Sheet 1

Ex 1 $\quad g \in \mathbb{R}^n$ radially symmetric, $g(x) = g_0(|x|)$.

1) g_0 is C^2: Let $r_0 \in [0,\infty)$ and $(r_n)_{n\in\mathbb{N}}$ sequence in $[0,\infty)\setminus\{r_0\}$ with $r_n \to r_0$ $(n\to\infty)$. Take any $\omega \in S^{n-1}$. Then $r_n \cdot \omega \xrightarrow{n\to\infty} r_0\omega$ in $\mathbb{R}^n$

and: $\quad |g_0(r_n) - g_0(r_0)| = |g(r_n\omega) - g(r_0\omega)| \longrightarrow 0 \quad (n\to\infty)$

Moreover: $\forall r_0 \in [0,\infty)$: $\left|\frac{g_0(r_n) - g_0(r_0)}{r_n - r_0}\right| = \left|\frac{g(r_n\omega) - g(r_0\omega)}{r_n - r_0}\right| \overset{\text{diff. of } g}{=} \left|\frac{g'(r_0\omega)\cdot[(r_n - r_0)\omega]}{r_n - r_0}\right| + o(1)$

$$= \underbrace{|g'(r_0\omega)[\omega]|}_{} + o(1)$$

independent of the choice of $\omega \in S^{n-1}$, as $\quad g'(r\omega)[\omega] = \lim\limits_{h\to 0} \frac{g(r\omega + h\cdot\omega) - g(r\omega)}{h}$

$$= \lim_{h\to 0} \frac{g((r+h)|\omega|) - g(r\omega)}{h}$$

$\Rightarrow$ limit exists and so does the derivative

$$\overset{\text{radial symmetry}}{=} \lim_{h\to 0} \frac{g((r+h)\tilde{\omega}) - g(r\tilde{\omega})}{h}$$

$$= g'(r_0\tilde{\omega})[\tilde{\omega}]$$

Furthermore one has

$$g_0'(0) = g'(0)[\omega] = g'(0)[-\omega] = -g'(0)[\omega] \iff g_0'(0) = g'(0)[\omega] = 0 \quad (\omega \in S^{n-1})$$

Second derivative is copying the calculation for the first one and spending one more (1).

$$\Delta g(x) = \sum_{k=1}^n \partial_{x_k}^2 g(x) = \sum_{k=1}^n \partial_{x_k}^2 g_0(|x|) = \sum_{k=1}^n \partial_{x_k}\left(g_0'(|x|)\cdot \frac{x_k}{|x|}\right)$$

$$= \sum_{k=1}^n g_0''(|x|)\cdot\frac{x_k^2}{|x|^2} + g_0'(|x|)\cdot\left(\frac{1}{|x|} - \frac{x_k^2}{|x|^3}\right)$$

$$= g_0''(|x|) + \left(\frac{n}{|x|} - \frac{1}{|x|}\right) g_0'(|x|)$$

$$= g_0''(|x|) + \frac{(n-1)}{|x|} g_0'(|x|)$$

Exercise 2 | $f \in C(\overline{B_R(0)})$, $g \in C(\partial B_R(0))$

$$\begin{cases} (-\Delta - \lambda)u = f & \text{in } B_R(0) \\ u = g & \text{on } \partial B_R(0) \end{cases}$$

i) radially symmetric solution for $f(x) = |x|$, $g(x) = R$.

Writing (with a slight abuse of notation) $u(x) = u(r)$ the equation reads

$$-u''(r) - (n-1)\frac{u'(r)}{r} - \lambda u(r) = r \quad , \quad \text{so for } r \in (0,R):$$

$$-ru''(r) - (n-1)u'(r) - \lambda r\, u(r) = r^2$$

Making the ansatz $u(r) = \sum_{k=0}^{\infty} \alpha_k r^k$ yields

$$-\sum_{k\geq 2} \alpha_k k(k-1) r^{k-1} - \sum_{k \geq 1} \alpha_k (n-1) k \cdot r^{k-1} - \lambda \sum_{k\geq 0} \alpha_k r^{k+1} = r^2 \qquad (r>0)$$

$$= -\sum_{k\geq 2} \alpha_k k(k-1) r^{k-1} - \sum_{k\geq 1} \alpha_k (n-1)k \cdot r^{k-1} - \lambda \sum_{k\geq 2} \alpha_{k-2} r^{k-1} = r^2 \qquad (r>0)$$

As u is radially symmetric: $u'(0) = \alpha_1 = 0$

For $k=1$ ~~this~~ the polynomial reads also $\alpha_1 = 0$

$\left(k=2 \quad -2\alpha_2 r^1 - 2\alpha_2 (n-1) r - \lambda \alpha_0 r = 0 \right)$

$k=3 \quad -6\alpha_3 r^2 - 3\alpha_3(n-1)r^2 - \lambda \underbrace{\alpha_1}_{=0} r^2 = \cancel{r^2}\ 1 \quad \Leftrightarrow \quad \alpha_3 = \frac{-1}{6+3(n-1)} = -\frac{1}{3(n+1)}$

$k \geq 4: \ -k(k-1)\alpha_k - k(n-1)\alpha_k = \lambda \alpha_{k-2}$

$= -k(k+n-2) \qquad \alpha_n =$

Hence:

$$\alpha_{2k} = \left(\prod_{l=1}^{k} \frac{+\lambda}{(2-2l-n)2l} \right) \cdot \alpha_0 \qquad k \geq 1$$

$$\alpha_{2k+1} = \left(\prod_{l=2}^{k} \frac{+\lambda}{(2-(2l+1)-n)(2l+1)} \right) \alpha_3 \qquad k \geq 2$$

$$= \left(+ \prod_{l=2}^{k} \frac{+\lambda}{(2l+n)(2l+1)} \right) \cdot \frac{(-1)}{3(n+1)} \qquad k \geq 2$$