Karlsruher Institut für Technologie (KIT)
Institut für Analysis
Dr. Rainer Mandel
Peter Rupp

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Boundary and Eigenvalue Problems

Exercise Sheet 11

Let H be Hilbert space and let $T : H \to H$ be a linear, bounded operator in H. Its resolvent set $\rho(T)$ is defined by

$$\rho(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is bijective}\}$$

whereas its spectrum is given by $\sigma(T) := \mathbb{C} \setminus \rho(T)$.

Exercise 26

Consider the shift operators in l^2:

$$L : l^2 \to l^2, \; (Lx)_n = x_{n+1} \qquad (n \geq 1),$$

and

$$R : l^2 \to l^2, \; (Rx)_1 = 0, \; (Rx)_n = x_{n-1} \qquad (n \geq 2).$$

For $T = L$ and $T = R$ compute

(a) the adjoint operator T^*

(b) point spectrum

$$\sigma_p := \{\lambda \in \sigma(T) : T - \lambda \text{ is not injective}\},$$

(c) continuous spectrum

$$\sigma_c := \{\lambda \in \sigma(T) : T - \lambda \text{ is not surjective and } Ran(T - \lambda) \text{ is dense in } l^2\},$$

(d) residual spectrum

$$\sigma_r := \{\lambda \in \sigma(T) : T - \lambda \text{ is not surjective and } Ran(T - \lambda) \text{ is not dense in } l^2\}.$$

HINT: Consider the point spectrum of the adjoint operator.

(e) For $\lambda \in \rho(T)$ compute the resolvent $(T-\lambda)^{-1}$.

Solution $T^* = R$ and $R^* = T$ is a simple computation.

Let $(x_k) \in l^2 \setminus \{0\}$ and $\lambda \in \mathbb{C}$ fullfill $(L-\lambda)(x_k) = 0$. That means

$$x_{k+1} - \lambda x_k = 0 \text{ for all } k \in \mathbb{N}$$

. Consequently $x_k = \lambda^{k-1} x_1$ which either converges for $x_1 = 0$ (which will yield the trivial sequence and is hence neglicable) or $|\lambda < 1$. Hence, $\sigma_p(L) = \{|\lambda| \leq 1\}$. For the rightshift, the eigenvalue equation reads

$$\lambda x_1 = 0, \qquad x_{k-1} - \lambda x_k = 0 \qquad (k \geq 2).$$

Hence (($x_1 = 0$ would yield the trivial element again). $\lambda = 0$ and thus, $\sigma_p(R) = \{0\}$.

For the residual spectrum note, that $(x_k) \in ker(T-\lambda)$ iff $(x_k) \in ran(T^* - \overline{\lambda})^\perp$. Therefore $\sigma_r(L) = \sigma_p(R) = \{0\}$ and $\sigma_r(R) = \sigma_p(L) = \{|\lambda| < 1\}$.

Now we first compute the resolvent operator. For $\lambda \in \rho$ we have $(L-\lambda)(x_k) = (y_k)$ iff $x_{k+1} - \lambda x_k = y_k$ for $k \in \mathbb{N}$. One can prove by induction that this yields to

$$x_k = \lambda^k x_1 + \sum_{l=1}^{k-1} \lambda^{k-1-l} y_l.$$

For $y = e_i$ with $e_i = (\delta_{ik})_{k \in \mathbb{N}}$, we have

$$((L-\lambda)(x_k))_k = \begin{cases} -\lambda^{k-1-i}, & k-1 < i \\ 0 & k-1 \geq i. \end{cases}$$

This makes sense for all. So $e_i \in ran(L-\lambda)$ for all $i \in \mathbb{N}$. Therefore: $\sigma_{ess}(L) \subset \{|\lambda| \geq 1\}$.

Moreover, for $|\lambda| > 1$ one can estimate

$$\|(L-\lambda)(x_k)\| \geq |\lambda| \|(x_k)\| - \|T x_k\| \geq (|\lambda| - 1)\|(x_k)\|.$$

As in the exercise class, this plus the denseness of the image implies, that $L - \lambda$ is (boundedly) invertible for $|\lambda| > 1$, so $\rho(L) \supset \{|\lambda| > 1\}$.

Finally consider the sequence

$$(x^l)_k := \frac{\lambda^k}{2^{\frac{k}{l}}}$$

Then we find:

$$\|(L-\lambda)(x^l_k)\| = \|(\frac{\lambda^{k+1}}{2^{\frac{k+1}{l}}} - \frac{\lambda^{k+1}}{2^{\frac{k}{l}}})\| = \|\frac{1}{2^{\frac{k}{l}}}(\frac{1}{2^{\frac{1}{l}}} - 1)\| = (\frac{1}{2^{\frac{1}{l}}} - 1)\|x^l_k\|$$

For $l \to \infty$ one finds $\|(L-\lambda)(x^l_k)\| / \|(x^l_k)\| \to 0$. Hence, if $L - \lambda$ was invertible, its inverse would not be bounded. This contradicts the open mapping theorem. Therefore: $\sigma_{ess}(L) \supset \{|\lambda| = 1\}$ We conclude: $\sigma_{ess}(L) = \{|\lambda| = 1\}$ and $\rho(L) = \{|\lambda| > 1\}$.

For $T = R$ and $\lambda \in \rho(R)$ one finds: $(R - \lambda)(x_k) = y_k$ iff $x_k = \frac{1}{\lambda^k} y_1 - \sum_{l=2}^{k} y_l \frac{1}{\lambda^{k-l+1}}$. Again, for e_i whe get:

$$x_k = \begin{cases} \frac{1}{\lambda^k} & i = 1 \\ -\frac{1}{\lambda^{k+1-i}} & 2 \leq i \leq k \\ 0 & 1 \leq k - 1 \leq i \end{cases}$$

is a preimage. Hence $R - \lambda$ has a dense range and thus, by a similar estimate as above: $\rho(R) \supset \{|\lambda| > 1\}$. As the spectrum of an operator is closed, we have $\sigma(R) = \{|\lambda| \leq 1\}$ and so $\sigma_{ess}(R) = \{|\lambda| = 1\}$, as the partition of the spectrum is disjoint.