

## Bases for subspaces and how to find them.

**Definition.** Let  $S$  be a subspace of  $\mathbb{R}^n$ . A set  $B$  of vectors is a basis for  $S$  if

1.  $\text{Span}(B) = S$ ,
2.  $B$  is linearly independent.

**Fact 1.** A basis of  $S$  is a largest collection of linearly independent vectors in  $S$ .

**Fact 2.** A basis of  $S$  is a smallest collection of vectors spanning  $S$ .

**Fact 3.** All bases of  $S$  have the same number of vectors.

### Row-reduction method to find a basis.

Let  $S = \text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\})$ .

- Arrange  $\mathbf{u}_i$ s as rows of a matrix, call it  $A$ .
- Find  $\text{rref}(A)$ .
- The set of nonzero rows of  $\text{rref}(A)$  forms a basis of  $S$ .

### Column selection method to find a basis.

Let  $S = \text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\})$ .

- Arrange  $\mathbf{u}_i$ s as columns of a matrix, call it  $A$ .
- Find  $\text{rref}(A)$ .
- Identify pivot columns of  $\text{rref}(A)$ .
- Select corresponding columns of  $A$  as a basis for  $S$ .

### Enlarging a given set of vectors to a basis of $S$ .

Let  $S = \text{Span}(\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\})$ . Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in S$  be a given set of linearly independent vectors.

Find a basis of  $S$  containing  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ .

- Arrange  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$  as columns of a matrix in THIS order.
- Apply column selection method to find a basis.
- Since  $\mathbf{u}_i$ s are linearly independent, they will correspond to pivot columns and thus will be selected in a basis.

### Examples

**Row-reduction** Let  $S = \text{Span}\{(1, 2, 3, 4), (0, -1, 2, 1), (1, 0, 0, 5)\}$ . Find a basis for  $S$ .

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 1 & 2 & 3 & 4 \\ 0 & -1 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 2 & 3 & -1 \\ 0 & -1 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 7 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 7 & 1 \end{bmatrix}$$

**Answer:** A basis of  $S$  is  $B = \{(1, 0, 0, 5), (0, -1, 2, 1), (0, 0, 7, 1)\}$ .

**Column-selection** Let  $S = \text{Span}\{(1, 2, 3, 4), (1, 1, 5, 5), (0, -1, 2, 1), (1, 0, 0, 5)\}$ . Find a basis for  $S$ .

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & -1 & 0 \\ 3 & 5 & 2 & 0 \\ 4 & 5 & 1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & -2 \\ 0 & 2 & 2 & -3 \\ 0 & 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, the column selection method chooses columns number 1, 2, 4 from the original matrix.

**Answer:** A basis of  $S$  is  $B = \{(1, 2, 3, 4), (1, 1, 5, 5), (1, 0, 0, 5)\}$ .

### Enlarging a given set of vectors to a basis.

Let  $S = \text{Span}\{(1, 2, 3, 4), (0, -1, 2, 1), (1, 0, 0, 5)\}$ . Let  $\mathbf{v}_1 = (1, 1, 5, 5)$ ,  $\mathbf{v}_2 = (0, 2, -4, -2)$ . Find a basis of  $S$  containing  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 2 & 2 & -1 & 0 \\ 5 & -4 & 3 & 2 & 0 \\ 5 & -2 & 4 & 1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & -1 & -1 \\ 0 & -4 & -2 & 2 & -5 \\ 0 & -2 & -1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the column selection method chooses columns number 1, 2, 5.

**Answer:** A basis of  $S$  is  $B = \{(1, 1, 5, 5), (0, 2, -4, -2), (1, 0, 0, 5)\}$ .