Exercise 27

Let $\lambda \in \operatorname{essrg}(a)$ $\Rightarrow$ $\forall n \in \mathbb{N}$: $\operatorname{vol}(\underbrace{\{x \in \mathbb{R}^n : |a(x) - \lambda| < \tfrac{1}{n}\}}_{A_n}) > 0$

Define $u_n \in L^2$ by $u_n = \begin{cases} \chi_{A_n}, & \text{if } \operatorname{vol}(A_n) < \infty \\ \chi_{\tilde{A}_n}, & \text{if } \operatorname{vol}(A_n) = \infty \text{ and } \tilde{A}_n \subset A_n \text{ bounded} \end{cases}$

$\Rightarrow \|(T-\lambda)u_n\| = \|(a-\lambda)u_n\| \leq \frac{1}{n} \cdot \|u_n\|.$

Assume, $(T-\lambda)^{-1}$ exists, then

$$\|(T-\lambda)^{-1}\| \geq \frac{\|(T-\lambda)^{-1}((T-\lambda)u_n)\|}{\|(T-\lambda)u_n\|} \geq n \longrightarrow \infty,$$

$\Rightarrow$ $(T-\lambda)^{-1}$ is not bounded $\lightning$ Open-mapping-theorem.

$\Rightarrow$ $(T-\lambda)$ is not invertible $\Rightarrow$ $\lambda \in \sigma(T)$.

Let $\lambda \notin \operatorname{essrg}(a)$ $\Rightarrow$ $\exists\, \varepsilon > 0$ $|a(x) - \lambda| > \varepsilon$ for almost all $x \in \mathbb{R}^n$

For $w \in L^2(\mathbb{R}^n)$ define $u := \frac{1}{a(x)-\lambda}\, w \in L^2$.

$\Rightarrow$ $(T-\lambda)u = w$ and $\left\|\frac{1}{a(x)-\lambda}\, w\right\| \leq \frac{1}{\varepsilon} \cdot \|w\|$ $\forall w \in L^2(\mathbb{R}^n)$

$\Rightarrow$ $(T-\lambda)$ is invertible. $\Rightarrow$ $\lambda \notin \sigma(T)$.

$\square$

Exercise 28

(a) $\Rightarrow$ (b) Define $u_N := \sum_{n \in \mathbb{N}}^{N} \langle u_n, \phi_n \rangle \phi_n$. By (a), $u_N \to_H u$

$\Rightarrow$ $\|u_N\|_H^2 \longrightarrow \|u\|_H^2$ i.e. $\langle u_N, u_N \rangle = \left\langle \sum_{n=1}^{N} \langle u_n, \phi_n \rangle \phi_n ; \sum_{k=1}^{N} \langle u_k, \phi_k \rangle \phi_k \right\rangle$

$= \sum_{k,n=1}^{N} \delta_{kl} \langle u_n, \phi_n \rangle \overline{\langle u_k, \phi_k \rangle} = \sum_{n=1}^{N} |\langle u_n, \phi_n \rangle|^2 \longrightarrow \|u\|_H^2$

(b) $\Rightarrow$ (c) Let $\langle u_n, \phi_n \rangle = 0$ $\forall n \in \mathbb{N}$ $\Rightarrow 0 = \sum_{n \in \mathbb{N}} |\langle u_n, \phi_n \rangle|^2 \overset{(b)}{=} \|u\|_H^2$

$\Rightarrow u = 0$

(c) $\Rightarrow$ (a)

$u_N = \sum_{n=1}^{N} \langle u_n, \phi_n \rangle_{,} \phi_n$ $\Rightarrow$ $\forall_{k \in \mathbb{N}}$ and $N \geq k$ one has

$$\langle u-u_N, \phi_k \rangle = \langle u - \sum_{n=1}^{N} \langle u, \phi_n \rangle \phi_n, \phi_k \rangle = \langle u, \phi_k \rangle - \sum_{n=1}^{N} \langle u, \phi_n \rangle \delta_{kn}$$

$$= 0$$

$$\Rightarrow \lim_{N \to \infty} \langle u - u_N, \phi_k \rangle = 0 \quad \forall k \in \mathbb{N} . \quad (*)$$

By the Bessel-inequality: $\sum_{k=1}^{N} |\langle u, \varphi_k \rangle|^2 = \sum_{k=1}^{N} \langle u, \cancel{\ } \langle u, \varphi_k \rangle \varphi_k \rangle$

$$\leq \|u\| \cdot \| \sum_{k=1}^{N} \langle u, \varphi_k \rangle \varphi_k \|$$

$$= \|u\| \cdot \left(\sum_{k=1}^{N} |\langle u, \varphi_k \rangle|^2 \right)^{1/2} \Rightarrow \sum_{k=1}^{N} |\langle u, \varphi_k \rangle|^2 \leq \|u\|^2$$

One finds: u_N is a Cauchy-sequence and hence konv. to $v \in H$

Then $(*)$ tells:

$$\langle u - v, \phi_k \rangle = 0 \quad \forall k \in \mathbb{N} \iff u - v = 0 \iff u = v$$

$\square$