Exercise 29

For $B(u,v) := \int_\Omega \nabla u \cdot \nabla v$ we get that $(\varphi_n)_{n\in\mathbb{N}}$ is an ONB in $L^2(\Omega)$ and $\left(\frac{\varphi_n}{\sqrt{\lambda_n}}\right)_{n\in\mathbb{N}} = \left(\frac{\varphi_n}{\sqrt{B(\varphi_n,\varphi_n)}}\right)_{n\in\mathbb{N}}$ is an ONB in $(H_0^1(\Omega), B(\cdot,\cdot))$. This implies for $u\in H_0^1(\Omega)$

$$\begin{aligned}
\infty > \; & B(u,u) \\
= \; & \sum_{n=1}^\infty \left| B\left(u, \frac{\varphi_n}{\sqrt{\lambda_n}}\right)\right|^2 \\
= \; & \sum_{n=1}^\infty \lambda_n^{-1} \, |B(u,\varphi_n)|^2 && (B \text{ is bilinear}) \\
= \; & \sum_{n=1}^\infty \lambda_n \, \langle u,\varphi_n\rangle_{L^2(\Omega)}^2 && ((\varphi_n,\lambda_n) \text{ is an eigenpair})
\end{aligned}$$

Vice versa, let $u\in L^2(\Omega)$ satisfy $\sum_{n=1}^\infty \lambda_n |\langle u,\varphi_n\rangle_{L^2(\Omega)}|^2 < \infty$.

Define
$$u_N := \sum_{n=1}^N \langle u,\varphi_n\rangle \varphi_n .$$

Then $u_N \to u$ in $L^2(\Omega)$ and $(u_N)_{N\in\mathbb{N}}$ is a Cauchy sequence in $(H_0^1(\Omega), B(\cdot,\cdot))$ because $(N,M\in\mathbb{N}, N>M)$

$$\begin{aligned}
\|u_N-u_M\|^2_{H_0^1(\Omega)} &= B(u_N-u_M, u_N-u_M) + \|u_N-u_M\|^2 \\
&= \sum_{n,m=M+1}^N \langle u,\varphi_n\rangle_{L^2(\Omega)} \langle u,\varphi_m\rangle_{L^2(\Omega)} \underbrace{B(\varphi_n,\varphi_m)}_{=\lambda_n\delta_{nm}} + \|u_N-u_M\|^2 \\
&= \sum_{n=M+1}^N |\langle u,\varphi_n\rangle|^2_{L^2(\Omega)} \cdot \lambda_n + \|u_N-u_M\|^2 \\
&\longrightarrow 0 \quad \text{as } N,M\to\infty
\end{aligned}$$

So $(u_N)_{N\in\mathbb{N}}$ also converges in $H_0^1(\Omega)$ to some $w\in H_0^1(\Omega)$.

Hence,

$$u = L^2(\Omega)\text{-}\lim_{N\to\infty} u_N \overset{\uparrow}{=} H_0^1(\Omega)\text{-}\lim_{N\to\infty} u_N = w \in H_0^1(\Omega).$$

<u>Notice</u>: convergence in $H_0^1(\Omega) \Rightarrow$ convergence in $L^2(\Omega)$

Exercise 30

(a) Corollary 13 of the lecture implies

$$u = \sum_{m \in \mathbb{N}} \frac{\langle f, \varphi_m \rangle_{L^2(\Omega)}}{\lambda_m - \lambda} \varphi_m .$$

So,

$$\|u\|_{L^2(\Omega)}^2 = \sum_{m \in \mathbb{N}} \left| \frac{\langle f, \varphi_m \rangle_{L^2(\Omega)}}{\lambda_m - \lambda} \right|^2$$

$$= \sum_{m \in \mathbb{N}} \frac{|\langle f, \varphi_m \rangle_{L^2(\Omega)}|^2}{(\lambda_m - \lambda)^2}$$

$$\leq \frac{1}{\min\{|\lambda_m - \lambda| : m \in \mathbb{N}\}^2} \cdot \underbrace{\sum_{k \in \mathbb{N}} |\langle f, \varphi_k \rangle_{L^2(\Omega)}|^2}_{= \|f\|_2^2}$$

(b) and (c) Set $G_N(x,y) := \sum_{m=1}^{N} \frac{\varphi_m(x)\varphi_m(y)}{\lambda_m - \lambda}$.

For $N, M \in \mathbb{N}_0$ with $N > M$ we have

$$\|G_N - G_M\|_{L^2(\Omega \times \Omega)}^2 = \int_\Omega \int_\Omega \left| \sum_{m=M+1}^{N} \frac{\varphi_m(x)\varphi_m(y)}{\lambda_m - \lambda} \right|^2 dx\, dy$$

$$= \int_\Omega \int_\Omega \sum_{m,k=M+1}^{N} \frac{\varphi_m(x)\varphi_m(y)\varphi_k(x)\varphi_k(y)}{(\lambda_m - \lambda)(\lambda_k - \lambda)} \, dx\, dy$$

$$= \sum_{m,k=M+1}^{N} \frac{1}{(\lambda_m - \lambda)(\lambda_k - \lambda)} \cdot \underbrace{\left(\int_\Omega \varphi_m(x)\varphi_k(x)\,dx \right)}_{= \delta_{mk}} \underbrace{\left(\int_\Omega \varphi_m(y)\varphi_k(y)\,dy \right)}_{\delta_{mk}}$$

$$= \sum_{m=M+1}^{N} \frac{1}{|\lambda_m - \lambda|^2} .$$

By Theorem 17 we know $c \leq \frac{\lambda_m}{m^{2/n}} \leq C$ for some positive numbers $c, C > 0$ and almost all $m \in \mathbb{N}$. In particular

$$\frac{1}{|\lambda_m - \lambda|^2} \sim m^{-\frac{4}{n}} \quad \text{as } m \to \infty .$$

(Recall that n is the space dimension.)

So $(G_N)_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega\times\Omega)$ if and only if $n\in\{1,2,3\}$. In this case there is a $G\in L^2(\Omega\times\Omega)$ such that $G_N\to G$ in $L^2(\Omega\times\Omega)$, hence

$$u_N(x) = \sum_{m=1}^{N} \frac{\langle f,\varphi_m\rangle_{L^2(\Omega)}}{\lambda_m-\lambda}\varphi_m(x)$$

$$= \int_\Omega \left(\sum_{m=1}^{N} \frac{\varphi_m(x)\varphi_m(y)}{\lambda_m-\lambda}\right) f(y)\,dy$$

$$= \int_\Omega G_N(x,y)\, f(y)\,dy$$

and

$$\left\| u - \int_\Omega G(\cdot,y)f(y)\,dy \right\|_{L^2(\Omega)}$$

$$\leq \|u-u_N\|_{L^2(\Omega)} + \left\|u_N - \int_\Omega G_N(\cdot,y)f(y)\,dy\right\|_{L^2(\Omega)}$$

$$+ \left\| \int_\Omega \big(G_N(\cdot,y)-G(\cdot,y)\big) f(y)\,dy \right\|_{L^2(\Omega)}$$

$$\leq \|u-u_N\|_{L^2(\Omega)} + 0 + \|G_N-G\|_{L^2(\Omega\times\Omega)}\|f\|_{L^2(\Omega)}$$

$$\longrightarrow 0 \quad \text{as } N\to\infty.$$

This gives (b) and (c) follows from above:

$$\|G\|^2_{L^2(\Omega\times\Omega)} = \lim_{N\to\infty} \|G_N\|^2_{L^2(\Omega\times\Omega)}$$

$$= \lim_{N\to\infty} \sum_{m=1}^{N} \frac{1}{|\lambda_m-\lambda|^2}$$

$$= \sum_{m\in\mathbb{N}} \frac{1}{|\lambda_m-\lambda|^2}$$

Exercise 31

In order to solve the initial boundary value problem for the wave equation we make the ansatz

$$u(x,t) = \sum_{n\in\mathbb{N}} A_n(t)\varphi_n(x)$$

and formally derive a formula for A_n.

From the PDE:

$$\begin{aligned} 0 &= u_{tt} - \Delta u \\ &= \sum_{n\in\mathbb{N}} \left[\frac{\partial^2}{\partial t^2}(A_n(t)\varphi_n(x)) - \Delta_x(A_n(t)\varphi_n(x)) \right] \\ &= \sum_{n\in\mathbb{N}} \left[A_n''(t)\varphi_n(x) - A_n(t)\cdot\lambda_n\varphi_n(x) \right] \\ &= \sum_{n\in\mathbb{N}} \left(A_n''(t) - \lambda_n A_n(t) \right)\varphi_n(x) \end{aligned}$$

This implies $A_n'' = \lambda_n A_n$ and hence (λ_n are positive)

$$A_n(t) = \alpha_n \cos(\sqrt{\lambda_n}\, t) + \beta_n \sin(\sqrt{\lambda_n}\, t)$$

From the initial data:

$$f(x) = u(x,0) = \sum_{n\in\mathbb{N}} A_n(0)\varphi_n(x) = \sum_{n\in\mathbb{N}} \alpha_n \varphi_n(x)$$

$$g(x) = u_t(x,0) = \sum_{n\in\mathbb{N}} A_n'(0)\varphi_n(x) = \sum_{n\in\mathbb{N}} \sqrt{\lambda_n}\,\beta_n\varphi_n(x)$$

This gives $\alpha_n = \langle f,\varphi_n\rangle_{L^2(\Omega)}$, $\beta_n = \dfrac{\langle g,\varphi_n\rangle_{L^2(\Omega)}}{\sqrt{\lambda_n}}$.

Conclusion:

$$u(x,t) = \sum_{n\in\mathbb{N}} \left(\cos(\sqrt{\lambda_n}\,t)\langle f,\varphi_n\rangle_{L^2(\Omega)} + \frac{\sin(\sqrt{\lambda_n}\,t)}{\sqrt{\lambda_n}}\langle g,\varphi_n\rangle_{L^2(\Omega)} \right)\varphi_n(x)$$