Exercise 8

i) For $u \in C^1(\Omega \setminus \{x_0\})$ the classical derivative $\partial_{x_i} u$ would be the candidate for the weak derivative.

Then for any $\varphi \in C_c^\infty(\Omega)$ one has

$$-\int_\Omega u \cdot \partial_{x_i}\varphi = \lim_{\varepsilon \to 0} -\int_{\Omega \setminus B_\varepsilon(x_0)} u \cdot \partial_{x_i}\varphi = \lim_{\varepsilon \to 0} \int_{\Omega \setminus B_\varepsilon(x_0)} \partial_{x_i} u \cdot \varphi$$

$$- \int_{\Omega \setminus B_\varepsilon(x_0)} \underbrace{\partial_{x_i}(u \cdot \varphi)}_{= \operatorname{div}\begin{pmatrix} 0 \\ u\cdot\varphi \\ 0 \end{pmatrix} \leftarrow i\text{-th component}}$$

$$\overset{\text{Gauß-Theorem}}{=} \lim_{\varepsilon \to 0} - \int_{\partial B_\varepsilon(x_0)} \nu \cdot \begin{pmatrix} 0 \\ u\cdot\varphi \\ 0 \end{pmatrix} + \int_\Omega \partial_{x_i} u \cdot \varphi$$

$\partial_{x_i} u \in L^1_{loc}$

Moreover, we have

$$\int_{\partial B_\varepsilon(x_0)} \nu \cdot \begin{pmatrix} 0 \\ u\cdot\varphi \\ 0 \end{pmatrix} \leq \int_{\partial B_\varepsilon(x_0)} |u \cdot \varphi| \leq C \int_{\partial B_\varepsilon(x_0)} |x - x_0|^{-\alpha} |\varphi|$$

$$= C \cdot \varepsilon^{-\alpha} \underbrace{\operatorname{vol}(\partial B_\varepsilon(x_0))}_{\leq C \cdot \varepsilon^{n-1}} \xrightarrow{\alpha < n-1} 0$$

ii) Take $n = 1$, $u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$ Then $|u(x)| \leq 1 = |x|^0$

and u is differentiable on $(\mathbb{R} \setminus \{0\})$. with $u'(x) = 0$ $(x \in \mathbb{R} \setminus \{0\})$

$u' \in L^1_{loc}$ and $u \in L^1_{loc}$ <u>but</u>

$$-\int u \varphi' = -\int_0^\infty \varphi' + \int_{-\infty}^0 \varphi' = 2\varphi(0) \neq \int 0 \cdot \varphi \quad \text{in general.}$$

b) i) $\mathcal{X} : (-1,1) \to \mathbb{R}$, $\mathcal{X}(x) = |x|$.

Let $\varphi \in C_c^\infty(-1,1)$, then:

$$-\int_{-1}^{1} \mathcal{X} \cdot \varphi' = \int_{-1}^{0} x \cdot \varphi' - \int_{0}^{1} x \cdot \varphi' = \underbrace{[x \cdot \varphi]_{-1}^{0}}_{=0} - \underbrace{[x \cdot \varphi]_{0}^{1}}_{=0} - \int_{-1}^{0} \varphi + \int_{0}^{1} \varphi$$

$$= \int_{-1}^{1} \mathcal{X}' \cdot \varphi \quad , \text{ where } \quad \mathcal{X}' = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

That this function is not w-differentiable we saw in a) ii)

ii) For $\varphi \in C_c^\infty(\mathbb{R}^2)$ $\varphi(0,0) \neq 0$ we have

$$-\int_{\mathbb{R}^2} f(x,y) \cdot \partial_x \varphi = -\int_{\mathbb{R} \times \mathbb{R}_+} \partial_x \varphi + \int_{\mathbb{R} \times \mathbb{R}_-} \partial_x \varphi - \int_{\mathbb{R}_+ \times \mathbb{R}} \partial_x \varphi + \int_{\mathbb{R}_- \times \mathbb{R}} \partial_x \varphi$$

$$= -0 + 0 + 2\int_{\mathbb{R}} \varphi(0,y)\, dy$$

$$\Rightarrow \partial_x f(x,y) = 0 \quad \forall x \neq 0 \Rightarrow f \equiv 0 \ \lightning$$

but: $$-\int f(x,y)\, \partial_{xy} \varphi = -\int_{\mathbb{R} \times \mathbb{R}_+} \partial_{xy} \varphi + \int_{\mathbb{R} \times \mathbb{R}_-} \partial_{xy} \varphi - \int_{\mathbb{R}_+ \times \mathbb{R}} \partial_{xy} \varphi + \int_{\mathbb{R}_- \times \mathbb{R}} \partial_{xy} \varphi$$

$$= 2\int_{\mathbb{R}} \partial_x \varphi(x,0) + 2 \int_{\mathbb{R}} \partial_y \varphi(y,0) = 0 \quad \forall \varphi \in C_c^\infty$$

$$\Rightarrow \partial_{xy} f \equiv 0$$

$u_\varepsilon \in C^\infty$ wo?

Exercise 9

a) $\Omega \subset \mathbb{R}^n$ open, $u \in L^1_{loc}(\mathbb{R})$ weakly differentiable, $G \in C^1(\mathbb{R})$
$G' \in L^\infty(\mathbb{R})$

Claim: $G(u)$ is weakly differentiable and $\partial_i(G(u)) = G'(u) \cdot \partial_i u$

Proof: Consider the mollificated version u_ε of u defined by

$$u_\varepsilon(x) = (\rho_\varepsilon * u)(x) = \int_{\mathbb{R}^n} \underbrace{\rho_\varepsilon(x-y)}_{\text{cpt supp, bounded.}} u(y)\,dy \quad ✓$$

Moreover u_ε is differentiable as

$$\lim_{h\to 0} \frac{u_\varepsilon(x+he_i) - u_\varepsilon(x)}{h} = \lim_{h\to 0} \int_{\mathbb{R}^n} \underbrace{\frac{\rho_\varepsilon(x-y+he_i) - \rho_\varepsilon(x-y)}{h}}_{\leq \|\rho_\varepsilon'\|_\infty \cdot \chi_{\operatorname{supp}\rho_\varepsilon}(x-y)} u(y)\,dy$$

dominated convergence
$$\overset{\Downarrow}{=} (\rho_\varepsilon' * u)(x).$$

Moreover: for $v_\varepsilon \to v$ in L^1_{loc}. Fix $K \subset\subset \mathbb{R}^n$ and choose a smooth fct $\tilde{v}$ in $L^1(K)$ approximating v. Then

$$\|v_\varepsilon - v\|_{L^1(K)} \leq \underbrace{\|v_\varepsilon - \tilde{v}_\varepsilon\|_{L^1(K)}}_{\leq \|v-\tilde{v}\|_{L^1(K)}} + \underbrace{\|\tilde{v}_\varepsilon - \tilde{v}_\varepsilon\|_{L^1(K)}}_{\text{conv. unif.}} + \|\tilde{v} - v\|_{L^1(K)}$$

$$\longrightarrow 0$$

Now $u_\varepsilon \in C^1 \Rightarrow G(u_\varepsilon)$ is classically differentiable with $\partial_i(G(u_\varepsilon)) = G'(u_\varepsilon) \cdot \partial_i u_\varepsilon$.

Moreover: $G(u_\varepsilon) \longrightarrow G(u)$ in L^1_{loc} as

$$\int_K |G(u_\varepsilon) - G(u)| \leq \|G'\|_\infty \cdot \int_K |u_\varepsilon - u| \longrightarrow 0.$$