Exercise 17

Let X be a Banach Space, $\mathcal{F} \subset X$

Assume for any $\varepsilon > 0$ $\exists$ $B_\varepsilon(x_n)_{n=1}^N$ s.t.

$\mathcal{F} \subset \bigcup_{n=1}^N B_\varepsilon(x_n)$.

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in X. For a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ with $\varepsilon_k \to 0$ $(k\to\infty)$ choose $(x_{n,k})_{n=1}^{N_k}$ s.t. $\bigcup_{n=1}^{N_k} B_\varepsilon(x_{n,k}) \supset \mathcal{F}$. For $k=0$ there is n_0^0 s.t. $u_n \in B_{\varepsilon_0}(x_{n_0^0})$ for infinetely many n. Choose a subsequence u_n^0 s.t. $(u_n^0) \in B_{\varepsilon_0}(x_{n_0^0})$ $\forall$ $n\in\mathbb{N}$. Next choose n_1^0, s.t. $u_n^0 \in B_{\varepsilon_1}(x_{n,1}^0)$ for infinitely many n and again choose a subsequence $(u_n^1)_{n\in\mathbb{N}}$ of u_n^0 with $u_n^1 \in B_{\varepsilon_1}(x_{n,1})$ for all $n\in\mathbb{N}$.

Proceed inductively. Note that $x_{n^0_{,m}} \in B_{2\varepsilon_{m-1}}(x_{n^0_{,m-1}})$ $\forall m \geq 1$ as $\varepsilon_m < \varepsilon_{m-1}$, otherwise $B_{\varepsilon_m}(x_{n^0_{,m}}) \cap B_{\varepsilon_{m-1}}(x_{n^0,m-1}) = \emptyset$

$\Rightarrow$ $x_{n^0_{,m}}$ is a Cauchy sequence and hence converges to some x. Then $u_n^n \to x$ as for any $\varepsilon > 0$ choose n s.t. $\varepsilon_n + \varepsilon_{n-1} < \frac{\varepsilon}{2}$ and $|u_n^n - x| \leq |u_n^n - x_{n^0_{,n}} + x_{n^0_{,n}} - x|$

$\leq \varepsilon_n + \varepsilon_n < \varepsilon$

Assume now there is a sequence $(x_n)_{n\in\mathbb{N}}$ in X without a converging subsequence.

Then there is $\varepsilon > 0$ and a subsequence $(x_n)_n$ s.t. $\|x_n - x_m\| \geq \varepsilon \quad \forall m \neq n$

$\Rightarrow$ Any finite cover of Balls with radius $\frac{\varepsilon}{2}$ cannot cover $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$.

Exercise 11

$\Omega \subset \mathbb{R}^n$ bounded, open, connected $\Gamma_0 \subset \partial\Omega$ with $\sigma(\Gamma_0) > 0$.

(a) Claim: $\forall\ u \in H^1_{\Gamma_0}(\Omega)$: $\int_{\Gamma_0} u \, d\sigma = 0$

Proof: Take $(u_n) \in C^\infty_{\Gamma_0}(\Omega)$ approximating u in H^1.

Then $\left|\int_{\Gamma_0} u \, d\sigma\right| \overset{\text{def. of trace}}{=} \int_{\Gamma_0} \lim_{n\to\infty} |u_n| \, d\sigma \leq \liminf_{n\to\infty} \int_{\Gamma_0} \underbrace{|u_n|}_{=0 \text{ on } \Gamma_0} = 0.$

(b) Assume $\sup_{u \in H^1_{\Gamma_0}(\Omega)\setminus\{0\}} \frac{\|u\|_{L^2}}{\|\nabla u\|_{L^2}} = \inf_{u \in H^1_{\Gamma_0}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|}{\|u\|} = 0$. Then

there is a sequence (u_n) in $H^1_{\Gamma_0}(\Omega)$ s.t. $\|u_n\|_{L^2} = 1$ and

$\|\nabla u_n\| \to 0$ As $H^1_{\Gamma_0}(\Omega) \subset L^2(\Omega)$ is cpt there is a $u \in H^1_{\Gamma_0}(\Omega)$

with $\nabla u = 0$ and $\|u\|_{L^2} = 1$. (as (u_n) bdd implies $u_n \underset{H^1_{\Gamma_0}}{\rightharpoonup} u$ for some

$u \overset{K\text{-}R}{\Rightarrow} u_n \to u$ in L^2)

$\Rightarrow$ u is constant $\Rightarrow$ $u = u|_{\Gamma_0} = 0$ ↯ $\|u\|_{L^2} = 1$

c) Define $M : H^1_{0}(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$, $M(u,v) = \int \nabla u \cdot \nabla v$

Then:

$|M(u,v)| \leq \|u\|_{H^1_0} \cdot \|v\|_{H^1_0}$

$M(u,u) = \int |\nabla u|^2$

and $\int f \cdot u \leq \|f\| \cdot \|u\| \leq C \cdot \|f\| \cdot \|\nabla u\| = C\|f\| \cdot \|u\|_{H^1_0}$

Lax-Milgram $\Rightarrow$ The bvp $\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

has a unique weak solution