NOTES ON WRONSKIANS

What follows is an account of what the Wronskian does and does not tell you, restricted to the case of two functions. The results generalize easily to sets of more than two functions. First an arbitrary pair of functions is considered, then further results are found when both functions are solutions to the same homogeneous equation. All functions will be assumed differentiable be defined on the same domain.

1. When f and g are arbitrary functions

Proposition 1. *If f and g are two differentiable functions whose Wronskian is nonzero* at any point*, then they are linearly independent.*

Proof. Assume $w[f\ g](x_0) \neq 0$ for some point x_0 in the domain. Suppose $c_1 f + c_2 g = 0$ for some scalars c_1,c_2. We aim to show that $c_1 = c_2 = 0$ is the only solution for c_1, c_2. By taking the derivative of the first equation, we produce the following system of equations:

$$\begin{aligned} c_1 f(x) + c_2 g(x) &= \quad 0 \\ c_1 f'(x) + c_2 g'(x) &= \quad 0, \end{aligned}$$

which are meant to hold for all x in the domain. But plugging in $x = x_0$, we get the following system, which we write as a matrix equation:

$$\begin{bmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The fact that the Wronskian is nonzero at x_0 means that the square matrix on the left is nonsingular, hence this equation has only the solution $c_1 = c_2 = 0$, so f and g are independent. □

Notice that in this case it was enough to find the Wronskain nonzero at any one point in the domain - it might not be nonzero everywhere (that is, it might be zero at some points in the domain). Also, we get "by pure logic", the contrapositive of this statement, which says that if f and g are dependent, then their Wronskian is zero *at all points* in the domain.

Next notice that for arbitrary f, g, the Wronskian DOES NOT provide a test for dependence. As a counterexample, let $f(x) = x|x|$ and $g(x) = x^2$. Then you can check that the Wronskian of f and g is ALWAYS zero, but these functions are INdependent!

2. When f and g are both solutions of a second order homogeneous equation

Proposition 2. *If f and g are both solutions to the equation $y'' + ay' + by = 0$ for some a and b, and if the Wronskian is zero at any point in the domain, then it is zero everywhere and f and g are dependent.*

Proof. I omit the tedious but straightforward calculation showing that the Wronskian $w(x)$ of f and g satisfies the first order ODE $y' + ay = 0$. Then by uniqueness of solutions, one concludes that if w is zero somewhere, it's zero everywhere. From this, we deduce that the system

$$\begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has a nontrivial solution, for each x. Then one has to argue that this solution does not depend on x, concluding that there are nonzero constants c_1, c_2 such that $c_1 f(x) + c_2 g(x) = 0$ *for all* x in the domain, i.e., f and g are dependent. □

In slightly more generality, it can be shown that any two *analytic* functions whose wronskian is everywhere zero are dependent. So in the above proposition, the hypothesis that f and g both solve the same linear homogeneous ODE just guaranteed that (a) they are both analytic, and (b) the Wronskian is *always* zero (in fact, (a) implies (b) anyway, but this is harder, I think).