Systems of equations.

1 What do you need to know

1. How to write a system in a matrix form.
2. How to write a system as a set of equations.
3. Find $rref$ of a matrix.
4. Write solution using $rref$.

2 Gaussian elimination

Row operations:
1. switch two rows,
2. add a constant multiple of one row to another row,
3. multiply a row by nonzero constant.

A matrix is in **reduced row echelon form** $rref$ if there is a special set of position called **pivot positions** such that:

1. a matrix has 1 in each pivot position.
2. all entries below, above and to the left of a pivot position are 0,
3. each non-zero row has exactly one pivot position,
4. pivot position in row i is strictly to the right of the pivot position in row $i+1$, i.e. pivot positions form a stair-case,
5. zero rows are last.

How to create $rref$ of a matrix using row operations:

1. Make sure that the row with leftmost nonzero element is first.
2. Make the leftmost non-zero element of the first row 1 - this will correspond to a pivot position.
3. Make all elements below the pivot equal to 0.

Iterate the above three steps for the matrix formed by remaining rows. After recursion is done for all rows, and all pivot positions are created, make 0's above each pivot.

3 Meaning of RREF for the systems

For the systems of linear equations: the non-pivot columns give free variables. If there is a row $0, 0, \ldots, 0, a$, where $a \neq 0$, then the system has no solutions.

4 Example of Gaussian elimination

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & \text{-1} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 1 & 5 & 0 & 0 \end{bmatrix} \rightarrow_{r_1 \leftrightarrow r_2} \begin{bmatrix} \mathbf{1} & 1 & 0 & 0 & 0 & \text{-1} & 0 \\ 0 & \mathbf{1} & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 1 & 5 & 0 & 0 \end{bmatrix} \rightarrow_{r_4 = r_4 - 4r_2}$$

$$\begin{bmatrix} \mathbf{1} & 1 & 0 & 0 & 0 & \text{-1} & 0 \\ 0 & \mathbf{1} & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & \text{-4} \end{bmatrix} \rightarrow_{r_4 \leftrightarrow r_3} \begin{bmatrix} \mathbf{1} & 1 & 0 & 0 & 0 & \text{-1} & 0 \\ 0 & \mathbf{1} & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 1 & 0 & \text{-4} \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \end{bmatrix} \rightarrow_{r_3 = r_3 - r_4}$$

$$\begin{bmatrix} \mathbf{1} & 1 & 0 & 0 & 0 & \text{-1} & 0 \\ 0 & \mathbf{1} & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \text{-4} \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \end{bmatrix} \rightarrow_{r_2 = r_2 - r_3} \begin{bmatrix} \mathbf{1} & 1 & 0 & 0 & 0 & \text{-1} & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \text{-4} \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \end{bmatrix} \rightarrow_{r_1 = r_1 - r_2}$$

$$\begin{bmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & \text{-1} & \text{-5} \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \text{-4} \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \end{bmatrix} = rref(A).$$

System, corresponding to this matrix:

$$\begin{cases} x_2 + x_5 = 1 \\ x_1 + x_2 - x_6 = 0 \\ x_5 = 0 \\ 4x_2 + x_4 + 5x_5 = 0 \end{cases} \approx \begin{cases} x_1 - x_6 = -5 \\ x_2 = 5 \\ x_4 = -4 \\ x_5 = 0. \end{cases}$$

Solutions:

$$S = \{(x_1, x_2, x_3, x_4, x_5, x_6) : \ \mathbf{x_6 = t}, \ \mathbf{x_3 = s}, \ x_1 = t{-}5, \ x_2 = 5, \ x_4 = -4, \ x_5 = 0, \quad t, s \in \mathbb{R}\}$$

$$= \{(t-5, 5, s, -4, 0, t): \quad t, s \in \mathbb{R}\}.$$