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# Boundary and Eigenvalue Problems

## Exercise Sheet 4

### Exercise 8

(a) Let $\Omega \subset \mathbb{R}^n$ be open and $u \in C^1(\Omega \setminus \{x_0\})$ for some $x_0 \in \Omega$ with $\partial_i u \in L^1_{loc}(\Omega)$. Assume furthermore, that there are $C, \alpha > 0$ such that

$$|u(x)| \leq C|x - x_0|^{-\alpha} \qquad (x \in \Omega \setminus \{x_0\}).$$

  (i) Show that $\partial_i u$ is a weak derivative to $u$ on $\Omega$ provided that $\alpha < n - 1$.

  (ii) Give an example for $\partial_i u$ not being a weak derivative in the case $\alpha \geq n - 1$.

(b) Consider the function

$$\chi : (-1, 1) \to \mathbb{R}, \ \chi(x) = |x|$$

  (i) Show that $\chi$ is weakly differentiable, but not twice weakly differentiable (i.e $\chi'$ is not weakly differentiable).

  (ii) Define $f : (-1, 1) \times (-1, 1) \to \mathbb{R}$ by $f(x, y) = \chi'(x) + \chi'(y)$. Show that $\partial^2_{xy} f$ exists (in the weak sense) although the first weak partial derivatives don't.

### Exercise 9

(a) Let $\Omega \subset \mathbb{R}^n$ be open, $u \in L^1_{loc}(\Omega)$ be weakly differentiable and let $G \in C^1(\mathbb{R})$ with $G' \in L^\infty(\mathbb{R})$. Show $G(u) \in L^1_{loc}(\Omega)$. Moreover, prove the 'chain rule' for weak derivatives: $G(u)$ is weakly differentiable and $\partial_i(G(u)) = G'(u)\partial_i u$ for all $i = 1, \dots, n$.
*Hint: It is useful to consider the mollified version $u_\varepsilon$ of $u$*

(b) Use (a) to prove that $u_+, u_-$ ($u_\pm := \max\{0, \pm u\}$) and $|u|$ are weakly differentiable provided that $u$ is weakly differentiable.
*Hint: Approximate $(\cdot)_\pm$ and $|\cdot|$ by a smooth version.*



— please turn around —

(c) Extend the chain rule to the following situation: Let $G \in C^1(\mathbb{R})$ and $u \in L^1_{loc}(\Omega)$ be weakly differentiable and such that $G'(u) \in L^p_{loc}(\Omega)$ and $\partial_i u \in L^{\frac{p}{p-1}}_{loc}(\Omega)$ for some $p \in [1, \infty]$. Assume furthermore that $G(u) \in L^p_{loc}(\Omega)$. Then $G(u)$ is weakly differentiable and $\partial_i(G(u)) = G'(u)\partial_i u$.

*Hint: Approximate $G$ and $u$ independently by truncated, bounded functions as follows: $G_{\tilde{R}}(z) := \varphi_{\tilde{R}}(z)G(z)$, $u_R(x) := \psi_R(u(x))$, where $\psi_R := \int_0^x \varphi_R(y)\, dy$ and $\varphi_R \in C^\infty(\mathbb{R})$, $\varphi_R(x) \in [0, 1]$, $\varphi_R(x) = 1$ for $x \in [-R, R]$, $\varphi_R = 0$ for $x \notin [-2R, 2R]$ and $|\nabla \varphi_R(x)| \leq C/R$ for some $C > 0$, see for example Proposition 3 from the lecture notes. Then use (a).*