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Boundary and Eigenvalue Problems

Exercise Sheet 10

Exercise 23

Let X be a Banach space. Show that if there exists $\varepsilon > 0$ and an uncountable set $(x_j)_{j\in J}$ in X such that $\|x_j - x_i\| \geq \varepsilon$ for $i \neq j$, then X is not separable.

Exercise 24

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain.

(a) Show that $L^p(\Omega)$ is separable for $p \in [1, \infty)$.
Hint: Use the Stone-Weierstraß theorem.

(b) Show that $L^\infty(\Omega)$ is not separable.
Hint: Use exercise 23.

Exercise 25

Show that every infinite dimensional separable Hilbert space $(X, \langle\cdot,\cdot\rangle_X)$ is isometrically isomorphic to $(l^2, \langle\cdot,\cdot\rangle_{l^2})$, where

$$l^2 := \{(x_k)_{k\in\mathbb{N}} : \sum_{k\in\mathbb{N}} |x_k|^2 < \infty\},$$

and

$$\langle (x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}}\rangle_{l^2} := \sum_{k\in\mathbb{N}} x_k\overline{y_k} \qquad ((x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}} \in l^2).$$

That is: Show that there is a linear, bijective map $\iota : X \to l^2$ such that $\langle \iota(x), \iota(y)\rangle_{l^2} = \langle x, y\rangle_X$
Hint: Use Parseval's identity.

— please turn around —

Exercise 26

Let X be a Hilbert space.

(a) Show that if X is separable, then any set M with the following property is countable: For all $x, y \in M$: If $x \neq y$, then $\langle x, y \rangle = 0$.

(b) Show that X is separable if and only if there exists a (countable) orthonormal basis.