Exercise 22

) Let $u \in C_0^\infty(\Omega)$, $F: \Omega \to \mathbb{R}^n$ C^1-vector field.

Then:

$$0 = \int_{\partial\Omega} F \cdot u^2 = \int_\Omega \operatorname{div}(F \cdot u^2) = \int \operatorname{div} F \cdot u^2 + 2\int F \cdot u \cdot \nabla u$$

$$\leq \int \operatorname{div} F \cdot u^2 + 2 \cdot \int \left[\tfrac{1}{2}(Fu)^2 + \tfrac{1}{2}|\nabla u|^2\right]$$

$$= \int \operatorname{div} F \cdot u^2 + \int |F|^2 u^2 + \int |\nabla u|^2$$

$$\iff \quad -\int (\operatorname{div} F + |F|^2) u^2 \leq \int |\nabla u|^2$$

b) For $F_\varepsilon(x) = \dfrac{tx}{|x|^2+\varepsilon}$, $t<0$, $\varepsilon>0$ one has

$$|F_\varepsilon(x)|^2 = \frac{t^2|x|^2}{(|x|^2+\varepsilon)^2}, \quad \nabla F_\varepsilon(x) = \frac{t \cdot n}{|x|^2+\varepsilon} - \frac{2x \cdot tx}{(|x|^2+\varepsilon)^2}$$

$$= \frac{t^2}{(|x|^2+\varepsilon)^2} - \frac{\varepsilon t^2}{(|x|^2+\varepsilon)^2} \qquad = \frac{tn}{(|x|^2+\varepsilon)} - \frac{2|x|^2 \cdot t}{(|x|^2+\varepsilon)^2}$$

$$= \frac{tn}{(|x|^2+\varepsilon)} - 2t \cdot \frac{1}{|x|^2+\varepsilon} + 2t\varepsilon \frac{1}{(|x|^2+\varepsilon)^2}$$

$$= \frac{(n-2)t}{(|x|^2+\varepsilon)} + 2t\varepsilon \cdot \frac{1}{(|x|^2+\varepsilon)^2}$$

Therefor one finds:

$$-\int (\operatorname{div} F + |F|^2) = -\int \left(\frac{n-2}{(|x|^2+\varepsilon)} t + 2t\varepsilon \frac{1}{(|x|^2+\varepsilon)^2} + \frac{t^2}{|x|^2+\varepsilon} - \varepsilon \frac{t^2}{(|x|^2+\varepsilon)^2}\right) u^2$$

$$= -\int \left(\frac{1}{|x|^2+\varepsilon} \cdot ((n-2)t + t^2) u^2 + \varepsilon \cdot \int \frac{2t - t^2}{(|x|^2+\varepsilon)^2} u^2\right) \leq \int |\nabla u|^2$$

Fatou's Lemma tells us

$$\int |\nabla u|^2 \geq \liminf_{n\to\infty} \int \dots \geq \int \liminf (\dots) = -\int_\Omega \frac{(n-2)t+t^2}{|x|^2} u^2 dx$$

$\forall t<0$

maximizing the RHS:

$$-(n-2)t - t^2 \leq -(n-2)\cdot -\frac{n-2}{2} - \left(-\frac{n-2}{2}\right)^2$$
$$= \frac{(n-2)^2}{4}$$

$$\Rightarrow \frac{(n-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx \leq \int |\nabla u|^2$$

For $u \in H_0^1(\Omega)$ pick a sequence $(\varphi_k)_{k\in\mathbb{N}}$ in $C_0^\infty(\Omega)$ with $\varphi_k \xrightarrow[H^1(\Omega)]{} u$. w.l.o.g $\varphi_k \to u$ pointwise a.e (otherwise take a subsequence. Then, Fatou's lemma again yields:

$$\frac{(n-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx = \frac{(n-2)^2}{4} \int_\Omega \liminf \frac{\varphi_k^2}{|x|^2} dx \leq \frac{(n-2)^2}{4} \liminf_{n\to\infty} \int_\Omega \frac{\varphi_k^2}{|x|^2}$$
$$\leq \liminf_{n\to\infty} \int_\Omega |\nabla \varphi_k|^2 = \int_\Omega |\nabla u|^2$$

c) Define $B[u,v] = \int_\Omega \nabla u \cdot \nabla v + \mu \frac{u\cdot v}{|x|^2} dx$.

Then:
$$|B[u,v]| \leq \|\nabla u\|_{L^2} \cdot \|\nabla v\|_{L^2} + \mu \left\|\frac{u}{x}\right\|_{L^2} \cdot \left\|\frac{v}{x}\right\|_{L^2}$$
$$\leq \left(1 + \frac{(n-2)^2}{4}\mu\right) \|\nabla u\| \cdot \|\nabla v\|$$

and for $\mu \geq 0$ $B[u,u] \geq \int |\nabla u|^2$

whereas for $\mu<0$:

$$B[u,u] = \int |\nabla u|^2 + \mu \int \frac{u^2}{|x|^2} \overset{\text{Hardy}}{\geq} \int |\nabla u|^2 + \mu \cdot \frac{(n-2)^2}{4} \int |\nabla u|^2$$
$$= \left(1-\mu \cdot \frac{(n-2)^2}{4}\right) \int |\nabla u|^2$$

$\Rightarrow$ B is coercive.

By Lax-Milgram the problem has a unique solution.

Exercise 23

Let $\varphi \in C_0^\infty(\Omega)$, for $u \in L^p(\Omega)$, $v \in L^q(\Omega)$ define mollificated $u_\varepsilon, v_\varepsilon$ functions $u_\varepsilon, v_\varepsilon$ for $\varepsilon < \varepsilon_0$, ε_0 small enough, on supp φ.

Then:

$$-\int u \cdot v \cdot \partial_{x_k} \varphi = -\lim_{\varepsilon \to 0} \lim_{\delta \to 0} \int u_\varepsilon \cdot v_\delta \cdot \partial_{x_k} \varphi = \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \int \partial_{x_k}(u_\varepsilon v_\delta)\varphi$$

$$= \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \int \underbrace{(\partial_{x_k} u_\varepsilon)}_{\in L^{\tilde{p}}(\Omega)} \cdot \underbrace{v_\delta}_{\to_{L^q} u} \cdot \underbrace{\varphi}_{\in L^\infty} + \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \int \underbrace{\partial_{x_k} v_\delta}_{\to_{L^{\tilde{q}}} \partial_{x_k} v} \cdot \underbrace{\varphi \cdot u_\varepsilon}_{\in L^p}$$

$\frac{1}{\tilde{p}} + \frac{1}{q} = \frac{1}{p} + \frac{1}{\tilde{q}} = 1$

$$= \lim_{\varepsilon \to 0} \int \underbrace{\partial_{x_k} u_\varepsilon}_{\to_{L^{\tilde{p}}} \partial_{x_k} u} \cdot \underbrace{v\varphi}_{\in L^q} + \lim_{\varepsilon \to 0} \int \underbrace{\partial_{x_k} v \cdot \varphi}_{\in L^{q'}} \underbrace{u_\varepsilon}_{\to_{L^p} u}$$

$$= \int \partial_{x_k} u \cdot v \cdot \varphi + \int \partial_{x_k} v \cdot \varphi u$$

$$= \int (\partial_{x_k} u \cdot v + \partial_{x_k} v \cdot u) \varphi$$

$\Rightarrow$ $u \cdot v$ is weakly differentiable and $\partial_{x_k}(u \cdot v) = \partial_{x_k} u \cdot v + u \cdot \partial_{x_k} v \in L^1_{loc}$

Moreover: $u \cdot v \in L^r$ by Hölder's inequality as

$\|u \cdot v\|_{L^r} \leq \|u\|_p \cdot \|v\|_q$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and

$\partial_{x_k}(u \cdot v) \in L^{\tilde{r}}$, as $\|\partial_{x_k} u \cdot v + \partial_{x_k} v \cdot u\|_{L^{\tilde{r}}} \leq \|\partial_{x_k} u \cdot v\|_{L^{\tilde{r}}} + \|u \cdot \partial_{x_k} v\|_{L^{\tilde{r}}}$

$$\leq C\left(\|\partial_{x_k} u \cdot v\|_{L^{\frac{\tilde{p}q}{\tilde{p}+q}}} + \|u \cdot \partial_{x_k} v\|_{L^{\frac{p\tilde{q}}{p+\tilde{q}}}} \right)$$

$$\leq C\left(\|\partial_{x_k} u\|_{L^{\tilde{p}}} \cdot \|v\|_{L^q} + \|u\|_{L^p} \cdot \|\partial_{x_k} v\|_{L^{\tilde{q}}} \right)$$