Fakultät für Mathematik
Institut für Analysis
Dr. Rainer Mandel

Seminar, Winter term 2018/19

# Boundary and Eigenvalue Problems

Dr. Rainer Mandel

## Data

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| Date: | Friday, 14:00-15:30 (preliminary) |
| Room: | Seminar room 3.068, building 20.30 (preliminary) |
| Participants: | Advanced Bachelor or Master students in mathematics |
| Requirements: | Boundary and Eigenvalue Problems or Partial Differential Equations |

## Subject

The seminar is intended to extend and deepen the participants' knowledge about boundary and eigenvalue problems. A list of possible topics includes:

1. **Absence of eigenvalues for certain Schrödinger operators**
   In 1958 Kato proved that if $V(x)$ converges to $V_\infty < 0$ sufficiently fast as $|x| \to \infty$, then the Schrödinger operator $-\Delta + V(x)$ does not have positive eigenvalues in the sense that
   $$-\Delta\psi + V(x)\psi = \lambda\psi \quad \text{in } \mathbb{R}^n, \quad \lambda > 0 \qquad \Rightarrow \quad \psi \notin L^2(\mathbb{R}^n).$$
   The student is asked to prove this result.
   **Task:** Present, illustrate and prove this result to be found on the first 11 pages of the given source.
   **Source:** Kato: Growth Properties of Solutions of the Reduced Wave Equation With a Variable Coefficient, Comm. Pure Appl. Math. 12 (1959) 403-425.
2. **Exponential decay of eigenfunctions**
   In this talk it is shown that certain eigenfunctions of Schrödinger operators $-\Delta + V(x)$ on $\mathbb{R}^n$ decay exponentially at infinity. To this end, the mathematical tool called Agmon's metric has to be introduced in order to prove the estimate $|\varphi(x)| \leq Ce^{-c|x|}$ for some explicit $c > 0$ and certain eigenfunctions $\varphi$ of the Schrödinger operator.
   **Task:** Introduce the concept of Agmon's metric and prove the result on exponential decay (Theorem 3.10).
   **Source:** Hislop, Sigal: Introduction to spectral theory (p.27-p.36), see also Pankov: Lecture Notes on Schrödinger Equations (p.113-p.119
3. **The Faber-Krahn inequality**
   The aim is to prove the Faber-Krahn inequality, which says that the first Dirichlet eigenvalue of the Laplacian on a given domain $\Omega \subset \mathbb{R}^n$ is larger than the first eigenvalue on a ball in $\mathbb{R}^n$ with the same volume, i.e.
   $$\lambda_1(\Omega) \geq \lambda_1(\Omega^*).$$
   Here, $\Omega^*$ denotes a(ny) ball with the same volume as $\Omega$, i.e. we have $|\Omega| = |\Omega^*|$.
   **Task:** Present the relevant facts about "Schwarz symmetrization" (Chapter 1), prove

Talenti's comparison principle (p.48-p.50) and apply this result to the proof of the Faber-Krahn inequality (p.83-p.88)
**Source:** Kesavan: Symmetrization & Applications

4. **The isoperimetric inequality and the best Sobolev constant for $p = 1$**
The aim of this talk is to determine the best Sobolev constant, i.e. the best possible $C > 0$ such that the Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq C \int_{\mathbb{R}^n} |\nabla u| \qquad \text{for all } u \in C_0^\infty(\mathbb{R}^n)$$

holds.
**Task:** Prove the "Isoperimetric Inequality" (p.19-27) and derive the formula for the best Sobolev constant (p.28-29, 44-46)
**Source:** Kesavan: Symmetrization & Applications, see also Ziemer: Weakly Differentiable Functions (p.76-p.83)

5. **The Polya-Szegö Inequality**
The student proves that the the symmetrized function $u^*$ of any given $u \in W^{1,p}(\Omega)$ lies in $W^{1,p}(\Omega^*)$ and even satisfies

$$\int_{\Omega^*} |\nabla u^*|^p \leq \int_{\Omega} |\nabla u|^p.$$

Here, $\Omega^*$ is as in the talk about the Faber-Krahn inequality.
**Task:** Review the relevant material about "Schwarz symmetrization" from Chapter 1 and prove the above-mentioned inequality (p.35-38). Discuss Remark 2.3.3, Theorem 2.3.2 and the use of the Polya-Szegö Inequality in an alternative proof of the Faber-Krahn-inequality and Sobolev's imbedding theorem.
**Source:** Kesavan: Symmetrization & Applications.

6. **The best Sobolev constant for $p \in (1, n)$**
In the talk the student should present Talenti's derivation of the Sobolev constant $C_{p,n}$ for $p \in (1, n)$ such that

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{pn}{n-p}}\right)^{\frac{n-p}{np}} \leq C_{p,n} \left(\int_{\mathbb{R}^n} |\nabla u|^p\right)^{\frac{1}{p}} \qquad \text{for all } u \in C_0^\infty(\mathbb{R}^n)$$

**Task:** Explain the author's ideas and proofs leading towards this inequality.
**Source:** Talenti: Best constant in Sobolev inequality. Ann. Mat. Pura Appl. (4) 110 (1976), 353-372. (Lemma 1 need not be proved in view of the previous talk, so there are actually 5 pages less to present! )

7. **Weyl's law**
The eigenvalues of the Dirichlet-Laplacian $\lambda_k(\Omega)$ on a bounded smooth domain $\Omega \subset \mathbb{R}^n$ tend to $\infty$ as $k \to \infty$. In 1912, H. Weyl found that

$$\frac{\lambda_k(\Omega)}{k^{n/2}} \to \frac{|\Omega|}{(2\pi)^n |B_1(0)|} \quad \text{as } k \to \infty$$

where $|B_1(0)|$ is the volume of the unit ball. This result should be presented in the talk.
**Task:** Prove and illustrate this result with the aid of examples.
**Source:** Courant, Hilbert: Methods of Mathematical Physics Vol I, Chapter VI, Theorem 16, see also Strauss: Partial Differential Equations, Kac: Can you hear the shape of a drum?

8. **Courant's Nodal Domain Theorem**
   This theorem states that the number of nodal domains (maximal open subsets of $\Omega$ where $u$ does not change sign) of any eigenfunction associated with the $k$-th Dirichlet eigenvalue of a symmetric elliptic operator is less or equal than $k$. This extends the fact that the first eigenvalue has a positive eigenfunction (so with exactly one nodal domain).
   **Task:** Prove this result and illustrate it with examples (p.1-p.11).
   **Source:** A. Ancona, B. Helffer, Hoffmann-Ostenhof: Nodal Domain Theorems a la Courant
9. **Eigenvalue problems for nonsymmetric elliptic operators**
   The typical eigenvalue theory allows to prove the existence of eigenvalues of symmetric elliptic operators such as the Laplacian. In this talk eigenvalue problems of the form
   $$-\sum_{i,j=1}^{n} a_{ij}\partial_{ij}u + \sum_{i=1}^{n} b_i\partial_i u + cu = \lambda u \quad \text{in } \Omega, \qquad u \in H_0^1(\Omega)$$
   are investigated.
   **Task:** Prove the existence and uniqueness (up to multiplication with a scalar) of a positive eigenfunction of such an operator. p.340-p.344,p.347 and Exercise 10
   **Source:** Evans, Partial Differential Equations
10. **Homogenization**
    In physical applications, the material properties can not be resolved exactly and one is interested in effective material laws. Mathematically this can be described by finding solutions $u_\varepsilon$ of boundary value problems such as
    $$-\operatorname{div}(A(\frac{\cdot}{\varepsilon})\nabla u_\varepsilon) = f \quad \text{in } \Omega, \qquad u_\varepsilon \in H_0^1(\Omega)$$
    and study their limit as $\varepsilon \to 0$.
    **Task:** Explain and prove results from homogenization theory, notably Theorem 8.3 and Theorem 8.5.
    **Source:** Chipot: Elliptic Equations (p.106-p.118)
11. **Liouville Theorems**
    The student studies linear elliptic BVPs on the whole space $\mathbb{R}^n$ that are of the form
    $$-\operatorname{div}(A\nabla u) + cu = 0 \quad \text{in } \mathbb{R}^n$$
    It shall be proved the nonexistence of nontrivial bounded solutions to this problem. These results show that a solution theory for such BVPs does not carry over from bounded domains to $\mathbb{R}^n$ and that the theory depends on the space dimension $n$.
    **Task:** Present first the classical Liouville Theorem ($A$ = identity matrix and $c = 0$) Theorem 19.4, then Theorem 19.9 and 19.10.
    **Source:** Chipot: Elliptic Equations (p.260-p.272)
12. **Morrey's Imbedding Theorem and the Moser-Trudinger Inequality**
    The aim is to prove the imbeddings
    $$W^{1,p}(\Omega) \to L^{\phi}(\Omega) \quad \text{if } p = n, \qquad W^{1,p}(\Omega) \to C^{0,1-\frac{n}{p}}(\overline{\Omega}) \quad \text{if } n < p < \infty$$
    where $\phi(t) = \exp(|t|^{\frac{n}{n-1}}) - 1$. Here, $L^{\phi}(\Omega)$ is a so-called Orlicz space.
    **Task:** Prove the corresponding inequalities from Theorem 7.10, Theorem 7.15, Theorem 7.17 (p.155-p.164)
    **Source:** Gilbarg, Trudinger: Elliptic Partial Differential Equations of 2nd order, see also Chipot: Elliptic Equations (p.224-p.227)

13. **(. . . Other topics you are interested in . . . )**

The topics will be selected according to the interests and preknowledge of the students; including other topics may be possible. If you are interested please come to a first meeting on

**Friday, 29.06.2018, 1:15 pm (i.e. 13:15 Uhr), room 2.067, building 20.30.**

## Contact

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