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Boundary and Eigenvalue Problems

Exercise Sheet 12

Throughout this exercise sheet let $\Omega \subset \mathbb{R}^n$ be C^1 and bounded, $(\lambda_k)_{k\in\mathbb{N}}$ are the eigenvalues associated with an ONB of eigenfunctions $(\varphi_k)_{k\in\mathbb{N}}$ of the Dirichlet problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Exercise 29

For $u \in L^2(\Omega)$ prove that

$$u \in H_0^1(\Omega) \qquad \Leftrightarrow \qquad \sum_{k\in\mathbb{N}} \lambda_k |\langle u, \varphi_k\rangle_{L^2(\Omega)}|^2 < \infty.$$

Exercise 30

Let $\lambda \notin \{\lambda_k : k \in \mathbb{N}\}$ and let $f \in L^2(\Omega)$. Let $u \in H_0^1(\Omega)$ be the weak solution of

$$\begin{cases} -\Delta u - \lambda u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

(a) Prove the spectral estimate:

$$\|u\|_{L^2(\Omega)} \le \frac{1}{\min\{|\lambda - \lambda_k| : k \in \mathbb{N}\}} \|f\|_{L^2(\Omega)}.$$

(b) For $n = 1, 2, 3$ show that there is a (Greens-) function $G \in L^2(\Omega \times \Omega)$ such that

$$u(x) = \int_\Omega G(x,y) f(y)\, dy \qquad (x \in \Omega)$$

(c) Compute $\|G\|_{L^2(\Omega\times\Omega)}$.

— please turn around —

Exercise 31

Use the spectral decomposition of $-\Delta$ to derive a formula for the solution to the Cauchy-problem of the wave equation

$$\begin{cases} \frac{\partial^2}{\partial t^2} u - \Delta u = 0 & \text{in } (0,\infty)\times\Omega \\ u(0,x) = f(x) & x \in \Omega \\ \frac{\partial}{\partial t} u(0,x) = g(x) & x \in \Omega. \end{cases}$$