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Boundary and Eigenvalue Problems

Exercise Sheet 1

Exercise 1

Let $g \in C^2(\mathbb{R}^n)$ be radially symmetric, i.e. of the form $g(x) = g_0(|x|)$ for some function $g_0 : \mathbb{R}_{\geq 0} \to \mathbb{R}$. Prove that $g_0 \in C^2([0,\infty))$ and $g_0'(0) = 0$. Moreover, calculate $\Delta g(x)$ in terms of $g_0(|x|)$, $g_0'(|x|)$, $g_0''(|x|)$.

Exercise 2

For $R > 0$, let $B_R(0) := \{x \in \mathbb{R}^n : |x| < R\} \subset \mathbb{R}^n$ be the ball with radius $R > 0$ centered at $0 \in \mathbb{R}^n$. For $f \in C(\overline{B_R(0)})$ and $g \in C(\partial B_R(0))$ consider the following eigenvalue problem

$$\begin{cases} -\Delta u - \lambda u = f & \text{in } B_R(0) \\ u = g & \text{on } \partial B_R(0) \end{cases} \tag{1}$$

(i) Find a radially symmetric solution to (1) for $f(x) = |x|$ and $g(x) = R$. Exercise 1 might be useful here.

(ii) Let $n = 3$, $f = 0$ and $g = 0$. Find the set of all $\lambda \in \mathbb{R}$ for which there exist nontrivial radially symmetric solutions to (1). Furthermore, determine all solutions to the problem for these λ.

(iii) Are there $n \in \mathbb{N}$, $n \geq 2$, radially symmetric functions f, g such that there is a solution u that is not radially symmetric?

Exercise 3

Consider the eigenvalue problem

$$\begin{cases} -\Delta u - \lambda u = 0 & \text{in } Q \\ u = 0 & \text{on } \partial Q \end{cases} \tag{2}$$

where $Q = (0, l_1) \times (0, l_2) \subset \mathbb{R}^2$ with $l_1, l_2 > 0$.

(i) Find $\lambda \in \mathbb{R}$ such that solutions $u \in C^2(Q)$ of the form $u(x, y) = v(x)w(y)$ exist.

(ii) Find $l_1, l_2 > 0$, , $\lambda \in \mathbb{R}$ such that λ is a multiple eigenvalue of (2), i.e. some $\lambda \in \mathbb{R}$ such that there are at least two linearly independent solutions to (2).