Boundary and Eigenvalue Problems

Exercise Sheet 10

Exercise 23
Let $X$ be a Banach space. Show that if there exists $\varepsilon > 0$ and an uncountable set $(x_j)_{j \in J}$ in $X$ such that $\|x_j - x_i\| \geq \varepsilon$ for $i \neq j$, then $X$ is not separable.

Exercise 24
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain.

(a) Show that $L^p(\Omega)$ is separable for $p \in [1, \infty)$.

*Hint:* Use the Stone-Weierstraß theorem.

(b) Show that $L^\infty(\Omega)$ is not separable.

*Hint:* Use exercise 23.

Exercise 25
Show that every infinite dimensional separable Hilbert space $(X, \langle \cdot, \cdot \rangle_X)$ is isometrically isomorphic to $(l^2, \langle \cdot, \cdot \rangle_{l^2})$, where

$$l^2 := \{(x_k)_{k \in \mathbb{N}} : \sum_{k \in \mathbb{N}} |x_k|^2 < \infty\};$$

and

$$\langle (x_k)_{k \in \mathbb{N}}, (y_k)_{k \in \mathbb{N}} \rangle_{l^2} := \sum_{k \in \mathbb{N}} x_k \bar{y}_k \quad ((x_k)_{k \in \mathbb{N}}, (y_k)_{k \in \mathbb{N}} \in l^2).$$

That is: Show that there is a linear, bijective map $\iota : X \to l^2$ such that $\langle \iota(x), \iota(y) \rangle_{l^2} = \langle x, y \rangle_X$

*Hint:* Use Parseval’s identity.
Exercise 26

Let $X$ be a Hilbert space.

(a) Show that if $X$ is separable, then any set $M$ with the following property is countable: For all $x, y \in M$: If $x \neq y$, then $\langle x, y \rangle = 0$.

(b) Show that $X$ is separable if and only if there exists a (countable) orthonormal basis.