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Dr. Rainer Mandel
Peter Rupp

Boundary and Eigenvalue Problems

Exercise Sheet 2

Exercise 4

(a) Find some sufficient condition for $b \in C^1([0,1])$ and $c \in C([0,1])$ such that for all $f \in C([0,1])$, $\gamma_0, \gamma_1 \in \mathbb{R}$ the boundary value problem

$$\begin{cases} -u'' + bu' + cu = & f \qquad \text{in } (0,1) \\ u'(0) = \gamma_0,\ u'(1) = & \gamma_1 \end{cases}$$

has a unique solution.

(b) Let $b \in C^1(\mathbb{R})$, $c \in C(\mathbb{R})$ be 1-periodic. Find a sufficient condition on b and c, such that for any 1-periodic $f \in C(\mathbb{R})$ the equation

$$-u'' + bu' + cu = f$$

has a unique 1-periodic solution.
Hint: Impose periodic boundary conditions $u(0) = u(1)$ and $u'(0) = u'(1)$.

Exercise 5

(a) Prove Wirtinger's inequality: For all $f \in C^1([0,\pi])$ with $f(0) = f(\pi) = 0$ one has

$$\int_0^\pi f(t)^2\,dt \le \int_0^\pi f'(t)^2\,dt$$

with equality if and only if $f(t) = a\,\sin(t)$ for some $a \in \mathbb{R}$.
Hint: Find a $\psi \in C^1(0,\pi)$ such that the following identity of (improper) integrals hold:

$$\int_0^\pi (f'(t)^2 - f(t)^2)\,dt = \int_0^\pi (f'(t) - f(t)\psi(t))^2\,dt$$

(b) Use (a) to find a $c_0 < 0$ such that the following improved solvability condition for the boundary value problem from the lecture holds: Let $c \in C([0,\pi])$ with $c(x) \ge c_0$, then for all $f \in C([0,\pi])$, $\gamma_0, \gamma_1 \in \mathbb{R}$ there is a unique solution to

$$\begin{cases} -u'' + cu = & f \qquad \text{in } (0,\pi) \\ u(0) = \gamma_0,\ u(\pi) = & \gamma_1 \end{cases}$$