Exercise 19

a) $(x_k)_{k\in\mathbb{N}}$, $(r_k)_{k\in\mathbb{N}}$, $r_k \to \infty$, $B_{r_k}(x_k) \subseteq \Omega$ $(k\in\mathbb{N})$

$$v_k(x) = v\left(\frac{x-x_k}{r_k}\right), \quad x\in\Omega, \quad v\in C_0^\infty(B_1(0))$$

$$\|u_k\|^p_{L^p(\Omega)} = \int_{B_{r_k}(x_k)} \left| v\left(\frac{x-x_k}{r_k}\right)\right|^p dx = \int_{B_{r_k}(0)} \left|v\left(\frac{x}{r_k}\right)\right|^p dx$$

$$= r_k^n \int_{B_1(0)} |v|^p dx \Rightarrow \|v_k\|_{L^p(\Omega)} = r_k^{\frac{n}{p}} \|v\|_{L^p(\Omega)}$$

$$\|\nabla v_k\|^p_{L^p} = \int_{B_{r_k}(x_k)} \left|\nabla\left(v\left(\frac{x-x_k}{r_k}\right)\right)\right|^p = r_k^{n-p} \int_{B_1(0)} |(\nabla v)(x)|^p dx = r_k^{n-p} \|\nabla v\|^p_{L^p(\Omega)}$$

$$\Rightarrow \|\nabla v_k\|_{L^p(\Omega)} = r_k^{\frac{n}{p}-1} \|\nabla v\|_{L^p(\Omega)}$$

Somit gilt:

$$\frac{\|v_k\|_{L^p}}{\|\nabla v_k\|_{L^p}} = \frac{r_k^{\frac{n}{p}} \|v\|_{L^p(\Omega)}}{r_k^{\frac{n}{p}-1} \|\nabla v\|_{L^p(\Omega)}} = r_k \cdot \frac{\|v\|_{L^p}}{\|\nabla v\|_{L^p}} \to \infty$$

b) If $B_{r_k}(x_k) \subseteq \Omega$ $\forall k\in\mathbb{N}$ and $r_k \to \infty$, then there is $(y_k)_{k\in\mathbb{N}}$ with $\|y_k\| \to \infty$ $(k\to\infty)$ and $B_1(y_k) \subseteq \Omega$ $(k\in\mathbb{N})$.

Choose $v\in C_0^\infty(B_1(0))\setminus\{0\}$ and define $v_k\in C_0^\infty(B_1(y_k))$ by $v_k(x) = v(x-x_k)$ $(x\in\Omega)$.

Then $\|v_k\|_{W^{1,q}} = \|v\|_{W^{1,q}}$, and hence it is a bounded sequence. Moreover $(1\leq p<\infty)$ $\|v_k\|_{L^p}$ is constant, and $v_k \rightharpoonup 0$ in L^p as $\forall \varphi\in C_c^\infty(\Omega)\subset (L^p)' \cong L^{p'}$ with $\frac{1}{p}+\frac{1}{p'}=1$ one has

$\int v_k\cdot\varphi = 0$ for k sufficiently large s.t. $\operatorname{supp}\varphi \cap \operatorname{supp} v_k = \emptyset$

As $\|v_k\|_{L^p} \not\to 0$ this convergence cannot be strong.

we assume $v \geq 0$

If $p=1$, we still have $\|v_k\|_{L^1} = \|v\|_{L^1} = \text{const.}$

choose $\varphi \in C_c^\infty(\mathbb{R}) \subset L^\infty \Rightarrow \int \varphi \cdot v_k \longrightarrow 0$, but for

$\varphi \equiv 1$ we have $\int \varphi\, v_k \overset{!!}{=} \int v_k = \int |v_k| = \|v_k\| \not\to 0$.

$\Rightarrow$ $v_k \not\rightharpoonup 0$ in L^1 and also not strongly.

(20) Let $\Omega \subset \mathbb{R}^n$ be connected and C^1

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases} \quad (*)$$

By the alternative theorem, the problem is solvable iff $f \perp N^*$, where $u^* \in N^* \Leftrightarrow u^*$ solves the adjoint problem

$$\begin{cases} -\Delta u^* = 0 \\ \partial_\nu u^* = 0 \, . \end{cases}$$

This is equivalent to $\int_\Omega -\Delta u^* \varphi = 0 \quad \forall \varphi \in C^\infty(\Omega)$

$\Leftrightarrow \int_\Omega \nabla u^* \cdot \nabla \varphi = 0 \quad \forall \varphi \in C^\infty(\Omega)$

$\Leftrightarrow u^*$ is constant.

Hence, $f \perp N^* \Leftrightarrow \int_\Omega f^* = 0$

In this case, $(*)$ is solvable, but not uniquely

21) Let $u \in C_0^\infty(\mathbb{R}^n)$, $h \in \mathbb{R}^n$

$$\Rightarrow \quad \|u(\cdot+h)-u\|_{L^p}^p = \int_{\mathbb{R}^n} |u(x+h)-u(x)|^p \, dx$$

$$\int_{\mathbb{R}^n} \left| \int_0^1 \frac{d}{dt} u(x+t\cdot h) \, dt \right|^p dx$$

$$= \int_{\mathbb{R}^n} \left| \int_0^1 \nabla u(x+th) \cdot h \, dt \right|^p dx$$

$$\overset{\text{Jensen}}{\leq} \int_{\mathbb{R}^n} \int_0^1 |\nabla u(x+th)|^p \, dt \, dx \cdot |h|^p$$

$$= \int_0^1 \int_{\mathbb{R}^n} |\nabla u(x+th)|^p \, dx \cdot dt \cdot h^p = |h|^p \int_0^1 \|\nabla u\|_p^p \, dt = |h|^p \cdot \|\nabla u\|_p^p$$

For $u \in W^{1,p}$, take $(\varphi_k)_{k \in \mathbb{N}}$ in $C_0^\infty(\mathbb{R}^n)$ with $\varphi_k \to_{W^{1,p}} u$. Then:

$$\|u(\cdot+h)-u\|_p \leq \|(u-\varphi_k)(\cdot+h)-(u-\varphi_k)u\|_p + \|\varphi_k(\cdot+h)-\varphi_k\|_p$$

$$\leq \|(u-\varphi_k)(\cdot+h)-(u-\varphi_k)u\|_p + |h| \, \|\nabla\varphi_k\|$$

$$\leq \underbrace{\|(u-\varphi_k)(\cdot+h)-(u-\varphi_k)u\|_p}_{\leq 2\|u-\varphi_k\|_p} + |h| \, \|\nabla u\| + |h| \, \|\nabla(\varphi_k-u)\|_p$$

$$\longrightarrow |h| \cdot \|\nabla u\|_p \quad (k \to \infty)$$

□