Exercise 23

Assume that X is seperable and let $(y_k)_{k\in\mathbb{N}}$ be a countable, dense subset. Then we have: $\forall \varepsilon>0$ for each $j\in J$ there is $k\in\mathbb{N}$ s.t. $\|x_i - y_k\| < \frac{\varepsilon}{2}$. As J is uncountable, there is i_1, i_2, k s.t. $\|x_{i_1} - y_k\| < \frac{\varepsilon}{2}$ and $\|x_{i_2} - y_k\| < \frac{\varepsilon}{2}$, otherwise we had an injective map $i: J \to \mathbb{N}$ and hence a surjective map $i^{-1}: i(\mathbb{N}) \to J$ which contradicts the uncountability.

But then we find:

$$\|x_{i_1} - x_{i_2}\| \le \|x_{i_1} - y_k\| + \|y_k - x_{i_2}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \;\lightning.$$

Exercise 24:

Stone-Weierstraß theorem: Let $E \subset C(K)$ be a subset $\emptyset \neq K \subset\subset \mathbb{R}^n$ such that:

1) $\forall p \in E \;\exists f \in C(K)$ s.t. $f(p) \neq 0$
2) $\forall p,q \in E,\; p\neq q \;\exists f \in C(K)$ s.t. $f(p) \neq f(q)$
3) $f, g \in E \Rightarrow f\cdot g \in C(E)$

Then E is dense in $C(K)$ w.r.t. $\|\cdot\|_\infty$

In particular: Polynomials over K are dense in $C(K)$.

a) For each $k\in\mathbb{N}$ define $\Omega_k := \{x\in\Omega : \operatorname{dist}(x, \partial\Omega) \ge \frac{1}{k}\}$

Then: Ω_k is compact and $\forall \varepsilon>0 \;\exists k\in\mathbb{N}$ s.t. $|\operatorname{vol}(\Omega_k) - \operatorname{vol}(\Omega)| < \varepsilon$.

Let $p\in[1,\infty)$. Claim: The polynomials with rational coefficients are dense in $L^p(\Omega)$.

Proof: Let $f\in L^p(\Omega)$ be arbitrary by monoton convergence:

$$\int_\Omega |f|^p - \int_{\Omega_k} |f|^p \to 0 \quad (k\to\infty).$$

Choose $k\in\mathbb{N}$ large enough s.t. $\int_\Omega |f|^p - \int_{\Omega_k} |f|^p < \frac{\varepsilon}{3}$

Now, according to Stone-Weierstraß, there is a polynomial p on Ω_k s.t. $\|p-f\|_{L^\infty(\Omega_k)} < \frac{\varepsilon}{3\,\mathrm{vol}(\Omega_k)^{\frac{1}{p}}}$. Now one approximates p by a rational coefficient polynomial as follows:

Write $p(x) = \sum_{k=1}^{N} p_k x^k$ and let $R>0$ be such that $B_R(0) \supset \Omega_k$,

choose $q_k \in \mathbb{Q}$ such that $|q_k - p_k| \leq \frac{\varepsilon}{3N R^k \cdot \mathrm{vol}(\Omega_k)^{\frac{1}{p}}}$.

Then: for $q(x) = \sum_{k=1}^{N} q_k x^k$ one has

$$\|p-q\|_{L^p(\Omega_k)} \leq \mathrm{vol}(\Omega_k)^{\frac{1}{p}} \|p-q\|_\infty \leq \sum_{k=1}^{N} \sup_{x\in\Omega_k} |p_k - q_k| x^k \cdot \mathrm{vol}(\Omega_k)^{\frac{1}{p}}$$

$$\leq \mathrm{vol}(\Omega_k)^{\frac{1}{p}} \cdot \sum_{k=1}^{N} R^k \cdot \frac{\varepsilon}{3N R^k \mathrm{vol}(\Omega_k)^{\frac{1}{p}}} = \frac{\varepsilon}{3}.$$

Then, summing up:

$$\|f-q\|_{L^p(\Omega)} \leq \|f - f|_{\Omega_k}\|_{L^p(\Omega)} + \|f|_{\Omega_k} - p\|_{L^p(\Omega_k)} + \|p-q\|_{L^p(\Omega_k)} < \varepsilon$$

$\square$

b) Choose an interval (a,b), such that $(a,b)\times \mathbb{R}^{n-1} \cap \Omega \neq \emptyset$.
For each $r \in (a,b)\cap(\mathbb{R}\setminus\mathbb{Q})$ define $f_r : \Omega \to \mathbb{R}$, $f_r(x) = \begin{cases} 1, & x_1 \geq r \\ 0, & x_1 < r, \end{cases}$

where $x = (x_1, \overline{x})$.

Then: $f_r \in L^\infty(\Omega)$ and for $r_1 < r_2$ one has:

$$(f_{r_1} - f_{r_2})(x) = \begin{cases} 0 & x_1 < r_1 \vee x_1 \geq r_2 \\ 1 & x_1 \in [r_1, r_2) \end{cases}$$

$\Rightarrow$ $\|f_{r_1} - f_{r_2}\|_{L^\infty} = 1$ and hence: $(f_r)_{r\in(a,b)\cap(\mathbb{R}\setminus\mathbb{Q})}$ satisfies the condition in Ex. 23 $\Rightarrow$ $L^\infty(\Omega)$ is not seperable.

Exercise 25

Let $(X, \langle\cdot,\cdot\rangle_X)$ be a seperable Hilbert space and let $(\varphi_k)_{k\in\mathbb{N}}$ be an orthonormal basis. Define

$$i: X \to \ell^2, \quad u \mapsto (\langle u, \varphi_k\rangle_X)_{(k\in\mathbb{N})}$$

isometry:

$$\begin{aligned} \langle i(u), i(v)\rangle_{\ell^2} &= \sum_{k=1}^{\infty} \langle u, \varphi_k\rangle_X \cdot \overline{\langle v, \varphi_k\rangle_X} \\ &= \sum_{k=1}^{\infty} \langle u, \langle v, \varphi_k\rangle_X \varphi_k\rangle \\ &= \langle u, \sum_{k=1}^{\infty} \langle v, \varphi_k\rangle_X \cdot \varphi_k\rangle \\ &= \langle u, v\rangle_X \end{aligned}$$

From isometry we have also injectivity.

Surjectivity. Let $(x_k)_{k\in\mathbb{N}} \in \ell^2$ be a sequence.

Define $u_N := \sum_{k=1}^{N} x_k \cdot \varphi_k$. Then $u_N \in X$ $(N\in\mathbb{N})$ and

$$\|u_N - u_M\|_X^2 \overset{\text{o.B.d.A. } M<N}{=} \left\|\sum_{k=M+1}^{N} x_k \cdot \varphi_k\right\|_X^2 \overset{\text{Pyth.}}{=} \sum_{k=M+1}^{N} x_k^2 \cdot \underbrace{\|\varphi_k\|_X^2}_{=1} \to 0 \ (M\to\infty)$$

$\Rightarrow$ $(u_N)_{N\in\mathbb{N}}$ is a Cauchy sequence and hence conv. in X.

Write $u = \lim_{N\to\infty} u_N = \sum_{k=1}^{\infty} x_k \varphi_k$. Then:

$$(i(u))_k = \langle u, \varphi_k\rangle_X = x_k \Rightarrow u \text{ is a preimage of } (x_k)_{k\in\mathbb{N}}.$$

$\Rightarrow$ surjectivity

$\square$

Exercise 26

a) Let X be a seperable Hilbert space. Assume one has an uncountable set $M \subset X$ s.t. $\forall x \neq y$ in M : $\langle x, y \rangle = 0$.

Then $M \setminus \{0\}$ is also uncountable. Consider the set $(x_k)_{k \in J}$ with $x_k = \frac{y_k}{\|y_k\|}$ $\forall y_k \in M$. Then we have:

$$\|x_k - x_j\|^2 = \|x_k\|^2 - 2\langle x_k, x_j \rangle + \|x_j\|^2 = 2 - \delta_{kj}$$

$\Rightarrow$ For $k \neq j$ one has $\|x_k - x_j\| \geq \sqrt{2}$

$\overset{\text{Ex 23}}{\Rightarrow}$ X is not seperable ↯

b) Let X be seperable, take a countable, dense set $(x_k)_{k \in \mathbb{N}}$. $x_0 = 0$, $x_k \neq 0$

and perform Gram-Schmidt-orthogonalization:

$$\tilde{\varphi}_k := x_k - \sum_{\ell=1}^{k-1} \frac{\langle x_k, \varphi_\ell \rangle}{\langle x_k, x_k \rangle} \cdot x_k, \quad \varphi_k := \frac{\tilde{\varphi}_k}{|\tilde{\varphi}|_k}$$

Then: $[\varphi_k : k \in \mathbb{N}]$ is dense in X, as for any $u \in X, \varepsilon > 0$ choose $k \in \mathbb{N}$ s.t. $\|x_k - u\|_X < \varepsilon$. By definition of $(\varphi_\ell)_{\ell \in \mathbb{N}}$, x_k is a linear combination of $(\varphi_\ell)_{\ell \in \mathbb{N}}$ (by induction) $\Rightarrow x_k \in [\varphi_k : k \in \mathbb{N}]$

$\Rightarrow (\varphi_k)_{k \in \mathbb{N}}$ is an ONB.

Let now $(\varphi_k)_{k \in \mathbb{N}}$ be an ONB. Then $\{\sum_{k=1}^{N} q_k \varphi_k \mid q_k \in \mathbb{Q}\}$ is dense in X and countable. Indeed let $u \in X$ be arbitrary.

For $\varepsilon > 0$ choose $N \in \mathbb{N}$, $(\alpha_k)_k$ in $\mathbb{R}$ s.t. $\|\sum_{k=1}^{N} \alpha_k \varphi_k - u\| < \frac{\varepsilon}{2}$.

Then choose $(q_k)_{k}$ in $\mathbb{Q}$ s.t. $|q_k - \alpha_k| < \frac{\varepsilon}{2\sqrt{N}}$. Then:

$$\|\sum_{k=1}^{N} q_k \varphi_k - u\|_X \leq \|\sum_{k=1}^{N} \alpha_k \varphi_k - \sum_{k=1}^{N} q_k \varphi_k\|_X + \frac{\varepsilon}{2} = \left(\sum_{k=1}^{N} |\alpha_k - q_k|^2\right)^{\frac{1}{2}} + \frac{\varepsilon}{2} < \varepsilon$$

□