Boundary and Eigenvalue Problems

Exercise Sheet 7

Exercise 16
Let \((H, \langle \cdot, \cdot \rangle)\) be a Hilbert space and let \(S, T : H \to H\) be linear operators. Decide whether the following statements are true or false and prove your claims.

(a) If \(S\) is bounded and \(T\) compact, then \(S \circ T\) and \(T \circ S\) are compact.
(b) If \(T\) is compact, then so is \(T^2\).
(c) If \(T^2\) is compact, then so is \(T\).
(d) If \(\dim(\text{Ran}(T)) < \infty\) and \(T\) is bounded, then \(T\) is compact.
(e) If \(\dim(\text{Ran}(T)) < \infty\), then \(T\) is compact.

Exercise 17
Let \(X\) be a Banach space and \(\mathcal{F} \subset X\). Show that the following two statements are equivalent:

(a) Any sequence in \(\mathcal{F}\) has a convergent subsequence in \(X\).
(b) For any \(\varepsilon > 0\) there is a finite cover \((B_\varepsilon(x_n))_{n=1}^N\) of open balls with radius \(\varepsilon\) given by \(B_\varepsilon(x_n) := \{x \in B : |x - x_n| < \varepsilon\} \subset X\) for some \(x_n \in X\) such that

\[
\mathcal{F} \subset \bigcup_{n=1}^N B_\varepsilon(x_n).
\]

Exercise 18
Let \(\Omega \subset \mathbb{R}^n\) be open, bounded and connected and let \(\Gamma_0 \subset \partial \Omega\) with positive surface measure.

(a) Show that for all \(u \in H^1_{\Gamma_0}(\Omega)\) one has

\[
\int_{\Gamma_0} u \, d\sigma = 0
\]
(b) Use the Rellich-Kondrachev theorem to show that there is a $C > 0$, such that for all $u \in H^1_{\Gamma_0}(\Omega)$ one has:

$$||u||_{L^2(\Omega)} \leq C \||\nabla u||_{L^2(\Omega)}$$

*Hint:* Proof by contradiction using that $\int_{\Omega} |\nabla u|^2 \, dx = 0$ implies $u$ is constant.

(c) Show that the boundary value problem

\[
\begin{cases}
-\Delta u = f & \text{in } \Omega \\
\quad u = 0 & \text{on } \Gamma_0 \\
\quad \partial_{\nu} u = 0 & \text{on } \Gamma_1 = \partial \Omega \setminus \Gamma_0.
\end{cases}
\]

has a unique (weak) solution $u \in H^1_{\Gamma_0}(\Omega)$. 