Karlsruher Institut für Technologie (KIT)
Institut für Analysis
Dr. Rainer Mandel
Peter Rupp

SoSe 2018
04.05.2018

# Boundary and Eigenvalue Problems

## Exercise Sheet 3

### Exercise 5

Consider the boundary value problem

$$\begin{cases} -u'' &= f \qquad\qquad \text{on } (0,a) \\ u(0) &= \gamma_0, \;\; u(a) - u'(a) = \gamma_1 \end{cases} \tag{1}$$

(a) Find all $a > 0$ such that for all $\gamma_0, \gamma_1 \in \mathbb{R}$ and $f \in C([0,a])$ the problem (1) has a unique solution $u$.

(b) Compute the Green's function in the affirmative case.

### Exercise 6

Consider the boundary value problem

$$\begin{cases} -u'' + u &= f \text{ in } (0,1) \\ u'(0) = u'(1) &= 0 \end{cases} \tag{2}$$

Show that for any $f \in C([0,1])$ the problem (2) is uniquely solvable and compute the Green's function to (2).

### Exercise 7

For a differential operator $L$ of order $m$ defined by

$$L\phi = \sum_{k=0}^{m} a_k \phi^{(k)}$$

with functions $a_k \in C^k([a,b])$ let $L^*$ be its (formal) adjoint operator, given by

$$L^*\phi = \sum_{k=0}^{m} (-1)^k (a_k \phi)^{(k)}.$$

Assume that $L = L^*$ holds.



— please turn around —

(a) For $m = 2$, classify all $a_0, a_1, a_2$, such that $L = L^*$.

(b) For any $f, g \in C^m([a,b])$ compute $\int_a^b Lf\, g\, dx - \int_a^b f\, Lg\, dx$ only in terms of the values (and the derivatives) of $f, g$ on the boundary $\{a, b\}$.

(c) Consider now periodic boundary value problem

$$\begin{cases} Lu = f & \text{on } (a,b) \\ u^{(\nu)}(b) - u^{(\nu)}(a) = 0 & \nu = 0, \dots, m-1. \end{cases} \tag{3}$$

and assume that $u = 0$ is the unique solution to the homogeneous problem. Let $G$ be the Green's function to (3) and

$$(L^{-1}f)(x) := \int_a^b G(x,\xi) f(\xi)\, d\xi$$

for $f \in C_0(a,b)$. Show that for all $f, g \in C_0(a,b)$ one has

$$\int_a^b (L^{-1}f)\, g\, dx = \int_a^b f\, (L^{-1}g)\, dx.$$

(d) Use (c) to show that $G(x,\xi) = G(\xi, x)$ for all $(x,\xi) \in [a,b] \times [a,b]$.