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Boundary and Eigenvalue Problems

Exercise Sheet 11

Let H be Hilbert space and let $T: H \to H$ be a linear, bounded operator in H. Its resolvent set $\rho(T)$ is defined by

$$\rho(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is bijective}\}$$

whereas its spectrum is given by $\sigma(T) := \mathbb{C} \setminus \rho(T)$.

Exercise 26

Consider the shift operators in l^2:

$$L : l^2 \to l^2, \; (Lx)_n = x_{n+1} \qquad (n \geq 1),$$

and

$$R : l^2 \to l^2, \; (Rx)_1 = 0, \; (Rx)_n = x_{n-1} \qquad (n \geq 2).$$

For $T = L$ and $T = R$ compute

(a) the adjoint operator T^*

(b) point spectrum

$$\sigma_p := \{\lambda \in \sigma(T) : T - \lambda \text{ is not injective}\},$$

(c) continuous spectrum

$$\sigma_c := \{\lambda \in \sigma(T) : T - \lambda \text{ is not surjective and } Ran(T - \lambda) \text{ is dense in } l^2\},$$

(d) residual spectrum

$$\sigma_r := \{\lambda \in \sigma(T) : T - \lambda \text{ is not surjective and } Ran(T - \lambda) \text{ is not dense in } l^2\}.$$

Hint: Consider the point spectrum of the adjoint operator.

(e) For $\lambda \in \rho(T)$ compute the resolvent $(T - \lambda)^{-1}$.

Exercise 27

Let $a \in L^\infty(\mathbb{R}^N)$. Define the essential range of a by

$$essrg(a) := \{\lambda \in \mathbb{C} : \forall \varepsilon > 0,\ vol(\{x \in \mathbb{R}^n : |a(x) - \lambda| < \varepsilon\}) > 0\}.$$

Show that the spectrum of the operator

$$T : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N) : Tu = a \cdot u$$

is given by $\sigma(T) = essrg(a)$.

Exercise 28

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let $(\phi_n)_{n\in\mathbb{N}}$ be an orthonormal system. Show that the following statements are equivalent:

(a) For all $u \in H$, $u = \sum_{n\in\mathbb{N}} \langle u, \phi_n \rangle \phi_n$, as a limit in H.

(b) For all $u \in H$ one has $\sum_{n\in\mathbb{N}} |\langle u, \phi_n \rangle|^2 = \|u\|^2$.

(c) $u \in H$ and $\langle u, \phi_n \rangle = 0$ for all $n \in \mathbb{N}$ implies $u = 0$.