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Boundary and Eigenvalue Problems

Exercise Sheet 7

Exercise 16

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let $S, T: H \to H$ be linear operators. Decide whether the following statements are true of false and prove your claims.

(a) If S is bounded and T compact, then $S \circ T$ and $T \circ S$ are compact.

(b) If T is compact, then so is T^2.

(c) If T^2 is compact, then so is T.

(d) If $\dim(\mathrm{Ran}(T)) < \infty$ and T is bounded, then T is compact.

(e) If $\dim(\mathrm{Ran}(T)) < \infty$, then T is compact.

Exercise 17

Let X be a Banach space and $\mathcal{F} \subset X$. Show that the following two statements are equivalent:

(a) Any sequence in $\mathcal{F}$ has a convergent subsequence in X.

(b) For any $\varepsilon > 0$ there is a finite cover $(B_\varepsilon(x_n))_{n=1}^N$ of open balls with radius ε given by $B_\varepsilon(x_n) := \{x \in B : |x - x_n| < \varepsilon\} \subset X$ for some $x_n \in X$ such that

$$\mathcal{F} \subset \bigcup_{n=1}^N B_\varepsilon(x_n).$$

Exercise 18

Let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected and let $\Gamma_0 \subset \partial\Omega$ with positive surface measure.

(a) Show that for all $u \in H^1_{\Gamma_0}(\Omega)$ one has

$$\int_{\Gamma_0} u \, d\sigma = 0$$

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(b) Use the Rellich-Kondrachev theorem to show that there is a $C > 0$, such that for all $u \in H^1_{\Gamma_0}(\Omega)$ one has:

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}$$

Hint: Proof by contradiction using that $\int_\Omega |\nabla u|^2\,dx = 0$ implies u is constant.

(c) Show that the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_0 \\ \partial_\nu u = 0 & \text{on } \Gamma_1 = \partial\Omega \setminus \Gamma_0. \end{cases}$$

has a unique (weak) solution $u \in H^1_{\Gamma_0}(\Omega)$.