1 Linear Algebra

- Linear independence. $v_1, \ldots, v_k$ are linearly independent if the equation $\alpha_1 v_1 + \cdots + \alpha_k v_k = 0$ has only trivial solution $\alpha_1 = \cdots = \alpha_k = 0$.

- Subspace $(U, +, \cdot)$ of a vector space $(V, +, \cdot)$ is a triple such that $U \subseteq V$, $\forall u, v \in U$, $u + v \in U$ and $\forall \lambda \in \mathbb{R}$ $\lambda \cdot u \in U$. Affine subspace is a set $W = U + u$, where $U$ is a subspace and $u \in V$. Subspace can be written as a Span.

- $\text{Span} \{v_1, \ldots, v_k\} = \{\alpha_1 v_1 + \cdots + \alpha_k v_k : \alpha_i \in \mathbb{R}\}$. Basic of a subspace $U$ is a set of vectors $B$ that are linearly independent and such that $\text{Span}(B) = U$. The number of elements in a basis is called dimension of $U$ and denoted $\dim(U)$. Equivalent definition of a basis: maximal subset of $U$ of linearly independent vectors, minimal subset spanning $U$.

To find a basis: write the subspace a span. Let $S = \text{Span} \{u_1, u_2, \ldots, u_k\}$. Arrange $u_i$s as columns of a matrix, call it $A$. Find $\text{rref}(A)$. Identify pivot columns of $\text{rref}(A)$. Select corresponding columns of $A$ as a basis for $S$.

- Equations of lines, planes, subspaces in normal form, in parametric form, norms, scalar product, vector product.

- Linear transformations. A map $L : U \rightarrow V$ is a linear transform if $L(u + v) = L(u) + L(v)$ and $L(\lambda u) = \lambda L(u)$ for all $u, v \in U$ and $\lambda \in \mathbb{R}$. Any linear transform is a matrix transform.

- Distances, projections, reflexions.
  - If $W$ is a subspace, $w \in W$ is closest to $v$ then $w - v$ is orthogonal to $W$.
  - If $w_1, \ldots, w_m$ is an orthonormal basis of $W$, then $\text{proj}_W v = (v \cdot w_1)w_1 + (v \cdot w_2)w_2 + \cdots + (v \cdot w_m)w_m$.
  - If $W = \{w : w \cdot a = \rho\}$, $||a|| = 1$, then $\text{dist}(v, W) = |v \cdot a - \rho|$.
  - Projection of a vector $y$ onto a vector $q$ is
    \[ \text{proj}_q(y) = (y \cdot q) \frac{q}{||q||^2}. \]
  - Distance from a point $A$ to a plane $P$ with normal vector $n$ and containing a point $Q$ is
    \[ \text{dist}(A, P) = \frac{|AQ \cdot n|}{n}. \]
  - Distance from a point $A$ to a line $L$ in space that has a direction vector $u$ and passes through a point $Q$ is
    \[ \text{dist}(A, L) = \frac{|AQ \times u|}{|u|}. \]
  - Distance between lines $L_1, L_2$ in space that have a direction vectors $u_1, u_2$ and pass through points $Q_1, Q_2$ is
    \[ \text{dist}(L_1, L_2) = \frac{|Q_1Q_2 \cdot (u_1 \times u_2)|}{|u_1 \times u_2|}. \]
  - Distance between two planes $P_1, P_2$ with normal equations $n \cdot x = d$ and $n \cdot x = e$ is
    \[ \text{dist}(P_1, P_2) = \frac{|e - d|}{|n|}. \]
  - Distance between a point $A$ and a line $L_{\text{plane}}$ in the plane with a normal vector $n$ and passing through a point $Q$ is
    \[ \text{dist}(A, L_{\text{plane}}) = \frac{|AQ \cdot n|}{|n|}. \]
- Projection of a point $A$ onto a line $L$ in space with direction vector $u$ and through the point $Q$.

$$\text{proj}_L(A) = \text{proj}_u(A - Q) + Q = ((A - Q) \cdot u)\frac{u}{||u||^2} + Q.$$ 

- Projection of a point $A$ onto a plane $P$ with a normal vector $n$ passing through a point $Q$. Let $d = \text{dist}(A, P)$. Then

$$\text{proj}_P(A) = A \pm d\left(\frac{n}{||n||}\right).$$ 

Check which belongs to the plane.

- Projection of a point $A$ onto a line $L$ in the plane with normal vector $n$. Let $d = \text{dist}(A, L)$. Then

$$\text{proj}_L(A) = A \pm d\left(\frac{n}{||n||}\right).$$ 

Check which belongs to the line.

- Matrix $A$.

  - Rank of $A$ is the number of pivots in the reduced row echelon form, or maximum number of linearly independent rows, or maximal number of linearly independent columns.
  
  - Determinant of $A$ for square $A$: definition via row reduction, many properties.
  
  - Eigenvalues, eigenvectors of $A$ for square $A$.
  
  - Determinant $\det(A) = 0$ off columns are linearly dependent iff rows are linearly dependent iff $Ax = 0$ has infinitely many solutions iff $\text{rank}(A) < n$ where $A$ is $n \times n$.
  
  - $Ax$ is a linear combination of columns with coefficients $x_1, \ldots, x_n$.
  
  - Matrix product, properties, inverse, calculations. Determinant of a product.
  
  - Diagonalization of a matrix.
  
  - Matrices of rotation, dilation, projection, reflexion.

- Systems: procedure, number of solutions, understanding what is the set of solutions. Ker, Im of a linear transform.

## 2 Differential equations

- Separable $g(y)dy = f(x)dx$. Integrate.

- Linear $y' + p(x)y = q(x)$. Find integrating factor $\mu(x) = e^{\int p(x)dx}$. Multiply the equation by $\mu$. After simplifications it becomes:

$$\left(\mu y\right)' = q\mu.$$ 

Integrate.

- Bernoulli $y' + p(x)y = q(x)y^n$. Substitution

$$y = v^{\frac{1}{n-1}}.$$ 

This reduced the equation to linear.

- Linear higher order constant coefficients homogeneous. $a_k y^{(k)} + a_{k-1} y^{(k-1)} + \cdots + a_0 y = 0$. Plug $y = e^{\lambda x}$. The equation reduces to $a_k \lambda^k + a_{k-1} \lambda^{k-1} + \cdots + a_0 \lambda = 0$. Find all $\lambda$’s. For each $\lambda_i$ find a solution $y_i$. General solution $y = c_1 y_1 + \cdots + c_k y_k$.

Cases: non-repeated real, repeated real, complex, repeated complex. Example: $\lambda_1 = 5, \lambda_2 = 5, \lambda_3 = 5, \lambda_4 = 1 - 6i, \lambda_5 = 1 + 6i, \lambda_6 = 1 - 6i, \lambda_7 = 1 + 6i, \lambda_8 = 0$. Then $y = c_1 e^{5x} + c_2 xe^{5x} + c_3 x^2 e^{5x} + c_4 e^{\pi} \cos(6x) + c_5 e^{\pi} \sin(6x) + c_6 xe^{\pi} \cos(6x) + c_7 xe^{\pi} \sin(6x) + c_8$. 

• Linear higher order non-homogeneous. \(a_k y^{(k)} + a_{k-1} y^{(k-1)} + \cdots + a_0 y = f(x)\).

\[ y_{\text{gen, nonhom}} = y_{\text{gen, hom}} + y_{\text{particular, nonhom.}} \]

To find particular non-homogeneous, use method of undetermined coefficients. For example, when \(f(x) = e^{3x}\), look for a solution candidate in the form \(y = Ae^{3x}\); when \(f(x) = \sin(3x)\), look for a solution candidate in the form \(y = A\sin(3x) + B\cos(3x)\); when \(f(x) = x^2\), look for a solution candidate in the form \(y = Ax^2 + Bx + C\).

For the second order, one could use the method of variation of parameters. If \(y_1, y_2\) is a fundamental solution of homogeneous DE, then the general solution is \(y = u_1 y_1 + u_2 y_2\), where \(u_1' = -\frac{y_2 f}{W(y_1,y_2)}, \quad u_2' = \frac{y_1 f}{W(y_1,y_2)}\).

• Reduction of order linear homogeneous. Given a solution \(y\), look for another solution in the form \(y_1 = v y\), plug \(y_1\) in the DE, obtain a lower order DE in terms of \(v\), solve it, back substitute.

• Euler equations: special linear non-constant coefficients. \(a_k y^{(k)} + a_{k-1} y^{(k-1)} + \cdots + a_0 y = f(x)\).

\[ y_{\text{gen, nonhom}} = y_{\text{gen, hom}} + y_{\text{particular, nonhom.}} \]

To find \(y_{\text{gen, hom}}\) use a candidate \(y = x^r\). Plug, obtain an equation on \(r\), find all \(r\), for repeated ones, multiply by \(\ln x\).

Example: \(r_1 = 2, r_2 = 2, r_3 = 2, r_4 = 0, r_5 = 4 - 7i, r_6 = 4 + 7i\). Then \(y = y_{\text{gen, homogeneous}} = c_1 x^2 + c_2 x^2 \ln x + c_3 x^2 \ln^2 x + c_4 + c_5 x^{(4-7i)} + c_6 x^{(4+7i)}\). Rearranging complex solutions, we get another way to write the solution. \(y = c_1 x^2 + c_2 x^2 \ln x + c_3 x^2 \ln^2 x + c_4 + c_5 x^4 \cos(7 \ln x) + c_6 x^4 \sin(7 \ln x)\).

• Systems of linear homogeneous DEs constant coefficients. \(y' = Ay\), where \(A\) is a square matrix and \(y\) is a vector function.

Look for solutions in the form \(y = e^{\lambda x} v\), where \(v\) is a vector. Plug in, get a new equation \(\lambda v = Av\), i.e., \(\lambda\) is an eigenvalue, \(v\) is a corresponding eigenvector. For each \(\lambda\) find a solution \(y_i\). Then a general solution is \(y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n\).

Eigenvalues \(\lambda = \alpha \pm i \beta\) with eigenvectors \(\hat{v} = \hat{a} \pm i \hat{b}\).

Then real solutions are given by

\[ u(x) = e^{\alpha x} (\cos(\beta x) \hat{a} - \sin(\beta x) \hat{b}) \]
\[ v(x) = e^{\alpha x} (\sin(\beta x) \hat{a} + \cos(\beta x) \hat{b}) \]

• Laplace transform method for solving linear DEs and systems of linear DEs with initial value conditions.

Apply Laplace transform to the DE with unknown \(y = y(x)\). The DE will be transferred into an algebraic equation with unknown \(U = L(y)\) and coefficients in terms of \(s\). Find \(U\) in terms of \(s\). Find \(y = L^{-1}(U)\) using the Table or convolution rules.

• The power series method for solving DEs. Given an initial value problem at \(x_0\), represent a solution as a power series expanded at \(x_0\). I.e., \(y = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots\). Plug in the DE. Starting with \(a_0, a_1\), find all other coefficients recursively.