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Boundary and Eigenvalue Problems

Exercise Sheet 8

Exercise 19

Let $\Omega \subseteq \mathbb{R}^n$ be an open set such that there are sequences $(x_k)_{k\in\mathbb{N}}$, $(r_k)_{k\in\mathbb{N}}$ with $r_k \to \infty$, and $B_{r_k}(x_k) \subseteq \Omega$ for all $k \in \mathbb{N}$. For a nontrivial function $v \in C_0^\infty(B_1(0))$ consider the sequence $(v_k)_{k\in\mathbb{N}}$ in $C_0^\infty(\Omega)$ defined by

$$v_k(x) = v\left(\frac{x - x_k}{r_k}\right) \qquad x \in \Omega.$$

Moreover, let $l \in \mathbb{N}$ and $1 \leq p, q < \infty$.

(a) Use $(v_k)_{k\in\mathbb{N}}$ to show that there is no Poincaré inequality

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}.$$

(b) Show that there are no compact embeddings of $W^{l,q}(\Omega) \hookrightarrow L^p(\Omega)$.

Exercise 20

Let $\Omega \subset \mathbb{R}^n$ be C^1, connected, open and bounded. Consider the Neumann problem

$$\begin{cases} -\Delta u &= f \qquad \text{in } \Omega \\ \partial_\nu u &= 0 \qquad \text{on } \partial\Omega. \end{cases}$$

Give a condition on $f \in L^2(\Omega)$ such that the Neumann problem has a weak solution $u \in H^1(\Omega)$. Is the solution unique?

Exercise 21

Show the following inequality used in the proof of the theorem of Kolmogorov-Riesz. For all $u \in W^{1,p}(\mathbb{R}^n)$ one has for $h \in \mathbb{R}^n$:

$$\|u(\cdot + h) - u\|_{L^p(\mathbb{R}^n)} \leq |h|\|\nabla u\|_{L^p(\mathbb{R}^n)}.$$