Exercise 10

$\Omega \subseteq \mathbb{R}^n$ open

a) Claim: $W^{k,p}(\Omega)$ is a Banach space $\forall k \in \mathbb{N}, p \in [1,\infty]$

Proof: Let $(u_m)_{m\in\mathbb{N}}$ be a Cauchy sequence. Then $(\partial^\alpha u_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in L^p for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le k$. Hence, there is u^α s.t. $\|\partial^\alpha u_m - u^\alpha\|_{L^p} \to 0$ $(m\to\infty)$. Now, we need to show: $u^\alpha = \partial^\alpha u$, where $u = u^0$.

Let $\varphi \in C_c^\infty(\Omega)$. Then:

$$\int \partial^\alpha u \ \varphi = (-1)^{|\alpha|} \int u \cdot \partial^\alpha \varphi = \lim_{m\to\infty} (-1)^{|\alpha|} \int u_m \ \partial^\alpha \varphi = \lim_{m\to\infty} \int \partial^\alpha u_m \cdot \varphi = \int u^\alpha \varphi$$

and therefore:

$$\|u_m - u\|_{W^{k,p}} = \sum_{|\alpha|\le k} \|\partial^\alpha (u_m - u)\|_{L^p} = \sum_{|\alpha|\le k} \|\partial^\alpha u_m - \partial^\alpha u\|_{L^p} \to 0$$

□

b) Claim: $W^{k,p}_{\Gamma_0}(\Omega)$ are Banach spaces for all $\Gamma_0 \subseteq \partial\Omega$, k,p as above

Proof: We show that $W^{k,p}_{\Gamma_0}(\Omega) \subseteq W^{k,p}(\Omega)$ are closed subspaces.

Let $(u_m)_{m\in\mathbb{N}}$ be a sequence in $W^{k,p}_{\Gamma_0}(\Omega)$ converging to $u \in W^{k,p}(\Omega)$

w.l.o.g $\|u_m - u\|_{W^{k,p}(\Omega)} \le \frac{1}{2^{m+1}}$. Choose now $\varphi_m \in C^\infty_{\Gamma_0}$ with $\|u_m - \varphi_m\|_{W^{k,p}} \le \frac{1}{2^{m+1}}$ $(m \in \mathbb{N})$. Then:

$$\|\varphi_m - u\| \le \|\varphi_m - u_m\| + \|u_m - u\| \le \frac{1}{2^m} \Rightarrow \varphi_m \xrightarrow[\|\cdot\|_{k,p}]{} u \ (m\to\infty)$$ and

$\varphi_m \in C^\infty_{\Gamma_0}$. Hence, we $\overline{C^\infty_{\Gamma_0}}^{\|\cdot\|_{k,p}} = W^{k,p}_{\Gamma_0}$

□

Exercise 11

$\Omega \subset B_R(0)$, $c \in L^\infty(\Omega)$

Claim: The bvp $\begin{cases} -\Delta u + x \cdot \nabla u + c(x) u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

has a unique solution provided that $c \geq \frac{N}{2} - \frac{1}{R^2} + \varepsilon$ for some $\varepsilon > 0$

Proof: Take any $\varphi \in C_c^\infty(\Omega)$, then the weak formulation we get by

$$\int -\Delta u \cdot \varphi + x \cdot \nabla u \cdot \varphi + c u \cdot \varphi = \int f \cdot \varphi$$

$$\Longleftrightarrow \underbrace{\int \nabla u \cdot \nabla \varphi - \int u \cdot \operatorname{div}(x \cdot \varphi) + c u \cdot \varphi}_{=: B(u,\varphi)} = \underbrace{\int f \cdot \varphi}_{=: F(\varphi)}$$

Boundedness of B:

$$B(u,\varphi) \leq \|u\|_{H^1} \cdot \|\varphi\|_{H^1} + R \cdot \|u\|_{H^1} \cdot \|\varphi\|_{L^2} + \|c\|_\infty \cdot \|u\|_{L^2} \cdot \|\varphi\|_{L^2} \quad ✓$$

Coercivity:

$$B(u,u) = \int |\nabla u|^2 + \int x \cdot \nabla u \cdot u + \int c u^2$$

$$= \int |\nabla u|^2 + \frac{1}{2} \int x \cdot \nabla(u^2) + \int c u^2$$

$$= \int |\nabla u|^2 - \frac{N}{2} \int u^2 + \int c u^2$$

$$= \int |\nabla u|^2 + \int \left(c - \frac{N}{2}\right) u^2$$

$$\geq \int |\nabla u|^2 + \left(\underline{c} - \frac{N}{2}\right) \int u^2 \quad , \left(\underline{c} - \frac{N}{2}\right) > 0 \rightsquigarrow \text{fertig, sonst}$$

$$\geq \int |\nabla u|^2 - \left(\frac{N}{2} - \underline{c}\right) \cdot R^2 \int |\nabla u|^2$$

$$= \underbrace{\left(1 - \frac{N}{2} R^2 + \underline{c} \cdot R^2\right)}_{\geq \delta} \int |\nabla u|^2$$

Für $\underline{c} > \frac{N}{2} - \frac{1}{R^2} + \varepsilon$

By Lax-Milgram the solution to the bvp exists and is unique □