Exercise 5

$$\begin{cases} -u'' = f & \text{on } (0,a) \\ u(0) = \gamma_0, \quad u(a) - u'(a) = \gamma_1 \end{cases} \qquad (1)$$

(a) The problem (1) has a unique solution for any RHS

$\Leftrightarrow$ The homogeneous problem has only the trivial solution.

The equation $-u''(x) = 0$ has a fundamental system $u_1(x) = 1$, $u_2(x) = x$. So any solution is of the form

$u(x) = \alpha x + \beta$.

Inserting in the bc yields $u(0) = \beta = 0$ and $\alpha \cdot a + \beta - \alpha = \alpha(a-1) = 0$. Hence $a = 1 \Leftrightarrow \exists$ nontrivial solution of the form $u(x) = \alpha x$ for any $\alpha \in \mathbb{R} \setminus \{0\}$.

(b) Properties of G: i) $G \in C([0,a] \times [0,a])$, ii) $\partial_x G \in C(\overline{\Delta}_1) \cap C(\overline{\Delta}_2)$,

$\partial_x G(x, x_-) - \partial_x G(x, x_+) = \frac{1}{-1}$

iii) $\partial_x^2 G \in C(\overline{\Delta}_1) \cap C(\overline{\Delta}_2)$

iv) $G(\cdot, t)$ satisfies the hom. ODE/bc exept at $\{x = t\}$

From i-iv) we get

$$G(x,t) = A_1(t) \cdot 1 + B_1(t) \cdot x \qquad (x < t)$$
$$G(x,t) = A_2(t) \cdot 1 + B_2(t) \cdot x \qquad (x > t).$$

From i) we get $G(t_-, t) = G(t_+, t) \Leftrightarrow A_1(t) + B_1(t) \cdot t = A_2(t) + B_2(t) \cdot t$

ii). $\partial_x G(x,x) - \partial_x G(x, x_+) = B_2 - B_1(x) = -1$,

iii) $G(0,t) = A_1(t) = 0$

iv) $G(a,t) - \partial_x G(a,t) = A_2(t) \cdot 1 + \underbrace{B_2(t) \cdot a - B_2(t)}_{= (a-1) B_2(t)} = 0$

Hence:

$$\left. \begin{aligned} B_1(t) \cdot t &= A_2(t) + B_2(t) \cdot t \\ B_1(t) \cdot t &= (B_2(t) + 1) \cdot t \\ A_2(t) &= (1-a) B_2(t) \end{aligned} \right\} \quad A_2(t) + Bt \text{ [illegible] } = 0$$

$\Rightarrow B_2(t) = \frac{t}{1-a}$

$\Rightarrow B_1(t) = \frac{1-a+t}{1-a}$

$$\Rightarrow \quad G(x,t) = \begin{cases} \left(1 + \frac{t}{1-a}\right)\cdot x, & x<t \\ t + \frac{t}{1-a} x & x > t \end{cases}$$

$$G(a,t) - \partial_x G(a,t)$$
$$= \left(t \cdot \left(\frac{1-a+a}{1-a}\right)\right) - \frac{t}{1-a} = 0 \checkmark$$

Exercise 6

$$\begin{cases} -u'' + u = f \\ u'(0) = u'(1) = 0 \end{cases} \quad (2)$$

$$-u''(x) + u(x) = 0 \quad \Leftrightarrow \quad u(x) = A \cdot \sinh(x) + B \cdot \cosh(x)$$
$$u'(x) = 0 \Leftrightarrow \quad A \cdot \underbrace{\cosh(0)}_{=1} = 0 \Leftrightarrow A = 0$$
$$u'(1) = 0 \Leftrightarrow \quad + B \cdot \sinh(1) = 0 \Leftrightarrow B = 0$$

Hence, the hom. problem is uniquely solvable.

$$G(x,t) = \begin{cases} A_1(t)\sinh(x) + B_1(t)\cosh(x) & (x < t) \\ A_2(t)\sinh(x) + B_2(t)\cosh(x) & (x > t) \end{cases}$$

$$\partial_x G(0,t) = A_1(t)\cosh(0) = A_1(t) = 0$$
$$\partial_x G(1,t) = A_2(t)\cosh(1) + B_2(t)\sinh(1) = 0 \quad \Leftrightarrow \quad A_2(t) = -\frac{\sinh(1)}{\cosh(1)} B_2(t)$$

$$\partial_x G(x,x_-) - \partial_x(x,x_+) = A_2(x)\cosh(x) + B_2(x)\sinh(x) - B_1(x)\sinh(x) = -1$$
$$G(x,x_-) - G(x,x_+) = A_2(x)\sinh(x) + B_2(x)\cosh(x) - B_1(x)\cosh(x) = 0$$

$$\Rightarrow \quad -\frac{\sinh(1)}{\cosh(1)}\cosh(x) B_2(x) + B_2(x)\sinh(x) - B_1(x)\sinh(x) = -1 \quad \Big| \cdot \cosh(x)$$
$$-\frac{\sinh(1)}{\cosh(1)}\sinh(x) B_2(x) + B_2\cosh(x) - B_1(x)\cosh(x) = 0 \quad \Big| \cdot \sinh(x)$$

$$-\frac{\sinh(1)}{\cosh(1)} B_2(x) = -\cosh(x)$$

$$\cosh(x)\cdot B_1(x)$$
$$\|$$

$$\Rightarrow \quad B_2(t) = \frac{\cosh(1)}{\sinh(1)}\cosh(t) \quad \Rightarrow \quad \left(\cosh(x) - \frac{\sinh(1)}{\cosh(1)}\sinh(x)\right)\cdot \frac{\cosh(1)}{\sinh(1)}\cosh(x)$$