LOCAL WELL-POSEDNESS OF A QUASILINEAR WAVE EQUATION

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Abstract. We study a quasilinear wave equation on a domain arising in nonlinear optics. We show local well-posedness for strong solutions using Kato's approach to quasilinear evolution equations.

1. Introduction

In this paper we show the local well-posedness of the nonlinear wave equation

$$\partial_t^2\big(u + K(u)\big) = \Delta u \tag{1.1}$$

for a function $u : [0, T] \times \Omega \to \mathbb{R}$ on a bounded domain $\Omega \subset \mathbb{R}^n$ for $n \in \{1, 2, 3\}$. This equation is used in physics to describe transport of light in a waveguide, including the nonlinear interaction of the electromagnetic amplitude with the waveguides' material. If for example $E \in \mathbb{R}^3$ is the electrical field and $P(E) \in \mathbb{R}^3$ the polarisation vector, then Maxwell equations result in the system of equations $\partial_t^2(E + P(E)) = -\nabla \times \nabla \times E = \Delta E - \nabla(\nabla \cdot E)$. There are special solutions of the form $E = (0, 0, u(x_1, x_2))$ which are given by our equation (1.1) with $K(u) = P((0, 0, u)) \cdot (0, 0, 1)$ for $\Omega \subset \mathbb{R}^2$.

For $K = 0$, the problem (1.1) becomes the linear wave equation. We assume that

$$K \in C^4(\mathbb{R}) \quad \text{and} \quad K'(0) > -1. \tag{1.2}$$

The most important case is the *Kerr model*

$$K(z) = \lambda z^3$$

for a parameter $\lambda \in \mathbb{R}$, see e.g. [MN04], [PNTB09].

Equation (1.1) will be supplemented by homogeneous Dirichlet boundary conditions on $\partial\Omega$ (for simplicity) and initial conditions for $u(0, \cdot)$ and $\partial_t u(0, \cdot)$.

In the present work we rewrite equation (1.1) by differentiating the left hand side which leads to the quasilinear wave equation

$$\partial_t^2 u = f(u)\Delta u + g(u)(\partial_t u)^2, \tag{1.3}$$

where we use (1.2) and introduce the functions $f, g : \mathbb{R} \to \mathbb{R}$ by

$$f(z) := \frac{1}{1 + K'(z)} \qquad \text{and} \qquad g(z) := \frac{-K''(z)}{1 + K'(z)} \tag{1.4}$$

2000 *Mathematics Subject Classification.* Primary: 35L72. Secondary: 47D06.

Key words and phrases. Quasilinear wave equation, Kerr nonlinearity, Kato's method, Kato stability, operator semigroups.

H. Gerner has been supported by the Deutsche Forschungsgemeinschaft in the framework of the Research Training Group 1294.

for all $z \in \mathbb{R}$. In the Kerr model $K(z) = \lambda z^3$ one obtains the expressions

$$f(z) = \frac{1}{1+3\lambda|z|^2} \qquad \text{and} \qquad g(z) = \frac{-6\lambda z}{1+3\lambda|z|^2}. \tag{1.5}$$

Observe that (1.2) yields

$$f, g \in C^2([-\rho,\rho]) \quad \text{and} \quad f \geq \delta \qquad \text{for some} \;\; \rho, \delta > 0. \tag{1.6}$$

This condition is assumed throughout this paper. For the Kerr nonlinearity in (1.5) with $\lambda < 0$ we can take any $\rho \in (0, (3|\lambda|)^{-1/2})$, but the constants below will explode as ρ approaches $(3|\lambda|)^{-1/2}$. Observe that in the 'defocusing' case $K(z) = \lambda z^3$ with $\lambda > 0$, the functions f and g together with their derivatives are bounded on $\mathbb{R}$, though we still have $f(z) \to 0$ as $|z| \to \infty$. We will comment on this case in Corollary 4.2.

We note that we cannot expect to obtain global strong solvability in the 'focusing' case $\lambda < 0$ in the Kerr model. In fact, for Neumann boundary conditions one easily constructs a solution $u(t,x) = \phi(t)$ which does not depend on space variables: For simplicity, take $\lambda = -\frac{1}{3}$ and the initial conditions $u(0) = 0$ and $\partial_t u(0) = 1$. Then the map $\psi : [0,1] \to [0, \frac{2}{3}]$, $\psi(s) = s - \frac{1}{3}s^3$, has the inverse $\phi : [0, \frac{2}{3}] \to [0,1]$ with $\phi'(t) \to \infty$ as $t \to \frac{2}{3}^-$. Setting $u(t,\cdot) = \phi(t)$, we obtain $u(0) = 0$, $\partial_t u(0) = 1$,

$$\partial_t^2 (u(t) - \tfrac{1}{3} u(t)^3) = \partial_t^2 \psi(\phi(t)) = 0 = \Delta u(t),$$

and the time derivative of this solution blows up as $t \to \frac{2}{3}^-$.

We solve (1.3) by means of Kato's approach to quasilinear evolution equations for initial values $(u_0, v_0) \in H^3(\Omega) \times H^2(\Omega)$. Since f and g are only defined near 0, we also have to impose that the initial values are small in $H^2(\Omega) \times H^1(\Omega)$, so that the first component is small in L^∞ by Sobolev's embedding in space dimensions $n \leq 3$. One could treat larger space dimensions n in Sobolev spaces of higher order. The case $n = 1$ is significantly simpler since then already H^1 embeds into L^∞ leading to a different, easier analytical setting, cf. Section 2. In view of the physical motivation and for the sake of conciseness, we restrict ourselves to the case $n \in \{2,3\}$, where the results also cover $n = 1$ without giving optimal results.[1] We note that the smallness condition can be dropped in the case $\lambda > 0$ in the Kerr model, see Corollary 4.2.

Our results cannot directly be deduced from Kato's fundamental well-posedness theorems in [Kat75] since the crucial dissipativity assumption (A1) of this paper does not hold in the standard norm of the basic space, say, $H^1(\Omega) \times L^2(\Omega)$. Kato has already noted in Remark 11.1 in [Kat75] that one has to employ state dependent norms to deal with this difficulty. In [HKM77], this approach has been used to solve quasilinear wave equations on $\mathbb{R}^n$. However, the relevant well-posedness Theorems I and II impose a smallness condition on the initial data in a higher norm than needed by our results. We adapt the techniques of [HKM77] to our situation and modify them to obtain the improved

[1] One can surely lower this regularity to $(u_0, v_0) \in H^{s+1}(\Omega) \times H^s(\Omega)$ for $s > \frac{n}{2}$, but to focus on the main difficulties we avoid fractional Sobolev spaces.

smallness condition. Also since in [HKM77] many proofs are only sketched, we present our our arguments in detail.

Alternatively one can rewrite the wave equation as a hyperbolic system of first order using e.g. the new state $z = (\partial_t u, f(u)^{1/2}\partial_1 u, \cdots, f(u)^{1/2}\partial_n u)$, see Section 16 in [Kat75]. In [Kat75a], Kato treated such systems for the spatial domain $\mathbb{R}^n$ and by means of state dependent norms. (See also Section 12 in [Kat75] for a special case.) Again, the initial values have to satisfy a smallness condition in a higher norm than in our paper, cf. Theorem II of [Kat75a]. But more importantly, the resulting first order system is still solved in H^3 (or H^s with $s > 5/2$, cf. Footnote 1), and hence one obtains solutions u in H^4, whereas our approach and the one of [HKM77] works for u in H^3. This drawback was already discussed on p. 62/63 of [Kat75].

Quasilinear wave equations have been studied intensively also by means of more direct methods, but mostly for full space problems. Here we refer to the monograph [Sog08], where more general equations have been treated in a setting of higher regularity, see in particular Theorem 1.4.1. On the full space $\mathbb{R}^n$ one can use Strichartz' estimates to reduce the necessary regularity for certain classes of quasilinear wave equations, see [ST05] and also [Sog08]. On a bounded domain these estimates are not available (at least not in their full power). Moreover, there are several results for related, but different systems on (partly exterior) domains, see e.g. [MS10], [Nak03], [Wei86], [Yao07] and the references therein.

In the second section we discuss the necessary prerequisites to rewrite (1.3) as a first order problem $\partial_t w(t) = A(w(t))w(t)$ and state a simplified version of our main result. Following Kato's work, one then considers the non-autonomous linear problem $\partial_t w(t) = A(\widetilde{w}(t))w(t)$ for a given function $\widetilde{w}$. Using Kato's paper [Kat70], we then obtain a solution $w = \Phi(\widetilde{w})$ of the linear problem. The crucial step are stability estimates in $H^{k+1} \times H^k$, $k \in \{0, 1, 2\}$, for products of the semigroups generated by $A(\widetilde{w}(t_j))$ which are derived using state depending norms, see Lemma 3.3. In Section 4 we then establish the local well-posedness of the initial value problem (1.3) (and thus of (1.1)) by a fixed point argument for the map $\widetilde{w} \mapsto \Phi(\widetilde{w})$.

2. Notation, analytical setting and main result

We first list some notation and assumptions used throughout this paper.

Notation. For Banach spaces $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$, we write $\mathcal{B}(X, Y)$ for the space of bounded linear mappings from X to Y endowed with the operator norm, where we put $\mathcal{B}(X) := \mathcal{B}(X, X)$. By $(A, \mathcal{D}(A))$ we denote a (possibly unbounded) operator A together with its domain of definition $\mathcal{D}(A)$. The closed ball in X with center x and radius r is designated by $\overline{B}_X(x, r)$. Throughout C stands for a generic positive constant.

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \in \{1, 2, 3\}$, with a boundary $\partial\Omega$ of class C^3. On Ω we work with the classical (real-valued) function spaces $C^k(\overline{\Omega})$ and $W^{k,p}(\Omega)$ for $k \in \mathbb{N}$ and $p \in [1, \infty]$. Their usual norms are written as $|\cdot|_{C^k}$ and $|\cdot|_{W^{k,p}}$ when Ω is clear from the context. The special notation $H^k(\Omega)$ is used for the Hilbert spaces $W^{k,2}(\Omega)$, and $H^k_0(\Omega)$ denotes the subspace of function in

$H^k(\Omega)$ with vanishing boundary trace. We often make use of the continuous embeddings

$$H^1(\Omega) \hookrightarrow L^p(\Omega) \qquad \text{and} \qquad H^2(\Omega) \hookrightarrow L^\infty(\Omega) \tag{2.1}$$

for $p \in [1,6]$ which result from Sobolev's embedding theorem on $\Omega \subset \mathbb{R}^n$, $n \in \{1,2,3\}$, and provide us with the estimates $|v|_{L^p} \le C\,|v|_{H^1}$ and $|v|_{L^\infty} \le C\,|v|_{H^2}$ for all appropriate v, where C only depends on Ω.

For functions $v : [0,T] \times \Omega \to \mathbb{R}$ depending also on the time variable, $C(0,T;X)$ is the Banach space of continuous functions $v : [0,T] \to X; t \mapsto v(t,\,\cdot\,)$, equipped with the norm $\|v\|_{C(0,T;X)} := \max_{t\in[0,T]} |v(t)|_X$, where X is (a closed subspace of) a suitable function space with respect to the spatial variable and $T > 0$. The space $C^k(0,T;X)$ is defined analogously. By $\mathrm{Lip}(0,T;X)$ we denote the space of Lipschitz continuous functions with norm $\|v\|_{\mathrm{Lip}(0,T;X)} := \|v\|_{C(0,T;X)} + [v]_{\mathrm{Lip}(0,T;X)}$, where

$$[v]_{\mathrm{Lip}(0,T;X)} := \sup_{0<t<t'<T} \frac{|v(t) - v(t')|_X}{|t-t'|}.$$

Analytical setting. We want to reformulate (1.3) as a first order (in time) initial value boundary problem to investigate the well-posedness within Kato's theory. To this aim, we employ the Dirichlet Laplace operator Δ_D in $L^2(\Omega)$ with domain $\mathcal{D}(\Delta_D) = H^2(\Omega) \cap H_0^1(\Omega)$. This selfadjoint operator gives rise to the scale of spaces

$$\mathcal{H}_0 := L^2(\Omega), \qquad \mathcal{H}_k := \mathcal{D}\big((-\Delta_D)^{k/2}\big)$$

with norms given by

$$|\varphi|_{\mathcal{H}_0} = |\varphi|_{L^2(\Omega)} \qquad \text{and} \qquad |\varphi|_{\mathcal{H}_k} = |(-\Delta_D)^{k/2}|_{L^2(\Omega)}$$

for $k \in \mathbb{N}$, where $\Delta_D : \mathcal{H}_{k+2} \to \mathcal{H}_k$ is an isometric isomorphism, cf. Section V.1.2 in [Ama95]. We write Δ_D for each realization $\Delta_D : \mathcal{H}_{k+2} \to \mathcal{H}_k$ of the Dirichlet Laplace. Usually we also omit subscript D. We recall the isomorphisms

$$\begin{aligned} &\mathcal{H}_1 \cong H_0^1(\Omega), \qquad \mathcal{H}_2 \cong H^2(\Omega) \cap H_0^1(\Omega), \\ &\mathcal{H}_3 \cong \{\varphi \in H^3(\Omega) \cap H_0^1(\Omega) \,:\, \Delta\varphi\lfloor_{\partial\Omega} = 0\}. \end{aligned}$$

(Here the cases $k = 1$ and $k = 2$ are well known, whereas the case $k = 3$ follows from the isomorphy of $\Delta_D : \mathcal{H}_3 \to \mathcal{H}_1$.)

Using $v := \partial_t u$ and $w := (u,v)^T \in \mathcal{H}_2 \times \mathcal{H}_1$, we write (1.3) as the first order system

$$\frac{\mathrm{d}}{\mathrm{d}t} w = \begin{pmatrix} 0 & I \\ f(u)\Delta_D & g(u)v \end{pmatrix} w = \begin{pmatrix} 0 & I \\ f(u)\Delta_D & 0 \end{pmatrix} w + \begin{pmatrix} 0 & 0 \\ 0 & g(u)v \end{pmatrix} w \tag{2.2}$$

with $w(0) = w_0 = (u(0,\,\cdot\,), \partial_t u(0,\,\cdot\,))$. To treat (2.2), we define the spaces

$$\mathcal{X}_k := \mathcal{H}_{k+1} \times \mathcal{H}_k$$

with norms given by $|(u,v)|^2_{\mathcal{X}_k} := |u|^2_{\mathcal{H}_{k+1}} + |v|^2_{\mathcal{H}_k}$ for $k \in \{0,1,2\}$, and we introduce the operators

$$\begin{aligned} A(w) &:= \begin{pmatrix} 0 & I \\ f(u)\Delta_D & g(u)v \end{pmatrix}, \qquad A_0(w) := \begin{pmatrix} 0 & I \\ f(u)\Delta_D & 0 \end{pmatrix}, \\ B(w) &:= \begin{pmatrix} 0 & 0 \\ 0 & g(u)v \end{pmatrix} \end{aligned} \tag{2.3}$$

for $w = (u,v) \in \mathcal{X}_1$. In view of (1.6), these operators are only defined if $|u|_{L^\infty} \le \rho$. Using Sobolev's embedding, we fix a number $r > 0$ (depending only on ρ and Ω) such that

$$|u|_{\mathcal{H}_2} \le r \quad \Longrightarrow \quad |u|_{L^\infty} \le \rho, \tag{2.4}$$

and restrict ourselves to u with $|u|_{\mathcal{H}_2} \le r$. The constants C and C_k below may depend on r. For this number $r > 0$ and any $R > 0$ we introduce the space

$$E(R) := \big\{\Psi \in \mathcal{X}_2 \,:\, |\Psi|_{\mathcal{X}_1} \le r,\ |\Psi|_{\mathcal{X}_2} \le R\big\} \subset \mathcal{X}_2. \tag{2.5}$$

We now state a simplified version of our main result Theorem 4.1 about

$$\begin{aligned} &\partial_t^2 u = f(u)\Delta_D u + g(u)(\partial_t u)^2, \quad t \in [0,T], \\ &u(0) = u_0, \qquad \partial_t u(0) = v_0. \end{aligned} \tag{2.6}$$

Theorem. *Assume that* (1.6) *holds and let* $w_0 = (u_0, v_0) \in \mathcal{X}_2$ *have a sufficiently small norm in* $\mathcal{X}_1$*. Then there is a time* $T > 0$ *and a function* $u = \partial_t v$ *and* $u \in C(0,T;\mathcal{H}_3) \cap C^1(0,T;\mathcal{H}_2) \cap C^2(0,T;\mathcal{H}_1)$ *satisfying* $|(u(t), \partial_t u(t))|_{\mathcal{X}_1} \le r$ *for* $t \in [0,T]$ *and* (2.6)*.*

Any other solution of (2.6) *in this class on a time interval* $[0,T']$ *coincides with* u *on* $[0, \min\{T,T'\})]$*. Moreover, the map* $(u_0,v_0) \mapsto (u, \partial_t u)$ *is Lipschitz from suitable bounded subsets of* $\mathcal{X}_2$ *to* $C(0,T;\mathcal{H}_2) \times C(0,T;\mathcal{H}_1)$*.*

Finally, if f *and* g *are given as in* (1.4) *for a function* K *fulfilling* (1.2)*, then the assertions also hold if we replace in* (2.6) *the PDE by* $\partial_{tt}(u + K(u)) = \Delta_D u$*.*

3. The linear non-autonomous problem

In Kato's approach to quasilinear problems one freezes a function $\widetilde{w} = (\widetilde{u}, \widetilde{v}) \in E(R)$, see (2.5), in the nonlinear part of (2.2) and then solves the resulting non–autonomous linear initial value problem

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} w(t) &= A(\widetilde{w}(t))w(t) = \big(A_0(\widetilde{w}) + B(\widetilde{w})\big)w(t), \qquad t \ge 0, \\ w(0) &= w_0. \end{aligned} \tag{3.1}$$

For the analysis of this equation, we introduce the weighted norm

$$|\Psi|^2_{\mathcal{X}_{0,\widetilde{u}}} = |(\varphi,\psi)|^2_{\mathcal{X}_{0,\widetilde{u}}} := \int_\Omega \Big\{ |\nabla\varphi|^2 + |\psi|^2 \frac{1}{f(\widetilde{u})} \Big\}\,\mathrm{d}x$$

for $\Psi = (\varphi,\psi) \in \mathcal{X}_0$. Note that $|\cdot|_{\mathcal{X}_{0,\widetilde{u}}}$ and $|\cdot|_{\mathcal{X}_0}$ are equivalent norms with

$$C_0^{-1} |\cdot|_{\mathcal{X}_0} \le |\cdot|_{\mathcal{X}_{0,\widetilde{u}}} \le C_0 \,|\cdot|_{\mathcal{X}_0} \tag{3.2}$$

where $C_0 > 0$ only depends on δ, ρ and f in (1.6). We first show that our operators generate (quasi-)contractive semigroups with respect to these norms.

Lemma 3.1. *Let $\widetilde{w} = (\widetilde{u}, \widetilde{v}) \in E(R)$. Then the operators $\pm(A_0(\widetilde{w}), \mathcal{X}_1)$ generate a contraction semigroup on $(\mathcal{X}_0, |\cdot|_{\mathcal{X}_0,\widetilde{u}})$.*

Proof. Let $\Psi = (\varphi, \psi) \in \mathcal{X}_1$. Integrating by parts, we compute

$$\langle A_0(\widetilde{w})\Psi, \Psi\rangle_{\mathcal{X}_0,\widetilde{u}} = \left\langle \begin{pmatrix} \psi \\ f(\widetilde{u})\Delta_D\varphi \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle_{\mathcal{X}_0,\widetilde{u}} = \int_\Omega (\nabla\psi\cdot\nabla\varphi + \psi\,\Delta\varphi)\,\mathrm{d}x = 0.$$

The operator $A_0(\widetilde{w})$ is thus dissipative on $(\mathcal{X}_0, |\cdot|_{\mathcal{X}_0,\widetilde{u}})$. It further has the inverse

$$A_0(\widetilde{w})^{-1} = \begin{pmatrix} 0 & \Delta_D^{-1}\left(\frac{1}{f(\widetilde{u})}\,\cdot\right) \\ I & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{X}_0, \mathcal{X}_1)$$

so that $\mu - A_0(\widetilde{w})$ is invertible for small $\mu > 0$. The result now follows from the Theorem of Lumer–Phillips, see e.g. Theorem 1.4,3 in [Paz83]. The case $-A_0(\widetilde{w})$ is treated in the same way. □

We can now treat the full operator $A(\widetilde{w})$ by a perturbation argument. (Note that $B(\widetilde{w})$ is not dissipative, in general.)

Lemma 3.2. *Let $\widetilde{w} = (\widetilde{u}, \widetilde{v}) \in E(R)$. Then there is a constant $C_1 > 0$ such that $(\pm A(\widetilde{w}) - C_1R, \mathcal{X}_1)$ generates a contraction semigroup on $(\mathcal{X}_0, |\cdot|_{\mathcal{X}_0,\widetilde{u}})$.*

Proof. Let $\Psi = (\varphi, \psi) \in \mathcal{X}_0$. The embedding (2.1) then yields

$$|B(\widetilde{w})\Psi|_{\mathcal{X}_0} = |g(\widetilde{u})\widetilde{v}\psi|_{L^2} \le |g|_{L^\infty}|\widetilde{v}|_{L^\infty}|\psi|_{L^2} \le C\,|\widetilde{w}|_{\mathcal{X}_2}|\Psi|_{\mathcal{X}_0} \le CR\,|\Psi|_{\mathcal{X}_0}.$$

In view of (3.2), we now obtain a constant $C_1 > 0$ such that $B(\widetilde{w})$ on $(\mathcal{X}_0, |\cdot|_{\mathcal{X}_0,\widetilde{u}})$ is bounded by C_1R. Using this bound and Lemma 3.1, we can apply the bounded perturbation theorem (see e.g. Theorem 3.1.1 in [Paz83]) which implies the assertion. □

To solve (3.1), we need time depending functions $\widetilde{w}(t) \in E(R)$, cf. (2.5). We thus introduce the space

$$E(T, R, L) := \big\{\Psi \in C\big(0, T; E(R)\big) \,:\, [\Psi]_{\mathrm{Lip}(0,T;\mathcal{X}_1)} \le L\big\} \tag{3.3}$$

for any $R \ge 1$, $L > 0$ and $T > 0$. The crucial concept in Kato's theory is the *stability* of the family of generators $(A(\widetilde{w}(t)))_{t\in[0,T]}$, as defined in the next lemma. We write $(\mathrm{e}^{\tau A})_{\tau\ge 0}$ for the C^0-semigroup generated by A on a Banach space X. Recall that the part $A_|$ of A in a continuously embedded Banach space $Y \hookrightarrow X$ is defined by $\mathcal{D}(A_|) = \{y \in \mathcal{D}(A) \cap Y : Ay \in Y\}$ and $A_|y = Ay$.

Lemma 3.3. *Let $\widetilde{w} \in E(T, R, L)$. Then the parts of $\pm A(\widetilde{w}(t))$, $t \in [0, T]$, in $\mathcal{X}_1$ and $\mathcal{X}_2$ generate C^0-semigroups. Moreover, the family $\big(\pm A(\widetilde{w}(t))\big)_{t\in[0,T]}$ is* stable *in $\mathcal{X}_k$ with constants $M = C_4\mathrm{e}^{C_2LT}$ and $\beta = C_3R^3$, i.e.,*

$$\Big\|\mathrm{e}^{\pm\tau_n A(\widetilde{w}(t_n))}\cdots\mathrm{e}^{\pm\tau_1 A(\widetilde{w}(t_1))}\Big\|_{\mathcal{B}(\mathcal{X}_k)} \le C_4\mathrm{e}^{C_2LT}\mathrm{e}^{C_3R^3(\tau_1+\cdots+\tau_n)}$$

for $k \in \{1, 2\}$, some constants $C_i > 0$, any decomposition $0 \le t_1 \le \cdots \le t_n \le T$ and all $\tau_j \ge 0$, where $n \in \mathbb{N}$ and $j = 1, \ldots, n$.

Proof. We show stability in $\mathcal{X}_0$ and $\mathcal{X}_2$ and then conclude the claimed stability on $\mathcal{X}_1$ by interpolation of the function spaces. We restrict ourselves to the case of $A(\widetilde{w}(t))$ since $-A(\widetilde{w}(t))$ is dealt with in the same way. Take $\widetilde{w} = (\widetilde{u}, \widetilde{v}) \in E(T, R, L)$.

1) To treat $\mathcal{X}_0$, let $t, s \in [0, T]$ and $\Psi = (\varphi, \psi) \in \mathcal{X}_0$. We first compare the norms with weights $\widetilde{u}(t)$ and $\widetilde{u}(s)$. Employing (1.6), we estimate

$$\begin{aligned} |\Psi|^2_{\mathcal{X}_0,\widetilde{u}(t)} &= \int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x + \int_\Omega |\psi|^2 \frac{f(\widetilde{u}(s)) - f(\widetilde{u}(t))}{f(\widetilde{u}(s)) f(\widetilde{u}(t))} \,\mathrm{d}x + \int_\Omega |\psi|^2 \frac{1}{f(\widetilde{u}(s))} \,\mathrm{d}x \\ &\le |\Psi|^2_{\mathcal{X}_0,\widetilde{u}(s)} + \frac{|f'|_{L^\infty(B_r)}}{\delta} |\widetilde{u}(t) - \widetilde{u}(s)|_{L^\infty(\Omega)} \int_\Omega |\psi^2| \frac{1}{f(\widetilde{u}(s))} \,\mathrm{d}x \\ &\le \big(1 + C_2' L |t-s|\big) |\Psi|^2_{\mathcal{X}_0,\widetilde{u}(s)} \le \mathrm{e}^{C_2' L|t-s|} |\Psi|^2_{\mathcal{X}_0,\widetilde{u}(s)} \end{aligned} \tag{3.4}$$

for a constant $C_2' \ge 0$. Set $C_2 = C_2'/2$, and also

$$\Pi_n := \mathrm{e}^{\tau_n A(\widetilde{w}(t_n))} \mathrm{e}^{\tau_{n-1} A(\widetilde{w}(t_{n-1}))} \cdots \mathrm{e}^{\tau_1 A(\widetilde{w}(t_1))}.$$

To control these products, we use that $\mathrm{e}^{\tau A(\widetilde{w}(t))}$ is bounded by $\mathrm{e}^{\tau C_1 R}$ in $|\cdot|_{\mathcal{X}_0,\widetilde{u}(t)}$ (see Lemma 3.2) and the above inequality (3.4). It follows

$$\begin{aligned} |\Pi_n \Psi|_{\mathcal{X}_0,\widetilde{u}(t_n)} &= |\mathrm{e}^{\tau_n A(\widetilde{w}(t_n))} \Pi_{n-1}\Psi|_{\mathcal{X}_0,\widetilde{u}(t_n)} \le \mathrm{e}^{\tau_n C_1 R} |\Pi_{n-1}\Psi|_{\mathcal{X}_0,\widetilde{u}(t_n)} \\ &\le \mathrm{e}^{\tau_n C_1 R} \mathrm{e}^{C_2 L (t_n - t_{n-1})} |\Pi_{n-1}\Psi|_{\mathcal{X}_0,\widetilde{u}(t_{n-1})} \\ &\le \cdots \le \mathrm{e}^{(\tau_n + \cdots + \tau_1) C_1 R} \mathrm{e}^{C_2 L (t_n - t_1)} |\Psi|_{\mathcal{X}_0,\widetilde{u}(t_1)}. \end{aligned}$$

The norm equivalence (3.2) then yields

$$|\Pi_n|_{\mathcal{B}(\mathcal{X}_0)} \le C_0^2 \mathrm{e}^{(\tau_n + \cdots + \tau_1) C_1 R} \mathrm{e}^{C_2 L (t_n - t_1)},$$

i.e., the asserted estimate on $\mathcal{X}_0$ even with constants $M = C_0^2 e^{C_2 L (t_n - t_1)}$ and $\beta = C_1 R$. (A similar argument can be found in Proposition 3.4 in [Kat70].)

2) Next, we show stability on $\mathcal{X}_2$. For this purpose we define the isometric isomorphism

$$S := \begin{pmatrix} \Delta_D & 0 \\ 0 & \Delta_D \end{pmatrix} : \mathcal{X}_2 \to \mathcal{X}_0,$$

and consider, suppressing in the following the variable t, and the subscript D,

$$S A(\widetilde{w}) S^{-1} = A(\widetilde{w}) + \begin{pmatrix} 0 & 0 \\ \Delta\big(f(\widetilde{u}) \,\cdot\,\big) - f(\widetilde{u})\Delta & \Delta\big(g(\widetilde{u})\widetilde{v}\Delta^{-1} \,\cdot\,\big) - g(\widetilde{u})\widetilde{v} \end{pmatrix}.$$

For convenience, we write for the rest of the proof $\widetilde{w} = w = (u, v) \in \mathcal{X}_2$. Observe that the multiplication by $f(u)$ or $g(u)v$ preserves the Dirichlet boundary condition. We set

$$F_{21}(w) = \Delta\big(f(u) \,\cdot\,\big) - f(u)\Delta, \quad F_{22}(w) = \Delta\big(g(u)v\Delta^{-1} \,\cdot\,\big) - g(u)v,$$

where $F_2(w) = (F_{21}(w), F_{22}(w))$ is considered as an operator from $\mathcal{X}_0$ to $L^2(\Omega)$ acting as $F_2(w)\Psi = (F_{21}(w)\varphi, F_{22}(w)\psi)$. We claim that $F_2(w)$ is uniformly bounded for $w = (u, v) \in \mathcal{X}_2$ with $|w|_{\mathcal{X}_2} \le R$, where we let $R \ge 1$.

To show the claim, we let $\Psi = (\varphi, \psi) \in \mathcal{X}_0$. The first component reads

$$F_{21}(w)\varphi = f''(u)|\nabla u|^2 \varphi + 2f'(u)\nabla u \cdot \nabla\varphi + f'(u)\Delta u\, \varphi,$$

while we obtain for the second component

$$F_{22}(w)\psi = \Big(g'(u)\Delta u\, v + g''(u)|\nabla u|^2 v + 2g'(u)\nabla u\cdot\nabla v + g(u)\Delta v\Big)\Delta^{-1}\psi \\ + 2\big(g(u)\nabla v + g'(u)v\nabla u\big)\cdot\nabla\Delta^{-1}\psi.$$

All the appearing nonlinear functions of u belong to $L^\infty(\Omega)$ and are bounded by a constant C depending on r, f and g, due to (1.6) and (2.4). Using Hölder's inequality and Sobolev's embedding (2.1), we then deduce

$$\begin{aligned} |F_{21}(w)\phi|_{L^2} &\le C\Big(\big|\,|\nabla u|^2\varphi\big|_{L^2} + \big|\,|\nabla u|\,|\nabla\varphi|\,\big|_{L^2} + \big|\Delta u\,\varphi\big|_{L^2}\Big)\\ &\le C\Big(|\nabla u|_{L^6}^2|\varphi|_{L^6} + |\nabla u|_{L^\infty}|\nabla\varphi|_{L^2} + |\Delta u|_{L^3}|\varphi|_{L^6}\Big)\\ &\le CR^2\,|\varphi|_{H^1} \end{aligned}$$

since we have $|u|_{H^3} \le R \le R^2$. To control $F_{22}(w)\psi$, we follow the same pattern and estimate

$$\begin{aligned} |F_{22}(w)\psi|_{L^2} &\le C\Big(|\Delta u|_{L^6}|v|_{L^6} + |\nabla u|_{L^6}^2|v|_{L^\infty} + |\nabla u|_{L^6}|\nabla v|_{L^6}\Big)|\Delta^{-1}\psi|_{L^6}\\ &\qquad + C\,|\Delta v|_{L^2}|\Delta^{-1}\psi|_{L^\infty} + C\big(|\nabla v|_{L^3} + |v|_{L^\infty}|\nabla u|_{L^3}\big)|\nabla\Delta^{-1}\psi|_{L^6}\\ &\le CR^3\,|\nabla\Delta^{-1}\psi|_{H^1} \le CR^3\,|\psi|_{L^2}, \end{aligned}$$

using $|u|_{H^3} + |v|_{H^2} \le CR$ and the regularisation properties of Δ^{-1} stated in Section 2. Setting $F(w)\Psi := (0, F_2(w)\Psi)$, we thus arrive at

$$|F(w)\Psi|_{\mathcal{X}_0} = |F_2(w)\Psi|_{L^2(\Omega)} \le CR^3\,|\Psi|_{\mathcal{X}_0}$$

for $w \in \mathcal{X}_2$. In view of (3.2), the operator $F(w(t))$ is bounded by C_3R^3 on $(\mathcal{X}_0, |\cdot|_{\mathcal{X}_{0,u(t)}})$ for a constant $C_3 \ge C_1$ (showing again the variable t).

Lemma 3.2 and the bounded perturbation theorem (see e.g. Theorem 3.1.1 in [Paz83]) thus imply that $SA(w(t))S^{-1}$ generates a semigroup $(T_t(\tau))_{\tau\ge 0}$ on $(\mathcal{X}_0, |\cdot|_{\mathcal{X}_{0,u(t)}})$ which is bounded by $\mathrm{e}^{C_3R^3\tau}$ for all $\tau \ge 0$ and $t \in [0,T]$. Proposition 2.4 of [Kat70] (or Theorem 4.5.8 in [Paz83]) then yields that $T_t(\tau)$ is equal to $S\mathrm{e}^{\tau A(w(t))}S^{-1}$ and that the part of $A(w(t))$ in $\mathcal{X}_2$ generates a C^0-semigroup on $\mathcal{X}_2$, which is the restriction of $\mathrm{e}^{\tau A(w(t))}$ to $\mathcal{X}_2$ and hence denoted by the same symbol. We now introduce equivalent norms on $\mathcal{X}_2$ given by $|\Psi|_{\mathcal{X}_{2,u(t)}} = |S\Psi|_{\mathcal{X}_{0,u(t)}}$. Estimate (3.4) and the just stated results yield

$$\begin{aligned} |\Psi|_{\mathcal{X}_{2,u(t)}} &= |S\Psi|_{\mathcal{X}_{0,u(t)}} \le \mathrm{e}^{C_2L|t-s|}|S\Psi|_{\mathcal{X}_{0,u(s)}} = \mathrm{e}^{C_2L|t-s|}|\Psi|_{\mathcal{X}_{2,u(s)}},\\ |\mathrm{e}^{\tau A(w(t))}\Psi|_{\mathcal{X}_{2,u(t)}} &= |S\mathrm{e}^{\tau A(w(t))}S^{-1}S\Psi|_{\mathcal{X}_{0,u(t)}} \le \mathrm{e}^{C_3R^3\tau}|\Psi|_{\mathcal{X}_{2,u(t)}} \end{aligned} \tag{3.5}$$

for $\Psi \in \mathcal{X}_2$, $\tau \ge 0$ and $t, s \in [0,T]$. Exactly as in step 1), we then deduce that $\big(A(w(t))\big)_{t\in[0,T]}$ is stable on $\mathcal{X}_2$ with constants $M = C_0^2\mathrm{e}^{C_2LT}$ and $\beta = C_3R^3$. The stability on $\mathcal{X}_1$ then follows by interpolation, see e.g. Theorem V.1.5.4 in [Ama95]. □

As the last major ingredient for the linear well-posedness result, we establish the Lipschitz continuity of the map $w \mapsto A(w)$.

Lemma 3.4. *There is a constant $C_5 > 0$ such that*

$$|A(w) - A(\overline{w})|_{\mathcal{B}(\mathcal{X}_2,\mathcal{X}_1)} \le C_5\,|w-\overline{w}|_{\mathcal{X}_1} \qquad \text{and} \qquad |A(w)|_{\mathcal{B}(\mathcal{X}_2,\mathcal{X}_1)} \le C_5$$

for all $w, \overline{w} \in E(R)$.

Proof. Let $w = (u,v)$ and $\overline{w} = (\overline{u},\overline{v})$ belong to $\mathcal{X}_1$ and take $\Psi = (\varphi,\psi) \in \mathcal{X}_2$. From (2.3) we deduce

$$\begin{aligned} |(A(w) - A(\overline{w}))\Psi|_{\mathcal{X}_1} &\le |(f(u) - f(\overline{u}))\Delta\varphi|_{\mathcal{H}^1} + |(g(u)v - g(\overline{u})\overline{v})\psi|_{\mathcal{H}^1} \\ &= \big|\nabla[(f(u) - f(\overline{u}))\Delta\varphi]\big|_{L^2} + \big|\nabla[(g(u)v - g(\overline{u})\overline{v})\psi]\big|_{L^2}. \end{aligned} \tag{3.6}$$

(Recall that we have chosen the norm $|\nabla\,\cdot\,|_{L^2}$ on $\mathcal{H}^1$.) The first term in the last line can be written in the form

$$\begin{aligned} \nabla\big((f(u) - f(\overline{u}))\Delta\varphi\big) &= \big(f'(u)\nabla u - f'(\overline{u})\nabla\overline{u}\big)\Delta\varphi + \big(f(u) - f(\overline{u})\big)\nabla(\Delta\varphi) \\ &= \big((f'(u) - f'(\overline{u}))\nabla u + f'(\overline{u})\nabla(u-\overline{u})\big)\Delta\varphi + \big(f(u) - f(\overline{u})\big)\nabla(\Delta\varphi). \end{aligned}$$

We estimate this term as in the proof of Lemma 3.3 and thus derive

$$\begin{aligned} &\big|\nabla\big((f(u) - f(\overline{u}))\Delta\varphi\big)\big|_{L^2} \\ &\quad\le C\Big(|u-\overline{u}|_{L^\infty}|\nabla u\,\Delta\varphi|_{L^2} + |\nabla(u-\overline{u})\Delta\varphi|_{L^2} + |u-\overline{u}|_{L^\infty}|\nabla(\Delta\varphi)|_{L^2}\Big) \\ &\quad\le C\Big(|u-\overline{u}|_{L^\infty}|\nabla u|_{L^3}|\Delta\varphi|_{L^6} + |\nabla(u-\overline{u})|_{L^3}|\Delta\varphi|_{L^6} + |u-\overline{u}|_{L^\infty}|\varphi|_{H^3}\Big) \\ &\quad\le C\big(1 + |u|_{H^2}\big)|u-\overline{u}|_{H^2}|\varphi|_{H^3}. \end{aligned}$$

This shows the required bound for the first summand in (3.6) since $|u|_{H^2} \le r$. For the second summand we obtain

$$\nabla\big((g(u)v - g(\overline{u})\overline{v})\psi\big) = \nabla\big(g(u)v - g(\bar{u})\overline{v}\big)\psi + \big(g(u)v - g(\overline{u})\overline{v}\big)\nabla\psi.$$

The gradient in the first term is expanded as

$$\begin{aligned} \nabla\big(g(u)v - g(\overline{u})\overline{v}\big) &= g'(u)\nabla u\,v - g'(\overline{u})\nabla\overline{u}\,\overline{v} + g(u)\nabla v - g(\overline{u})\nabla\overline{v} \\ &= (g'(u) - g'(\overline{u}))\nabla u\,v + g'(\overline{u})(\nabla u\,v - \nabla\overline{u}\,\overline{v}) \\ &\qquad + (g(u) - g(\overline{u}))\nabla v + g(\overline{u})\nabla(v-\overline{v}) \\ &= (g'(u) - g'(\overline{u}))\nabla u\,v + g'(\overline{u})\nabla(u-\overline{u})v + g'(\overline{u})\nabla\overline{u}\,(v-\overline{v}) \\ &\qquad + (g(u) - g(\overline{u}))\nabla v + g(\overline{u})\nabla(v-\overline{v}). \end{aligned}$$

As before, we estimate

$$\begin{aligned} &\big|\nabla\big(g(u)v - g(\overline{u})\overline{v}\big)\psi\big|_{L^2} \\ &\quad\le C\Big(|u-\overline{u}|_{L^\infty}|\nabla u\,v\psi|_{L^2} + |\nabla(u-\overline{u})v\psi|_{L^2} + |\nabla\overline{u}\,(v-\overline{v})\psi|_{L^2} \\ &\qquad\quad + |u-\overline{u}|_{L^\infty}|\nabla v\,\psi|_{L^2} + |\nabla(v-\overline{v})\psi|_{L^2}\Big) \\ &\quad\le C\Big(|u-\overline{u}|_{L^\infty}|\nabla u|_{L^6}|v|_{L^6}|\psi|_{L^6} + |\nabla(u-\overline{u})|_{L^6}|v|_{L^6}|\psi|_{L^6} \\ &\qquad\quad + |\nabla\overline{u}|_{L^6}|v-\overline{v}|_{L^6}|\psi|_{L^6} + |u-\overline{u}|_{L^\infty}|\nabla v|_{L^2}|\psi|_{L^\infty} + |\nabla(v-\overline{v})|_{L^2}|\psi|_{L^\infty}\Big) \\ &\quad\le C(1 + |u|_{H^2}^2 + |v|_{H^1}^2)|w-\overline{w}|_{\mathcal{X}_1}\,|\psi|_{H^2}. \end{aligned}$$

For the remaining term we write

$$g(u)v - g(\overline{u})\overline{v} = (g(u) - g(\overline{u}))v + g(\overline{u})(v - \overline{v}),$$

and conclude correspondingly

$$\begin{aligned}\big|(g(u)v - g(\overline{u})\overline{v})\nabla\psi\big|_{L^2} &\le |(g(u) - g(\overline{u}))v\nabla\psi|_{L^2} + |g(\overline{u})(v - \overline{v})\nabla\psi|_{L^2}\\ &\le C\Big(|u - \overline{u}|_{L^\infty}|v|_{L^3}|\nabla\psi|_{L^6} + |v - \overline{v}|_{L^3}|\nabla\psi|_{L^6}\Big)\\ &\le C\big(1 + |v|_{H^1}\big)|w - \overline{w}|_{\mathcal{X}_1}\,|\psi|_{H^2}.\end{aligned}$$

Using also $|v|_{H^1} \le r$, we obtain the first claim. The second inequality follows immediately by setting $\overline{w} := 0$. □

Corollary 3.5. *Let $\widetilde{w} \in E(T,R,L)$. Then $A(\widetilde{w}(t)) \in \mathcal{B}(\mathcal{X}_2,\mathcal{X}_1)$ for all $t \in [0,T]$ and the map $t \mapsto A(\widetilde{w}(t))$ is Lipschitz-continuous with*

$$[A(\widetilde{w})]_{\mathrm{Lip}(0,T;\mathcal{B}(\mathcal{X}_2,\mathcal{X}_1))} \le C_5 L.$$

Proof. We use Lemma 3.4 with $w = \widetilde{w}(t)$ and $\overline{w} = \widetilde{w}(s)$, yielding the bound

$$|A(\widetilde{w}(t)) - A(\widetilde{w}(s))|_{\mathcal{B}(\mathcal{X}_2,\mathcal{X}_1)} \le C_5|\widetilde{w}(t) - \widetilde{w}(s)|_{\mathcal{X}_1} \le C_5 L\,|t - s|. \qquad \square$$

For $\widetilde{w} \in E(T,R,L)$, we have thus established the properties:

(1) $A(\widetilde{w}(t))$ generates a C^0-semigroup on $\mathcal{X}_1$ for all $t \in [0,T]$.
(2) $\big(A(\widetilde{w}(t))\big)_{t\in[0,T]}$ is $\mathcal{X}_1$-stable.
(3) The parts of $A(\widetilde{w}(t))$ in $\mathcal{X}_2$ generate a C^0-semigroup for each t and $\big(A(\widetilde{w}(t))\big)_{t\in[0,T]}$ is $\mathcal{X}_2$-stable.
(4) $A(\widetilde{w}(t)) \in \mathcal{B}(\mathcal{X}_2,\mathcal{X}_1)$ for each t and the map $t \mapsto A(\widetilde{w}(t))$ is norm-continuous in $\mathcal{B}(\mathcal{X}_2,\mathcal{X}_1)$.
(5) $\mathcal{X}_2$ is a Hilbert space. There are equivalent Hilbert norms $|\cdot|_t$ on $\mathcal{X}_2$ such that $A(\widetilde{w}(t)) - C_3R^3$ is contractive for $|\cdot|_t$ and $|\Psi|_t \le \mathrm{e}^{C_2L|t-s|}|\Psi|_s$ for all $t,s \in [0,T]$ and $\Psi \in \mathcal{X}_2$.
(6) The analogous assertions hold for the family $(-A(\widetilde{w}(T-t)))_{t\in[0,T]}$.

(The second part of property (5) follows from (3.5).) Combined with the results in [Kat70], these facts yield the well-posedness of (3.1), as recorded in the next proposition. We call a family $U(t,s)$, $0 \le s \le t \le T$, of bounded linear operators on a Banach space X an *evolution family* if it is strongly continuous in (t,s), $U(s,s) = I$ and $U(t,s) = U(t,r)U(r,s)$ for all $0 \le s \le r \le t \le T$.

Proposition 3.6. *Let $\widetilde{w},\overline{w} \in E(T,R,L)$ and assume that* (1.6) *holds. Then there is a unique evolution family $U_{\widetilde{w}}(t,s)$ for $0 \le s \le t \le T$ in $\mathcal{B}(\mathcal{X}_1)$, where*

(a) *$U_{\widetilde{w}}(t,s)\mathcal{X}_2 \subset \mathcal{X}_2$, $U_{\widetilde{w}}(t,s)$ is strongly continuous on $\mathcal{X}_2$ in (t,s) and*

$$\|U_{\widetilde{w}}(t,s)\|_{\mathcal{B}(\mathcal{X}_k)} \le C_4 \exp\big((C_2L + C_3R^3)(t-s)\big), \qquad k = 1,2;$$

(b) *the derivatives*

$$\begin{aligned}\partial_t U_{\widetilde{w}}(t,s)w_0 &= A(\widetilde{w}(t))U_{\widetilde{w}}(t,s)w_0,\\ \partial_s U_{\widetilde{w}}(t,s)w_0 &= -U_{\widetilde{w}}(t,s)A(\widetilde{w}(s))w_0\end{aligned}$$

exist in $\mathcal{X}_1$ and are continuous in (t,s);

(c) *we have in* $\mathcal{X}_1$

$$U_{\widetilde{w}}(t,s)w_0 - w_0 = \int_s^t A(\widetilde{w}(\tau))U_{\widetilde{w}}(\tau,s)w_0 \,\mathrm{d}\tau,$$

$$U_{\widetilde{w}}(t,s)w_0 - U_{\overline{w}}(t,s)w_0 = \int_s^t U_{\widetilde{w}}(t,\tau)\big(A(\overline{w}(\tau)) - A(\widetilde{w}(\tau))\big)U_{\overline{w}}(\tau,s)w_0 \,\mathrm{d}\tau$$

for all $0 \le s \le t \le T$ *and* $w_0 \in \mathcal{X}_2$.

Proof. We first note that (c) follows from (b) by integration. Let $\widetilde{w}$ belong to $E(T,R,L)$. Properties (1)–(4) are the hypotheses of Theorem 4.1 in [Kat70] (or of Theorem 5.3.1 in [Paz83]). This result shows the existence and uniqueness of an evolution family $U_{\widetilde{w}}(t,s)$ on $\mathcal{X}_1$ satisfying the asserted estimate on $\mathcal{X}_1$ and the second differential equation in (b). Theorems 5.1 and 5.2 and equation (5.2) of [Kat70] require only two more assumptions which are somewhat weaker than (5). Among other points, they say that (for each $w_0 \in \mathcal{X}_2$ and $t \in [0,T]$)

- $U_{\widetilde{w}}(t,s)$ leaves invariant $\mathcal{X}_2$ with the bound in (a),
- the map $[0,t_0] \to \mathcal{X}_2$; $s \mapsto U_{\widetilde{w}}(t_0,s)w_0$, is continuous,
- and $U_{\widetilde{w}}(t,s)w_0 \to w_0$ in $\mathcal{X}_2$ as $(t,s) \to (t_0,t_0)$, with $0 \le s \le t \le T$.

Using also property (6), we can further apply Theorem 7.7.13 of [Fat83] (or Remark 5.3 of [Kat70]), which yields the first equation in (b) and the continuity of $t \mapsto V(t) := A(\widetilde{w}(t))U_{\widetilde{w}}(t,s)w_0$ in $\mathcal{X}_1$, for $w_0 \in \mathcal{X}_2$, $s \in [0,T]$ and $t \in [s,T]$.

To check the remaining continuity assertions, we first note that the domain of the part of $A(\widetilde{w}(t))$ in $\mathcal{X}_1$ is equal to $\mathcal{X}_2$ and that its graph norm is equivalent to that of $\mathcal{X}_2$ uniformly in $t \in [0,T]$. Indeed, let $(u,v) \in \mathcal{X}_1$ satisfy

$$A(\widetilde{w}(t))(u,v) = (v, f(\widetilde{u}(t))\Delta_D u + g(\widetilde{u}(t))\widetilde{v}(t)v) = (\varphi,\psi) \in \mathcal{X}_1.$$

Since $\widetilde{w} \in E(T,R,L)$, we derive $v \in \mathcal{H}_2$, $\Delta_D u = f(\widetilde{u}(t))^{-1}(\psi - g(\widetilde{u}(t))\widetilde{v}(t)v) =: h(t) \in \mathcal{H}_1$ and that $|h(t)|_{\mathcal{H}_1}$ is uniformly bounded. Hence, $u = \Delta_D^{-1}h(t)$ belongs to $\mathcal{H}_3$ and its norm in $\mathcal{H}_3$ is bounded independently of $t \in [0,T]$.

For a fixed $\lambda > C_3R^3$, the resolvent $(\lambda - A(\widetilde{w}(t)))^{-1}$ thus belongs to $\mathcal{B}(\mathcal{X}_1,\mathcal{X}_2)$ and is uniformly bounded in this space for $t \in [0,T]$. This fact and (4) yield that the resolvent is continuous as a map from $[0,T]$ to $\mathcal{B}(\mathcal{X}_1,\mathcal{X}_2)$. So the continuity of V seen above implies that $t \mapsto U_{\widetilde{w}}(t,s)w_0$ is continuous in $\mathcal{X}_2$.

We now proceed as indicated in Remark 5.4 of [Kat70]. Let $0 \le s_0 < t_0 \le T$ and $(t_n,s_n) \to (t_0,s_0)$ for $0 \le s_n \le t_n \le T$. Fix $\tau \in (s_0,t_0)$. For large n we have $t_n > \tau > s_n$ and so $U_{\widetilde{w}}(t_n,s_n)w_0 = U_{\widetilde{w}}(t_n,\tau)U_{\widetilde{w}}(\tau,s_n)w_0$ tends to $U_{\widetilde{w}}(t_0,s_0)w_0$ in $\mathcal{X}_2$ by the established strong continuity of $U_{\widetilde{w}}(t,s)$ in t and in s separately. We have thus shown the strong continuity of $(t,s) \mapsto U_{\widetilde{w}}(t,s)$ in $\mathcal{X}_2$. Combined with (4), it implies the remaining parts of (b). □

4. The nonlinear problem

Having solved the linear nonautonomous problem, we proceed with the main result of this paper, we can now solve the nonlinear problem by a fixed point method. Recall the definition of $r > 0$ in (2.4) and that we treat space dimensions $n \le 3$. The constant C_4 is taken from Lemma 3.3.

Theorem 4.1. *Let* (1.6) *be true and* $w_0 = (u_0, v_0) \in \mathcal{X}_2$ *satisfy* $|w_0|_{\mathcal{X}_1} \le \eta r/C_4$ *for some* $\eta \in (0,1)$. *Then there is a time* $T = T(\eta, r, |w_0|_{\mathcal{X}_2}) > 0$ *and a function* $w \in C^1(0,T;\mathcal{X}_1) \cap C(0,T;\mathcal{X}_2)$ *with* $|w(t)|_{\mathcal{X}_1} \le r$ *for* $t \in [0,T]$ *that solves*

$$\frac{\mathrm{d}}{\mathrm{d}t} w(t) = A(w(t))w(t), \quad t \in [0,T], \qquad w(0) = w_0. \tag{4.1}$$

If we set $w(t) = (u(t), v(t))$, *then* $u = \partial_t v$ *and* $u \in C(0,T;\mathcal{H}_3) \cap C^1(0,T;\mathcal{H}_2) \cap C^2(0,T;\mathcal{H}_1)$ *satisfies* $|(u(t), \partial_t u(t))|_{\mathcal{X}_1} \le r$ *and*

$$\begin{aligned} \partial_t^2 u &= f(u)\Delta_D u + g(u)(\partial_t u)^2, \quad t \in [0,T], \\ u(0) &= u_0, \qquad \partial_t u(0) = v_0. \end{aligned} \tag{4.2}$$

Any other solution of (4.2) *in this class on a time interval* $[0, T']$ *coincides with* u *on* $[0, \min\{T, T'\})]$. *Moreover, the map*

$$\overline{B}_{\mathcal{X}_1}(0, \tfrac{\eta r}{C_4}) \cap \overline{B}_{\mathcal{X}_2}(0; R) \to C(0,T;\mathcal{H}_2) \times C(0,T;\mathcal{H}_1); \quad (u_0, v_0) \mapsto (u, \partial_t u),$$

is Lipschitz for every $R > 0$, *where* $T = T(\eta, r, R)$.

Finally, if f *and* g *are given as in* (1.4) *for a function* K *fulfilling* (1.2), *then the assertions also hold if we replace in* (4.2) *the PDE by* $\partial_{tt}(u + K(u)) = \Delta_D u$.

Proof. 1) We construct a solution of (4.1) by means of the contraction principle. Let $w_0 \in \mathcal{X}_2$ with $|w_0|_{\mathcal{X}_1} \le \eta r/C_4$ for some $\eta \in (0,1)$. We fix numbers $R = 2C_4|w_0|_{\mathcal{X}_2}$ and $L = RC_5$ for the constants C_4 and C_5 from Lemmas 3.3 and 3.4, respectively. We further fix a sufficiently small final time $T > 0$ such that

$$\mathrm{e}^{(C_2 L + C_3 R^3)T} \le \min\{2, 1/\eta\} \quad \text{and} \quad T \le \frac{1}{2C_4 L}. \tag{4.3}$$

For these parameters we define the set $E_T := E(T, R, L)$ as in (3.3) and endow it with the metric

$$d\big(w, \overline{w}\big) := \|w - \overline{w}\|_{C(0,T;\mathcal{X}_1)}.$$

Then $(C(0,T;\mathcal{X}_1), d)$ is a complete normed space, and the subset of $w \in C(0,T;\mathcal{X}_1)$ with $|w(t)|_{\mathcal{X}_1} \le r$ for all $t \in [0,T]$ and $[w]_{\mathrm{Lip}(0,T;\mathcal{X}_1)} \le L$ is closed in $(C(0,T;\mathcal{X}_1), d)$. The reflexivity of $\mathcal{X}_2$ yields that every ball in $\mathcal{X}_2$ is weakly closed. Thus, if a sequence $\{\Psi_k\}_k \subset E(R)$ converges in $\mathcal{X}_1$ to some Ψ, then $\Psi \in E(R)$. As a result, (E_T, d) is complete. For $w \in E_T$ we define

$$\Phi_{w_0}(w)(t) := U_w(t, 0)w_0,$$

where U_w is given by Proposition 3.6. We look for a fixed point of Φ_{w_0} on E_T

To show that Φ_{w_0} maps E_T into itself, we first note that Proposition 3.6 and (4.3) yield

$$|\Phi_{w_0}(w)(t)|_{\mathcal{X}_k} \le C_4 \mathrm{e}^{(C_2 L + C_3 R^3)T} |w_0|_{\mathcal{X}_k} \le \begin{cases} r & \text{for } k = 1, \\ R & \text{for } k = 2, \end{cases} \tag{4.4}$$

for all $t \in [0,T]$. For the Lipschitz bound for $t \mapsto \Phi_{w_0}(w)(t)$, we estimate

$$\begin{aligned} \big|U_w(t,0)w_0 - U_w(t',0)w_0\big|_{\mathcal{X}_1} &\le \int_{t'}^{t} |A(w(s))U_w(s,0)w_0|_{\mathcal{X}_1}\,\mathrm{d}s \\ &\le \sup_{s\in[0,T]} \big\{ |A(w(s))|_{\mathcal{B}(\mathcal{X}_2,\mathcal{X}_1)} |U_w(s,0)w_0|_{\mathcal{X}_2} \big\} \, |t - t'| \end{aligned}$$

$$\leq C_5 C_4 \mathrm{e}^{(C_2 L + C_3 R^3)T} |w_0|_{\mathcal{X}_2} |t-t'| \leq L\,|t-t'|,$$

for $0 < t' < t < T$, using Proposition 3.6, Lemma 3.4 and (4.3). Hence, $[\Phi_{w_0}(w)]_{\mathrm{Lip}(0,T;\mathcal{X}_1)} \leq L$ and $\Phi_{w_0}(w) \in E_T$.

To prove that Φ_{w_0} is a strict contraction in $(C(0,T;\mathcal{X}_1)$, we take $\widetilde{w}, \overline{w} \in E_T$. Proposition 3.6 Lemma 3.4 and (4.3) yield as above that

$$\begin{aligned}
&|U_{\widetilde{w}}(t,0)w_0 - U_{\overline{w}}(t,0)w_0|_{\mathcal{X}_1} \\
&\quad \leq \int_0^t \big| U_{\widetilde{w}}(t,s)\big(A(\widetilde{w}(s)) - A(\overline{w}(s))\big) U_{\overline{w}}(s,0) w_0 \big|_{\mathcal{X}_1} \,\mathrm{d}s \\
&\quad \leq T \sup_{s\in[0,T]} \big\{ |U_{\widetilde{w}}(t,s)|_{\mathcal{B}(\mathcal{X}_1)} |A(\widetilde{w}(s)) - A(\overline{w}(s))|_{\mathcal{B}(\mathcal{X}_2,\mathcal{X}_1)} |U_{\overline{w}}(s,0)|_{\mathcal{B}(\mathcal{X}_2)} \big\} \, |w_0|_{\mathcal{X}_2} \\
&\quad \leq T C_4^2 \mathrm{e}^{(C_2 L + C_3 R^3)T} C_5 \,\|\widetilde{w} - \overline{w}\|_{C(0,T;\mathcal{X}_1)} \, |w_0|_{\mathcal{X}_2} \\
&\quad \leq C_4 L T \,\|\widetilde{w} - \overline{w}\|_{C(0,T;\mathcal{X}_1)} \leq \frac{1}{2} \,\|\widetilde{w} - \overline{w}\|_{C(0,T;\mathcal{X}_1)}, \qquad t \in [0,T].
\end{aligned}$$

The contraction principle now gives a (unique) function $w \in E_T$ satisfying $w = U_w(\,\cdot\,,0)w_0$, or, $\frac{\mathrm{d}}{\mathrm{d}t} w(t) = A(w(t))w(t)$ for $t \in (0,T)$ and $w(0) = w_0$, due Proposition 3.6.

2) Let there be a solution $\overline{w} \in C^1(0,T';\mathcal{X}_1) \cap C(0,T';\mathcal{X}_2)$ of (4.1) with $\overline{w}(0) = w_0$ and $|\overline{w}(t)|_{\mathcal{X}_1} \leq r$ for all $t \in [0,T']$ and some $T' > 0$. Proposition 3.6 and (4.1) now yield

$$\overline{w}(t) - w(t) = \int_0^t \partial_s \big(U_w(t,s)\overline{w}(s)\big) \mathrm{d}s = \int_0^t U_w(t,s)\big(A(\overline{w}(s)) - A(w(s))\big)\overline{w}(s) \mathrm{d}s$$

for all $t \in [0, \min\{T,T'\}]$. Proposition 3.6 and Lemma 3.4 lead to

$$|\overline{w}(t) - w(t)|_{\mathcal{X}_1} \leq C_5 M e^{\beta T} \,\|\overline{w}\|_{C(0,T';\mathcal{X}_2)} \int_0^t |\overline{w}(s) - w(s)|_{\mathcal{X}_1} \,\mathrm{d}s$$

for some $M, \beta > 0$ so that $w = \overline{w}$ on $[0, \min\{T,T'\}]$ by Gronwall's inequality.

3) We show the continuous dependence of w on the initial data. Let $w_0, \overline{w}_0 \in \overline{B}_{\mathcal{X}_1}(0, \eta r/C_4) \cap \overline{B}_{\mathcal{X}_2}(0,R)$ where $\eta \in (0,1)$ and $R > 0$. The first part of the proof provides fixed points $w(\,\cdot\,, w_0) = w = \Phi_{w_0} w$ and $\overline{w}(\,\cdot\,, \overline{w}_0) = \overline{w} = \Phi_{\overline{w}_0} \overline{w}$ in E_T which solve (4.1). (Note that we can take the same T in view of (4.3).) The strict contractivity and Proposition 3.6 then imply

$$\begin{aligned}
\big| w(t,w_0) - \overline{w}(t,\overline{w}_0) \big|_{\mathcal{X}_1} &= \big| U_w(t,0)w_0 - U_{\overline{w}}(t,0)\overline{w}_0 \big|_{\mathcal{X}_1} \\
&\leq \big| \Phi_{w_0}(w)(t) - \Phi_{w_0}(\overline{w})(t) \big|_{\mathcal{X}_1} + \big| U_{\overline{w}}(t,0)(w_0 - \overline{w}_0) \big|_{\mathcal{X}_1} \\
&\leq \frac{1}{2} \big| w(t,w_0) - \overline{w}(t,\overline{w}_0) \big|_{\mathcal{X}_1} + M \mathrm{e}^{\beta T} \big| w_0 - \overline{w}_0 \big|_{\mathcal{X}_1}
\end{aligned}$$

for some $M, \beta > 0$. This gives

$$\big| w(t,w_0) - \overline{w}(t,\overline{w}_0) \big|_{\mathcal{X}_1} \leq 2M \mathrm{e}^{\beta T} \big| w_0 - \overline{w}_0 \big|_{\mathcal{X}_1}$$

for $t \in [0,T]$, as asserted.

4) Finally, we transfer the results to the second order problems. If $w = (u,v)$ solves (4.1), then $v = \partial_t u$ and it easily follows that u belongs to $C(0,T;\mathcal{H}_3) \cap C^1(0,T;\mathcal{H}_2) \cap C^2(0,T;\mathcal{H}_1)$ and satisfies (4.2). Conversely, if u belongs to this

space and satisfies $|(u(t), \partial_t u(t)|_{\mathcal{X}_1} \leq r$ and (4.2) on $[0,T]$, then $w = (u, \partial_t u)$ is contained $C^1(0,T;\mathcal{X}_1) \cap C(0,T;\mathcal{X}_2)$ and solves (4.1). This equivalence then implies the remaining assertions concerning (4.2). In a similar way, we derive the last part of the theorem. □

An inspection of the proofs yields that for $f, g \in C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$ with $f > 0$, the smallness condition in $\mathcal{X}_1$ can be dropped. (Observe that we still bound the norms in $\mathcal{X}_2$ by R.) In view of (1.4) we thus obtain our final result.

Corollary 4.2. *In the setting of Theorem 4.1 we consider the Kerr nonlinearity* $k(z) = \lambda z^3$ *for some* $\lambda > 0$. *Then we can drop in the theorem all restrictions involving* r.

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