 u)\overline{v}\,\mathrm{d}s = \int_0^1 u\partial^2\overline{v}\,\mathrm{d}s + [\overline{v}\partial u - u\partial\overline{v}]_0^1 = (u|\partial^2 v);$$

i.e., $(\partial^2, W^{2,2}(0,1)) \subseteq A_0'$. Conversely, take $v \in D(A_0')$. For $u \in C_c^\infty(0,1)$ we have $\overline{u} \in C_c^\infty(0,1) \subseteq \mathrm{D}(A)$ and hence obtain

$$\int_0^1 (\partial^2 \overline{u})\overline{v}\,\mathrm{d}s = (A_0\overline{u}|v) = (\overline{u}|A_0'v) = \int_0^1 \overline{u}\overline{A_0'v}\,\mathrm{d}s.$$

After complex conjugation, we see that $v \in D^{2,2}(0,1) \cap X$ and $\partial^2 v = A_0' v \in X$. The function v thus belongs to $W^{2,2}(0,1)$ by Example 3.36. □

Theorem 4.9. *Let A be densely defined and selfadjoint on the Hilbert space X and let B be symmetric with $\mathrm{D}(A) \subseteq \mathrm{D}(B)$. Assume there are constants $c > 0$ and $\delta \in [0, 1/2)$ such that $\|Bx\| \leqslant c\|x\| + \delta\|Ax\|$ for all $x \in \mathrm{D}(A)$. Then the operator $A + B$ with $\mathrm{D}(A + B) = \mathrm{D}(A)$ is selfadjoint.*

Proof. By Theorem 4.7, the number it belongs to $\rho(A)$ for all $t \in \mathbb{R}\backslash\{0\}$. Take $\varepsilon \in (0, 1 - 2\delta) \subseteq (0,1)$ and $x \in X$. Using (4.3), we estimate

$$\begin{aligned}\|B(\mathrm{i}tI - A)^{-1}x\| &\leqslant \delta\|A(\mathrm{i}tI - A)^{-1}x\| + c\,\|(\mathrm{i}tI - A)^{-1}x\| \\ &= \delta\,\|\mathrm{i}t(\mathrm{i}tI - A)^{-1}x - x\| + c\,\|(\mathrm{i}tI - A)^{-1}x\| \\ &\leqslant \delta\Big(\frac{|t|}{|t|} + 1\Big)\|x\| + \frac{c}{|t|}\|x\| \leqslant (1-\varepsilon)\|x\|,\end{aligned}$$

whenever $|t| \geqslant \frac{c}{1-2\delta-\varepsilon}$. Theorem 1.27 now implies that $\pm it \in \rho(A + B)$ for such t. Moreover, $A + B$ is symmetric since

$$((A + B)x|y) = (Ax|y) + (Bx|y) = (x|Ay) + (x|By) = (x|(A + B)y)$$

for all $x, y \in \mathrm{D}(A)$. So, $A + B$ is selfadjoint due to Theorem 4.7. □

Actually, in the above theorem it suffices to assume that $\delta < 1$, see Theorem X.13 in [**RS**].

Example 4.10. Let $X = L^2(\mathbb{R}^3)$ and $A = \Delta$ with $\mathrm{D}(A) = W^{2,2}(\mathbb{R}^3)$. Set $Vu(x) = \frac{b}{|x|_2}u(x)$ for $x \in \mathbb{R}^3$, $u \in \mathrm{D}(A)$ and some $b \in \mathbb{R}$. Then $A + V$ with domain $\mathrm{D}(A)$ is selfadjoint.

Proof. Since $kp = 4 > d = 3$, we have $\mathrm{D}(A) \hookrightarrow C_0(\mathbb{R}^3)$ by Theorem 3.17. Let $0 < \varepsilon \leqslant 1$. Using polar coordinates and Example 3.41, we compute

$$\begin{aligned}\int_{\mathbb{R}^3} |Vu|^2\,\mathrm{d}x &= b^2\int_{B(0,\varepsilon)} \frac{|u(x)|^2}{|x|_2^2}\,\mathrm{d}x + b^2\int_{\mathbb{R}^3\backslash B(0,\varepsilon)} \frac{|u(x)|^2}{|x|_2^2}\,\mathrm{d}x \\ &\leqslant c\,\|u\|_\infty^2 \int_0^\varepsilon \frac{r^2}{r^2}\,\mathrm{d}r + \frac{b^2}{\varepsilon^2}\int_{\mathbb{R}^3\backslash B(0,\varepsilon)} |u|^2\,\mathrm{d}x \\ &\leqslant c\varepsilon\|u\|_{2,2}^2 + \frac{b^2}{\varepsilon^2}\|u\|_2^2 \leqslant c\varepsilon\|Au\|_2^2 + c\varepsilon\|u\|_2^2 + \frac{b^2}{\varepsilon^2}\|u\|_2^2,\end{aligned}$$

for constants $c > 0$ independent of $u \in \mathrm{D}(A)$ and ε. Moreover, V is symmetric on $\mathrm{D}(A)$ since

$$(Vu|v) = \int_{\mathbb{R}^3} \frac{b}{|x|_2}u(x)\overline{v(x)}\,\mathrm{d}x = (u|Vv)$$

for all $u, v \in \mathrm{D}(A)$. For small $\varepsilon > 0$, Theorem 4.9 implies that $A + V$ is selfadjoint. □

The spectra $\sigma(A+V)$ and $\sigma_p(A+V)$ and the eigenfunctions of $A+V$ are computed in §7.3.4 of [**Tr**], where $b > 0$. The above operator $A = \Delta + V$ is used in physics to describe the hydrogen atom, see also Example 4.19.

4.2. The spectral theorems

Hermitian matrices are unitarily equivalent to diagonal matrices and thus very easy to treat. In this section we establish the infinite dimensional analogues of this basic result from linear algebra. The following results can be extended to normal operators, and the separability assumption made below can be removed. See e.g. Corollaries X.2.8 and X.5.4 in [**DS2**] or Theorems 13.24, 13.30 and 13.33 in [**Ru2**].

Let $T \in \mathcal{B}(Z)$ for a Banach space Z and $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ be a complex polynomial. We then define the operator polynomial

$$p(T) = a_0 I + a_1 T + \ldots + a_n T^n \in \mathcal{B}(Z). \tag{4.4}$$

This gives a map $p \mapsto p(T)$ from the space of polynomials to $\mathcal{B}(Z)$. For selfadjoint T on a Hilbert space one can extend this map to all $f \in C(\sigma(T))$, as seen in the next theorem. We set $p_1(z) = z$.

Theorem 4.11 (Continuous functional calculus). *Let $T \in \mathcal{B}(X)$ be selfadjoint on a Hilbert space X. There is exactly one map $\Phi_T : C(\sigma(T)) \to \mathcal{B}(X)$; $f \mapsto f(T)$ satisfying*

(C1) $(\alpha f + \beta g)(T) = \alpha f(T) + \beta g(T)$,
(C2) $\|f(T)\| = \|f\|_\infty$ *(hence, Φ_T is injective),*
(C3) $\mathbb{1}(T) = I$ *and* $p_1(T) = T$,
(C4) $(fg)(T) = f(T)g(T) = g(T)f(T)$,
(C5) $f(T)' = \overline{f}(T)$

for all $f, g \in C(\sigma(T))$ and $\alpha, \beta \in \mathbb{C}$. In particular, we have $\Phi_T(p) = p(T)$ for each polynomial p, where $p(T)$ is given by (4.4).

Proof. We first show the properties (C1)-(C5) for polynomials $p(t) = a_0 + a_1 t + \ldots + a_n t^n$ and $q(t) = b_0 + b_1 t + \ldots + b_n t^n$ with $t \in \mathbb{R}$ and the map $p \mapsto p(T)$ defined by (4.4), where any $a_j, b_j \in \mathbb{C}$ may be equal to 0. Clearly, (C1) and (C3) are true in this case, and $p(T)' = \sum_{j=0}^n \overline{a}_j (T^j)' = \overline{p}(T)$ since $T = T'$. Moreover,

$$(pq)(T) = \sum_{m=0}^{2n} \Big(\sum_{\substack{0 \leqslant j,k \leqslant n \\ j+k=m}} a_j b_k \Big) T^m = \sum_{j=0}^n a_j T^j \sum_{k=0}^n b_j T^k = p(T)q(T)$$

and so $(pq)(T) = (qp)(T) = q(T)p(T)$; i.e., (C4) is shown for polynomials. Properties (C4) and (C5) imply that $p(T)$ is normal. Hence, Proposition 4.3 and Lemma 4.12 below imply

$$\|p(T)\| = \max\{|\lambda| \,\big|\, \lambda \in \sigma(p(T))\} = \max\{|\lambda| \,\big|\, \lambda \in p(\sigma(T))\} = \|p\|_\infty.$$

Let $f \in C(\sigma(T))$. Since $\sigma(T) \subseteq \mathbb{R}$ is compact, Weierstraß' approximation theorem yields real polynomials such that $p_n \to \operatorname{Re} f$ and $q_n \to \operatorname{Im} f$ in $C(\sigma(T))$ as $n \to \infty$, hence $p_n + \mathrm{i} q_n \to f$. We can thus extend the map $p \mapsto p(T)$ to a linear isometry $\Phi_T : f \mapsto f(T)$ from $C(\sigma(T))$ to $\mathcal{B}(X)$. By continuity, Φ_T also satisfies (C4) and (C5) on $C(\sigma(T))$.

If there is another map $\Psi : C(\sigma(T)) \to \mathcal{B}(X)$ satisfying (C1)-(C5), then $\Psi(p) = p(T) = \Phi_T(p)$ for all polynomials by (C1), (C3) and (C4), so that $\Psi = \Phi_T$ by continuity and density. □

We observe that we have actually shown uniqueness in then calss of linear and continuous maps $\Psi : C(\sigma(T)) \to \mathcal{B}(X)$ fulfilling (C3) and (C4).

LEMMA 4.12. *Let $T \in \mathcal{B}(Z)$ for a Banach space Z and let p be a polynomial. Then $\sigma(p(T)) = p(\sigma(T))$.*

PROOF. See Theorem 5.3 below or the exercises. □

COROLLARY 4.13. *Let $T \in \mathcal{B}(X)$ be selfadjoint and $f \in C(\sigma(T))$. Then the following asssertions hold.*

(C6) If $Tx = \lambda x$ for some $x \in X$ and $\lambda \in \mathbb{C}$, then $f(T)x = f(\lambda)x$.
(C7) The operator $f(T)$ is normal.
(C8) $\sigma(f(T)) = f(\sigma(T))$ (spectral mapping theorem).
(C9) $f(T)$ is selfadjoint if and only if f is real-valued.

PROOF. Take a sequence of polynomials p_n converging to f uniformly. Let $Tx = \lambda x$. Property (C6) holds for a polynomial since

$$p(T)x = \sum_{j=0}^{n} a_j T^j x = \sum_{j=0}^{n} a_j \lambda^j x = p(\lambda)x.$$

Using (C2), we obtain

$$f(T)x = \lim_{n\to\infty} p_n(T)x = \lim_{n\to\infty} p_n(\lambda)x = f(\lambda)x.$$

From (C5) and (C4) we infer $f(T)f(T)' = f(T)\overline{f}(T) = \overline{f}(T)f(T) = f(T)'f(T)$ so that $f(T)$ is normal.

We next show (C8). If $\mu \notin f(\sigma(T))$, then $g = \frac{1}{\mu - f}$ is an element of $C(\sigma(T))$. Thus (C3) and (C4) yield

$$(\mu I - f(T))g(T) = g(T)(\mu I - f(T)) = (g(\mu\mathbb{1} - f))(T) = \mathbb{1}(T) = I.$$

Hence, $\mu \in \rho(f(T))$. Let $\mu = f(\lambda)$ for some $\lambda \in \sigma(T)$. Then $\mu_n := p_n(\lambda)$ belongs to $\sigma(p_n(T))$ for all $n \in \mathbb{N}$ by Lemma 4.12. As above, the operators $\mu_n I - p_n(T)$ tend to $\mu I - f(T)$ in $\mathcal{B}(X)$. Suppose that $\mu I - f(T)$ was invertible. Then also $\mu_n I - p_n(T)$ would be invertible for large n by Theorem 1.27. This is impossible, and so $\mu \in \sigma(f(T))$.

For the last assertion, observe that $f(T) = f(T)'$ if and only if $(f - \overline{f})(T) = 0$ if and only if $f - \overline{f} = 0$, because Φ_T is injective. □

COROLLARY 4.14. *Let $n \in \mathbb{N}$ and $T = T' \in \mathcal{B}(X)$ with $\sigma(T) \subseteq \mathbb{R}_+$ (in this case one writes $T = T' \geqslant 0$). Then there is a unique selfadjoint operator $W \in \mathcal{B}(X)$ with $\sigma(W) \subseteq \mathbb{R}_+$ such that $W^n = T$.*

PROOF. Consider $w(t) = t^{1/n}$ for $t \in \sigma(T) \subseteq \mathbb{R}_+$ and define $W := w(T)$. Then $W^n = w^n(T) = p_1(T) = T$. Properties (C8) and (C9) imply that $W = W' \geqslant 0$. For the proof of the uniqueness of W, we refer to Korollar VII.1.16 in [**We**] or the exercises. □

We next use basic properties of orthogonal projections and orthonormal bases which are discussed in, e.g., Chapter 3 of [**Sc2**].

THEOREM 4.15 (Compact case). *Let X be a Hilbert space with* $\dim X = \infty$*, $T \in \mathcal{B}(X)$ be compact and selfadjoint, and A be densely defined, closed and selfadjoint on X having a compact resolvent. Then the following assertions hold.*

(a) i) There is an index set $J \in \{\varnothing, \mathbb{N}, \{1, \ldots, N\} \,|\, N \in \mathbb{N}\}$ and eigenvalues $\lambda_j \neq 0$, $j \in J$, such that $\sigma(T) = \{0\} \,\dot\cup\, \{\lambda_j \,|\, j \in J\} \subseteq \mathbb{R}$ where $\lambda_j \to 0$ as $j \to \infty$ if $J = \mathbb{N}$.

ii) There is an orthonormal basis of $\mathrm{N}(T)^\perp = \overline{\mathrm{R}(T)}$ *consisting of eigenvectors of T for the eigenvalues λ_j.*

iii) The eigenspace $E_j(T) := \mathrm{N}(\lambda_j I - T)$ is finite-dimensional and the orthogonal projection P_j onto $E_j(T)$ commutes with T, for each $j \in J$.

iv) The sum $T = \sum_{j\in J} \lambda_j P_j$ converges in $\mathcal{B}(X)$.

(b) i) We have $\sigma(A) = \sigma_p(A) = \{\mu_n \,|\, n \in \mathbb{N}\} \subseteq \mathbb{R}$ with $|\mu_n| \to \infty$ as $n \to \infty$.

ii) There is an orthonormal basis of X consisting of eigenvectors of A.

iii) The eigenspaces $E_n(A) = \mathrm{N}(\mu_n I - A)$ are finite dimensional and the orthogonal projections Q_n onto $E_n(A)$ satisfy $Q_n \mathrm{D}(A) \subseteq \mathrm{D}(A)$ and $AQ_n x = Q_n Ax$ for all $x \in \mathrm{D}(A)$ and $n \in \mathbb{N}$.

iv) The sum $Ax = \sum_{n=1}^\infty \mu_n Q_n x$ converges in X for all $x \in \mathrm{D}(A)$.

PROOF. 1) Theorem 2.11, 2.16 and 4.7 show the assertions i) in (a) and (b), except for the claim that $\sigma(A)$ is infinite, and also that $\dim E_j(T)$ and $\dim E_n(A)$ are finite for all j and n. Let $x, y \in \mathrm{D}(A)$ be eigenvectors of A for eigenvalues $\mu_n \neq \mu_k$. Then

$$\mu_n\,(x|y) = (Ax|y) = (x|Ay) = \mu_k\,(x|y),$$

so that $(x|y) = 0$. Similarly, one sees that $E_j(T) \perp E_k(T)$ if $j \neq k$. By the Gram-Schmidt procedure, each eigenspace $E_j(T)$ and $E_n(A)$ has an orthonormal basis of eigenvectors for $\lambda_j \neq 0$ and μ_n, respectively. The union of these bases gives orthonormal sets $\mathcal{B}_T$ and $\mathcal{B}_A$.

2) Let $J = \mathbb{N}$ in a), the other cases are treated similarly. Observe that $\mathcal{B}_T \subseteq \mathrm{R}(T)$. Set $\mathbb{1}_j = \mathbb{1}_{\{\lambda_j\}} \in C(\sigma(T))$ and $\varphi_n = \mathbb{1}_1 + \ldots + \mathbb{1}_n$ for every $j, n \in \mathbb{N}$. We then have $\mathbb{1}_j(T)^2 = \mathbb{1}_j^2(T) = \mathbb{1}_j(T)$. Moreover, (C4) and (C9) imply that $T\mathbb{1}_j(T) = \mathbb{1}_j(T)T$, $\mathbb{1}_j(T)' = \mathbb{1}_j(T)$ and

$$(\lambda_j I - T)\mathbb{1}_j(T) = ((\lambda_j \mathbb{1} - p_1)\mathbb{1}_j)(T) = ((\lambda_j - \lambda_j)\mathbb{1}_j)(T) = 0.$$

If $\lambda_j v = Tv$ for some $v \in X$, we further deduce $\mathbb{1}_j(T)v = \mathbb{1}_j(\lambda_j)v = v$ from (C6). As a result, $\mathbb{1}_j(T)$ is a selfadjoint projection onto $E_j(T)$. For $x \in X$ and $y \in \mathrm{N}(\mathbb{1}_j(T))$, we then obtain $(\mathbb{1}_j(T)x|y) = (x|\mathbb{1}_j(T)y) = 0$ so that $\mathbb{1}_j(T)$ is orthogonal; i.e., $\mathbb{1}_j(T) = P_j$ and $\varphi_n(T) = P_1 + \ldots + P_n$ for all $j, n \in \mathbb{N}$. We have shown iii) in (a).

Since $TP_j = \lambda_j P_j$, the operator $\varphi_n(T)T$ is a partial sum of the series in part iv). Employing also (C2), we thus derive iv) from

$$\|T - \varphi_n(T)T\| = \|(p_1 - \varphi_n p_1)(T)\| = \|p_1 - \varphi_n p_1\|_\infty = \sup_{j \geqslant n+1} |\lambda_j| \longrightarrow 0,$$

as $n \to \infty$. It also follows that $\varphi_n(T)y \in \operatorname{lin} \mathcal{B}_T$ converges to y as $n \to \infty$ for all $y \in \overline{\mathrm{R}(T)}$. Therefore, $\mathcal{B}_T$ is an orthonormal basis of $\overline{\mathrm{R}(T)}$ due to e.g. Theorem 3.15 in [**Sc2**]. Finally, (2.1) shows that $\overline{\mathrm{R}(T)} = {}^\perp\mathrm{N}(T') = \mathrm{N}(T)^\perp$ because $T = T'$ and X is reflexive.

3) Fix $t \in \rho(A) \cap \mathbb{R}$. Then $R(t,A)' = R(t,A') = R(t,A)$ by (4.2), and this operator is compact and has a trivial kernel. By 2), $X = \mathrm{N}(R(t,A))^{\perp}$ possesses an orthonormal basis of eigenvectors w of $R(t,A)$ for the eigenvalues $\lambda \neq 0$. Proposition 1.20 yields the eigenvector $v = \lambda R(t,A)w$ of A for the eigenvalue $\mu = t - \lambda^{-1}$. Since $\dim X = \infty$ and the eigenspaces of $R(t,A)$ are finite dimensional, A has infinitely many distinct eigenvalues; i.e., part i) in (b) is shown.

Let $x \in X$ be orthogonal to all eigenvectors of A. For the above v, we obtain $0 = (x|v) = (x|\lambda R(t,A)w) = \lambda\,(R(t,A)x|w)$. Since the eigenvectors w span X, we infer that $R(t,A)x = 0$ and hence $x = 0$. Consequently, $\mathcal{B}_A$ is a basis of X and part ii) of (b) is true.

4) Let $\{v_{n,1}, \cdots, v_{m_n}\}$ be eigenvectors of A forming an orthonormal basis of $E_n(A)$ for $n \in \mathbb{N}$. From step 3) we then deduce

$$Q_n x = \sum_{j=1}^{m_n} (x|v_{n,j})\, v_{n,j} \qquad \text{and} \qquad x = \sum_{n=1}^{\infty} Q_n x$$

for all $x \in X$. For $x \in \mathrm{D}(A)$ it follows

$$\begin{aligned} Q_n A x &= \sum_{j=1}^{m_n} (Ax|v_{n,j})\, v_{n,j} = \sum_{j=1}^{m_n} (x|Av_{n,j})\, v_{n,j} = \sum_{j=1}^{m_n} (x|\mu_n v_{n,j})\, v_{n,j} \\ &= \sum_{j=1}^{m_n} (x|v_{n,j})\, \mu_n v_{n,j} = \sum_{j=1}^{m_n} (x|v_{n,j})\, A v_{n,j} = AQ_n X. \end{aligned}$$

We thus conclude

$$A\sum_{k=1}^{n} Q_k x = \sum_{k=1}^{n} Q_k A x \longrightarrow \sum_{k=1}^{\infty} Q_k A x = \sum_{k=1}^{\infty} AQ_k x = \sum_{k=1}^{\infty} \mu_k Q_k x,$$

as $n \to \infty$, so that the closedness of A yields the last assertion. □

Remark 4.16. a) In the above proof we also obtain that

$$Ax = \sum_{n=1}^{\infty} \mu_n \sum_{j=1}^{m_n} (x|v_{n,j})\, v_{n,j}$$

for all $x \in \mathrm{D}(A)$. An analogous result holds for T.

b) Let T be selfadjoint and compact such that $\mathrm{N}(T)$ is separable. Let $\{z_k \,|\, k \in J_0\}$ be an orthonormal basis for $\mathrm{N}(T)$, where $J_0 \subseteq \mathbb{Z}_-$ could be empty. Denote by λ_l the non-zero eigenvalues of T (repeated according to their multiplicity) with corresponding orthonormal basis of eigenvectors $\{w_l \,|\, l \in J_1\}$. The union $\{b_j \,|\, j \in J'\}$ of $\{z_k \,|\, k \in J_0\}$ and $\{w_l \,|\, l \in J_1\}$ is an orthonormal basis of X. By e.g. Theorem 3.18 in [**Sc2**], the map

$$\Phi : X \to \ell^2(J'); \quad \Phi x = \big(\,(x|b_j)\,\big)_{j \in J'},$$

is unitary with $\Phi^{-1}((\xi_j)_{j\in J'}) = \sum_{j \in J'} \xi_j b_j$. Moreover, the transformed operator $\Phi T \Phi^{-1}$ acts on $\ell^2(J')$ as the multiplication operator

$$\Phi T \Phi^{-1}(\xi_j) = \Phi T \sum_{j\in J'} \xi_j b_j = \Phi \sum_{j \in J'} \lambda_j \xi_j b_j = (\lambda_j \xi_j)_{j \in J'},$$

where $\lambda_j := 0$ if $j \in J_0$. Hence, $\Phi T \Phi^{-1}$ is represented as infinite diagonal matrix with diagonal elements λ_j. Analogous results hold for A from Theorem 4.15, see e.g. Theorems 4.5.1-4.5.3 in [**Tr**]. $\Diamond$

If one wants to extend part a) of the above remark to the non-compact case one is led to the so-called spectral measures, see the exercises. We generalize below part b).

Theorem 4.17 (Multiplicator representation, bounded case). *Let $T \in \mathcal{B}(X)$ be selfadjoint on a separable Hilbert space X. Then there is a σ–finite measure space $(\Omega, \mathcal{A}, \mu)$, a measurable function $h : \Omega \to \sigma(T)$ and a unitary operator $U : X \to L^2(\mu)$ such that*

$$Tx = U^{-1}hUx \qquad \textit{for all} \;\; x \in X.$$

Proof (partly sketched). 1) Let $v_1 \in X\backslash\{0\}$. We define the linear subspaces

$$Y_1 := \{f(T)v_1 \,|\, f \in C(\sigma(T))\} \quad \text{and} \quad X_1 = \overline{Y}_1$$

of X. Since $Tf(T)v_1 = (p_1 f)(T)v_1 \in Y_1$ for every $f \in C(\sigma(T))$, we obtain $TY_1 \subseteq Y_1$ and so $TX_1 \subseteq X_1$. We introduce the map

$$\varphi_1 : C(\sigma(T)) \to \mathbb{C}; \quad \varphi_1(f) = (f(T)v_1|v_1) \,,$$

which is linear and bounded because $|\varphi_1(f)| \leqslant \|f(T)\|\|v_1\|^2 = \|f\|_\infty \|v_1\|^2$ due to (C2). If $f \geqslant 0$, then $\sigma(f(T)) \subseteq \mathbb{R}_+$ by (C8). So we can deduce from Corollary 4.14 that

$$(f(T)v_1|v_1) = \big(f(T)^{1/2}v_1\big|f(T)^{1/2}v_1\big) = \|f(T)^{1/2}v_1\|^2 \geqslant 0.$$

The Riesz representation theorem of $C(\sigma(T))^*$ now gives a positive measure μ_1 on $\mathcal{B}(\sigma(T))$ such that

$$\varphi_1(f) = \int_{\sigma(T)} f \,\mathrm{d}\mu_1,$$

for all $f \in C(\sigma(T)) \subseteq L^2(\mu_1)$, see Theorem 2.14 of [**Ru1**]. For $x = f(T)v_1 \in Y_1$, we define $V_1 x := f \in L^2(\mu_1)$. We compute

$$\begin{aligned}\|V_1 x\|^2_{L^2(\mu_1)} &= \int_{\sigma(T)} |f|^2 \,\mathrm{d}\mu_1 = \varphi_1(\overline{f} f) = \big((\overline{f} f)(T)v_1|v_1\big) \\ &= \big(f(T)' f(T)v_1|v_1\big) = (f(T)v_1|f(T)v_1) = \|x\|^2_X.\end{aligned}$$

In particular, if $x = f(T)v_1 = g(T)v_1$ for some $g \in C(\sigma(T))$, then

$$\|f - g\|_2^2 = \|(f(T) - g(T))v_1\|_X^2 = 0,$$

and so $f = g$ in $L^2(\mu_1)$. As a result, $V_1 : Y_1 \to L^2(\mu_1)$ is a linear isometric map and can be extended to a linear isometry $U_1 : X_1 \to L^2(\mu_1)$.

Observe that $C(\sigma(T)) \subseteq \mathrm{R}(U_1)$. Riesz' theorem also yields that μ_1 is regular.[1] Theorem 3.14 in [**Ru1**] then shows that $C(\sigma(T))$ is dense in $L^2(\mu_1)$,

[1] A positive measure μ is called *regular*, if if satisfies

$$\mu(B) = \inf\{\mu(\mathcal{O}) \,|\, B \subseteq \mathcal{O}, \; \mathcal{O} \text{ is open}\} = \sup\{\mu(K) \,|\, K \subseteq B, K \text{ is compact}\}$$

for all B in the corresponding Borel σ–algebra.

so that U_1 has dense range. Hence, the isometry U_1 is bijective and thus unitary by e.g. Proposition 5.52 in [**Sc2**]. Finally, we compute

$$U_1Tf(T)v_1 = V_1(p_1f)(T)v_1 = p_1f = p_1U_1f(T)v_1,$$

for $f \in C(\sigma(T))$. By density, it follows $Tx = U_1^{-1}p_1U_1x$ for all $x \in X_1$.

2)[2] We are done if there is an $v_1 \in X$ such that $X_1 = X$. In general this is not true. Using Zorn's Lemma (see Theorem I.2.7 of [**DS1**]), we instead find an orthonormal system of spaces X_n as in step 1) which span X. To that aim, we introduce the collection $\mathcal{E}$ of all sets E having as elements at most countably many closed subspaces $X_j \subset X$ of the type constructed in step 1) such that $X_i \perp X_j$ for all $X_i \neq X_j$ in E. The system $\mathcal{E}$ is ordered via inclusion of sets. Let $\mathcal{C}$ be a chain in $\mathcal{E}$; i.e., a subset of $\mathcal{E}$ such that $E \subseteq F$ or $F \subseteq E$ for all $E, F \in \mathcal{C}$. We put $C = \bigcup_{E\in\mathcal{C}} E$. Cleary, $E \subseteq C$ for all $E \in \mathcal{C}$. Let $Y, Z \in C$. Then Y and Z are closed subspaces of X as constructed in 1) and there are sets $E, F \in \mathcal{C}$ such that $Y \in E$ and $Z \in F$. We may assume that $E \subseteq F$ and so $X, Y \in F$. The subspaces Y and Z are thus orthogonal (if $Y \neq Z$). As a result, C contains pairwise orthogonal subspaces of X. If $x \perp y$ have norm 1, then $\|x - y\|^2 = \|x\|^2 + \|y\|^2 = 2$. The separability of X then implies that at most countably many subspaces belong to C, so that C is an element of $\mathcal{E}$ and hence an upper bound of $\mathcal{C}$. Zorn's Lemma now gives a maximal element $M = \{X_j \,|\, j \in J\}$ in $\mathcal{E}$, where $J \subset \mathbb{N}$ and X_j are pairwise orthogonal subspaces as constructed in step 1).

Assume that there is a vector $v \in X$ being orthogonal to all X_j. Let Z be the closed span of all vectors $f(T)v$ with $f \in C(\sigma(T))$. Let $g \in C(\sigma(T))$ and $x = g(T)v_i \in X_i$ for some $i \in J$, where v_i generates X_i as in step 1). We then obtain

$$(x|f(T)v) = (f(T)'g(T)v_i|v) = ((\overline{f}g)(T)v_i|v) = 0$$

since $(\overline{f}g)(T)v_i \in X_i$. By density, it follows that Z is orthogonal to all X_j and thus $M \cup \{Z\} \in \mathcal{E}$. The maximality on M now implies that Z belongs to M, so that $v = 0$. Consequently, X is the closed linear span of the elements in the orthogonal subspaces X_j. (One writes $X = \bigoplus_{j\in J} X_j$.) We now define

$$\Omega = \bigcup_{j\in J} \sigma(T) \times \{j\} \subseteq \mathbb{R}^2, \quad \mathcal{A} = \mathcal{B}(\Omega), \quad \mu(A) = \sum_{j\in J} \mu_j(A_j)$$

with $A_j \times \{j\} = A \cap (\sigma(T) \times \{j\})$ and

$$h(\lambda \times \{j\}) = \lambda, \qquad Ux = \sum_{j\in J} U_j x_j,$$

where $\lambda \times \{j\} \in \Omega$, U_j is given as in step 1), and $x = \sum_{j\in J} x_j$ and $x_j \in X_j$. It can be seen that Ω, $\mathcal{A}$, μ, h and U satisfy the assertions. □

We add an observation to the above proof. Let $\lambda \in \sigma(T)\backslash\sigma_p(T)$. Then $\Lambda = \bigcup_{j\in J}\{\lambda \times \{j\}\} \subset \Omega$ is a μ–null set. In fact, otherwise the characteristic function f of any subset of Λ with measure in $(0, \infty)$ would be a non-zero element of $L^2(\mu)$. Hence, $x = U^{-1}f \neq 0$ would be an eigenvector of T for the eigenvalue λ, since $Tx = U^{-1}\lambda f = \lambda x$. We further note that in the proof

[2] Not shown in the lectures.

of Theorem 4.17 one could also take $\Omega = \bigcup_{j\in J} \sigma(T_j)\times\{j\}$, where $T_j = T|_{X_j}$. It can be shown that $\sigma(T) = \bigcup_{j\in J} \sigma(T_j)$.

The above representation of bounded selfadjoint operators as multiplication operators now leads to a multiplication representation and to a B_b–functional calculus for (possibly) unbounded selfadjoint operators A. Here $B_b(\sigma(A))$ is the Banach space of bounded Borel functions on $\sigma(A)$ endowed with the supremum norm. We use this space instead of $L^\infty(\sigma(A))$ to avoid certain technical problems. Set $r_\lambda(z) = (\lambda - z)^{-1}$ for $z \in \mathbb{C}\backslash\{\lambda\}$.

Theorem 4.18 (Unbounded A). *Let A be a closed, densely defined and selfadjoint operator on a separable Hilbert space X. Then the following assertions hold.*

(a) There is a σ–finite measure space $(\Omega, \mathcal{A}, \mu)$, a measurable function $h : \Omega \to \sigma(A)$ and a unitary operator $U : X \to L^2(\mu)$ such that

$$\mathrm{D}(A) = \{x \in X \,|\, hUx \in L^2(\mu)\} \quad \textit{and} \quad Ax = U^{-1}hUx.$$

(b) There is a contractive map $\Psi_A : B_b(\sigma(A)) \to \mathcal{B}(X)$; $\Psi_T(f) = f(A)$, satisfying (C1) and (C3)–(C5), where $p_1(T) = T$ in (C3) is replaced by $r_\lambda(A) = R(\lambda, A)$ for $\lambda \in \rho(A)$. Moreover, if $f_n \in B_b(\sigma(A))$ are uniformly bounded and converge to $f \in B_b(\sigma(T))$ pointwise, then $f_n(A)x \to f(A)x$ as $n \to \infty$ for all $x \in X$. Finally, for $x \in \mathrm{D}(A)$ and $f \in B_b(\sigma(A))$ the vector $f(A)x$ belongs to $\mathrm{D}(A)$ and $Af(A)x = f(A)Ax$.

Proof. a) We additionally assume that $\sigma(A) \neq \mathbb{R}$ and fix $t \in \rho(A)\cap\mathbb{R}$.[3] Then $R(t, A) \in \mathcal{B}(X)$ is selfadjoint and can be represented as $R(t, A) = U^{-1}mU$ on a space $L^2(\Omega, \mu)$ as in Theorem 4.17. Recall that Proposition 1.20 yields $\sigma(A) = t - [\sigma(R(t, A))\backslash\{0\}]^{-1}$. Set

$$h(\lambda \times \{j\}) = t - \frac{1}{m(\lambda \times \{j\})} = t - \frac{1}{\lambda} \in \sigma(A),$$

for $j \in J$ and $\lambda \in \sigma(R(t, A))\backslash\{0\}$. The sets $\{0 \times \{j\}\}$ have μ–measure 0 in view of the obervation before the theorem, due to the injectivity of $R(t, A)$. We can thus extend h by 0 to a measurable function on Ω.

Let $x \in \mathrm{D}(A)$. We put $y = tx - Ax \in X$. Since $x = R(t, A)y = U^{-1}mUy$, we obtain

$$hUx = hmUy = (tm - \mathbb{1})Uy \in L^2(\mu),$$
$$U^{-1}hUx = tU^{-1}mUy - y = tx - y = Ax.$$

If $x \in X$ satisfies $hUx \in L^2(\mu)$, then we put $y = U^{-1}(t\mathbb{1} - h)Ux \in X$ and obtain $mUy = (tm - mh)Ux = Ux$. Therefore, $x = U^{-1}mUy = R(t, A)y$ belongs to $\mathrm{D}(A)$, and part (a) is proved.

b) We define $\Psi_A : f \mapsto f(A)$ by

$$f(A)x = U^{-1}(f \circ h)Ux$$

for $f \in B_b(\sigma(A))$ and $x \in X$. We further set $M_f\varphi = (f \circ h)\varphi$ for $\varphi \in L^2(\mu)$. It is straightforward to check that $f(A) \in \mathcal{B}(X)$, Ψ_T is linear, $\mathbb{1}(A) = I$ and (C5) is true. Let $\lambda \in \rho(A)$. We have

$$hUr_\lambda(A)x = h(r_\lambda \circ h)Ux = h(\lambda\mathbb{1} - h)^{-1}Ux \in L^2(\mu)$$

[3] If $\sigma(A) = \mathbb{R}$, we instead take $t = \mathrm{i}$ and use below the version of Theorem 4.17 for the normal operator $R(\mathrm{i}, A)$ given in e.g. Satz VII.1.25 in [**We**].

for all $x \in X$. So part (a) yields that $r_\lambda(A)X \subseteq \mathrm{D}(A)$ and $(\lambda I - A)r_\lambda(A) = I$. Similarly, one sees that $r_\lambda(A)(\lambda x - Ax) = x$ for all $x \in \mathrm{D}(A)$, and so (C3) is shown. The contractivity follows from $\|f(A)\| = \|M_f\| \leqslant \|f\|_\infty$. For property (C4) we observe that

$$(fg)(A)x = U^{-1}(f \circ h)(g \circ h)Ux = U^{-1}(f \circ h)UU^{-1}(g \circ h)Ux = f(A)g(A)x,$$

where $f, g \in B_b(\sigma(A))$ and $x \in X$.

Let $f, f_n \in B_b(\sigma(A))$ be uniformly bounded by c such that $f_n \to f$ pointwise as $n \to \infty$. For every $x \in X$, we have $f_n(A)x - f(A)x = U^{-1}((f_n - f) \circ h)Ux$. Since $(f_n - f) \circ h \to 0$ pointwise and $|((f_n - f) \circ h)Ux| \leqslant 2c\,|Ux|$, Lebesgue's convergence theorem shows that $((f_n - f) \circ h)Ux$ tends to 0 in $L^2(\mu)$ and so $f_n(A)x \to f(A)x$ in X as $n \to \infty$.

Let $x \in \mathrm{D}(A)$. The above results yield that

$$g := h(f \circ h)Ux = (f \circ h)UU^{-1}hUx = (f \circ h)UAx \in L^2(\mu).$$

On the other hand, $g = hUU^{-1}(f \circ h)Ux = hUf(A)x$. Part (a) thus implies that $f(A)x \in \mathrm{D}(A)$ and

$$Af(A)x = U^{-1}hUf(A)x = U^{-1}g = U^{-1}(f \circ h)UAx = f(A)Ax. \qquad \square$$

We conclude with one of the most important applications of the above theorem.

Example 4.19 (Schrödinger's equation). Let X be a Hilbert space and H be a closed, densely defined and selfadjoint operator on X. For a given $u_0 \in \mathrm{D}(H)$ we claim that there is exactly one function $u \in C^1(\mathbb{R}, X) \cap C(\mathbb{R}, [\mathrm{D}(H)])$ solving

$$\tfrac{\mathrm{d}}{\mathrm{d}t}u(t) = -\mathrm{i}Hu(t), \quad t \in \mathbb{R}, \qquad u(0) = u_0. \tag{4.5}$$

(H is called Hamiltonian.) The solution is given by $u(t) = T(t)u_0$ for unitary operators $T(t)$ on X satisfying $T(0) = I$, $T(t)T(s) = T(s)T(t)$ and $T(t)^{-1} = T(-t)$ for $t, s \in \mathbb{R}$. Moreover, $t \mapsto T(t)x \in X$ continuous for $t \in \mathbb{R}$ and all $x \in X$. An example for this setting is $X = L^2(\mathbb{R}^3)$ and $H = -(\Delta + \frac{b}{|x|_2})$ with $\mathrm{D}(H) = W^{2,2}(\mathbb{R}^3)$, see Example 4.10.

Proof. For $t \in \mathbb{R}$, we consider the bounded function $f_t(\xi) = \mathrm{e}^{-\mathrm{i}t\xi}$, $\xi \in \mathbb{R}$. Theorem 4.18 allows us to define $T(t) = f_t(H) \in \mathcal{B}(X)$. Since $f_0 = \mathbb{1}$ and $f_t f_s = f_{t+s}$, we obtain $T(0) = I$ and $T(t)T(s) = T(s)T(t)$ for $t, s \in \mathbb{R}$. With $s = -t$ it follows that $T(t)$ has the inverse $T(-t)$. Moreover, $T(t)$ is unitary since $T(t)' = \overline{f_t}(H) = f_{-t}(H) = T(-t)$. Because of $\|f_t\|_\infty = 1$ and the continuity of $t \mapsto f_t(\xi)$ for fixed $\xi \in \mathbb{R}$, the function $\mathbb{R} \ni t \mapsto T(t)z \in X$ is continuous for each $x \in X$.

Let $u_0 \in \mathrm{D}(H)$. We set $y = \tau u_0 - Hu_0$ for some $\tau \in \rho(H)$, so that $u_0 = R(\tau, H)y = r_\tau(H)y$. We then obtain

$$\begin{aligned} \frac{1}{t-s}(T(t)u_0 - T(s)u_0) &= \frac{1}{t-s}(f_t(H) - f_s(H))r_\tau(H)y \\ &= \left(\frac{1}{t-s}(f_t - f_s)r_\tau\right)(H)y =: g_{t,s}(H)y \end{aligned}$$

for all $t \neq s$. Observe that $g_{t,s}(\xi) \to \frac{-\mathrm{i}\xi}{\tau - \xi} f_s(\xi) =: m(\xi)f_s(\xi)$ as $t \to s$ for all $\xi \in \sigma(H)$ and $\|g_{t,s}\|_\infty \leqslant \|m\|_\infty = \sup_{\xi \in \sigma(H)} |\frac{\xi}{\tau - \xi}| < \infty$ for all $t \neq s$. So

Theorem 4.18 shows that there exists

$$\tfrac{\mathrm{d}}{\mathrm{d}t}T(t)u_0 = m(H)T(t)(\tau u_0 - Hu_0) = T(t)m(H)y.$$

We further compute

$$\begin{aligned} m(T)y &= U(m\circ h)U^{-1}y = U((-\mathrm{i}p_1 r_\tau)\circ h)U^{-1}y \\ &= -U\mathrm{i}hU^{-1}U(r_\tau \circ h)U^{-1}y = -\mathrm{i}HR(\tau,H)y = -\mathrm{i}Hu_0 \end{aligned}$$

by means of Theorem 4.18. Hence, we arrive at

$$\tfrac{\mathrm{d}}{\mathrm{d}t}T(t)u_0 = -\mathrm{i}T(t)Hu_0 = -\mathrm{i}HT(t)u_0,$$

using Theorem 4.18 once more. Due to these equations, $u = T(\cdot)u_0$ belongs to $C^1(\mathbb{R}_+, X) \cap C(\mathbb{R}_+, [\mathrm{D}(H)])$ and solves (4.5).

Let $v \in C^1(\mathbb{R}_+, X) \cap C(\mathbb{R}_+, [\mathrm{D}(H)])$ be another solution of (4.5). For $0 \leqslant s \leqslant t$ and $h \neq 0$ we compute

$$\begin{aligned} &\tfrac{1}{h}(T(t-s-h)v(s+h) - T(t-s)v(s)) - T(t-s)(v'(s) + \mathrm{i}Hv(s)) \\ &= T(t-s-h)\big(\tfrac{1}{h}(v(s+h)-v(s)) - v'(s)\big) + (T(t-s-h) - T(t-s))v'(s) \\ &\quad - \tfrac{1}{-h}(T(t-s-h) - T(t-s))v(s) - T(t-s)\mathrm{i}Hv(s) \longrightarrow 0 \end{aligned}$$

as $h \to 0$. For any $y \in X$ we thus obtain $\frac{\mathrm{d}}{\mathrm{d}s}\,(T(t-s)v(s)|y) = 0$, since $v' = -\mathrm{i}Hv$. Consequently,

$$(T(t)x|y) = (T(t)v(0)|y) = (T(0)v(t)|y) = (v(t)|y)\ ,$$

which gives $u(t) = v(t)$ for all $t \geqslant 0$. Thus the 'strongly continuous unitary group' $(T(t))_{t\in\mathbb{R}}$ solves (4.5) uniquely. $\square$

CHAPTER 5

Holomorphic functional calculi

We come back to the case of Banach spaces X and Y. We want to introduce functional calculi for non-selfadjoint operators on X, using now complex curve integrals.

5.1. The bounded case

Let $U \subseteq \mathbb{C}$ be open, $g : U \to Y$ be holomorphic (i.e., complex differentiable) and $\Gamma \subseteq U$ be a piecewise C^1 curve with parametrization $\gamma : [a,b] \to U$. This means that there are $a = a_0 < a_1 < \cdots < a_m = b$ such that $\gamma \in C^1([a_{k-1}, a_k], \mathbb{C})$ for all $k \in \{1, \dots, m\}$, where γ does not need to be continuous. We write $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_n$ with $\Gamma_k = \gamma_k([a_{k-1}, a_k])$ and define the curve integral

$$\int_\Gamma g \,\mathrm{d}z = \int_a^b g(\gamma(t))\gamma'(t)\,\mathrm{d}t := \sum_{k=1}^m \int_{a_{k-1}}^{a_k} g(\gamma(t))\gamma'(t)\,\mathrm{d}t$$

as a Banach space-valued Riemann integral (having the same definition, results and proofs as for $Y = \mathbb{R}$ in Analysis 1). Using Riemann sums, one easily checks that $T \int_\Gamma g \,\mathrm{d}z = \int_\Gamma Tg \,\mathrm{d}z$ for all $T \in \mathcal{B}(Y,Z)$ and Banach spaces Z. The index of a closed curve (i.e., $\gamma(a) = \gamma(b)$) is given by

$$n(\Gamma, z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\mathrm{d}w}{w - z}$$

for all $z \in \mathbb{C} \backslash \Gamma$. The index is the number of times that Γ winds around z, counted with orientation.

If Γ is closed and $n(\Gamma, z) = 0$ for all $z \notin U$, then Cauchy's integral theorem and formula

$$\int_\Gamma g \,\mathrm{d}z = 0, \tag{5.1}$$

$$n(\Gamma, z)g(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{g(w)}{w - z}\,\mathrm{d}w, \tag{5.2}$$

are valid for all $z \in U \backslash \Gamma$. This fact is shown for $Y = \mathbb{C}$ in Theorems IV.5.4 and IV.5.7 of [**Co1**]. For a general Banach space Y, the formulas (5.1) and (5.2) are thus satisfied by functions $z \mapsto \langle g(z), y^* \rangle$ for every $y^* \in Y^*$. Hence, $\langle \int_\Gamma g \,\mathrm{d}z, y^* \rangle = 0$ for all $y^* \in Y^*$ so that a corollary of the Hahn–Banach theorem yields (5.1) in Y. Similarly one deduces (5.2).

For compact, non-empty subsets $K \subseteq \mathbb{C}$, we introduce the space

$$H(K) = \{f : \mathrm{D}(f) \to \mathbb{C} \,\big|\, K \subseteq \mathrm{D}(f) \subseteq \mathbb{C},\ \mathrm{D}(f) \text{ is open}, f \text{ is holomorphic}\}.$$

Let $K \subseteq U \subseteq \mathbb{C}$, K be compact, and U be open. By Proposition VIII.1.1 in [**Co1**] and its proof there exists an *admissible curve* Γ *for* K *and* U (or, *in*

$U\backslash K$) which means that $\Gamma \subseteq U\backslash K$ is piecewise C^1, $n(\Gamma, z) = 1$ for all $z \in K$ and $n(\Gamma, z) = 0$ for all $z \in \mathbb{C}\backslash U$.

Let $T \in \mathcal{B}(X)$, $f \in H(\sigma(T))$, and Γ be admissible for $\sigma(T)$ and $\mathrm{D}(f)$. We then define

$$f(T) := \frac{1}{2\pi \mathrm{i}} \int_\Gamma f(\lambda) R(\lambda, T) \,\mathrm{d}\lambda \in \mathcal{B}(X). \tag{5.3}$$

This integral exists in the Banach space $\mathcal{B}(X)$ since $\lambda \mapsto f(\lambda)R(\lambda, T)$ is holomorphic on $\rho(T) \cap \mathrm{D}(f) \supseteq \Gamma$. Writing $R(\lambda, T)$ as "$\frac{1}{\lambda - T}$" one sees the similarity of (5.3) and (5.2), but here $R(\lambda, T)$ does not exist on $\sigma(T)$, whereas in (5.2) the function $w \mapsto \frac{1}{w-z}$ is defined on $\mathbb{C}\backslash\{z\}$.

If Γ' is another admissible curve for $\sigma(T)$ and $\mathrm{D}(f)$, then we set $\Gamma'' = \Gamma \cup (-\Gamma')$, where "$-$" denotes the inversion of the orientation. We then have

$$n(\Gamma'', z) = n(\Gamma, z) - n(\Gamma', z) = \begin{cases} 1 - 1 = 0, & z \in \sigma(T), \\ 0 - 0 = 0, & z \in \mathbb{C}\backslash \mathrm{D}(f). \end{cases}$$

So we can apply (5.1) on $U = \mathrm{D}(f)\backslash\sigma(T)$ obtaining

$$0 = \int_{\Gamma''} f(\lambda)R(\lambda, T) \,\mathrm{d}\lambda = \int_\Gamma f(\lambda)R(\lambda, T) \,\mathrm{d}\lambda - \int_{\Gamma'} f(\lambda)R(\lambda, T) \,\mathrm{d}\lambda.$$

Consequently, (5.3) does not depend on the choice of the admissible curve. We recall that $r_\lambda(z) = \frac{1}{\lambda - z}$ and $p_1(z) = z$ for $\lambda, z \in \mathbb{C}$ with $\lambda \neq z$.

THEOREM 5.1. *Let $T \in \mathcal{B}(X)$ for a Banach space X. Then the map*

$$\Phi_T : H(\sigma(T)) \to \mathcal{B}(X), \qquad f \mapsto f(T),$$

defined by (5.3) is linear and satisfies

(H1) $\|f(T)\| \leqslant c \sup_{\lambda \in \Gamma} |f(\lambda)|$ *for a constant* $c = c(\Gamma, T) > 0$,
(H2) $\mathbb{1}(T) = I$, $p_1(T) = T$, $r_\lambda(T) = R(\lambda, T)$,
(H3) $f(T)g(T) = g(T)f(T) = (fg)(T)$,
(H4) $f(T)^* = f(T^*)$,
(H5) if $f_n \to f$ *uniformly on compact subsets of* $\mathrm{D}(f)$*, then* $f_n(T) \to f(T)$ *in* $\mathcal{B}(X)$ *as* $n \to \infty$,

for all $\lambda \in \rho(T)$ and $f, g, f_n \in H(\sigma(T))$ with $\mathrm{D}(f_n) = \mathrm{D}(f)$ for every $n \in \mathbb{N}$.

Φ_T is the only linear map from $H(\sigma(T))$ to $\mathcal{B}(X)$ satisfying (H1)–(H3). For a polynomial p, the operators $p(T)$ in (5.3) and in (4.4) coincide. If X is a Hilbert space and $T = T'$, the above Φ_T is the restriction of the map Φ_T from Theorem 4.11.

PROOF. It is clear that $f \mapsto f(T)$ is linear. Property (H1) follows from

$$\|f(T)\| \leqslant \frac{1}{2\pi} \ell(\Gamma) \sup_{\lambda\in\Gamma} \|R(\lambda, T)\| \sup_{\lambda\in\Gamma} |f(\lambda)| =: c(\Gamma, T) \sup_{\lambda\in\Gamma} |f(\lambda)|.$$

Replacing here $f(T)$ by $f(T) - f_n(T) = (f - f_n)(T)$ we also deduce (H5). To check (H4), we recall that $\sigma(T) = \sigma(T^*)$ and $R(\lambda, T)^* = R(\lambda, T^*)$ from Theorem 1.24. Hence,

$$f(T)^* = \frac{1}{2\pi \mathrm{i}} \int_\Gamma f(\lambda) R(\lambda, T)^* \,\mathrm{d}\lambda = f(T^*).$$

We next show (H3). We choose a bounded open set $U \subseteq \mathbb{C}$ such that $\sigma(T) \subseteq U \subseteq \overline{U} \subseteq \mathrm{D}(f) \cap \mathrm{D}(g)$ and admissible curves Γ_f in $U\backslash\sigma(T)$ and

Γ_g in $(\mathrm{D}(f) \cap \mathrm{D}(g)) \backslash \overline{U}$. We then have $n(\Gamma_f, \mu) = 0$ for all $\mu \in \Gamma_g \subseteq \mathbb{C} \backslash U$ and $n(\Gamma_g, \lambda) = 1$ for all $\lambda \in \Gamma_f \subseteq \overline{U}$. Using the resolvent equation, Fubini's theorem in $\mathcal{B}(X)$ (see e.g. Theorem X.6.16 in [**AE**]) and (5.2) in $\mathbb{C}$, we compute

$$\begin{aligned} f(T)g(T) &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_f} f(\lambda) R(\lambda, T) \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_g} g(\mu) R(\mu, T) \,\mathrm{d}\mu \,\mathrm{d}\lambda \\ &= \left(\frac{1}{2\pi \mathrm{i}}\right)^2 \int_{\Gamma_f} \int_{\Gamma_g} f(\lambda) g(\mu) \frac{1}{\mu - \lambda} (R(\lambda, T) - R(\mu, T)) \,\mathrm{d}\mu \,\mathrm{d}\lambda \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_f} f(\lambda) R(\lambda, T) \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_g} \frac{g(\mu)}{\mu - \lambda} \,\mathrm{d}\mu \,\mathrm{d}\lambda \\ &\quad + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_g} g(\mu) R(\mu, T) \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_f} \frac{f(\lambda)}{\lambda - \mu} \,\mathrm{d}\lambda \,\mathrm{d}\mu \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_f} f(\lambda) g(\lambda) R(\lambda, T) \,\mathrm{d}\lambda = (fg)(T). \end{aligned}$$

This identity also yields $(fg)(T) = (gf)(T) = g(T)f(T)$.

To check (H2), we take $f = \mathbb{1}$ with $D(f) = \mathbb{C}$. We choose $\Gamma_0 = \partial B(0, 2\|T\|)$. Theorem 1.16 then leads to

$$\begin{aligned} \mathbb{1}(T) &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_0} R(\lambda, T) \,\mathrm{d}\lambda = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_0} \sum_{n=0}^{\infty} T^n \lambda^{-n-1} \,\mathrm{d}\lambda \\ &= \sum_{n=0}^{\infty} T^n \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_0} \lambda^{-n-1} \,\mathrm{d}\lambda = I, \end{aligned}$$

since the series converges in $\mathcal{B}(X)$ uniformly on Γ_0 and $\int_{\Gamma_0} \lambda^{-m} \,\mathrm{d}\lambda$ is equal to $2\pi \mathrm{i}$ if $m = 1$ and equal to 0 for $m \in \mathbb{Z} \backslash \{1\}$. The property $p_1(T) = T$ is shown similarly. For $\lambda \in \rho(T)$, consider $f_\lambda(z) = \lambda - z$. The previous results imply $f_\lambda(T) = \lambda I - T$ and $f_\lambda(T) r_\lambda(T) = r_\lambda(T) f_\lambda(T) = (r_\lambda f_\lambda)(T) = \mathbb{1}(T) = I$ so that $r_\lambda(T) = R(\lambda, T)$.

Let $\Psi : H(\sigma(T)) \to \mathcal{B}(X)$ be any linear map satisfying the assertions (H1)–(H3). The linearity, (H2) and (H3) imply that $\Psi(p) = p(T) = \Phi_T(p)$ for every polynomial p. If $f \in H(\sigma(T))$ and $\Gamma \subseteq D(f) \backslash \sigma(T)$ is admissible, then there are polynomials p_n converging uniformly to f on the compact set Γ. Hence, $p_n(T) \to \Psi(f)$ by (H1) and thus $\Psi = \Phi_T$. The final assertion is shown similarly. □

EXAMPLE 5.2. Let $E = C(K)$ for a compact set $K \subseteq \mathbb{R}^d$ and let $m \in C(K)$. We define $M\varphi = m\varphi$ for $\varphi \in E$. Proposition 1.14 shows that $M \in \mathcal{B}(E)$, $\sigma(M) = m(K)$, and $R(\lambda, M)\varphi = \frac{1}{\lambda - m}\varphi$ for all $\lambda \in \rho(M)$.

Let $f \in H(m(K))$, Γ be an admissible curve in $D(f) \backslash m(K)$, $\varphi \in E$, and $x \in K$. Using that the map $\psi \mapsto \psi(x)$ is continuous and linear from E to $\mathbb{C}$ and Cauchy's formula (5.2) for $\mu = m(x) \in m(K)$, we compute

$$\begin{aligned} (f(M)\varphi)(x) &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} f(\lambda) R(\lambda, M) \varphi \,\mathrm{d}\lambda (x) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} f(\lambda) (R(\lambda, M)\varphi)(x) \,\mathrm{d}\lambda \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{f(\lambda)}{\lambda - m(x)} \,\mathrm{d}\lambda \, \varphi(x) = f(m(x))\varphi(x). \end{aligned}$$

As a result, $f(M)\varphi = (f \circ m)\varphi$ is also a multiplication operator. $\Diamond$

THEOREM 5.3. *Let $T \in \mathcal{B}(X)$ and $f \in H(\sigma(T))$. We then have the spectral mapping theorem*

$$\sigma(f(T)) = f(\sigma(T)).$$

PROOF. Let $\mu \notin f(\sigma(T))$. After possibly shrinking $\mathrm{D}(f)$, the map $g = \frac{1}{\mu\mathbb{1}-f}$ belongs to $H(\sigma(T))$. The properties of the calculus yield

$$g(T)(\mu I - f(T)) = (\mu I - f(T))g(T) = (g(\mu\mathbb{1} - f))(T) = I,$$

so that $\mu \in \rho(f(T))$.

Conversely, let $\mu = f(\lambda)$ for some $\lambda \in \sigma(T)$. We set $g(z) = \frac{f(z)-\mu}{z-\lambda}$ for $z \in \mathrm{D}(f)\backslash\{\lambda\}$ and $g(\lambda) = f'(\lambda)$. Since g is bounded, it is holomorphic on $\mathrm{D}(f)$ by Riemann's theorem on removable singularities. Moreover, $g(z)(z-\lambda) = f(z) - \mu$ for all $z \in \mathrm{D}(f)$ and so

$$(\lambda I - T)g(T) = g(T)(\lambda I - T) = (g(\lambda\mathbb{1} - p_1))(T) = (\mu\mathbb{1} - f)(T) = \mu I - f(T).$$

Because the operator $(\lambda I - T)$ is not surjective or not injective, $\mu I - f(T)$ has the respective properties. Hence, μ is contained in $\sigma(f(T))$. $\square$

The next result can in particular be used to solve linear ordinary differential equations governed by matrices A.

EXAMPLE 5.4. Let $A \in \mathcal{B}(X)$. For $t \in \mathbb{R}$ we set $f_t(z) = \mathrm{e}^{tz}$, $z \in \mathbb{C}$, and define $\mathrm{e}^{tA} = f_t(A) \in \mathcal{B}(X)$. As in Example 4.19, one sees that

$$\mathrm{e}^{(t+s)A} = \mathrm{e}^{tA}\mathrm{e}^{sA} = \mathrm{e}^{sA}\mathrm{e}^{tA}, \quad \mathrm{e}^{0A} = I, \quad [\mathrm{e}^{tA}]^{-1} = \mathrm{e}^{-tA}$$

for all $t, s \in \mathbb{R}$. Moreover, $t \mapsto \mathrm{e}^{tA}$ belongs to $C^1(\mathbb{R}, \mathcal{B}(X))$ with

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tA} = A\mathrm{e}^{tA} = \mathrm{e}^{tA}A.$$

Hence, the map $u(t) = \mathrm{e}^{tA}u_0$ is the unique solution in $C^1(\mathbb{R}, X)$ of

$$\tfrac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t), \quad t \in \mathbb{R}, \qquad u(0) = u_0,$$

where $u_0 \in X$ is given. Theorem 5.3 further yields

$$r\big(\mathrm{e}^{tA}\big) = \max\big\{|\mu| \,\big|\, \mu \in \sigma(\mathrm{e}^{tA}) = \mathrm{e}^{t\sigma(A)}\big\} = \max\big\{\mathrm{e}^{t\,\mathrm{Re}\,\lambda} \,\big|\, \lambda \in \sigma(A)\big\} = \mathrm{e}^{ts(A)}$$

for the *spectral bound* $s(A) := \max\{\mathrm{Re}\,\lambda \,|\, \lambda \in \sigma(A)\}$. Therefore, if $s(A) < 0$ (i.e., $\sigma(A) \subseteq \{\lambda \,|\, \mathrm{Re}\,\lambda < 0\}$), then we deduce from Theorem 1.16 that

$$1 > r(e^A) = \lim_{n\to\infty} \|(e^A)^n\|^{1/n} = \lim_{n\to\infty} \|\mathrm{e}^{nA}\|^{1/n}.$$

So we can fix an index $N \in \mathbb{N}$ with $\|\mathrm{e}^{NA}\| =: q < 1$. Writing any given $t \geqslant 0$ as $t = kN + \tau$ for some $k \in \mathbb{N}_0$ and $0 < \tau \leqslant N$ we estimate

$$\|\mathrm{e}^{tA}\| = \|\big(\mathrm{e}^{NA}\big)^k\mathrm{e}^{\tau A}\| \leqslant q^k\,\|\mathrm{e}^{\tau A}\| \leqslant \max_{0\leqslant\tau\leqslant N}\|\mathrm{e}^{\tau A}\|\exp\Big(Nk\frac{\ln q}{N}\Big) \leqslant M\mathrm{e}^{-wt}$$

where $w := -\frac{\ln q}{N} > 0$ and $M := \max_{0\leqslant\tau\leqslant N}\|\mathrm{e}^{\tau A}\|\,\mathrm{e}^{|\ln q|}$. So spectral information on the given operator A implies the exponential decay

$$\|u(t)\| \leqslant M\mathrm{e}^{-wt}\|x\|, \qquad t \geqslant 0,$$

of the solutions u. $\Diamond$

Let $S \in \mathcal{B}(X)$ and $P = P^2 \in \mathcal{B}(X)$ be a projection with $SP = PS$. Set $X_1 = \mathrm{R}(P)$ and $X_2 = \mathrm{N}(P)$. Then, $X = X_1 \oplus X_2$ by e.g. Lemma 2.16 of [**Sc2**]. Moreover, if $y = Px \in \mathrm{R}(P)$, then $Sy = SPx = PSx \in \mathrm{R}(P)$. If $x \in \mathrm{N}(P)$, then $PSx = SPx = 0$ and so $Sx \in \mathrm{N}(P)$. As a result, S leaves invariant X_1 and X_2 and the restrictions $S_{|X_j} \in \mathcal{B}(X_j)$ are well defined.

Theorem 5.5 (Spectral projection). *Let $T \in \mathcal{B}(X)$ and $\sigma(T) = \sigma_1 \dot\cup \sigma_2$ for two disjoint closed sets $\sigma_j \neq \emptyset$ in $\mathbb{C}$. Then there is a projection $P \in \mathcal{B}(X)$ with $f(T)P = Pf(T)$ for all $f \in H(\sigma(T))$ such that $\sigma(T_j) = \sigma_j$ for $j \in \{1, 2\}$, where $T_j = T_{|X_j} \in \mathcal{B}(X_j)$, $X_1 = \mathrm{R}(P)$ and $X_2 = \mathrm{N}(P)$. Moreover, $X = X_1 \oplus X_2$ and $R(\lambda, T_j) = R(\lambda, T)_{|X_j}$ for $\lambda \in \rho(T) = \rho(T_1) \cap \rho(T_2)$. We further have*

$$P = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} R(\lambda, T)\, \mathrm{d}\lambda, \tag{5.4}$$

where Γ_1 is an admissible curve for σ_1 and any open set $U_1 \supseteq \sigma_1$ such that $\overline{U_1} \cap \sigma_2 = \emptyset$.

Proof. There are open sets U_j with $\overline{U}_1 \cap \overline{U}_2 = \emptyset$ and $\sigma_j \subseteq U_j$ for $j \in \{1, 2\}$. Define $h \in H(\sigma(T))$ by $h = 1$ on U_1 and $h = 0$ on U_2. We set $P = h(T) \in \mathcal{B}(X)$. We then deduce $P^2 = h^2(T) = h(T) = P$ and $f(T)P = Pf(T)$ for all $f \in H(\sigma(T))$ from (H3) and (H2). The above remarks show that $X = X_1 \oplus X_2$ and that the operators $T_j = T_{|X_j} \in \mathcal{B}(X_j)$ are well defined.

The formula (5.4) follows by choosing $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_j are admissible curves for σ_j and U_j for $j \in \{1, 2\}$. Let $\lambda \notin \sigma_1$. We may shrink U_1 so that $\lambda \notin U_1$ since P does not depend on the choice of Γ and thus not on the choice of U_1. We define $g(z) = \frac{1}{\lambda - z}$ for $z \in U_1$ and $g(z) = 0$ for $z \in U_2$. Then $g \in H(\sigma(T))$ and

$$g(T)(\lambda I - T) = (\lambda I - T)g(T) = ((\lambda \mathbb{1} - p_1)g)(T) = h(T) = P.$$

Setting $R = g(T)_{|X_1} \in \mathcal{B}(X_1)$, we thus obtain

$$R(\lambda I_{X_1} - T_1) = (\lambda I_{X_1} - T_1)R = I_{X_1}.$$

This means that $\lambda \in \rho(T_1)$, and so $\sigma(T_1) \subseteq \sigma_1$. Similarly, one sees that $\sigma(T_2) \subseteq \sigma_2$. In particular, $\sigma(T_1)$ and $\sigma(T_2)$ are disjoint. Let $\lambda \in \rho(T_1) \cap \rho(T_2)$. Given $x \in X$, we have unique $x_1 \in X_1$ and $x_2 \in X_2$ such that $x = x_1 + x_2$. If $\lambda x - Tx = 0$, then $0 = \lambda x_1 - T_1 x_1 + \lambda x_2 - T_2 x_2 \in X_1 \oplus X_2$ so that x_j belongs to $\mathrm{N}(\lambda I - T_j) = \{0\}$ for $j \in \{1, 2\}$; i.e., $x = 0$. Given $y \in X$, we define $x_j = R(\lambda, T_j) y_j \in X_j$ for $j \in \{1, 2\}$. Setting $x = x_1 + x_2$, we derive

$$\lambda x - Tx = \lambda x_1 - T_1 x_1 + \lambda x_2 - T_2 x_2 = y_1 + y_2 = y.$$

We have proved that $\lambda \in \rho(T)$, $R(\lambda, T)y = R(\lambda, T_1)y_1 + R(\lambda, T_2)y_2$, $R(\lambda, T)_{|X_j} = R(\lambda, T_j)$, and

$$\sigma(T) = \sigma_1 \dot\cup \sigma_2 \subseteq \sigma(T_1) \dot\cup \sigma(T_2) \subseteq \sigma_1 \dot\cup \sigma_2,$$

which implies that $\sigma(T_j) = \sigma_j$ for $j \in \{1, 2\}$. □

We use the above concept to refine the results in Example 5.4 about the long-term behavior of e^{tA}.

Example 5.6 (Exponential dichotomy). In the setting of Example 5.4, assume that $\sigma(A) \cap \mathrm{i}\mathbb{R} = \varnothing$. Hence, $\sigma(A) = \sigma_1 \dot{\cup} \sigma_2$ where $\sigma_1 \subseteq \{\lambda \in \mathbb{C} \,\big|\, \operatorname{Re}\lambda < 0\}$ and $\sigma_2 \subseteq \{\lambda \in \mathbb{C} \,\big|\, \operatorname{Re}\lambda > 0\}$. Let P be the spectral projection of A for σ_1 and define A_1 and A_2 as the restrictions of A to $X_1 = \mathrm{R}(P)$ and $X_2 = \mathrm{N}(P)$, respectively, as in Theorem 5.5. Then $(\mathrm{e}^{tA})_{|X_j} = \mathrm{e}^{tA_j} : X_j \to X_j$ and there are constants $\delta, N > 0$ such that

$$\|\mathrm{e}^{tA_1}\| \leqslant N\mathrm{e}^{-\delta t} \quad \text{and} \quad \|\mathrm{e}^{-tA_2}\| \leqslant N\mathrm{e}^{-\delta t}, \qquad t \geqslant 0.$$

In other words, X can be decomposed into e^{tA}-invariant subspaces on which e^{tA} decays exponentially in forward and in backward time, respectively.

Proof. Let Γ_1 be given as in Theorem 5.5. For $x \in X_1$ we compute

$$\begin{aligned} \mathrm{e}^{tA}x = \mathrm{e}^{tA}Px = (f_t h)(A)x &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} \mathrm{e}^{t\lambda} R(\lambda, A)x \,\mathrm{d}\lambda \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} \mathrm{e}^{t\lambda} R(\lambda, A_1)x \,\mathrm{d}\lambda = \mathrm{e}^{tA_1}x, \end{aligned}$$

where $f_t(\lambda) = \mathrm{e}^{t\lambda}$ for $t \in \mathbb{R}$ and h is given by the proof of Theorem 5.5. In the same way one derives $\mathrm{e}^{tA}x = \mathrm{e}^{tA_2}x$ for all $x \in X_2$ and $t \geqslant 0$. Since $\sigma(A_1) = \sigma_1$, we have $s(A_1) < 0$ and so Example 5.4 shows that $\|\mathrm{e}^{tA}x_1\| \leqslant M\mathrm{e}^{-\omega t}\|x_1\|$ for all $t \geqslant 0$ and $x_1 \in X_1$ and some constants $M, \omega > 0$.

Moreover, $\sigma(A_2) = \sigma_2$ and so $s(-A_2) < 0$. Note that the curve $\tilde{\Gamma} = \{-\lambda \,\big|\, \lambda \in \Gamma\}$ is admissible for $\sigma(-A) = -\sigma(A)$. Substituting $\mu = -\lambda$, we conclude that

$$\mathrm{e}^{-tA} = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \mathrm{e}^{-t\lambda}(\lambda I - A)^{-1} \,\mathrm{d}\lambda = \frac{1}{2\pi \mathrm{i}} \int_{\tilde{\Gamma}} \mathrm{e}^{t\mu}(\mu I - (-A))^{-1} \,\mathrm{d}\mu = \mathrm{e}^{t(-A)}$$

for all $t \in \mathbb{R}$. For $x_2 \in X_2$ we thus obtain

$$\mathrm{e}^{-tA_2}x_2 = \mathrm{e}^{-tA}x_2 = \mathrm{e}^{t(-A)}x_2 = \mathrm{e}^{t(-A_2)}x_2$$

so that $\|\mathrm{e}^{-tA_2}x_2\| \leqslant M'\mathrm{e}^{-\omega' t}\|x_2\|$ for all $t \geqslant 0$ and some $M', \omega' > 0$. $\square$

5.2. Sectorial operators

We extend the above results to certain unbounded operators A, restricting ourselves to the exponential e^{tA}.[1] For $\phi \in (0, \pi]$ we define the open *sector*

$$\Sigma_\phi = \{\lambda \in \mathbb{C}\backslash\{0\} \,\big|\, |\arg \lambda| < \phi\}.$$

We also set $\Sigma_{\pi/2} =: \mathbb{C}_+$ and $\mathbb{C}_- = -\mathbb{C}_+$. Note that $\Sigma_\pi = \mathbb{C}\backslash\mathbb{R}_-$.

Definition 5.7. *A closed operator A is called* sectorial of angle $\phi \in (0, \pi]$ *if there is a constant $K > 0$ such that $\Sigma_\phi \subseteq \rho(A)$ and*

$$\|R(\lambda, A)\| \leqslant \frac{K}{|\lambda|}, \qquad \lambda \in \Sigma_\phi.$$

In the literature several small variations of the above definition are used. Note that a sectorial operator of angle ϕ is also sectorial of angle $\phi' \in (0, \phi)$. In applications often arise operators A such that $A - \omega I$ is sectorial for some $\omega \in \mathbb{R}$, cf. Remark 5.13.

[1] See [**KW**] for a detailed study of a corresponding functional calculus.

EXAMPLE 5.8. Let X be a Hilbert space and A be closed, densely defined and selfadjoint on X. We further suppose that $\sigma(A) \subseteq \mathbb{R}_-$. Then A is sectorial of any angle $\phi \in (\frac{\pi}{2}, \pi)$.

PROOF. Let $\phi \in (\frac{\pi}{2}, \pi)$ and $\lambda \in \Sigma_\phi$. Since $R(\lambda, A)' = R(\overline{\lambda}, A)$, the operator $R(\lambda, A)$ is normal. Propositions 4.3 and 1.20 then yield

$$\|R(\lambda, A)\| = r(R(\lambda, A)) = d(\lambda, \sigma(A))^{-1} \leqslant d(\lambda, \mathbb{R}_-)^{-1} = \begin{cases} \frac{1}{|\lambda|}, & \operatorname{Re}\lambda \geqslant 0, \\ \frac{1}{|\operatorname{Im}\lambda|}, & \operatorname{Re}\lambda < 0. \end{cases}$$

If $\operatorname{Re}\lambda < 0$, we can write $\lambda = |\lambda|\mathrm{e}^{\pm i\theta}$ for some $\theta \in (\frac{\pi}{2}, \phi)$. We then have $\frac{|\operatorname{Im}\lambda|}{|\lambda|} = |\sin\theta| \geqslant \sin\phi > 0$, and thus

$$\|R(\lambda, A)\| \leqslant \frac{\frac{1}{\sin\phi}}{|\lambda|} =: \frac{K_\phi}{|\lambda|}, \qquad \lambda \in \Sigma_\phi. \qquad \square$$

Note that $K_\phi \to \infty$ as $\phi \to \pi$ in the above example.

EXAMPLE 5.9. Let $X = C([0,1])$ and $Au = u''$ with $D(A) = \{u \in C^2([0,1]) \,|\, u(0) = u(1) = 0\}$. Then A is sectorial of any angle $\phi < \pi$.

PROOF. We first recall from Example 3.40 that $\sigma(A) = \sigma_p(A) = \{-\pi^2 k^2 \,|\, k \in \mathbb{N}\}$. Let $\lambda \in \Sigma_\pi$ and $\lambda = \mu^2$ for some $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu > 0$. Let $f \in X$. There exists a unique function $u \in \mathrm{D}(A)$ and $\lambda u - Au = f$; i.e.,

$$u \in C^2([0,1]), \qquad u'' = \mu^2 u - f \quad \text{on} \quad [0,1], \qquad u(0) = u(1) = 0.$$

The solution of this ordinary boundary value problem is given by

$$\begin{aligned} u(t) &= a\mathrm{e}^{\mu t} + b\mathrm{e}^{-\mu t} + \frac{1}{2\mu}\int_0^1 \mathrm{e}^{-\mu|t-s|} f(s)\,\mathrm{d}s, \qquad 0 \leqslant t \leqslant 1, \\ u(0) &= a + b + \frac{1}{2\mu}\int_0^1 \mathrm{e}^{-\mu s} f(s)\,\mathrm{d}s = 0, \\ u(1) &= a\mathrm{e}^{\mu} + b\mathrm{e}^{-\mu} + \frac{\mathrm{e}^{-\mu}}{2\mu}\int_0^1 \mathrm{e}^{\mu s} f(s)\,\mathrm{d}s = 0, \end{aligned}$$

where the complex numbers $a = a(f, \mu)$ and $b = b(f, \mu)$ still have to be determined. The two boundary conditions above are equivalent to

$$\begin{aligned} \begin{pmatrix} a(f,\mu) \\ b(f,\mu) \end{pmatrix} &= \frac{1}{\mathrm{e}^{-\mu} - \mathrm{e}^{\mu}} \begin{pmatrix} \mathrm{e}^{-\mu} & -1 \\ -e^{\mu} & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2\mu}\int_0^1 \mathrm{e}^{-\mu s} f(s)\,\mathrm{d}s \\ -\frac{\mathrm{e}^{-\mu}}{2\mu}\int_0^1 \mathrm{e}^{\mu s} f(s)\,\mathrm{d}s \end{pmatrix} \\ &= \frac{1}{2\mu(\mathrm{e}^{-\mu} - \mathrm{e}^{\mu})} \begin{pmatrix} \mathrm{e}^{-\mu}\int_0^1 (\mathrm{e}^{\mu s} - \mathrm{e}^{-\mu s}) f(s)\,\mathrm{d}s \\ \int_0^1 (\mathrm{e}^{\mu}\mathrm{e}^{-\mu s} - \mathrm{e}^{-\mu}\mathrm{e}^{\mu s}) f(s)\,\mathrm{d}s \end{pmatrix}. \end{aligned}$$

We thus obtain $\lambda \in \rho(A)$ and

$$R(\mu^2, A)f(t) = a(f,\mu)\mathrm{e}^{\mu t} + b(f,\mu)\mathrm{e}^{-\mu t} + \frac{1}{2\mu}\int_0^1 \mathrm{e}^{-\mu|t-s|} f(s)\,\mathrm{d}s$$

for all $\mu^2 = \lambda \in \mathbb{C}\backslash\mathbb{R}_-$, $\operatorname{Re}\mu > 0$, $f \in X$ and $t \in [0,1]$.

Fix $\phi \in (\frac{\pi}{2}, \pi)$. Take $\lambda \in \Sigma_\phi$ and thus $\mu \in \Sigma_{\frac{\phi}{2}}$. Hence $\mu = |\mu|\,\mathrm{e}^{\mathrm{i}\theta}$ with $0 \leqslant |\theta| < \frac{\phi}{2}$ and $\operatorname{Re}\mu = |\mu|\cos\theta \geqslant |\mu|\cos\frac{\phi}{2}$. So we can estimate

$$\|R(\lambda, A)f\|_\infty \leqslant |a(f,\mu)|\,\mathrm{e}^{\operatorname{Re}\mu} + |b(f,\mu)| + \frac{\|f\|_\infty}{2|\mu|}\sup_{t\in[0,1]}\int_{t-1}^t \mathrm{e}^{-\operatorname{Re}\mu|r|}\,\mathrm{d}r$$

$$\begin{aligned}
&\leqslant \frac{\|f\|_\infty}{2|\mu|\,|e^{\mu}-\mathrm{e}^{-\mu}|}\Big(\int_0^1 \big(\mathrm{e}^{\operatorname{Re}\mu s}+\mathrm{e}^{-\operatorname{Re}\mu s}\big)\,\mathrm{d}s \\
&\quad + \int_0^1 (\mathrm{e}^{\operatorname{Re}\mu}\mathrm{e}^{-\operatorname{Re}\mu s}+\mathrm{e}^{-\operatorname{Re}\mu}\mathrm{e}^{\operatorname{Re}\mu s})\,\mathrm{d}s\Big) + \frac{\|f\|_\infty}{|\mu|\operatorname{Re}\mu} \\
&= \frac{\|f\|_\infty}{2\operatorname{Re}\mu|\mu||e^{\mu}-\mathrm{e}^{-\mu}|}\big((\mathrm{e}^{\operatorname{Re}\mu}-1+1-\mathrm{e}^{-\operatorname{Re}\mu}) \\
&\quad + \mathrm{e}^{\operatorname{Re}\mu}(1-\mathrm{e}^{-\operatorname{Re}\mu})+\mathrm{e}^{-\operatorname{Re}\mu}(\mathrm{e}^{\operatorname{Re}\mu}-1)\big) + \frac{\|f\|_\infty}{|\mu|\operatorname{Re}\mu} \\
&\leqslant \frac{\frac{1}{\cos(\phi/2)}}{|\mu|^2}\|f\|_\infty\Big(\frac{(\mathrm{e}^{\operatorname{Re}\mu}-\mathrm{e}^{-\operatorname{Re}\mu})+(\mathrm{e}^{\operatorname{Re}\mu}-\mathrm{e}^{-\operatorname{Re}\mu})}{2(\mathrm{e}^{\operatorname{Re}\mu}-\mathrm{e}^{-\operatorname{Re}\mu})}+1\Big) \\
&= \frac{\frac{2}{\cos(\phi/2)}}{|\lambda|}\|f\|_\infty. \qquad \square
\end{aligned}$$

Note that $\overline{D(A)} = \{u \in X \,|\, u(0) = u(1) = 0\}$ is not equal to X for the above 'Dirichlet-Laplacian', cf. Example 1.19 in [**Sc2**].

Similarly one can show that the 'Neumann-Laplacian' $A_1 u = u''$ with

$$D(A_1) = \{u \in C^2([0,1]) \,|\, u'(0) = u'(1) = 0\}$$

is sectorial for every angle $\phi < \pi$ with on $X = C([0,1])$. Moreover, its spectrum is given by $\sigma(A_1) = \sigma_p(A_1) = \{-\pi^2 k^2 \,|\, k \in \mathbb{N}_0\}$ with eigenfunctions $u_k(t) = \cos(k\pi t)$. (See exercises.) Here $D(A_1)$ is dense in X.

Example 5.10. Let $X = L^p(\mathbb{R})$, $1 \leqslant p < \infty$, and $Au = u'$ for $\mathrm{D}(A) = W^{1,p}(\mathbb{R})$. Then A is sectorial of any angle $\phi \in (0, \frac{\pi}{2})$.

Proof. Example 3.37 says that $\sigma(A) = \mathrm{i}\mathbb{R}$ and $\|R(\lambda, A)\| \leqslant \frac{1}{\operatorname{Re}\lambda}$ for $\operatorname{Re}\lambda > 0$. If $\phi \in (0, \frac{\pi}{2})$ and $\lambda \in \Sigma_\phi$, we have $|\operatorname{Re}\lambda| \geqslant |\lambda|\cos\phi$ and hence

$$\|R(\lambda, A)\| \leqslant \frac{1/\cos\phi}{|\lambda|}.$$

Because of its spectrum, A is not sectorial of angle $\phi = \frac{\pi}{2}$. $\square$

Let $1 < p < \infty$, $U \subseteq \mathbb{R}^d$ be open and bounded with ∂U of class C^2, and $X = L^p(\mathbb{R}^d)$. The operators

$$\begin{aligned}
A_0 &= \Delta, \quad \mathrm{D}(A_0) = W^{2,p}(U) \cap W_0^{1,p}(U), \\
A_1 &= \Delta, \quad \mathrm{D}(A_1) = \{u \in W^{2,p}(U) \,|\, \partial_\nu u = \sum_{j=1}^d \nu_j \operatorname{tr} \partial_j u = 0 \quad \text{on } \partial U\},
\end{aligned}$$

are sectorial on X with angle $\phi > \pi/2$. Here $\operatorname{tr} : W^{1,p}(U) \to L^p(\partial U)$ is the trace operator and ν is the outer unit normal. There are variants for the spaces $X = L^1(U)$ and $X = C(\overline{U})$ as well as for more general differential operators and boundary conditions. See e.g. Chapter 3 in [**Lu**] and also [**Ta**]. To obtain these results, one has to rely on methods from partial differential equations. One can however define the operators A_j in a more abstract way (in the 'weak sense') and show their sectoriality by purely functional analytic methods, cf. [**Ou**]. In this case, however, the domains are not known explicitly.

Let A be sectorial of angle $\phi \in (\frac{\pi}{2}, \pi)$ with constant K. Take any $r > 0$ and $\theta \in (\frac{\pi}{2}, \phi)$. We define

$$\begin{aligned}
\Gamma_1 &= \{\lambda = \gamma_1(s) = (-s)\mathrm{e}^{-\mathrm{i}\theta} \,\big|\, -\infty < s \leqslant -r\}, \\
\Gamma_2 &= \{\lambda = \gamma_2(\alpha) = r\mathrm{e}^{\mathrm{i}\alpha} \,\big|\, -\theta \leqslant \alpha \leqslant \theta\}, \\
\Gamma_3 &= \{\lambda = \gamma_3(s) = s\mathrm{e}^{\mathrm{i}\theta} \,\big|\, r \leqslant s < \infty\}, \\
\Gamma &= \Gamma(r, \theta) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3.
\end{aligned}$$

For $t > 0$, we introduce the operator

$$\mathrm{e}^{tA} = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \mathrm{e}^{t\lambda} R(\lambda, A)\,\mathrm{d}\lambda = \lim_{R\to\infty} \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_R} \mathrm{e}^{t\lambda} R(\lambda, A)\,\mathrm{d}\lambda, \tag{5.5}$$

where $\Gamma_R = \Gamma \cap \overline{B}(0, R)$ for $R > r$. We first have to show that the limit in (5.5) exists in $\mathcal{B}(X)$.

LEMMA 5.11. *Under the above assumptions, the integral in* (5.5) *converges absolutely in* $\mathcal{B}(X)$ *and gives an operator* $\mathrm{e}^{tA} \in \mathcal{B}(X)$ *which does not depend on the choice of* $r > 0$ *and* $\theta \in (\frac{\pi}{2}, \phi)$. *Moreover,* $\|\mathrm{e}^{tA}\| \leqslant M$ *for all* $t > 0$ *and a constant* $M = M(K, \theta) > 0$.

PROOF. Since $\|R(\lambda, A)\| \leqslant \frac{K}{|\lambda|}$ on Γ, we can estimate

$$\begin{aligned}
\left| \int_{\Gamma_R} \|\mathrm{e}^{t\lambda} R(\lambda, A)\|\,\mathrm{d}\lambda \right| &\leqslant K \int_r^R \frac{\exp(ts \operatorname{Re} \mathrm{e}^{-\mathrm{i}\theta})}{|s\mathrm{e}^{-\mathrm{i}\theta}|} |\mathrm{e}^{-\mathrm{i}\theta}|\,\mathrm{d}s \\
&\quad + K \int_{-\theta}^{\theta} \frac{\exp(tr \operatorname{Re} \mathrm{e}^{\mathrm{i}\alpha})}{|r\mathrm{e}^{\mathrm{i}\alpha}|} |\mathrm{i}r\mathrm{e}^{\mathrm{i}\alpha}|\,\mathrm{d}\alpha \\
&\quad + K \int_r^R \frac{\exp(ts \operatorname{Re} \mathrm{e}^{\mathrm{i}\theta})}{|s\mathrm{e}^{\mathrm{i}\theta}|} |\mathrm{e}^{\mathrm{i}\theta}|\,\mathrm{d}s \\
&\leqslant K \left(2\int_r^\infty \frac{\mathrm{e}^{ts\cos\theta}}{s}\,\mathrm{d}s + \int_{-\theta}^{\theta} \mathrm{e}^{tr\cos\alpha}\,\mathrm{d}\alpha \right) \\
&\leqslant K \left(2\int_{rt|\cos\theta|}^\infty \frac{\mathrm{e}^{-\sigma}}{\sigma} (-t\cos\theta) \frac{\mathrm{d}\sigma}{-t\cos\theta} + 2\theta\mathrm{e}^{tr} \right) \\
&=: Kc(r, t, \theta),
\end{aligned}$$

for all $R, t > 0$, where we substituted $\sigma = -st\cos\theta$. Thus the limit in (5.5) exists absolutely in $\mathcal{B}(X)$ by the majorant criterium, and $\|\mathrm{e}^{tA}\| \leqslant Kc(r, t, \theta)$. If we take $r = 1/t$, then $c(1/t, t, \theta) =: c(\theta)$ does not depend on $t > 0$.

So it remains to check that the integral in (5.5) is independent of $r > 0$ and $\theta \in (\frac{\pi}{2}, \phi)$. To this aim, we define $\Gamma' = \Gamma(r', \theta')$ for some $r' > 0$ and $\theta' \in (\frac{\pi}{2}, \phi)$, where we may assume that $\theta' \geqslant \theta$. We further set $\Gamma'_R = \Gamma' \cap \overline{B}(0, R)$ and choose $R > r, r'$. Let C_R^+ and C_R^- be the circle arcs from the endpoint of Γ_R to that of Γ'_R in $\{\operatorname{Im} \lambda > 0\}$ and $\{\operatorname{Im} \lambda < 0\}$, respectively. (If $\theta = \theta'$, then $C_R^\pm$ contain just one point.) Then $S_R = \Gamma_R \cup C_R^+ \cup (-\Gamma'_R) \cup (-C_R^-)$ is a closed curve in the starshaped domain Σ_ϕ. So (5.1) shows that

$$\int_{S_R} \mathrm{e}^{t\lambda} R(\lambda, A)\,\mathrm{d}\lambda = 0.$$

We further estimate

$$\Big\| \int_{C_R^+} \mathrm{e}^{t\lambda} R(\lambda, A)\,\mathrm{d}\lambda \Big\| \leqslant \int_\theta^{\theta'} \mathrm{e}^{tR\,\mathrm{Re}\,\mathrm{e}^{\mathrm{i}\alpha}} \frac{K}{|R\mathrm{e}^{\mathrm{i}\alpha}|} |\mathrm{i}R\mathrm{e}^{\mathrm{i}\alpha}|\,\mathrm{d}\alpha \leqslant K(\theta'-\theta)\mathrm{e}^{tR\cos\theta} \;\to 0,$$

as $R \to \infty$, and analogously for C_R^-. So we conclude that

$$\begin{aligned}\int_\Gamma \mathrm{e}^{t\lambda} R(\lambda, A)\,\mathrm{d}\lambda &= \lim_{R\to\infty} \int_{\Gamma_R} \mathrm{e}^{t\lambda} R(\lambda, A)\,\mathrm{d}\lambda = \lim_{R\to\infty} \int_{\Gamma'_R} \mathrm{e}^{t\lambda} R(\lambda, A)\,\mathrm{d}\lambda \\ &= \int_{\Gamma'} \mathrm{e}^{t\lambda} R(\lambda, A)\,\mathrm{d}\lambda. \end{aligned}$$ □

We next establish some of the fundamental properties of the operators e^{tA}. In view of these results one calls $(\mathrm{e}^{tA})_{t\geqslant 0}$ the 'holomorphic semigroup generated by A'. Actually, the theorem admits a converse. We refer to Section 2.1 of [**Lu**] for this and other facts.

Theorem 5.12. *Let A be sectorial of angle $\phi > \frac{\pi}{2}$. Define e^{tA} as in (5.5) for $t > 0$, and set $\mathrm{e}^{0A} = I$. Then the following assertions hold.*

(a) $\mathrm{e}^{tA}\mathrm{e}^{sA} = \mathrm{e}^{sA}\mathrm{e}^{tA} = \mathrm{e}^{(t+s)A}$ for all $t, s \geqslant 0$.

(b) The map $t \mapsto \mathrm{e}^{tA}$ belongs to $C^1((0,\infty), \mathcal{B}(X))$. Moreover, $\mathrm{e}^{tA}X \subseteq \mathrm{D}(A)$, $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tA} = A\mathrm{e}^{tA}$ and $\|A\mathrm{e}^{tA}\| \leqslant \frac{C}{t}$ for a constant $C > 0$ and all $t > 0$. We also have $A\mathrm{e}^{tA}x = \mathrm{e}^{tA}Ax$ for all $x \in \mathrm{D}(A)$ and $t \geqslant 0$.

(c) Let $x \in X$. Then $\mathrm{e}^{tA}x$ converges as $t \to 0$ in X if and only if $x \in \overline{\mathrm{D}(A)}$. In this case, $\mathrm{e}^{tA}x$ tends to x as $t \to 0$.

Proof. (a) Let $t, s > 0$. Take $0 < r < r'$ and $\frac{\pi}{2} < \theta' < \theta < \phi$. Set $\Gamma = \Gamma(r, \theta)$ and $\Gamma' = \Gamma(r', \theta')$. Using the resolvent equation and Fubini's theorem, we compute

$$\begin{aligned}\mathrm{e}^{tA}\mathrm{e}^{sA} &= \frac{1}{(2\pi\mathrm{i})^2} \int_\Gamma \mathrm{e}^{t\lambda} \int_{\Gamma'} \mathrm{e}^{s\mu} R(\lambda, A)R(\mu, A)\}dd\mu\,\mathrm{d}\lambda \\ &= \frac{1}{2\pi\mathrm{i}} \int_\Gamma \mathrm{e}^{t\lambda} R(\lambda, A) \frac{1}{2\pi\mathrm{i}} \int_{\Gamma'} \frac{\mathrm{e}^{s\mu}}{\mu-\lambda}\,\mathrm{d}\mu\,\mathrm{d}\lambda \\ &\quad + \frac{1}{2\pi\mathrm{i}} \int_{\Gamma'} \mathrm{e}^{s\mu} R(\mu, A) \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{\mathrm{e}^{t\lambda}}{\lambda-\mu}\,\mathrm{d}\lambda\,\mathrm{d}\mu. \end{aligned}$$

Fix $\lambda \in \Gamma$ and take $R > \max\{r, r', |\lambda|\}$. We define $C'_R = \{z = R\mathrm{e}^{\mathrm{i}\alpha} \,|\, \theta' \leqslant \alpha \leqslant 2\pi - \theta'\}$ and $S'_R = \Gamma'_R \cup C'_R$. Since $n(S'_R, \lambda) = 1$, Cauchy's formula (5.2) yields

$$\frac{1}{2\pi\mathrm{i}} \int_{S'_R} \frac{\mathrm{e}^{s\mu}}{\mu-\lambda}\,\mathrm{d}\mu = \mathrm{e}^{s\lambda}.$$

As in Lemma 5.11, we further compute

$$\int_{\Gamma'_R} \frac{\mathrm{e}^{s\mu}}{\mu-\lambda}\,\mathrm{d}\mu \;\longrightarrow\; \int_{\Gamma'} \frac{\mathrm{e}^{s\mu}}{\mu-\lambda}\,\mathrm{d}\mu \qquad \text{and}$$

$$\Big| \int_{C'_R} \frac{\mathrm{e}^{s\mu}}{\mu-\lambda}\,\mathrm{d}\mu \Big| \leqslant 2\pi R \sup_{\mu\in C'_R} \frac{\mathrm{e}^{s\,\mathrm{Re}\,\mu}}{|\mu-\lambda|} \leqslant \mathrm{e}^{sR\cos\theta'} \frac{2\pi R}{R-|\lambda|} \;\longrightarrow\; 0$$

as $R \to \infty$. Consequently,

$$\mathrm{e}^{s\lambda} = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma'} \frac{\mathrm{e}^{s\mu}}{\mu-\lambda}\,\mathrm{d}\mu.$$

Closing Γ_R with the circle arc $C_R = \{z = Re^{i\alpha} \,|\, \theta \leqslant \alpha \leqslant 2\pi - \theta\}$ for sufficiently large $R > r$, one verifies in the same way that

$$0 = \int_\Gamma \frac{e^{\lambda t}}{\lambda - \mu} \, d\lambda.$$

We thus conclude that

$$e^{tA} e^{sA} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} e^{s\lambda} R(\lambda, A) \, d\lambda = e^{(t+s)A} = e^{sA} e^{tA}.$$

(b) Let $x \in X$, $t > 0$, $\varepsilon > 0$, and $R > r$. Observe that the Riemann sums for $\int_{\Gamma_R} e^{t\lambda} R(\lambda, A) \, d\lambda$ converge in $[D(A)]$ since $\lambda \mapsto R(\lambda, A)$ is continuous in $\mathcal{B}(X, [D(A)])$. We thus obtain

$$\begin{aligned} A \int_{\Gamma_R} e^{\lambda t} R(\lambda, A) \, d\lambda &= \int_{\Gamma_R} e^{t\lambda} A R(\lambda, A) \, d\lambda \\ &= \int_{\Gamma_R} e^{\lambda t} \lambda R(\lambda, A) \, d\lambda - \int_{\Gamma_R} e^{t\lambda} \, d\lambda \, I. \end{aligned} \tag{5.6}$$

Take again $C_R = \{\mu = Re^{i\alpha} \,|\, \theta \leqslant \alpha \leqslant 2\pi - \theta\}$. Using (5.1), one shows as in part (a) that

$$\left| \int_{\Gamma_R} e^{t\lambda} \, d\lambda \right| = \left| - \int_{C_R} e^{t\lambda} \, d\lambda \right| \leqslant 2\pi R \sup_{\theta \leqslant \alpha \leqslant 2\pi - \theta} e^{tR\cos\alpha} \leqslant 2\pi R \, e^{\varepsilon R \cos\theta} \longrightarrow 0$$

as $R \to \infty$, uniformly for $t \geqslant \varepsilon$. Moreover, as in the proof of Lemma 5.11 (with $r = 1/t$) we estimate

$$\begin{aligned} \left| \int_{\Gamma_R} \| \lambda e^{t\lambda} R(\lambda, A) \| \, d\lambda \right| &\leqslant K \Big(2 \int_{\frac{1}{t}}^\infty \frac{s}{s} e^{ts\cos\theta} \, ds + \int_{-\theta}^{\theta} r e^{\cos\alpha} d\alpha \Big) \\ &\leqslant \frac{2K}{t|\cos\theta|} + \frac{2eK\theta}{t} =: \frac{C'}{t}. \end{aligned}$$

Therefore, the right hand side of (5.6) converges to

$$\int_\Gamma \lambda e^{\lambda t} R(\lambda, A) \, d\lambda$$

as $R \to \infty$. Since A is closed, it follows that $e^{tA} X \subseteq D(A)$ and

$$A e^{tA} = \frac{1}{2\pi i} \int_\Gamma \lambda e^{\lambda t} R(\lambda, A) \, d\lambda, \qquad \| A e^{tA} \| \leqslant \frac{C'}{2\pi t}$$

for all $t > 0$. In a similar way one sees that

$$\left| \int_{\Gamma \setminus \Gamma_R} \lambda e^{t\lambda} R(\lambda, A) \, d\lambda \right| \leqslant 2K \int_R^\infty e^{ts\cos\theta} \, ds \leqslant \frac{2K}{\varepsilon |\cos\theta|} e^{R\varepsilon\cos\theta} \longrightarrow 0$$

as $R \to \infty$, uniformly for $t \geqslant \varepsilon$. As a result,

$$\int_{\Gamma_R} \lambda e^{\lambda t} R(\lambda, A) \, d\lambda = \frac{d}{d\lambda} \int_{\Gamma_R} e^{\lambda t} R(\lambda, A) \, d\lambda$$

converges in $\mathcal{B}(X)$ uniformly for $t \geqslant \varepsilon$, and so $t \mapsto e^{tA} \in \mathcal{B}(X)$ is continuously differentiable for $t > 0$ with $\frac{d}{dt} e^{tA} = A e^{tA}$. For $x \in D(A)$, we further obtain

$$A e^{tA} x = \lim_{R \to \infty} \frac{1}{2\pi i} \int_{\Gamma_R} e^{\lambda t} R(\lambda, A) A x \, d\lambda = e^{tA} A x.$$

(c) Let $x \in \mathrm{D}(A)$, $R > r$, and $t > 0$. As in part (a), Cauchy's formula (5.2) yields

$$\frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\mathrm{e}^{\lambda t}}{\lambda} \,\mathrm{d}\lambda = \lim_{R\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_R} \frac{\mathrm{e}^{\lambda t}}{\lambda - 0} \,\mathrm{d}\lambda = 1$$

Observing that $\lambda R(\lambda, A)x - x = R(\lambda, A)Ax$, we conclude that

$$\mathrm{e}^{tA}x - x = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{e}^{\lambda t}\Big(R(\lambda, A) - \frac{1}{\lambda}\Big)x \,\mathrm{d}\lambda = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\mathrm{e}^{\lambda t}}{\lambda} R(\lambda, A)Ax \,\mathrm{d}\lambda.$$

Since the integrand is bounded by $\frac{c}{|\lambda|^2}$ on Γ for all $t \in (0, 1]$, Lebesgue's convergence theorem implies the existence of the limit

$$\lim_{t\to 0} \mathrm{e}^{tA}x - x = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{1}{\lambda} R(\lambda, A)Ax \,\mathrm{d}\lambda =: z.$$

Let $K_R = \{R\mathrm{e}^{\mathrm{i}\alpha} \,\big|\, -\theta \leqslant \alpha \leqslant \theta\}$. Cauchy's theorem (5.1) shows that

$$\int_{\Gamma_R \cup (-K_R)} \frac{1}{\lambda} R(\lambda, A)Ax \,\mathrm{d}\lambda = 0.$$

Since also

$$\left\| \int_{-K_R} \frac{1}{\lambda} R(\lambda, A)Ax \,\mathrm{d}\lambda \right\| \leqslant \frac{2\pi RK}{R^2} \, \|Ax\| \;\longrightarrow\; 0$$

as $R \to \infty$, we arrive at $z = 0$. Because of the uniform boundedness of e^{tA}, it follows that $\mathrm{e}^{tA}x \to x$ as $t \to 0$ for all $x \in \overline{\mathrm{D}(A)}$.

Conversely, if $\mathrm{e}^{tA}x \to y$ as $t \to 0$, then $y \in \overline{\mathrm{D}(A)}$ by part (b). Moreover, $R(1, A)\mathrm{e}^{tA}x = \mathrm{e}^{tA}R(1, A)x$ tends to $R(1, A)x$ as $t \to 0$, since $R(1, A)x \in \mathrm{D}(A)$. We thus obtain $R(1, A)y = R(1, A)x$, and so $x = y \in \overline{\mathrm{D}(A)}$. □

Remark 5.13. Let $A - \omega I = A_\omega$ be sectorial of angle greater than $\frac{\pi}{2}$ for some $\omega \in \mathbb{R}$. We then compute

$$\mathrm{e}^{\omega t}\mathrm{e}^{tA_\omega} = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{e}^{t(\lambda+\omega)} R(\lambda + \omega, A) \,\mathrm{d}\lambda = \frac{1}{2\pi \mathrm{i}} \int_{\omega+\Gamma} \mathrm{e}^{\mu t} R(\mu, A) \,\mathrm{d}\mu =: \mathrm{e}^{tA}$$

for all $t > 0$. It is easy to see that $\mathrm{e}^{tA} = \mathrm{e}^{\omega t}\mathrm{e}^{tA_\omega}$ has the analogous properties as in the case $\omega = 0$. ◊

We can now easily solve the evolution equation (5.7) governed by a sectorial operator A with angle $\phi > \pi/2$. One calls such problems 'of parabolic type' since diffusion problems are typical applications, see Example 5.15. In contrast to the Schrödinger equation in Example 4.19 we can allow for initial values in $\overline{\mathrm{D}(A)}$.

Corollary 5.14. *Let A be sectorial of angle $\phi > \frac{\pi}{2}$ and let $u_0 \in \overline{\mathrm{D}(A)}$. Then $u(t) = \mathrm{e}^{tA}u_0$, $t \geqslant 0$, is the unique solution in $C^1((0, \infty), X) \cap C((0, \infty), [\mathrm{D}(A)]) \cap C(\mathbb{R}_+, X)$ of the initial value problem*

$$u'(t) = Au(t), \quad t > 0, \qquad u(0) = u_0. \tag{5.7}$$

Proof. Existence follows from Theorem 5.12 and Remark 5.13. Let v be another solution of (5.7). Let $0 < \varepsilon \leqslant s \leqslant t - \varepsilon < t$. Theorem 5.12 then implies that

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{(t-s)A}v(s) = -\mathrm{e}^{(t-s)A}Av(s) + \mathrm{e}^{(t-s)A}v'(s) = 0.$$

As in Example 4.19, this fact yields $e^{(t-\varepsilon)A}v(\varepsilon) = e^{\varepsilon A}v(t-\varepsilon)$. Letting $\varepsilon \to 0$, one obtains that $e^{tA}u_0 = v(t)$ since $\tau \mapsto e^{\tau A}x$ is continuous for $\tau \geqslant 0$ and each $x \in \overline{D(A)}$. □

We only give one of the possible examples.

Example 5.15. Let $X = C([0,1])$ and $A\varphi = \varphi''$ with $D(A) = \{\varphi \in C^2([0,1]) \,|\, \varphi(0) = \varphi(1) = 0\}$. Let $u_0 \in C_0(0,1) = \overline{D(A)}$. Then the function $u(t) = e^{tA}u_0$, $t \geqslant 0$, belongs to

$$C(\mathbb{R}_+, C([0,1])) \cap C(0,\infty), C^2([0,1])) \cap C^1((0,\infty), C([0,1]))$$

and uniquely solves the partial differential equation

$$\begin{aligned} \partial_t u(t,x) &= \partial_{xx} u(t,x), \qquad t > 0,\ x \in [0,1], \\ u(t,0) &= u(t,1) = 0, \qquad t > 0, \\ u(0,x) &= u_0(x), \qquad x \in [0,1]. \end{aligned} \tag{5.8}$$

◊

The formula (5.5) and Theorem 5.12 allow to establish a theory for parabolic problems which is similar to the case of ordinary differential equations, see e.g. [**He**] and [**Lu**]. We only add a basic result on the longterm behavior that extends Example 5.4. The next theorem can be applied to (5.8) thanks to Example 5.9.

Theorem 5.16. *Let A be sectorial of angle $\phi > \pi/2$ and $s(A) = \sup\{\operatorname{Re}\lambda \,|\, \lambda \in \sigma(A)\} < -\delta < 0$. Then there is a constant $N \geqslant 1$ such that $\|e^{tA}\| \leqslant N e^{-\delta t}$ for all $t \geqslant 0$.*[2]

Proof. The assumptions imply that $A_{-\delta} = A + \delta I$ is sectorial of some angle $\psi \in (\pi/2, \phi)$. Take $\Gamma = \Gamma(r,\theta)$ with $r > 0$ and $\theta \in (\pi/2, \psi)$. Remark 5.13 then yields that $e^{tA} = e^{-\delta t} e^{tA_{-\delta}}$, where however e^{tA} is defined by the curve integral (5.5) on the shifted path $\Gamma' := -\delta + \Gamma$. Lemma 5.11 shows that $e^{tA_{-\delta}}$ is uniformly bounded for $t \geqslant 0$. It thus remains to verify that

$$\int_{-\delta+\Gamma} e^{t\mu} R(\mu, A)\, d\mu = \int_\Gamma e^{t\lambda} R(\lambda, A)\, d\lambda, \qquad t > 0. \tag{5.9}$$

To this end, let $R > r$ and $S_R^\pm$ be the horizontal line segments connecting the end points on Γ_R and Γ'_R in $\{\operatorname{Im}\lambda > 0\}$ and $\{\operatorname{Im}\lambda < 0\}$, respectively. Let $C_R = \Gamma_R \cup S_R^+ \cup (-\Gamma') \cup (-S_R^-)$. This path is contained in $\rho(A)$ and $n(C_R, z) = 0$ for all $z \in \sigma(A)$. Cauchy's theorem (5.1) now implies

$$\int_{C_R} e^{t\lambda} R(\lambda, A)\, d\lambda = 0.$$

Observe that the segments $S_R^\pm$ have fixed length δ and that $\operatorname{Re}\lambda \leqslant R\cos\theta < 0$ and $|\lambda| \geqslant R$ for all $\lambda \in S_R^\pm$. Because of $\theta < \phi$, the sets $S_R^\pm$ belong to Σ_ϕ for all sufficiently large R. We can thus estimate

$$\Big\| \int_{S_R^\pm} e^{t\lambda} R(\lambda, A)\, d\lambda \Big\| \leqslant \frac{\delta K}{R}\, e^{Rt\cos\theta}.$$

Since the right-hand side vanishes as $R \to \infty$, we have shown (5.9). □

[2] Not shown in the lectures.

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