Karlsruher Institut für Technologie (KIT) SoSe 2018
Institut für Analysis 15.06.2018
Dr. Rainer Mandel
Peter Rupp

Boundary and Eigenvalue Problems

Exercise Sheet 9

Exercise 22

Let $\Omega \subset \mathbb{R}^n$ with $n \geq 3$.

(a) Show that for $u \in C_0^\infty(\Omega)$ and a C^1-vector field $F : \overline{\Omega} \to \mathbb{R}^n$ one has:

$$\int_\Omega -(\nabla \cdot F + |F|^2)u^2 \, dx \leq \int_\Omega |\nabla u|^2 \, dx \tag{1}$$

(b) Prove Hardy's inequality

$$\int_\Omega \frac{u^2}{|x|^2} \, dx \leq \left(\frac{2}{n-2}\right)^2 \int_\Omega |\nabla u|^2 \, dx \tag{2}$$

by using (1) for the vector fields $F_\varepsilon(x) = \frac{tx}{|x|^2+\varepsilon}$ for some (fixed) $t < 0$ and $\varepsilon > 0$.

(c) Show that for $\mu > -(\frac{2}{n-2})^2$ and $f \in L^2(\Omega)$ the boundary value problem

$$\begin{cases} -\Delta u + \frac{\mu}{|x|^2}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique weak solution $u \in H_0^1(\Omega)$.

Exercise 23

Let $1 \leq p, \tilde{p}, q, \tilde{q} \leq \infty$, $\Omega \subset \mathbb{R}^n$ and $u \in L^p(\Omega)$, $v \in L^q(\Omega)$ with weak partial derivatives $\partial_k u \in L^{\tilde{p}}(\Omega)$ and $\partial_k v \in L^{\tilde{q}}(\Omega)$ for some $k \in \{1, \dots, n\}$. Moreover, let $r, \tilde{r} \geq 1$ be given by

$$r^{-1} := p^{-1} + q^{-1}, \quad \tilde{r}^{-1} := \max\{\tilde{p}^{-1} + q^{-1}, \, p^{-1} + \tilde{q}^{-1}\}.$$

Then $u \cdot v \in L^r(\Omega)$ is weakly differentiable w.r.t k and one has

$$\partial_k(u \cdot v) = \partial_k u \cdot v + u \cdot \partial_k v \in L^{\tilde{r}}(\Omega).$$