1 Linear Algebra

- Linear independence. $v_1, \dots, v_k$ are linearly independent iff the equation $\alpha_1 v_1 + \cdots + \alpha_k v_k = 0$ has only trivial solution $\alpha_1 = \cdots = \alpha_k = 0$.
- Subspace $(U, +, \cdot)$ of a vector space $(V, +, \cdot)$ is a triple such that $U \subseteq V$, $\forall u, v \in U$, $u + v \in U$ and $\forall \lambda \in \mathbb{R}$ $\lambda \cdot u \in U$. Affine subspace is a set $W = U + u$, where U is a subspace and $u \in V$. Subspace can be written as a Span.
- $Span(\{v_1, \dots, v_k\}) = \{\alpha_1 v_1 + \cdots + \alpha_k v_k : \alpha_i \in \mathbb{R}\}$. Basic of a subspace U is a set of vectors B that are linearly independent and such that $Span(B) = U$. The number of elements in a basis is called dimension of U and denoted $dim(U)$. Equivalent definition of a basis: maximal subset of U of linearly independent vectors, minimal subset spanning U.

  To find a basis: write the subspace a span. Let $S = Span(\{\mathbf{u_1}, \mathbf{u_2}, \dots, \mathbf{u_k}\})$. Arrange $\mathbf{u_i}$s as columns of a matrix, call it A. Find $rref(A)$. Identify pivot columns of $rref(A)$. Select corresponding columns of A as a basis for S.
- Equations of lines, planes, subspaces in normal form, in parametric form, norms, scalar product, vector product.
- Linear transformations. A map $L : U \to V$ is a linear transform if $L(u + v) = L(u) + L(v)$ and $L(\lambda u) = \lambda L(u)$ for all $u, v \in U$ and $\lambda \in \mathbb{R}$. Any linear transform is a matrix transform.
- Distances, projections, reflexions.
  - If W is a subspace, $w \in W$ is closest to v then $w - v$ is orthogonal to W.
  - If $w_1, \dots, w_m$ is an orthonormal basis of W, then $proj_W v = (v \cdot w_1)w_1 + (v \cdot w_2)w_2 + \cdots (v \cdot w_m)w_m$.
  - If $W = \{w : w \cdot a = \rho\}$, $||a|| = 1$, then $dist(v, W) = |v \cdot a - \rho|$.
  - Projection of a vector y onto a vector q is
    $$proj_q(y) = (y \cdot q)\frac{q}{||q||^2}.$$
  - Distance from a point A to a plane $\mathcal{P}$ with normal vector n and containing a point Q is
    $$dist(A, \mathcal{P}) = \frac{|AQ \cdot n|}{n}.$$
  - Distance from a point A to a line L in space that has a direction vector u and passes through a point Q is
    $$dist(A, L) = \frac{|AQ \times u|}{|u|}.$$
  - Distance between lines L_1, L_2 in space that have a direction vectors u_1, u_2 and pass through points Q_1, Q_2 is
    $$dist(L_1, L_2) = \frac{|Q_1Q_2 \cdot (u_1 \times u_2)|}{|u_1 \times u_2|}.$$
  - Distance between two planes P_1, P_2 with normal equations $n \cdot x = d$ and $n \cdot x = e$ is
    $$dist(P_1, P_2) = \frac{|e - d|}{|n|}.$$
  - Distance between a point A and a line L_{plane} in the plane with a normal vector n and passing through a point Q is
    $$dist(A, L_{plane}) = \frac{|AQ \cdot n|}{|n|}.$$

– Projection of a point A onto a line L in space with direction vector u and through the point Q.

$$proj_L(A) = proj_u(A - Q) + Q = ((A - Q) \cdot u)\frac{u}{||u||^2} + Q.$$

– Projection of a point A onto a plane P with a normal vector n passing through a point Q. Let $d = dist(A, P)$. Then

$$proj_P(A) = A \pm d(\frac{n}{|n|}).$$

Check which belongs to the plane.

– Projection of a point A onto a line L in the plane with normal vector n. Let $d = dist(A, L)$. Then

$$proj_L(A) = A \pm d(\frac{n}{|n|}).$$

Check which belongs to the line.

- Matrix A.
  - Rank of A is the number of pivots in the reduced row echelon form, or maximum number of linearly independent rows, or maximal number of linearly independent columns.
  - Determinant of A for square A: definition via row reduction, many properties.
  - Eigenvalues, eigenvectors of A for square A.
  - Determinant $det(A) = 0$ off columns are linearly dependent iff rows are linearly dependent iff $Ax = 0$ has infinitely many solutions iff $rank(A) < n$ where A is $n \times n$.
  - Ax is a linear combination of columns with coefficients $x_1, \ldots, x_n$.
  - Matrix product, properties, inverse, calculations. Determinant of a product.
  - Diagonalization of a matrix.
  - Matrices of rotation, dilation, projection, reflexion.
- Systems: procedure, number of solutions, understanding what is the set of solutions. Ker, Im of a linear transform.

2 Differential equations

- Separable $g(y)dy = f(x)dx$. Integrate.
- Linear $y' + p(x)y = q(x)$. Find integrating factor $\mu(x) = exp(\int p(x)dx)$. Multiply the equation by μ. After simplifications it becomes:

$$(y\mu)' = q\mu.$$

  Integrate.
- Bernoulli $y' + p(x)y = q(x)y^n$. Substitution

$$y = v^{\frac{1}{1-n}}.$$

  This reduced the equation to linear.
- Linear higher order constant coefficients homogeneous. $a_k y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_0 y = 0$. Plug $y = e^{\lambda x}$. The equation reduces to $a_k\lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_0\lambda = 0$. Find all λ's. For each λ_i find a solution y_i. General solution $y = c_1 y_1 + \cdots + c_k y_k$.

  Cases: non-repeated real, repeated real, complex, repeated complex. Example: $\lambda_1 = 5, \lambda_2 = 5, \lambda_3 = 5, \lambda_4 = 1 - 6i, \lambda_5 = 1 + 6i, \lambda_6 = 1 - 6i, \lambda_7 = 1 + 6i, \lambda_8 = 0$. Then $y = c_1 e^5 x + c_2 x e^{5x} + c_3 x^2 e^{5x} + c_4 e^x \cos(6x) + c_5 e^x \sin(6x) + c_6 x e^x \cos(6x) + c_7 x e^x \sin(6x) + c_8$.

- Linear higher order non-homogeneous. $a_k y^{(k)} + a_{k-1} y^{(k-1)} + \cdots + a_0 y = f(x)$.

$$y_{gen.nonhom} = y_{gen.hom} + y_{particular.nonhomog.}.$$

  To find particular non-homogeneous, use method of *undetermined coefficients*. For example, when $f(x) = e^{3x}$, look for a solution candidate in the form $y = Ae^{3x}$; when $f(x) = \sin(3x)$, look for a solution candidate in the form $y = A\sin(3x) + B\cos(3x)$; when $f(x) = x^2$, look for a solution candidate in the form $y = Ax^2 + Bx + C$.

  For the second order, one could use the method of variation of parameters. If y_1, y_2 is a fundamental solution of homogeneous DE, then the general solution is $y = u_1 y_1 + u_2 y_2$, where $u_1' = -\frac{y_2 f}{W(y_1,y_2)}$, $u_2' = \frac{y_1 f}{W(y_1,y_2)}$.

- Reduction of order linear homogeneous. Given a solution y, look for another solution in the form $y_1 = vy$, plug y_1 in the DE, obtain a lower order DE in terms of v, solve it, back substitute.

- Euler equations: special linear non-constant coefficients. $a_k y^{(k)} + a_{k-1} y^{(k-1)} + \cdots + a_0 y = f(x)$.

$$y_{gen.nonhom} = y_{gen.hom} + y_{particular.nonhomog.}.$$

  To find $y_{gen.hom}$ use a candidate $y = x^r$. Plug, obtain an equation on r, find all r, for repeated ones, multiply by $\ln x$.

  Example: $r_1 = 2, r_2 = 2, r_3 = 2, r_4 = 0, r_5 = 4 - 7i, r_6 = 4 + 7i$. Then $y = y_{gen.homogeneous} = c_1 x^2 + c_2 x^2 \ln x + c_3 x^2 \ln^2 x + c_4 + c_5 x^{(4-7i)} + c_6 x^{(4+7i)x}$. Rearranging complex solutions, we get another way to write the solution. $y = c_1 x^2 + c_2 x^2 \ln x + c_3 x^2 \ln^2 x + c_4 + c_5 x^4 \cos(7\ln x) + c_6 x^4 \sin(7\ln x)$.

- Systems of linear homogeneous DEs constant coefficients. $y' = Ay$, where A is a square matrix and y is a vector function.

  Look for solutions in the form $y = e^{\lambda x} v$, where v is a vector. Plug in, get a new equation $\lambda v = Av$, i.e., λ is an eigenvalue, v is a corresponding eigenvector. For each λ_i find a solution y_i. Then a general solution is $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$.

  Eigenvalues $\lambda = \alpha \pm i\beta$ with eigenvectors $\bar{v} = \bar{a} \pm i\bar{b}$.

  Then real solutions are given by

$$u(x) = e^{\alpha x}(\cos(\beta x)\bar{a} - \sin(\beta x)\bar{b})$$
$$v(x) = e^{\alpha x}(\sin(\beta x)\bar{a} + \cos(\beta x)\bar{b})$$

- Laplace transform method for solving linear DEs and systems of linear DEs with initial value conditions.

  Apply Laplace transform to the DE with unknown $y = y(x)$. The DE will be transferred into an algebraic equation with unknown $U = L(y)$ and coefficients in terms of s. Find U in terms of s. Find $y = L^{-1}(U)$ using the Table or convolution rules.

- The power series method for solving DEs. Given an initial value problem at x_0, represent a solution as a power series expanded at x_0. I.e., $y = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$. Plug in the DE. Starting with a_0, a_1, find all other coefficients recursively.