Exercise 13

For $u \in L^q(\mathbb{R}^n)$ one has for any $\alpha > 0$

$$\|u(\alpha\cdot)\|_{L^q}^q = \int |u(\alpha\cdot)|^q \, dx = \frac{1}{\alpha^n}\int |u(y)|^q \, dy = \alpha^{-n}\|u\|_q^q,$$

hence $\|u(\alpha\cdot)\|_q = \alpha^{-\frac{n}{q}}\|u\|_q$.

We also have

$$\|u(\alpha\cdot)\|_{W^{k,p}} = \left(\sum_{|\beta|\le k} \|\partial^\beta u(\alpha\cdot)\|_p^p\right)^{\frac{1}{p}} = \left(\sum_{|\beta|\le k} |\alpha|^{\beta p} \|(\partial^\beta u)(\alpha\cdot)\|_{L^p}^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{|\beta|\le k} \alpha^{\beta p - n}\|\partial^\beta u\|_{L^p}^p\right)^{\frac{1}{p}} \begin{cases} \le |\alpha|^{\overbrace{k-\frac{n}{p}}^{>0 \text{ by assumption}}} \|u\|_{W^{k,p}}, & \alpha \ge 1 \\ \le |\alpha|^{-\frac{n}{p}}\|u\|_{W^{k,p}}, & \alpha \le 1 \end{cases}$$

Assume now there is a continuous embedding $W^{k,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$

Then we have: For $\alpha \ge 1$:

$$\|u\|_{L^q} = |\alpha|^{\frac{n}{q}}\|u(\alpha\cdot)\|_{L^q} \le C|\alpha|^{\frac{n}{q}}\|u(\alpha\cdot)\|_{W^{k,p}} \le C|\alpha|^{\frac{n}{q}+k-\frac{n}{p}}\|u\|_{W^{k,p}}$$

For all $\alpha \ge 1$.

$$\Rightarrow \frac{n}{q} + k - \frac{n}{p} \ge 0 \iff n + kq - \frac{n}{p}q \ge 0$$

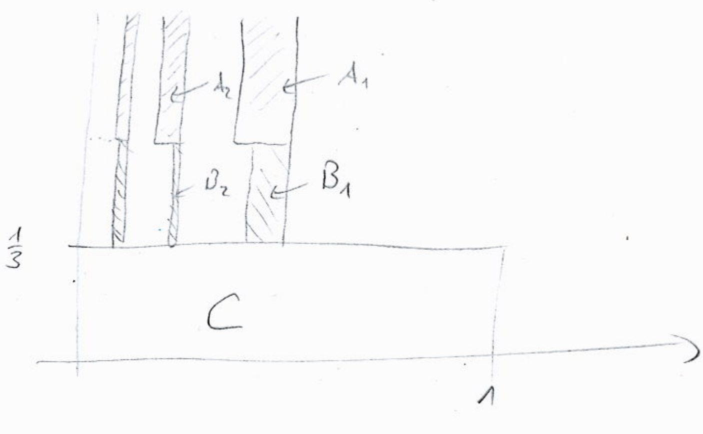
$$\iff \frac{n}{\frac{n}{p} - k} \ge q$$

$$\iff \frac{np}{n-kp} \ge q$$

If $\alpha \le 1$, then $\|u\|_q \le C \cdot |\alpha|^{\frac{n}{q}-\frac{n}{p}}\|u\|_{W^{k,p}} \Rightarrow q \ge p$.

Exercise 14

$b_m \overset{!}{\ge} 4$



$$u(x,y) = \begin{cases} 0 & (x,y)\in C \\ (3y-1)a_m & (x,y)\in B_m \\ a_m & (x,y)\in A_m \end{cases}$$ , $u(x,y)$ is continuous and differentiable a.e.

$$\|u\|_{L^p}^p = \sum_{m=1}^{\infty} a_m^p \int_{\frac{1}{3}}^{\frac{2}{3}} \int_{(1-\frac{1}{b_m})2^{-m}}^{2^{-m}} (3y-1)^p \, dx\, dy + \sum_{m=1}^{\infty} a_m^p \int_{\frac{2}{3}}^{1} \int_{\frac{3}{4}2^{-m}}^{2^{-m}} 1 \, dx\, dy$$

$$= \sum_{m=1}^{\infty} a_m^p \cdot \frac{1}{b_m} 2^{-m} \cdot \frac{1}{3(p+1)} \Big[(3y-1)^{p+1}\Big]_{\frac{1}{3}}^{\frac{2}{3}} + a_m^p \cdot \frac{1}{12} \cdot 2^{-m}$$

$$= \sum_{m=1}^{\infty} \frac{a_m^p}{b_m} \cdot \frac{2^{-m}}{3(p+1)} + \frac{a_m^p \cdot 2^{-m}}{12}$$

$$\|\nabla u\|_{L^p}^p = \sum_{m=1}^{\infty} a_m^p \cdot \int_{\frac{1}{3}}^{\frac{2}{3}} \int_{(1-\frac{1}{b_m})2^{-m}}^{2^{-m}} 3^p \, dx\, dy = \sum_{m=1}^{\infty} a_m^p \cdot \frac{1}{3} \cdot \left(\frac{1}{b_m}\right) 2^{-m}$$

Angenommen $\|u\|_{L^p}$ und $\|\nabla u\|_{L^p}$ existieren $\sum_{m=1}^{\infty} \frac{a_m^p}{2^m}$, $\sum_{m=1}^{\infty} \frac{a_m^p}{b_m} \cdot 2^{-m}$ konvergiert,

$\|u\|_{L^q}$ konvergiert nicht $\Leftarrow$ $\sum_{m=1}^{\infty} \frac{a_m^q}{2^m}$ divergiert $\wedge$ $\sum_{m=1}^{\infty} \frac{a_m^q}{b_m} \cdot 2^{-m}$ konvergiert.

Wähle $a_m = 2^{\frac{2m}{p+q}}$, $b_m = 2^{m^2}$, dann gilt:

$a_m^p = a_m^{\overset{<1}{\frac{2p}{p+q}} \cdot m}$, da $p<q$, $\qquad a_m^q = a_m^{\overset{>1}{\frac{2q}{p+q}} \cdot m}$

$\Rightarrow \sum \frac{a_m^p}{2^m}$ konv. , $\qquad \Rightarrow \sum \frac{a_m^q}{2^m}$ div

und $\frac{a_m^q}{b_m} 2^{-m}$, $\frac{a_m^p}{b_m} \cdot 2^{-m}$ konvergiert.