Exercise 4

a) According to the alternative theorem from the lecture it is equivalent to show, that the homogeneous problem has only the trivial solution. Consider

$$\begin{cases} -u'' + bu' + cu = 0 & \text{in } (0,1) \\ u'(0) = u'(1) = 0 \end{cases} \qquad (1)$$

For a solution u test the equation with u and we get:

$$-\int_0^1 u''\cdot u + \int_0^1 bu'\cdot u + \int_0^1 cu^2 = 0$$

$$\Leftrightarrow -\left[u'\cdot u\right]_0^1 + \int_0^1 |u'|^2 + \frac{1}{2}\int_0^1 b\cdot(u^2)' + \int_0^1 cu^2 = 0$$

$$\Leftrightarrow \int_0^1 (u')^2 - \frac{1}{2}\int_0^1 b'\cdot u^2 + \frac{1}{2}\left[bu^2\right]_0^1 + \int_0^1 cu^2 = 0$$

As one has no information about u the failure of this identity would hold if $(c-\frac{1}{2}b')\leq 0$, $b(0)=b(1)=0$ for non-trivial u.

Testing with u' yields:

$$-\int_0^1 u''\cdot u' + \int_0^1 bu'^2 + \int_0^1 cu'\cdot u = 0$$

$$\Leftrightarrow \underbrace{-\frac{1}{2}\left[u'^2\right]_0^1}_{=0} + \int_0^1 bu'^2 + \frac{1}{2}\int_0^1 (u^2)'\cdot c = 0.$$

If c is differentiable this would mean $c(0)=c(1)=0$ and c' has a sign $\Rightarrow c\equiv 0$ and b should have a sign.

Hence we have:

If $\begin{cases} c-\frac{1}{2}b \geq 0 & , b(0)=b(1)=0 \text{ there is no nontrivial solution to (1).} \\ b \neq 0 \ \forall x, \ c\equiv 0 & \text{there is no nontrivial solution to (1)} \end{cases}$

b) Consider the homogeneous, periodic bvp

$$\begin{cases} -u'' + bu' + cu = 0 \\ u(0)-u(1) = u'(0) - u'(1) = 0 \end{cases} \quad (2)$$

Testing with u implies:

$$-\int_0^1 u'' \cdot u + \int_0^1 bu' \cdot u + \int_0^1 cu^2 = \underbrace{-[u' \cdot u]_0^1}_{u'(1)\cdot u(1) - u'(0)\cdot u(0) = 0} + \int_0^1 u'^2 + \frac{1}{2}\int_0^1 b(u^2)' + \int_0^1 cu^2$$

$$= \int_0^1 u'^2 - \frac{1}{2}\int b' u^2 + \int cu^2 + \underbrace{[bu^2]_0^1}_{=0, \text{ as } b \text{ is periodic.}}$$

$\overset{u' \text{ nontrivial}}{\Rightarrow}$ $\int (c - \frac{1}{2}b')u^2 \geq 0$ provided that $(c - \frac{1}{2}b') \geq 0$ a.e.

$\rightsquigarrow$ If $c - \frac{1}{2}b' \geq 0$ a.e then (2) has no nontrivial solution.

Testing with u' implies

$$-\frac{1}{2}\int_0^1 ((u')^2)' + \int_0^1 bu'^2 + \frac{1}{2}\int_0^1 (u^2)' = \frac{1}{2}[u'^2]_0^1 + \frac{1}{2}[u^2]_0^1 + \int bu'^2$$

$= \int bu'^2 \neq 0$ if $b(x) \neq 0 \ \forall x \in (0,1)$

$\rightsquigarrow$ If $b(x) \neq 0 \ \forall x \in (0,1)$ then there is no nontrivial solution to (2)

Exercise 5

a) Assume there is $\eta \in C^1(0,\pi)$ s.t. for all $f \in C^1([0,\pi])$ one has $f(0)=f(\pi)=0$

$$\int_0^\pi (f'(t)^2 - f(t)^2)\,dt = \int_0^\pi (f'(t) - f(t)\eta(t))^2\,dt \geq 0,$$ then:

$$\int_0^\pi (f'(t)^2 - f(t)^2)\,dt = \int_0^\pi (f'(t)^2 - 2f'(t)f(t)\eta(t) + f(t)^2\eta(t)^2)$$

$$\Leftrightarrow \int_0^\pi (f'(t)^2 - f(t)^2)\,dt = \int_0^\pi (f'(t)^2 - (f^2)'\eta(t) + f(t)^2\eta(t)^2)$$

$$= \int_0^\pi f'(t)^2 + \int_0^\pi (f^2\eta'(t) + f(t)^2\eta(t)^2) - \lim_{\delta,\varepsilon\to 0}[f^2\cdot\eta]_\varepsilon^{\pi-\delta}$$

$$\Leftrightarrow 0 = \int (1 + \eta' + \eta^2) f^2 - \lim_{\delta,\varepsilon\to 0}[f^2\eta]_\varepsilon^{\pi-\delta}$$

Assume η solves $1 + \eta^2 + \eta' = 0$ in $(0,\pi)$

That means:
$$\frac{\eta'}{1+\eta^2} = -1$$

$$\Leftrightarrow \arctan(\eta) = -t + c$$

$$\Leftrightarrow \eta(t) = \tan(c-t).$$

As η should exist on $(0,\pi)$ this means: $c = \frac{\pi}{2}$

We still need to show that $\lim_{x\to 0} f^2\eta = \lim_{t\to\pi} f^2\eta$.

We have $\lim_{x\to 0} f^2\eta = \lim_{x\to 0} \dfrac{-f^2(x)\cdot\sin(t-\frac{\pi}{2})}{\cos(t-\frac{\pi}{2})} \overset{\text{l'Hôpital}}{=} \lim_{x\to 0}\dfrac{-2f'(x)\cdot f(x)\sin(t-\frac{\pi}{2}) - f^2\cos(t-\frac{\pi}{2})}{-\sin(t-\frac{\pi}{2})} = 0$

as $\lim_{t\to 0} f'(t)$ exists.

Same for $\lim_{x\to\pi}$