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SoSe 2018
25.05.2018

Boundary and Eigenvalue Problems

Exercise Sheet 6

Exercise 13

Prove the following result concerning the optimality of the admissible exponents in the embedding theorem of Sobolev: Assume $1 \leq p < \frac{n}{k}$, $k \in \mathbb{N}, n \in \mathbb{N}$ and there is a continuous embedding $W^{k,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, then one necessarily has

$$p \leq q \leq \frac{np}{n-kp}.$$

Hint: For a function $u \in L^q(\mathbb{R}^n)$ and $\alpha > 0$ consider $\|u(\alpha\cdot)\|_{W^{k,p}(\mathbb{R}^n)}$.

Exercise 14

Consider the open and bounded set $\Omega \subset \mathbb{R}^2$ defined by $\Omega := C \cup \bigcup_{m=1}^{\infty}(A_m \cup B_m)$ and

$$C := (0,1) \times \left(0, \frac{1}{3}\right), \quad B_m := \left(\frac{b_m - 1}{b_m} 2^{-m}, 2^{-m}\right) \times \left[\frac{1}{3}, \frac{2}{3}\right], \quad A_m := \left(\frac{3}{4} 2^{-m}, 2^{-m}\right) \times \left(\frac{2}{3}, 1\right).$$

where $b_m > 2$ for all $m \in \mathbb{N}$. Define

$$u(x,y) = \begin{cases} 0 & , \text{ if } (x,y) \in C \\ (3y-1)a_m & , \text{ if } (x,y) \in B_m \\ a_m & , \text{ if } (x,y) \in A_m \end{cases}$$

and for any given $q > p$ choose sequences $(a_m)_{m\in\mathbb{N}}$, $(b_m)_{m\in\mathbb{N}}$ such that $|\nabla u|$, $u \in L^p(\Omega)$ but $u \notin L^q(\Omega)$.

Exercise 15

Show that the operator $E : W^{1,p}(\mathbb{R}^{n-1} \times (0,\infty)) \to W^{1,p}(\mathbb{R}^n)$ defined by

$$E\phi(x) := \begin{cases} \phi(x) & , \text{ if } x_n \geq 0 \\ \phi(x', -x_n) & , \text{ if } x_n < 0, \end{cases}$$

defines an extension operator on $\mathbb{R}^{n-1} \times (0,\infty)$.