Karlsruhe Institute of Technology

Numerical Methods for Partial Differential Equations

• hp Finite Element Methods (1d) •

Lecture notes by

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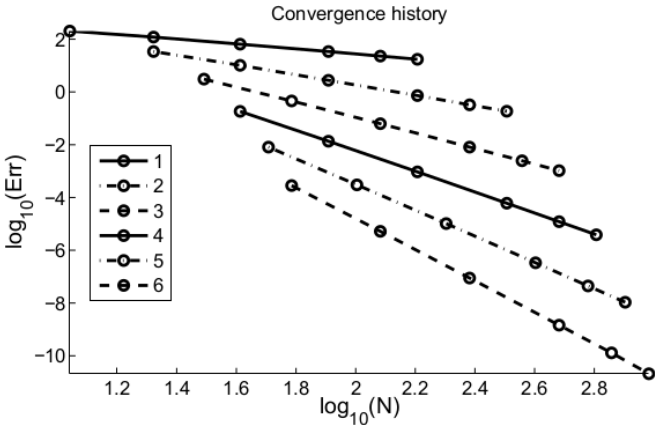

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1 Introduction

1.1 The model problem

Let $\Omega \in \mathbb{R}^d$ $(d = 1, 2, 3)$ be an open and bounded domain. For given functions

$$a : \overline{\Omega} \to \mathbb{R}_{>0}, \quad f : \Omega \to \mathbb{R}, \quad g : \partial\Omega \to \mathbb{R}$$

seek

$$u : \overline{\Omega} \to \mathbb{R}$$

that solves the *boundary value problem*

$$-\boldsymbol{\nabla}\cdot\big(a\boldsymbol{\nabla}u\big) = f \qquad \text{in } \Omega, \tag{1.1}$$

$$u = g \qquad \text{on } \partial\Omega. \tag{1.2}$$

For the special case $a = 1$ equation (1.1) reads

$$-\Delta u = f \qquad \text{in } \Omega. \tag{1.3}$$

The boundary value problem (1.1)–(1.2), resp. (1.3) and (1.2) (called *Poisson problem*), is a prototype problem that arises in many applications. In general, there are no easily evaluable analytical solution formulas for u. Thus u has to be computed numerically.

1.2 The variational formulation

1.2.1 The weak formulation

Multiplying (1.1) with *test functions* $v \in \mathcal{D}$ — these are compactly in Ω supported and infinitely often differentiable functions — and integrating by parts (*Gauss' theorem*) yields

$$\int_\Omega a\boldsymbol{\nabla}u \cdot \boldsymbol{\nabla}v = \int_\Omega fv \qquad \text{for all } v \in \mathcal{D}. \tag{1.4}$$

Defining

$$B(v, w) := \int_\Omega a\boldsymbol{\nabla}v \cdot \boldsymbol{\nabla}w, \tag{1.5}$$

$$F(v) := \int_\Omega fv, \tag{1.6}$$

we can formally write the boundary value problem (1.1), (1.2) in the form:

$$\begin{aligned} &\text{Seek } u : \overline{\Omega} \to \mathbb{R} \text{ with } u_{\lfloor\partial\Omega} = g \text{ and} \\ &B(u, v) = F(v) \qquad \text{for all } v \in \mathcal{D}. \end{aligned} \tag{1.7}$$

(1.7) is called the *weak formulation* of the boundary value problem (1.1), (1.2) and u will be called *weak solution*. (The term "weak" refers to the fact that only first order derivatives are used to define u, while the original problem requires two derivatives, and, as we will see below, these derivatives are not even the classical ones.)

1.2.2 Functions with bounded energy

We have to establish reasonable conditions on u (and thus on data a, f, g, Ω), so that the formulation (1.7) makes sense.

(i) For Ω we require Gauss' theorem to be applicable, which is for example the case if $\partial\Omega$ is Lipschitz-continuous.
(ii) We require a to be a bounded function.
(iii) In order that the *energy* $B(u,u)$ is well defined, we require (in view of (ii)) that

$$u \in H^1(\Omega) := \Big\{ v : \Omega \to \mathbb{R} \,:\, \int_\Omega \{|v|^2 + |\boldsymbol{\nabla} v|^2\} < \infty \Big\}. \tag{1.8}$$

$H^1(\Omega)$ is a complete linear function space, i.e, a *Banach space*. It can be shown that the restriction $v\lfloor_{\partial\Omega}$ is well defined for $v \in H^1(\Omega)$ and that

$$V := H^1_0(\Omega) := \Big\{ v \in H^1(\Omega) \,:\, v\lfloor_{\partial\Omega} = 0 \Big\} \tag{1.9}$$

is a closed (and thus complete) subspace of $H^1(\Omega)$. On V we use the norm

$$\|v\|_V := \Big(\int_\Omega |\boldsymbol{\nabla} v|^2 \Big)^{1/2} \qquad \text{for all } v \in V. \tag{1.10}$$

In fact, $\|v\|_V = 0$ implies $v = 0$ in V due to *Poincaré inequality*

$$\Big(\int_\Omega |v|^2 \Big)^{1/2} \le C_\Omega \Big(\int_\Omega |\boldsymbol{\nabla} v|^2 \Big)^{1/2} \qquad \text{for all } v \in V. \tag{1.11}$$

C_Ω is a positive constant that depends solely on Ω. Since V is the closure of $\mathcal{D}$ under the norm $\|\,.\,\|_V$, we now reformulate (1.7) as:

$$\begin{aligned} &\text{Seek } u \in H^1(\Omega) \text{ with } u\lfloor_{\partial\Omega} = g \text{ and} \\ &B(u,v) = F(v) \qquad \text{for all } v \in V. \end{aligned} \tag{1.12}$$

1.2.3 Reduction to homogeneous boundary conditions

If we assume that there exists a function $\widetilde{g} \in H^1(\Omega)$ with

$$\widetilde{g}\lfloor_{\partial\Omega} = g, \tag{1.13}$$

then we can decompose u into the form

$$u = u_0 + \widetilde{g}$$

and instead of (1.12) we consider the equation

$$B(u_0, v) = F(v) - B(\widetilde{g}, v) =: \widetilde{F}(v) \qquad \text{for all } v \in V \tag{1.14}$$

for $u_0 \in V$. In other words, we may assume (keeping in mind the assumption (1.13)) without loss of generality the *homogeneous boundary condition*

$$u = 0 \qquad \text{on } \partial\Omega. \tag{1.15}$$

Finally, our problem reads:

$$\begin{aligned} &\text{Seek } u \in V \text{ such that} \\ &B(u,v) = F(v) \qquad \text{for all } v \in V. \end{aligned} \tag{1.16}$$

1.3 Existence of solutions

Theorem 1 (Lax–Milgram theorem). *Let V be a complete normed space and let $B : V \times V \to \mathbb{R}$ be a* symmetric bilinear form *with*

$$|B(v,w)| \le C_B \|v\|_V \|w\|_V \qquad \text{for all } v,w \in V \qquad (\text{Continuity}), \tag{1.17}$$

$$B(v,v) \ge c_B \|v\|_V^2 \qquad \text{for all } v \in V \qquad (\text{Coercivity}) \tag{1.18}$$

for some positive constants c_B, C_B. If $F : V \to \mathbb{R}$ is a given linear continuous mapping, i.e.,

$$|F(v)| \le C_F \|v\|_V \qquad \text{for all } v \in V \tag{1.19}$$

for a positive constant C_F, then there exists a unique $u \in V$ satisfying (1.16) and it satisfies the a priori bound

$$\|u\|_V \le \frac{C_F}{c_B}. \tag{1.20}$$

Proof. See [BS94, Ch. 2]. The estimate follows immediately from

$$c_B \|u\|_V^2 \le B(u,u) = F(u) \le C_F \|u\|_V.$$

□

Remark 2.

(i) The smallest constant C_F that guarantees (1.19) is

$$\|F\|_{V^*} := \sup_{v \in V \setminus \{0\}} \left\{ \frac{|F(v)|}{\|v\|_V} \right\}. \tag{1.21}$$

The set of all linear continuous mappings $V \to \mathbb{R}$, equipped with the norm $\|\cdot\|_{V^*}$, is called the *dual space* V^* of V. In case of (1.9) it is also denoted by $H^{-1}(\Omega)$.

(ii) The estimate (1.20) does not only give an a priori bound on solutions of (1.16), but also describes the sensitivity of u to changes in F: For inputs F_1, $F_2 \in V^*$ (cf. (i)) with corresponding solutions $u_1, u_2 \in V$ one gets by linearity of the problem the estimate

$$\|u_1 - u_2\|_V \le \frac{1}{c_B} \|F_1 - F_2\|_{V^*}.$$

(iii) The result of Theorem 1 can also be established for nonsymmetric forms B [BS94, Ch. 2].

(iv) Since V is, by (1.17) and (1.18), also a complete normed space with the norm

$$|||u||| := \sqrt{B(u,u)} \qquad (\textit{Energy norm}), \tag{1.22}$$

that stems from the scalar product $(v,w) := B(v,w)$, i.e., a *Hilbert space*. In this case we would have $c_B = C_B = 1$.

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1.4 Application to the model problem

Assume that $a : \Omega \to \mathbb{R}$ is a piecewise continuous function that satisfies $0 < a_0 \le a(x) \le A_0$ for all $x \in \overline{\Omega}$ and that $\int_\Omega |f|^2 < \infty$. Then B from (1.5) satisfies

$$|B(v,w)| \le \int_\Omega a\,|\boldsymbol{\nabla} v|\,|\boldsymbol{\nabla} w| \le A_0 \Big(\int_\Omega |\boldsymbol{\nabla} v|^2 \int_\Omega |\boldsymbol{\nabla} w|^2\Big)^{1/2}$$

for all $v, w \in V$ by the Cauchy–Schwarz inequality. Moreover,

$$B(v,v) = \int_\Omega a|\boldsymbol{\nabla} v|^2 \ge a_0 \int_\Omega |\boldsymbol{\nabla} v|^2$$

for all $v \in V$ and, for F as in (1.6) and with (1.11),

$$|F(v)| \le \Big(\int_\Omega |f|^2 \int_\Omega |v|^2\Big)^{1/2} \le C_\Omega \Big(\int_\Omega |f|^2 \int_\Omega |\boldsymbol{\nabla} v|^2\Big)^{1/2}.$$

Thus we can apply Theorem 1 to obtain existence of a unique solution to the boundary value problem (1.1), (1.15). This solution satisfies the a priori bound

$$\Big(\int_\Omega |\boldsymbol{\nabla} u|^2\Big)^{1/2} \le \frac{C_\Omega}{\sqrt{a_0}} \Big(\int_\Omega |f|^2\Big)^{1/2}.$$

In case of nonhomogeneous boundary conditions this will read

$$\Big(\int_\Omega |\boldsymbol{\nabla} u|^2\Big)^{1/2} \le \frac{C_\Omega}{a_0} \Big(\int_\Omega |f|^2\Big)^{1/2} + \frac{2}{\sqrt{a_0}} \Big(\int_\Omega a|\boldsymbol{\nabla} \widetilde{g}|^2\Big)^{1/2}$$

in view of (1.14).

1.5 References

This topic is developed in many modern books on numerical methods for partial differential equations such as [BS94] [Sch98] [KA02] [LT03] [GR05] (to name only a few). The underlying abstract principles are from functional analysis, see, e.g., [Alt16] (Dt. [Alt99]).

2 The Galerkin Method

2.1 Approximation by a finite dimensional problem

For $N \in \mathbb{N}$, let V_N be a finite dimensional subspace of V with $\dim(V_N) = N$. Then we define $u_N \in V_N$ to be the solution of

$$\begin{gathered} \text{Seek } u_N \in V_N \text{ such that} \\ B(u_N, v_N) = F(v_N) \qquad \text{for all } v_N \in V_N \end{gathered} \tag{2.1}$$

with B and F as in Theorem 1. This is well-defined by $V_N \subset V$ and, moreover, existence of a unique solution u_N is guaranteed by Theorem 1 as before. However, problem (2.1) is a finite dimensional one. Let $\mathcal{B}_N$ be a basis of V_N. Then (2.1) is equivalent to

$$B(u_N, \phi) = F(\phi) \qquad \text{for all } \phi \in \mathcal{B}_N$$

and inserting the representation

$$u_N = \sum_{\psi \in \mathcal{B}_N} \alpha_\psi \psi$$

with coefficients $\alpha_\psi \in \mathbb{R}$, we obtain

$$\sum_{\psi \in \mathcal{B}_N} B(\psi, \phi)\alpha_\psi = F(\phi) \qquad \text{for all } \phi \in \mathcal{B}_N.$$

This is a linear system of equations of the form

$$Su_N = b \tag{2.2}$$

with

$$\begin{aligned} S &:= [B(\psi, \phi)]_{\phi,\psi \in \mathcal{B}_N} \in \mathbb{R}^{N,N}, \\ b &:= [F(\phi)]_{\phi \in \mathcal{B}_N} \in \mathbb{R}^N, \\ u_N &:= [\alpha_\psi]_{\psi \in \mathcal{B}_N} \in \mathbb{R}^N. \end{aligned} \tag{2.3}$$

(Here we employ the abuse of notation $u_N \in V_N \leftrightarrow u_N \in \mathbb{R}^N$ for the given basis $\mathcal{B}_N$). Note that S is symmetric positive definite by assumption on B, since by definition

$$Sv_N \cdot w_N = B(v_N, w_N) \qquad \text{for all } v_N, w_N \in V_N.$$

2.2 A priori error estimates

Theorem 3 (Céa theorem). *If u and u_N are respective solutions of (1.16) and (2.1) under the assumptions of Theorem 1, then*

$$\|u - u_N\|_V \le \frac{C_B}{c_B} \inf_{v_N \in V_N} \{\|u - v_N\|_V\}. \tag{2.4}$$

The quantity $C_S := C_B/c_B$ is called stability constant, *while the infimum measures the best possible approximation to u with functions in V_N.*

Proof. By the solution properties of (1.16), (2.1) we obtain for arbitrary $v_N \in V_N$

$$\begin{aligned} c_B\|u - u_N\|_V^2 &\le B(u - u_N, u - u_N) = B(u - u_N, u - v_N) \\ &\le C_B\|u - u_N\|_V\|u - v_N\|_V. \end{aligned}$$

If $\|u - u_N\|_V = 0$, the assertion is true. Otherwise we arrive at

$$\|u - u_N\|_V \le \frac{C_B}{c_B}\|u - v_N\|_V.$$

Since v_N was arbitrary, we can take the infimum on the right hand side. □

Remark 4.

(i) The aim of the Galerkin method is to find an approximation space V_N such that u_N can be efficiently computed from (2.2) and the infimum in (2.4) gets rapidly small for increasing N. A necessary and sufficient condition for convergence of the finite element method is that

$$\lim_{N\to\infty} \inf_{v_N \in V_N} \{\|v - v_N\|_V\} \to 0 \qquad \text{for all } v \in V.$$

The estimate (2.4) states that the error in the energy norm is, up to the stability constant, comparable to the best approximation to u in V_N. Therefore one calls u_N *quasi-optimal*. Using however the energy norm (1.22), one finds from $c_B = C_B = 1$ that actually

$$|||u - u_N||| = \inf_{v_N \in V_N} \{|||u - v_N|||\},$$

i.e., u_N *is* the best approximation to u in V_N in the energy norm.

(ii) Analogously to finite difference methods where one proves

$$\text{“stability and consistency} \quad \Longrightarrow \quad \text{convergence”},$$

the result (2.4) can be read as

$$\text{“stability and approximability} \quad \Longrightarrow \quad \text{convergence”}.$$

Note that it is understood here, that the stability constant does not depend on N.

(iii) The estimate (2.4) is treated further by explicitly defining a continuous linear *interpolation operator* $P_N : V \to V_N$. Then one utilises

$$\inf_{v_N \in V_N} \{\|u - v_N\|_V\} \leq \|u - P_N u\|_V$$

and proves estimates on $\|u - P_N u\|_V$ for certain classes of functions u containing possible solutions.

(iv) Usually, one does not work with the form B on V_N but with an approximation B_N (due to numerical integration for example). To analyse this situation one would use the estimates of Strang instead of Céa [Sch98, Ch. 2.4].

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2.3 Application to the model equation

Taking the example (1.5), (1.6) we get

$$S_{\phi\psi} := \int_\Omega a \boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\phi,$$
$$b_\phi := \int_\Omega f\phi.$$

By Section 1.4 we can take $c_B = a_0$ and $C_B = A_0$, hence

$$C_S = \frac{A_0}{a_0} = \frac{\sup_{x\in\Omega}\{a(x)\}}{\inf_{x\in\Omega}\{a(x)\}}.$$

However, measuring the error using $|\!|\!| \cdot |\!|\!|$ as in (1.22), we would obtain the best possible bound $C_S = 1$.

2.4 Generalized functions at the right hand side

In view of the weak formulation one can generalise the right hand side in (1.1) to be $f - \boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{f}}$ for some $\widetilde{\boldsymbol{f}} : \Omega \to \mathbb{R}^d$ with only $\int_\Omega |\widetilde{\boldsymbol{f}}|^2 < \infty(!)$ by setting

$$F(v) = \int_\Omega \{fv + \widetilde{\boldsymbol{f}} \cdot \boldsymbol{\nabla} v\}$$

in the weak formulation (1.5) of the homogeneous boundary value problem (1.15). We then can proceed as in Section 1.4 to get a solution u with a priori bound

$$\Big(\int_\Omega |\boldsymbol{\nabla} u|^2\Big)^{1/2} \leq \frac{1}{\sqrt{a_0}}\Big(C_\Omega\Big(\int_\Omega |f|^2\Big)^{1/2} + \Big(\int_\Omega |\widetilde{\boldsymbol{f}}|^2\Big)^{1/2}\Big).$$

2.5 References

See Section 1.5.

3 The Finite Element Method

In the following we study the case of one space dimension, $d = 1$. Without loss of generality we can assume that $\Omega := (0,1)$. The boundary value problem (1.1), (1.15) reads: Seek $u : [0,1] \to \mathbb{R}$ with

$$\begin{aligned} -\big(au'\big)' &= f \qquad \text{in } (0,1), \\ u(0) = u(1) &= 0, \end{aligned}$$

and the weak formulation specifies $u \in V = H_0^1(0,1)$ with

$$\int_0^1 au'v' = \int_0^1 fv \qquad \text{for all } v \in H_0^1(0,1). \tag{3.1}$$

The idea of the finite element method consists of decomposing Ω into small simple subregions and to approximate u locally with polynomials.

3.1 Linear finite elements

We define the *grid* $\overline{\mathcal{G}}_n \subset \overline{\Omega}$ for $n \in \mathbb{N}$, to be a set of distinct *grid points*

$$\overline{\mathcal{G}}_n := \big\{x_0, \dots, x_{n+1} \,:\, 0 = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1\big\}.$$

Especially, the set of *interior grid points* is

$$\mathcal{G}_n := \overline{\mathcal{G}}_n \cap \Omega = \{x_1, \dots, x_n\}. \tag{3.2}$$

A *decomposition* of Ω is the set of intervals

$$\mathcal{K}_n := \Big\{K = [x_{k-1}, x_k] \,:\, k \in \{1, \dots, n+1\}\Big\}. \tag{3.3}$$

For $K \in \mathcal{K}_n$ let $h_K := |K|$ be the *local grid size* and $h_{\mathcal{K}_n} = \max\{h_K : K \in \mathcal{K}_n\}$ be the *global grid size*. By h we denote the *grid size function* of $\mathcal{K}_n$, defined by $h(x) := h_K$ for $x \in K$. We now let

$$\mathcal{S}_0^1(\mathcal{K}_n) := \Big\{v \in C^0(\overline{\Omega}) \,:\, v\big\lfloor_K \in \mathbb{P}_1 \text{ for all } K \in \mathcal{K}_n \text{ and } v(0) = v(1) = 0\Big\} \tag{3.4}$$

be the space of *linear finite elements*. It is a linear function space that is spanned by the set $\{\psi_z\}_{z \in \mathcal{G}_n} \subset \mathcal{S}_0^1(\mathcal{K}_n)$ consisting of the *nodal basis functions* ψ_z, uniquely defined through $\psi_z(z') := \delta_{z,z'}$ for all $z \in \mathcal{G}_n$, $z' \in \overline{\mathcal{G}}_n$ with $N := \dim\big(\mathcal{S}_0^1(\mathcal{K}_n)\big) = n$. We will use the notation $V_N^1 := \mathcal{S}_0^1(\mathcal{K}_n)$. Each function $v_N \in V_N^1$ has the unique representation

$$v_N(x) = \sum_{z \in \mathcal{G}_n} \alpha_z \psi_z(x) \qquad \text{for all } x \in \overline{\Omega}$$

with coefficients $\alpha_z \in \mathbb{R}$. Again, we will not distinguish notationally between the function $v_N \in V_N^1$ and the corresponding vector $[v_N(z)]_{z \in \mathcal{G}_n} = [\alpha_z]_{z \in \mathcal{G}_n} \in \mathbb{R}^N$.

3.2 Finite elements of higher polynomial order

On the decomposition $\mathcal{K}_n$ as in Section 3.1 we define for $p \in \mathbb{N}_{\geq 1}$,

$$\mathcal{S}_0^p(\mathcal{K}_n) := \Big\{v \in C^0(\overline{\Omega}) \,:\, v\big\lfloor_K \in \mathbb{P}_p \text{ for all } K \in \mathcal{K}_n \text{ and } v(0) = v(1) = 0\Big\}. \tag{3.5}$$

Prescribing for $v \in \mathcal{S}_0^p(\mathcal{K}_n)$ values $v(z)$ for $z \in \mathcal{G}_n$, we are left with $p-1$ free coefficients on each interval, that is $(n+1)(p-1)$ in total. Hence $n + (n+1)(p-1) = np + p - 1$ coefficients determine any $v \in \mathcal{S}_0^p(\mathcal{K}_n)$ uniquely. With $N := \dim\big(\mathcal{S}_0^p(\mathcal{K}_n)\big) = np + p - 1$, we will use the notation $V_N^p := \mathcal{S}_0^1(\mathcal{K}_n)$.

The aim is now to find a local representation of the polynomials on K. For this, it is convenient to show that we can consider the situation on the *reference element* $\widehat{\Omega} := [-1,1]$ instead.

3.2.1 Transformation to the reference element

Let $K \in \mathcal{K}_n$ be given by $[x_{k-1}, x_k]$ for some $k \in \{1, \dots, n+1\}$. Then we define

$$\begin{aligned} Q_K : \widehat{\Omega} &\to K \\ Q_K(\xi) &:= \frac{1}{2}(x_{k-1} + x_k) + \frac{1}{2}(x_k - x_{k-1})\xi \\ &= \frac{1}{2}(1-\xi)x_{k-1} + \frac{1}{2}(1+\xi)x_k. \end{aligned}$$

Hence $Q_K \in \mathbb{P}_1$ with $Q_K' = \frac{1}{2}(x_k - x_{k-1}) = \frac{1}{2}h_K > 0$ and therefore $Q_K^{-1} : K \to \widehat{\Omega}$ exists and $Q_K^{-1} \in \mathbb{P}_1$. This can be utilised to rewrite (3.5) into the form

$$\begin{aligned} \mathcal{S}_0^p(\mathcal{K}_n) = \Big\{ v \in C^0(\overline{\Omega}) \,:\, v\lfloor_K = w_K \circ Q_K^{-1} \text{ with } w_K \in \mathbb{P}_p(\widehat{\Omega}) \text{ for all } K \in \mathcal{K}_n \\ \text{and } v(0) = v(1) = 0 \Big\}, \end{aligned} \tag{3.6}$$

since composition with Q_K^{-1} preserves the polynomial degree.

3.2.2 Polynomial bases on $\widehat{\Omega}$

There are several possibilities to define a basis for $\mathbb{P}_p$ on $\widehat{\Omega}$.

(i) *Monomials*. $\phi_l(\xi) := \xi^l$ for $l = 0, \dots, p$. However, for given $v \in \mathbb{P}_p$ the computation of the coefficients α_l in the representation $v = \sum_{l=0}^p \alpha_l \phi_l$ leads to an ill conditioned problem (even for moderate p).

(ii) *Newton polynomials*. Choose a uniform decomposition of $\widehat{\Omega}$ with $p+1$ points $\xi_0, \dots, \xi_p$ with $-1 =: \xi_0 < \cdots < \xi_p := 1$ and choose ϕ_l by $\phi_l(\xi_j) := \delta_{l,j}$. However, evaluation of $\phi_l(\xi)$ and $\phi_l'(\xi)$ for $\xi \neq \xi_l$ is costly.

(iii) *Orthogonal polynomials*. Given an appropriate scalar product $(\,.\,,\,.\,)$ on $\mathbb{P}_p$, one can construct a sequence of polynomials $\{\phi_l\}_{l \geq 0}$ with $\phi_l \in \mathbb{P}_l$ and $(\phi_l, \phi_j) = \delta_{l,j}$. Such a sequence generally satisfies a *recurrence relation* of the form

$$\beta_{l+1}\phi_{l+1}(\xi) = \big(\xi - \alpha_{l+1}\big)\phi_l(\xi) - \beta_l \phi_{l-1}(\xi) \qquad \text{for all } \xi \in \widehat{\Omega} \tag{3.7}$$

for sequences of numbers $\{\alpha_l\}_{l \geq 1}$ and $\{\beta_l\}_{l \geq 1}$. Such a recurrence relation can be used to efficiently evaluate $\phi_l(\xi)$ for $l = 0, \dots, p$ and $\xi \in \widehat{\Omega}$. A reasonable choice for a scalar product is

$$\big(v, w\big)_1 := \int_{\widehat{\Omega}} v' w',$$

since this diagonalises the matrix S in (2.2) for our model problem (3.1) in case of constant a. For convenience, we define further

$$\big(v, w\big)_0 := \int_{\widehat{\Omega}} vw.$$

3.2.3 The Legendre and integrated Legendre polynomials

Let $L_0 := 1$ and define the *Legendre polynomials* $L_l \in \mathbb{P}_l(\widehat{\Omega})$ for $l > 0$ by applying the Gram–Schmidt procedure to get

$$\big(L_l, L_j\big)_0 \overset{!}{=} \frac{1}{l + \frac{1}{2}}\, \delta_{l,j}.$$

Note, that this definition yields the $(.\,,.)_0$-orthonormalized Legendre polynomials only by a factor $1/\sqrt{l+1/2}$. For example, the first four polynomials obtained in this way are

$$L_0(\xi) = 1,\ L_1(\xi) = \xi,\ L_2(\xi) = \frac{1}{2}\big(3\xi^2 - 1\big),\ L_3(\xi) = \frac{1}{2}\xi\big(5\xi^2 - 3\big)$$

(cf. the graphs in Figure 1 (left)). They satisfy the boundary value problem

$$\begin{aligned} -\big((1-\xi^2)L_l'\big)' + l(l+1)L_l &= 0 \qquad \text{in } \widehat{\Omega}, \\ L_l(-1) = (-1)^l,\ L_l(1) &= 1, \end{aligned} \tag{3.8}$$

and the recurrence relation

$$\frac{l+1}{2l+1}L_{l+1}(\xi) = \xi L_l(\xi) - \frac{l}{2l+1}L_{l-1}(\xi) \tag{3.9}$$

(use (3.7) with $\alpha_{l+1} := 0$, $\beta_l := l/\sqrt{4l^2-1}$ and $\phi_l := \sqrt{l+1/2}\,L_l$). The *integrated Legendre polynomials* $\mathit{I\!L}_l \in \mathbb{P}_l$ are defined by

$$\mathit{I\!L}_0(\xi) := 1 \quad \text{and} \quad \mathit{I\!L}_l(\xi) := \int_{-1}^{\xi} L_{l-1} \qquad \text{for all } l \geq 1, \tag{3.10}$$

the first four of which are

$$\mathit{I\!L}_0(\xi) = 1,\ \mathit{I\!L}_1(\xi) = \xi + 1,\ \mathit{I\!L}_2(\xi) = \frac{1}{2}\big(\xi^2 - 1\big),\ \mathit{I\!L}_3(\xi) = \frac{1}{2}\xi\big(\xi^2 - 1\big).$$

It is immediate, that for $l \geq 2$

$$\mathit{I\!L}_l(-1) = 0 \quad \text{and} \quad \mathit{I\!L}_l(1) = \int_{-1}^{1} L_{l-1} = \big(L_{l-1}, L_0\big)_0 = 0.$$

From [Sch98, p. 360] we obtain after integration the relation

$$\mathit{I\!L}_l = \frac{1}{2l-1}\big(L_l - L_{l-2}\big) \qquad \text{for all } l \geq 2 \tag{3.11}$$

between the Legendre polynomials and the integrated Legendre polynomials.

The *basis of integrated Legendre polynomials* consists, for $l \geq 2$, of the rescaled functions

$$\widehat{\psi}_l := \sqrt{l - \frac{1}{2}}\ \mathit{I\!L}_l \tag{3.12}$$

and we obtain, for $l, j \geq 2$,

$$\widehat{\psi}_l(-1) = 0, \quad \widehat{\psi}_l(1) \sim \mathit{I\!L}_l(1) = 0, \tag{3.13}$$

$$\big(\widehat{\psi}_l, \widehat{\psi}_j\big)_1 = \sqrt{\Big(l - \frac{1}{2}\Big)\Big(j - \frac{1}{2}\Big)}\,\big(L_{l-1}, L_{j-1}\big)_0 = \frac{l - \frac{1}{2}}{(l-1) + \frac{1}{2}}\,\delta_{l,j} = \delta_{l,j}. \tag{3.14}$$

However, for $l \in \{0, 1\}$ we let

$$\widehat{\psi}_0(\xi) := \frac{1}{2}(1 - \xi) \quad \text{and} \quad \widehat{\psi}_1(\xi) := \frac{1}{2}(1 + \xi). \tag{3.15}$$

Then, for $l \in \{0, 1\}$ and $j \geq 2$

$$\big(\widehat{\psi}_l, \widehat{\psi}_j\big)_1 = \widehat{\psi}_l' \int_{-1}^{1} \widehat{\psi}_j' = 0, \tag{3.16}$$

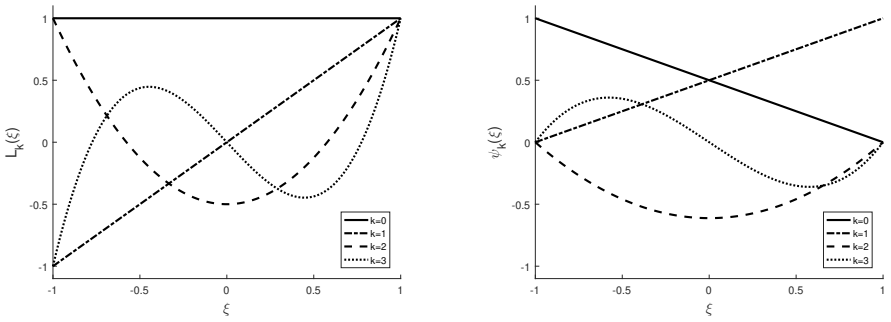

Figure 1: Graphs for $L_0, \ldots, L_3$ (left) and $\widehat{\psi}_0, \ldots, \widehat{\psi}_3$ (right).

while

$$\left[\left(\widehat{\psi}_l, \widehat{\psi}_j\right)_1\right]_{l,j\in\{0,1\}} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{3.17}$$

The graphs of $\widehat{\psi}_0, \ldots, \widehat{\psi}_3$ are shown in Figure 1 (right).

A basis for V_N^p is then constructed as follows: We transform these functions back to K and extend them in an appropriate way onto Ω. We arrive at a basis consisting of the (piecewise) linear basis functions (from Section 3.1) located at all interior mesh points, which are supported on two adjacent intervals, and the (piecewise) higher order basis functions that are supported on single intervals in $\mathcal{K}_n$.

3.2.4 Basis function evaluation

Each polynomial v in $\mathbb{P}_p$ has the unique representation

$$v(\xi) = v(-1)\widehat{\psi}_0(\xi) + v(1)\widehat{\psi}_1(\xi) + \sum_{l=2}^{p} \alpha_l \widehat{\psi}_l(\xi).$$

Hence two problems will arise in practice:

- Given $\alpha_2, \ldots, \alpha_p$, compute $v(\xi)$ or $v'(\xi)$ for one (or several) ξ efficiently, i.e., evaluate $\widehat{\psi}_l(\xi)$ or $\widehat{\psi}_l'(\xi)$ efficiently.
- Given $v \in \mathbb{P}_p$, compute $\alpha_2, \ldots, \alpha_p$.

The first task will be done by iteratively computing the values $L_l(\xi)$ and $\mathit{IL}_l(\xi)$ from (3.9) and (3.11), respectively, and then using (3.12) to get $\widehat{\psi}_l(\xi)$ and $\widehat{\psi}_l'(\xi)$. Since each iteration consists of a constant number of floating point operations (*flop*), the total cost to provide $\left[\widehat{\psi}_l'(\xi)\right]_{0\le l\le p}$ and $\left[\widehat{\psi}_l(\xi)\right]_{0\le l\le p}$ for a given ξ is $\mathcal{O}(p)$.

The second task is performed as follows: by orthogonality we have immediately

$$\alpha_l = \frac{\left(v, \widehat{\psi}_l\right)_1}{\left(\widehat{\psi}_l, \widehat{\psi}_l\right)_1} \qquad \text{for } l = 2, \ldots, p.$$

Now one chooses a quadrature rule in p points on $\widehat{\Omega}$ that is exact on polynomials of degree $2p-1$, so that for $l = 2, \ldots, p$

$$\alpha_l = \frac{1}{\left(\widehat{\psi}_l, \widehat{\psi}_l\right)_1} \sum_{j=1}^{p} \omega_j v'(\xi_j)\widehat{\psi}_l'(\xi_j),$$

and computes the values $\widehat{\psi}_l'(\xi_j)$ as in the first part. Examples for rules that meet this requirement are the *Gauss–Legendre quadrature rule* and the *Gauss–Lobatto quadrature rule* with p points [SM03, Ch. 10.5, 10.6]. Note that no rule with less than p points can have this property. Computation of $[\alpha_l]_{0\le l\le p}$ requires $\mathcal{O}(p^2)$ flop and p times the work account of $\xi \mapsto v'(\xi)$. The matrices $\big[\widehat{\psi}_l'(\xi_j)\big]_{l,j}$ and $\big[\widehat{\psi}_l(\xi_j)\big]_{l,j}$ are best computed in advance and stored (at the cost of $\mathcal{O}(p^2)$ flop).

3.2.5 Varying polynomial order

Given a vector $\boldsymbol{p} = [p_K]_{K\in\mathcal{K}_n} \in \mathbb{N}^{n+1}$ with $p_K \ge 1$, we let

$$\mathcal{S}_0^{\boldsymbol{p}}(\mathcal{K}_n) := \Big\{ v \in C^0(\overline{\Omega}) \,:\, v\lfloor_K \in \mathbb{P}_{p_K} \text{ for all } K \in \mathcal{K}_n \text{ and } v(0) = v(1) = 0 \Big\}. \tag{3.18}$$

Besides the n function values on $\mathcal{G}_n$ one has $p_K - 1$ additional degrees of freedom on each interval $K \subset \mathcal{K}_n$. The total number of coefficients that uniquely determine v is $N := \dim\big(\mathcal{S}_0^{\boldsymbol{p}}(\mathcal{K}_n)\big) = n + \sum_{K\in\mathcal{K}_n}\{p_K - 1\} = \sum_{K\in\mathcal{K}_n} p_K - 1$. With this definition of N in mind we write $V_N^{\boldsymbol{p}} := \mathcal{S}_0^{\boldsymbol{p}}(\mathcal{K}_n)$.

3.3 The discrete equations

3.3.1 The discrete equations for linear elements

We number the nodal basis functions from Section 3.1 according to the numbering in (3.2), writing $\psi_i \equiv \psi_{x_i}$. Thus the stiffness matrix S and the load vector b for the problem (3.1) are given by

$$\begin{aligned} S_{ij} &= \int_\Omega a\psi_j'\psi_i' = \sum_{K\in\mathcal{K}_n} \int_K a\psi_j'\psi_i', \\ b_i &= \int_\Omega f\psi_i = \sum_{K\in\mathcal{K}_n} \int_K f\psi_i. \end{aligned} \tag{3.19}$$

These integrals are computed as follows:

- The integrals on K are transformed to $\widehat{\Omega}$, where the evaluation is done by a quadrature rule.
- We compute the contributions on K and add them up.

In the present case of $p = 1$, we can choose the midpoint rule without spoiling the order of convergence proved later. On each K, one has to compute three integrals for S (by symmetry) and the integral for b. Thus the setup of the system needs $\mathcal{O}(N)$ flop. The matrix has the following properties:

- S is symmetric and positive definite,
- S is tridiagonal and weakly diagonally dominant.

The equations are therefore stably and efficiently solved by Gaussian elimination within $\mathcal{O}(N)$ flop.

3.3.2 The discrete equations for higher order elements

Following the guideline of the previous section we have to compute

$$\int_{\widehat{\Omega}} a(Q_K)\,\widehat{\psi}_l'\widehat{\psi}_m' \quad \text{and} \quad \int_{\widehat{\Omega}} f(Q_K)\,\widehat{\psi}_l \tag{3.20}$$

for $l, m \in \{0, \dots, p\}$. For approximation we choose a quadrature rule that is exact on polynomials of degree $2p_K + 1$. This requires a Gauss–Legendre rule with $p_K + 1$ points or a Gauss–Lobatto rule with $p_K + 2$ points. In general, the local matrix $\widehat{S}$ is filled with non-zero elements, except when a is constant, in which case the off-diagonal elements vanish (except $\widehat{S}_{01} = \widehat{S}_{10}$) by construction of the basis.

Each linear basis function is *coupled* (i.e., has in general non-zero matrix entries) with the two adjacent ones and all higher order basis functions in the two adjacent intervals. The higher order basis functions are coupled with those in the same interval only. We number

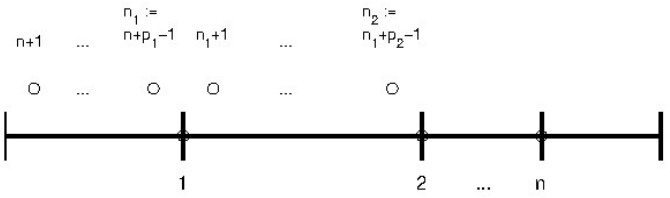

Figure 2: Numbering of the unknowns for the p-method.

the basis functions as shown in Figure 2 and decompose the vector of unknowns into the linear (L) and higher order part (P): $u_N = [u_\mathrm{L}; u_\mathrm{P}]$. This induces a *block decomposition*

$$S = \begin{bmatrix} S_\mathrm{LL} & S_\mathrm{LP} \\ S_\mathrm{PL} & S_\mathrm{PP} \end{bmatrix}$$

of the stiffness matrix S. S_LL is the tridiagonal matrix from Section 3.3.1. In the row of S_LP that corresponds to the grid point between K and K' (or the corresponding column of S_PL), we will find (in general) $p_K + p_{K'} - 2$ non-zero entries. S_PP is block-diagonal with $(p_K - 1) \times (p_K - 1)$ blocks for each interval $K \in \mathcal{K}_n$. The total number of non-zero elements of S is thus about

$$\begin{aligned} 3n + \sum_{K \in \mathcal{K}_n} 2(p_K - 1) + \sum_{K \in \mathcal{K}_n} (p_K - 1)^2 &= 3n + \sum_{K \in \mathcal{K}_n} \{p_K^2 - 1\} \\ &\approx 2n + \sum_{K \in \mathcal{K}_n} p_K^2, \end{aligned}$$

i.e., $\mathcal{O}(np^2)$ for constant p. The total cost for computing S on each K is $\mathcal{O}(p_K^3)$ (the p_K^2 integrals (3.20) are approximated as in Section 3.2.4 with precomputed values $\big[\widehat{\psi}_l(\xi_j)\big]_{l,j}$, $\big[\widehat{\psi}_l'(\xi_j)\big]_{l,j}$) and therefore $\mathcal{O}\big(\sum_{K \in \mathcal{K}_n} p_K^3\big)$ in total. For a uniform p-method, as an example, we have $N \approx np$, a total storage of $\mathcal{O}(np^2) = \mathcal{O}(pN)$, and a total work of $\mathcal{O}(np^3) = \mathcal{O}(p^2 N)$.

3.3.3 Solution techniques

Static condensation An equation with matrix S_LL is stably and efficiently solved by Gaussian elimination in $\mathcal{O}(N)$ flop. For $|\boldsymbol{p}|_\infty = 1$ the equations are easily solved this way with $\mathcal{O}(n = N)$ flop. For $|\boldsymbol{p}|_\infty > 1$, we write the system in the partitioned form

$$\begin{aligned} S_\mathrm{LL} u_\mathrm{L} + S_\mathrm{LP} u_\mathrm{P} &= b_\mathrm{L}, \\ S_\mathrm{PL} u_\mathrm{L} + S_\mathrm{PP} u_\mathrm{P} &= b_\mathrm{P}. \end{aligned}$$

Since S_PP is a block-diagonal matrix, it is easily inverted using $\mathcal{O}(\sum_{K \in \mathcal{K}_n} p_K^3)$ flop (i.e., $\mathcal{O}(np^3) = \mathcal{O}(p^2 N)$ for uniform p) and with storage as for S_PP. We thus solve the second

equation and insert it into the first one

$$u_{\mathrm{P}} = S_{\mathrm{PP}}^{-1}(b_{\mathrm{P}} - S_{\mathrm{PL}}u_{\mathrm{L}}),$$
$$\widetilde{S}_{\mathrm{LL}}u_{\mathrm{L}} := \big(S_{\mathrm{LL}} - S_{\mathrm{LP}}S_{\mathrm{PP}}^{-1}S_{\mathrm{PL}}\big)u_{\mathrm{L}} = b_{\mathrm{L}} - S_{\mathrm{LP}}S_{\mathrm{PP}}^{-1}b_{\mathrm{P}}.$$

This procedure is simply a Block–Gaussian elimination. $\widetilde{S}_{\mathrm{LL}}$ is called *Schur-complement* and is again a tridiagonal and symmetric $n \times n$-matrix. Therefore, it is best solved with Gaussian elimination ($\mathcal{O}(n)$ flop).

Principally we can exchange the role of u_{L} and u_{P} to get the Schur complement $\widetilde{S}_{\mathrm{PP}} := S_{\mathrm{PP}} - S_{\mathrm{PL}}S_{\mathrm{LL}}^{-1}S_{\mathrm{LP}}$. However, S_{LL}^{-1} is a full $n \times n$-matrix and hence it should only explicitly be formed for small n. If one uses an iterative method to solve for u_P, one could use Gaussian elimination for solving the systems with S_{LL} in each step (since $d_{\mathrm{L}} = S_{\mathrm{LL}}^{-1}S_{\mathrm{LP}}u_{\mathrm{P}}$ is obtained solving $S_{\mathrm{LL}}d_{\mathrm{L}} = S_{\mathrm{LP}}u_{\mathrm{P}}$), which is cheap compared to the application of the other matrices.

Preconditioned conjugate gradient method Since S is symmetric and positive we can apply the conjugate gradient method, but preconditioning is inevitable. In case of constant (or piecewise constant with respect to $\mathcal{K}_n$) a, the coupling matrices $S_{\mathrm{LP}} = S_{\mathrm{PL}}^{\dagger}$ vanish and S_{PP} is diagonal. In the general case, when $(a\lfloor_K)'$ is bounded for all $K \in \mathcal{K}_n$, we thus propose to use

$$B^{-1} := \begin{bmatrix} S_{\mathrm{LL}}^{-1} & 0 \\ 0 & \operatorname{diag}(S_{\mathrm{PP}})^{-1} \end{bmatrix}$$

as a preconditioner. We show below that $\operatorname{cond}(B^{-1}S)$ is bounded independently of h and $\boldsymbol{p}$ if $h_{\mathcal{K}_n}$ is small enough, depending on a and $\boldsymbol{p}$. So the number of conjugate gradient iterations to reach a certain tolerance is then bounded independently of h and $\boldsymbol{p}$. The operation $v \mapsto Sv$ needs $\mathcal{O}(\sum_{K\in\mathcal{K}_n} p_K^2)$ flop, while $v \mapsto B^{-1}v$ needs $\mathcal{O}(N)$ flop. For the method with uniform p we therefore need $\mathcal{O}(pN)$ flop.

Theorem 5. *For piecewise differentiable a and sufficiently small h, depending on a and $\boldsymbol{p}$, the condition number of $B^{-1}S$ is uniformly bounded in h and $\boldsymbol{p}$.*

Proof. Let $B_{\mathrm{L}} := S_{\mathrm{LL}}$ and $B_{\mathrm{P}} := \operatorname{diag}(S_{\mathrm{PP}})$. For $v_N = [v_{\mathrm{L}}; v_{\mathrm{P}}] \in V_N^{\boldsymbol{p}}$ we then have

$$Sv_N \cdot v_N = B_{\mathrm{L}}v_{\mathrm{L}} \cdot v_{\mathrm{L}} + 2S_{\mathrm{LP}}v_{\mathrm{P}} \cdot v_{\mathrm{L}} + B_{\mathrm{P}}v_{\mathrm{P}} \cdot v_{\mathrm{P}} + \big(S_{\mathrm{PP}} - B_{\mathrm{P}}\big)v_{\mathrm{P}} \cdot v_{\mathrm{P}}.$$

If we can show that

$$2\big|S_{\mathrm{LP}}v_{\mathrm{P}} \cdot v_{\mathrm{L}}\big| \le 2\gamma_{\mathrm{L}}\big(B_{\mathrm{L}}v_{\mathrm{L}} \cdot v_{\mathrm{L}}\ B_{\mathrm{P}}v_{\mathrm{P}} \cdot v_{\mathrm{P}}\big)^{1/2} \le \gamma_{\mathrm{L}}\big(B_{\mathrm{L}}v_{\mathrm{L}} \cdot v_{\mathrm{L}} + B_{\mathrm{P}}v_{\mathrm{P}} \cdot v_{\mathrm{P}}\big),$$
$$\big|\big(S_{\mathrm{PP}} - B_{\mathrm{P}}\big)v_{\mathrm{P}} \cdot v_{\mathrm{P}}\big| \le \gamma_{\mathrm{P}}B_{\mathrm{P}}v_{\mathrm{P}} \cdot v_{\mathrm{P}},$$

for numbers $\gamma_{\mathrm{L}}, \gamma_{\mathrm{P}} \in (0,1)$ with $\gamma_{\mathrm{L}} + \gamma_{\mathrm{P}} < 1$, that are independent of h and $\boldsymbol{p}$, then we can conclude

$$Sv_N \cdot v_N \le \max\{1+\gamma_{\mathrm{L}}, 1+\gamma_{\mathrm{P}}\}Bv_N \cdot v_N \le 2Bv_N \cdot v_N$$

and

$$Sv_N \cdot v_N \ge (1-\gamma_{\mathrm{L}})B_{\mathrm{L}}v_{\mathrm{L}} \cdot v_{\mathrm{L}} + (1-\gamma_{\mathrm{L}}-\gamma_{\mathrm{P}})B_{\mathrm{P}}v_{\mathrm{P}} \cdot v_{\mathrm{P}} \ge (1-\gamma_{\mathrm{L}}-\gamma_{\mathrm{P}})Bv_N \cdot v_N.$$

This spectral equivalence between the bilinear forms S and B implies the asserted condition number estimate.

For $K \in \mathcal{K}_n$ with $p_K > 1$, the local partition $v_{\mathrm{P};K} := v_{\mathrm{P}}\lfloor_K = [v_2; \ldots; v_{p_K}]$ represents a function of the form $\sum_{i=2}^{p_K} v_i\psi_i$, where ψ_i stands for the local integrated Legendre basis

function of order p_K. By their orthogonality and the regularity hypotheses on a we get with $a_K := \int_K a/|K|$

$$\begin{aligned}\Big|(S_{\mathrm{PP}} - B_{\mathrm{P}})v_{\mathrm{P};K} \cdot v_{\mathrm{P};K}\Big| &= \Big|\sum_{i=2}^{p_K}\sum_{j=2\,:\,j\neq i}^{p_K}\int_K (a-a_K)v_i\psi_i' v_j\psi_j'\Big| \\ &\le \frac{1}{2}\sup_{x\in K}\Big\{\frac{a'(x)}{a(x)}\Big\}\,h_K\,\sum_{i=2}^{p_K}\sum_{j=2}^{p_K}\Big(\int_K a|v_i\psi_i'|^2\int_K a|v_j\psi_j'|^2\Big)^{1/2} \\ &\le \frac{1}{2}\sup_{x\in K}\Big\{\frac{a'(x)}{a(x)}\Big\}\,h_K(p_K-1)\,\sum_{i=2}^{p_K}\int_K a|v_i\psi_i'|^2 \\ &= \frac{1}{2}\sup_{x\in K}\Big\{\frac{a'(x)}{a(x)}\Big\}\,h_K(p_K-1)\,B_{\mathrm{P}}v_{\mathrm{P};K}\cdot v_{\mathrm{P};K}.\end{aligned}$$

After summation over $K \in \mathcal{K}_n$ we obtain the required estimate with

$$\gamma_{\mathrm{P}} := \frac{1}{2}\max_{K\in\mathcal{K}_n}\Big\{\sup_{x\in K}\Big\{\frac{a'(x)}{a(x)}\Big\}\,h_K(p_K-1)\Big\}.$$

Using the same idea, we further get

$$\begin{aligned}\Big|S_{\mathrm{LP}}v_{\mathrm{P};K}\cdot v_{\mathrm{L};K}\Big| &= \Big|\sum_{i=2}^{p_K}\int_K (a-a_K)v_{\mathrm{L}}'\psi_i'\Big| \\ &\le \frac{1}{2}\sup_{x\in K}\Big\{\frac{a'(x)}{a(x)}\Big\}\,h_K\,\|\sqrt{a}v_{\mathrm{L}}'\|_{L^2(K)}\sum_{i=2}^{p_K}\Big(\int_K a|v_i\psi_i'|^2\Big)^{1/2} \\ &\le \frac{1}{2}\sup_{x\in K}\Big\{\frac{a'(x)}{a(x)}\Big\}\,h_K\sqrt{p_K-1}\,\Big(B_{\mathrm{L}}v_{\mathrm{L};K}\cdot v_{\mathrm{L};K}\;B_{\mathrm{P}}v_{\mathrm{P};K}\cdot v_{\mathrm{P};K}\Big)^{1/2}\end{aligned}$$

and the required estimate follows with

$$\gamma_{\mathrm{L}} := \frac{1}{2}\max_{K\in\mathcal{K}_n}\Big\{\sup_{x\in K}\Big\{\frac{a'(x)}{a(x)}\Big\}\,h_K\sqrt{p_K-1}\Big\} < \gamma_{\mathrm{P}}.$$

The assertion is thus true if h_K is small enough, depending on $a\lfloor_K$ and p_K. □

3.4 References

This section is based on [Sch98, Sect. 3.1, 3.2], except Theorem 5.

4 Error Estimates for the hp–Method

4.1 Notation

For a domain $G \subset \mathbb{R}^d$ we let

$$L^2(G) := \Big\{v : G\to\mathbb{R} \,:\, \|v\|_{L^2(G)} := \Big(\int_G |v|^2\Big)^{1/2} < \infty\Big\} \qquad \textit{(Lebesgue space)}$$

and for $m \in \mathbb{N}$

$$H^m(G) := \Big\{v : G\to\mathbb{R} \,:\, \|v\|_{H^m(G)} := \Big(\sum_{s=0}^{m}\|\boldsymbol{\nabla}^s v\|_{L^2(G)}^2\Big)^{1/2} < \infty\Big\} \qquad \textit{(Sobolev spaces).}$$

For a decomposition $\mathcal{K}$ of Ω and a vector $\boldsymbol{m} = [m_K]_{K\in\mathcal{K}}$ we let

$$H^{\boldsymbol{m}}(\mathcal{K}) := \left\{ v : \Omega \to \mathbb{R} \,:\, v_{\lfloor K} \in H^{m_K}(K) \text{ for all } K \in \mathcal{K} \right\} \quad \textit{(“broken” Sobolev spaces).}$$

Aim: As a consequence of Theorem 3 we have to estimate

$$\inf_{v_N \in V_N^{\boldsymbol{p}}} \left\{ \|v - v_N\|_V \right\}$$

for a given $v \in V$. In view of Remark 4.(iii), we will define an *interpolation operator*

$$\Pi_N^{\boldsymbol{p}} : L^2(\Omega) \to V_N^{\boldsymbol{p}}$$

and show estimates of the form

$$\|v - \Pi_N^{\boldsymbol{p}} v\|_V \le C(N, v).$$

Moreover, we want to have the dependencies on N and $\boldsymbol{p}$ explicitly.

Example

We consider the linear finite element method. If $a = 1$ in (3.1) and if the decomposition $\mathcal{K}_n$ is *uniform* ($h = h_K$ for all $K \in \mathcal{K}_n$), then approximation with linear finite elements yields a solution $u_N \in V_N^1$ ($N = n \le 1/h$) with

$$\|u - u_N\|_V \le \frac{1}{\pi} h \|u''\|_{L^2(\Omega)} \le \frac{1}{\pi}\frac{1}{N}\|u''\|_{L^2(\Omega)}.$$

The dependency on $1/N$ appears with the optimal exponent for piecewise linear approximations, i.e., one. This result is easily proved using the interpolant $P_N : H^1(\Omega) \to V_N^1$, $v \mapsto \sum_{z\in\mathcal{G}_n} v(z)\psi_z$ [SM03, Ch. 14.4].

4.2 Legendre series expansion

Assume that $v \in L^2(\widehat{\Omega})$. Then with $\alpha_l := (l + 1/2)\int_{\widehat{\Omega}} L_l v$ one obtains the convergence $\|v - \sum_{l=0}^p \alpha_l L_l\|_{L^2(\widehat{\Omega})} \to 0$ for $p \to \infty$. Especially, Parseval's identity gives

$$\|v\|^2_{L^2(\widehat{\Omega})} = \sum_{l=0}^{\infty} \frac{1}{l + \frac{1}{2}} |\alpha_l|^2.$$

Since $\operatorname{span}\{L_l \,:\, 0 \le l \le p\} = \mathbb{P}_p$, one immediately has

$$\inf_{v_p \in \mathbb{P}_p} \|v - v_p\|^2_{L^2(\widehat{\Omega})} = \sum_{l=p+1}^{\infty} \frac{1}{l + \frac{1}{2}} |\alpha_l|^2,$$

where the minimum is attained for $v_p := \sum_{l=0}^p \alpha_l L_l$. For this choice one obviously has $\|v_p\|_{L^2(\widehat{\Omega})} \le \|v\|_{L^2(\widehat{\Omega})}$ for all $p \in \mathbb{N}$. This motivates the interpolation operator with respect to the integrated Legendre polynomials that is introduced in the next section.

4.3 A projection onto $\mathbb{P}_p$

Let $v \in H^1(\widehat{\Omega})$ and

$$v' = \sum_{l=0}^{\infty} \beta_l L_l$$

be the Legendre series for $v' \in L^2(\widehat{\Omega})$, so that $\beta_l = (l+1/2)\int_{\widehat{\Omega}} L_l v'$. Then define for $p \in \mathbb{N}$

$$\widehat{\Pi}^p v := v(-1) + \sum_{l=1}^{p} \beta_{l-1} \mathbb{L}_l \tag{4.1}$$

with $\mathbb{L}_l$ from (3.10). Then, one has immediately for all $v \in H^1(\widehat{\Omega})$

$$\begin{aligned} &\|(\widehat{\Pi}^p v)'\|_{L^2(\widehat{\Omega})} \le \|v'\|_{L^2(\widehat{\Omega})}, \\ &\widehat{\Pi}^p v(-1) = v(-1) \text{ and } \widehat{\Pi}^p v(1) = v(1), \\ &\widehat{\Pi}^p v = v \quad \text{for all } v \in \mathbb{P}_p. \end{aligned}$$

The first and third statement follows from Section 4.2 while the second statement follows from

$$\begin{aligned} \widehat{\Pi}^p v(1) &= v(-1) + 2\beta_0 = v(-1) + 2\frac{1}{2}\int_{-1}^{1} L_0 v' \\ &= v(-1) + \big(v(1) - v(-1)\big) = v(1). \end{aligned}$$

Note: Using the previously defined basis $\{\widehat{\psi}_l\}_{l\ge 0}$ from Section 3.2.3, the series in (4.1) becomes

$$\widehat{\Pi}^p v = v(-1)\widehat{\psi}_0 + v(1)\widehat{\psi}_1 + \sum_{l=2}^{p} \frac{\beta_{l-1}}{\sqrt{l-\frac{1}{2}}}\widehat{\psi}_l.$$

Lemma 6 (Projection error for $\widehat{\Pi}^p$). *Let $s, p \in \mathbb{N}$ with $1 \le s \le p$. Then there is a linear projection operator $\widehat{\Pi}^p : H^1(\widehat{\Omega}) \to \mathbb{P}_p$ so that for all $v \in H^1(\widehat{\Omega})$*

$$\|v - \widehat{\Pi}^p v\|_{L^2(\widehat{\Omega})} \le \Big\|\frac{v - \widehat{\Pi}^p v}{\sqrt{1-(\,.\,)^2}}\Big\|_{L^2(\widehat{\Omega})} \le \frac{1}{\sqrt{p(p+1)}}\|(v - \widehat{\Pi}^p v)'\|_{L^2(\widehat{\Omega})}$$

and for all $v \in H^{s+1}(\widehat{\Omega})$

$$\|(v - \widehat{\Pi}^p v)'\|_{L^2(\widehat{\Omega})} \le \Big(\frac{(p-s)!}{(p+s)!}\Big)^{1/2}\|v^{[s+1]}\|_{L^2(\widehat{\Omega})}.$$

Proof. Step 1: The Legendre polynomials fulfill the orthogonality relations

$$\int_{\widehat{\Omega}} (1-\xi^2)^s L_l^{[s]}(\xi) L_j^{[s]}(\xi)\,\mathrm{d}\xi = \frac{1}{l+\frac{1}{2}}\frac{(l+s)!}{(l-s)!}\,\delta_{l,j}$$

for $0 \le s \le l, j$.
Proof of step 1: [Sch98, p. 71].
Step 2: Let $w \in H^s(\widehat{\Omega})$ and $w = \sum_{l=0}^{\infty} \alpha_l L_l$, then

$$\int_{\widehat{\Omega}} (1-\xi^2)^s |w^{[s]}(\xi)|^2\,\mathrm{d}\xi = \sum_{l=s}^{\infty} \frac{1}{l+\frac{1}{2}}\frac{(l+s)!}{(l-s)!}|\alpha_l|^2.$$

Proof of step 2: Compute the left hand side using step 1.
Step 3: We take $\widehat{\Pi}^p$ from Section 4.3. Given v, one gets with the previous steps that for

all $s \le p$

$$\begin{aligned}\|(v-\widehat{\Pi}^p v)'\|^2_{L^2(\widehat{\Omega})} &= \Big\|\sum_{l=p}^{\infty}\beta_l L_l\Big\|^2_{L^2(\widehat{\Omega})} = \sum_{l=p}^{\infty}\frac{1}{l+\frac12}|\beta_l|^2\\ &= \sum_{l=p}^{\infty}\frac{1}{l+\frac12}\frac{(l-s)!}{(l+s)!}\frac{(l+s)!}{(l-s)!}|\beta_l|^2\\ &\le \frac{(p-s)!}{(p+s)!}\int_{\widehat{\Omega}}(1-\xi^2)^s|(v')^{[s]}(\xi)|^2\,\mathrm{d}\xi\\ &\le \frac{(p-s)!}{(p+s)!}\int_{\widehat{\Omega}}|v^{[s+1]}|^2.\end{aligned}$$

Step 4: An additional orthogonality relation for the integrated Legendre polynomials is

$$\int_{\widehat{\Omega}}\frac{1}{1-\xi^2}\mathit{I\!L}_{l+1}\mathit{I\!L}_{j+1} = \frac{1}{l+\frac12}\frac{1}{l(l+1)}\delta_{l,j}.$$

Proof of step 4: From the differential equation (3.8) one gets by integration

$$-\mathit{I\!L}_{l+1} = \frac{1}{l(l+1)}(1-\xi^2)L_l'$$

and thus with step 1

$$\begin{aligned}\int_{\widehat{\Omega}}\frac{1}{1-\xi^2}\mathit{I\!L}_{l+1}\mathit{I\!L}_{j+1} &= \frac{1}{l(l+1)}\frac{1}{j(j+1)}\int_{\widehat{\Omega}}(1-\xi^2)L_l'L_j'\\ &= \frac{1}{l+\frac12}\frac{1}{l^2(l+1)^2}\frac{(l+1)!}{(l-1)!}\delta_{l,j}\\ &= \frac{1}{l+\frac12}\frac{1}{l(l+1)}\delta_{l,j}.\end{aligned}$$

Step 5: Using step 4 gives

$$\begin{aligned}\|v-\widehat{\Pi}^p v\|^2_{L^2(\widehat{\Omega})} &\le \int_{\widehat{\Omega}}\frac{1}{1-\xi^2}|v-\widehat{\Pi}^p v|^2 = \int_{\widehat{\Omega}}\frac{1}{1-\xi^2}\Big|\sum_{l=p}^{\infty}\beta_l\mathit{I\!L}_{l+1}\Big|^2\\ &= \sum_{l=p}^{\infty}\frac{1}{l+\frac12}\frac{1}{l(l+1)}|\beta_l|^2 \le \frac{1}{p(p+1)}\|(v-\widehat{\Pi}^p v)'\|^2_{L^2(\widehat{\Omega})},\end{aligned}$$

and this completes the proof. □

Theorem 7 (Projection error for $\Pi^{\boldsymbol{p}}_N$). *Let $\mathcal{G}$ and $\mathcal{K}$ be decompositions as in Section 3.1 and $V^{\boldsymbol{p}}_N$ as in Section 3.2.5 with prescribed polynomial degrees $\boldsymbol{p} = [p_K]_{K\in\mathcal{K}}$. Then there is a linear projection operator $\Pi^{\boldsymbol{p}}_N : H^1(\Omega)\cap H^{\boldsymbol{m}}(\mathcal{K}) \to V^{\boldsymbol{p}}_N$ with properties*

$$\begin{aligned}&\Pi^{\boldsymbol{p}}_N v = v && \text{for all } v\in V^{\boldsymbol{p}}_N,\\ &\Pi^{\boldsymbol{p}}_N v(z) = v(z) && \text{for all } v\in H^1(\Omega) \text{ and all } z\in\mathcal{G},\\ &\|(\Pi^{\boldsymbol{p}}_N v)'\|_{L^2(\Omega)} \le \|v'\|_{L^2(\Omega)} && \text{for all } v\in H^1(\Omega),\end{aligned}$$

so that for $s_K\in\mathbb{N}$ with $0\le s_K\le \min\{m_K-1, p_K\}$ and all $v\in H^1(\Omega)\cap H^{\boldsymbol{m}}(\mathcal{K})$

$$\|v-\Pi^{\boldsymbol{p}}_N v\|^2_{L^2(\Omega)} \le \sum_{K\in\mathcal{K}_n}\big(\frac12 h_K\big)^{2(s_K+1)}\frac{1}{p_K(p_K+1)}\frac{(p_K-s_K)!}{(p_K+s_K)!}\|(v-\Pi^{\boldsymbol{p}}_N v)^{[s_K+1]}\|^2_{L^2(K)} \tag{4.2}$$

and

$$\|(v-\Pi^{\boldsymbol{p}}_N v)'\|^2_{L^2(\Omega)} \le \sum_{K\in\mathcal{K}_n}\big(\frac12 h_K\big)^{2s_K}\frac{(p_K-s_K)!}{(p_K+s_K)!}\|v^{[s_K+1]}\|^2_{L^2(K)}. \tag{4.3}$$

Proof. The proof proceeds elementwise on $\mathcal{K}$. On $K \subset \mathcal{K}$ we define for $v \in H^1(K)$, using the transformation Q_K from Section 3.2.1,

$$\Pi_N^{\boldsymbol{p}} v\lfloor_K := \widehat{\Pi}_N^{\boldsymbol{p}_K}(v \circ Q_K) \circ Q_K^{-1}.$$

By Lemma 6 we have

$$\|(v - \Pi_N^{\boldsymbol{p}} v) \circ Q_K\|^2_{L^2(\widehat{\Omega})} \leq \frac{1}{p_K(p_K+1)} \frac{(p_K - s_K)!}{(p_K + s_K)!} \|(v \circ Q_K)^{[s_K+1]}\|^2_{L^2(\widehat{\Omega})}$$

and after transformation to K

$$\|v - \Pi_N^{\boldsymbol{p}} v\|^2_{L^2(K)} \leq \Big(\frac{1}{2} h_K\Big)^{2(s_K+1)} \frac{1}{p_K(p_K+1)} \frac{(p_K - s_K)!}{(p_K + s_K)!} \|v^{[s_K+1]}\|^2_{L^2(K)}.$$

Summing up yields (4.2) and by the same method of proof we get also (4.3). The remaining assertions follow from the properties of $\widehat{\Pi}^p$ in Section 4.3. □

4.4 Comparison between the h– and p–method

Theorem 8. *Assume uniform refinement in space ($h_K = h$ for all K) and constant polynomial degree ($p_K = p$ for all K) in Theorem 7. Let $v \in H^{m_0}(\Omega)$ for some $m_0 \geq 2$. If we fix $p_0 < m_0$, we obtain*

$$\|(v - \Pi_N^{p_0} v)'\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{4\pi p_0}} \Big(\frac{1}{N}\Big)^{p_0} \|v\|_{H^{m_0}(\Omega)}. \tag{4.4}$$

If we, on the other hand, fix h_0 and vary $p \geq m_0$, we obtain for large p, say $p \geq 2m_0$,

$$\|(v - \Pi_N^{p} v)'\|_{L^2(\Omega)} \leq 2\Big(\frac{2e}{N}\Big)^{m_0-1} \|v\|_{H^{m_0}(\Omega)}. \tag{4.5}$$

The second variant yields the better estimate for given N, since $p_0 < m_0$.

Proof. Applying Theorem 7 to the prescribed situation we get

$$\|(v - \Pi_N^{p_0} v)'\|_{L^2(\Omega)} \leq \Big(\frac{1}{2} h\Big)^s \Big(\frac{(p-s)!}{(p+s)!}\Big)^{1/2} \|v^{[s+1]}\|_{L^2(\Omega)}$$

for $s \in \mathbb{N}$ with $0 \leq s \leq \min\{m_0 - 1, p_0\}$. For the following we use *Stirling's formula*

$$k! = \sqrt{2\pi k}\Big(\frac{k}{e}\Big)^k e^{\mu_k}$$

for $k \to \infty$ with $0 < \mu_k < 1/(12k)$ [1]. If $p_0 < m_0$, we can take $s = p_0$ to obtain

$$\frac{(p_0 - s)!}{(p_0 + s)!} = \frac{1}{(2p_0)!} \leq \frac{1}{\sqrt{4\pi p_0}} \Big(\frac{e}{2p_0}\Big)^{2p_0}$$

and thus

$$\begin{aligned} \|(v - \Pi_N^{p_0} v)'\|_{L^2(\Omega)} &\leq \frac{1}{\sqrt{4\pi p_0}} \Big(\frac{1}{2} h\Big)^{p_0} \Big(\frac{e}{2p_0}\Big)^{p_0} \|u^{[p_0+1]}\|_{L^2(\Omega)} \\ &= \frac{1}{\sqrt{4\pi p_0}} \Big(\frac{e}{4}\Big)^{p_0} \Big(\frac{h}{p_0}\Big)^{p_0} \|u^{[p_0+1]}\|_{L^2(\Omega)}. \end{aligned}$$

Note that $u^{[p_0+1]}$ exists since $p_0 + 1 \leq m_0$. The number of unknowns is now $N \approx p_0 n \approx p_0/h$, i.e., $h/p_0 \approx 1/N$, and this shows (4.4).

[1] `https://de.wikipedia.org/wiki/Stirlingformel` (26.01.2018).

To show (4.5), we obtain with $s = m_1 := \max\{m_0 - 1, 1\} < p$

$$\frac{(p-m_1)!}{(p+m_1)!} \le 2\frac{\sqrt{p-m_1}}{\sqrt{p+m_1}}\frac{\left(\frac{p-m_1}{e}\right)^{p-m_1}}{\left(\frac{p+m_1}{e}\right)^{p+m_1}} \le 2e^{2m_1}\Big(\frac{1}{p^2-m_1^2}\Big)^{m_1}$$
$$\le 2\Big(\frac{1}{1-(m_1/p)^2}\Big)^{m_1}\Big(\frac{e}{p}\Big)^{2m_1} \le 2\Big(\frac{2e}{p}\Big)^{2m_1}$$

if $p \ge 2m_1$. Thus we get in this case, since $N \approx pn_0 \approx p/h_0$,

$$\|(v - \Pi_N^p v)'\|_{L^2(\Omega)} \le 2\Big(2e\frac{h_0}{p}\Big)^{m_0-1}\|u^{[m_0]}\|_{L^2(\Omega)} \le 2\Big(\frac{2e}{N}\Big)^{m_0-1}\|u^{[m_0]}\|_{L^2(\Omega)}.$$

□

4.5 Interpolation estimates in case of certain singular functions

As a prototype example for a singular function one can consider the function

$$u : [0,1] \to \mathbb{R}$$
$$u(x) := (x - x_0)_+^\alpha$$

(with $a_+ := \max\{a, 0\}$) for some $x_0 \in [0,1]$ and $\alpha \in (\frac{1}{2}, 1)$. One can prove (and provide suitable definitions) that it makes sense to write (4.5), for constant h and p, as

$$\|(v - \Pi_N^p v)'\|_{L^2(\Omega)} \le C_s\Big(\frac{1}{N}\Big)^{s-1}\|v\|_{H^s(\Omega)}$$

for real (!) $s \in [1, p)$. For the example above

$$\|u\|_{H^s(\Omega)}^2 = \int_{x_0}^1 |x - x_0|^{2(\alpha-s)}\,\mathrm{d}x < \infty$$
$$\text{iff} \quad \alpha - s > -\frac{1}{2} \quad \Longleftrightarrow \quad s < \alpha + \frac{1}{2} \text{ and } s - 1 < \alpha - \frac{1}{2}.$$

Hence the convergence is controlled by

$$\|(u - \Pi_N^p u)'\|_{L^2(\Omega)} \le C_\alpha\Big(\frac{1}{N}\Big)^{\alpha-1/2} \tag{4.6}$$

for $\alpha \in (\frac{1}{2}, 1)$ [Sch98, Sect. 3.3.4]. It can be proved that this estimate can be improved to $(1/N)^{2(\alpha-1/2)}$ if $x_0 \in \mathcal{G}_n$ [Sch98, Sect. 3.3.5].

4.6 Interpolation estimates in case of analytical functions

Theorem 9. *Let $v : \widehat{\Omega} \to \mathbb{R}$ be the restriction of an analytic function $E_r \to \mathbb{C}$, where E_r is, for $r := a + b > 1$, an ellipse with foci $\{\pm 1\}$ and axes length a and b. Then there is $v_p \in \mathbb{P}_p$ and a constant C_r such that*

$$v_p(\pm 1) = v(\pm 1),$$
$$p\|v - v_p\|_{L^2(\widehat{\Omega})} \le \|(v - v_p)'\|_{L^2(\widehat{\Omega})} \le C_r\sqrt{2p}\,e^{-\log(r)p}.$$

Application of this result yields the following: Let $v : \Omega := [0,1] \to \mathbb{R}$ and $\mathcal{K} = \mathcal{K}_1 = \{[0,1]\}$. Then there exists $v_N \in V_N^p$ (i.e., $N = p + 1$) and a constant D_r such that

$$\|(v - v_N)'\|_{L^2(\Omega)} \le D_r e^{-\gamma_r N}, \tag{4.7}$$

that is, the interpolant converges exponentially.

Proof.
Estimate for $v - v_p$: We assign to v the interpolant v_p as in Section 4.3, so that

$$v_p' = \sum_{l=0}^{p-1} \beta_l L_l,$$

with $\beta_l := (l+1/2)\int_{\widehat{\Omega}} L_l v'$ for $l \geq 0$. By assumption on v, one can prove [Dav75, Thm. 12.4.7] that $|\beta_l| \leq C_r(l+1/2)r^{-l}$ for some $C_r > 0$ and from this

$$\begin{aligned}\|(v-v_p)'\|^2_{L^2(\widehat{\Omega})} &= \sum_{l=p}^{\infty} \frac{1}{l+\frac{1}{2}}|\beta_l|^2 \leq C_r^2 \sum_{l=p}^{\infty} \big(l+\frac{1}{2}\big) r^{-2l} = C_r^2 \sum_{l=p}^{\infty} \big(l+\frac{1}{2}\big) e^{-2\log(r)l} \\ &\leq 2C_r^2 \int_p^{\infty} \big(x+\frac{1}{2}\big) e^{-2\log(r)x}\,\mathrm{d}x \leq 2C_r^2 p\, e^{-2\log(r)p}.\end{aligned}$$

The estimate for $\|v - v_p\|_{L^2(\Omega)}$ then follows from Lemma 6.

Estimate for $v - v_N$: We let $v_N := v_p \circ Q^{-1}_{[0,1]}$, where v_p is the approximation of $\widehat{v} = v \circ Q_{[0,1]}$ on $\widehat{\Omega}$. By application of the previous result we first get

$$\|(v-v_N)'\|^2_{L^2(\Omega)} \leq 2\|(\widehat{v}-v_p)'\|^2_{L^2(\widehat{\Omega})} \leq 4C_r^2(N-1)e^{-2\log(r)(N-1)}.$$

The function $s(z) := z\exp(-\kappa z)$ has its global maximum at $z = 1/\kappa$, so that we have

$$(N-1)e^{-\log(r)(N-1)} \leq \frac{1}{e\log(r)}.$$

Hence

$$\|(v-v_N)'\|_{L^2(\widehat{\Omega})} \leq \frac{2C_r}{\sqrt{e\log(r)}}\, e^{-\frac{1}{2}\log(r)(N-1)} \leq \frac{2C_r\sqrt{r}}{\sqrt{\log(r)}}\, e^{-\frac{1}{2}\log(r)N},$$

and our assertion holds with $D_r := 2C_r\sqrt{r}/\sqrt{\log(r)}$ and $\gamma_r := \frac{1}{2}\log(r)$. □

4.7 The *hp* finite element method

A priori meshes Typical problems may have singular points, but are piecewise analytic. Since interpolation estimates are local we may vary the local cell sizes h_K and local polynomial degrees p_K to get the best possible estimates (combing the two previous considerations). Such an optimisation has been considered for the case $u(x) = x^\alpha$. The result is: Use a geometrically refined (towards 0) grid $\mathcal{G}_n := \{0, q^n, q^{n-1}, \ldots, q, 1\}$ with some factor $q \in (0,1)$ and use polynomial degrees $p_k := \lfloor 1 + s(k-1) \rfloor$ for $k = 1, \ldots, n+1$, linearly growing from left to right with slope $s > 0$. Then it holds

$$\|(u-u_N)'\|_{L^2(\widehat{\Omega})} \leq Ce^{-\gamma\sqrt{N}}.$$

An explicit example is $\alpha = 0.7$, where $s = 2\alpha - 1 = 0.4$, $q = (\sqrt{2}-1)^2 \approx 0.7$, $\gamma \approx 0.788$ [BG96] [Sch98, Sect. 3.3.6].

Grids of these types can also used for problems in $\mathbb{R}^2$, $\mathbb{R}^3$. On such grids one can prove error estimates with $\mathcal{O}(e^{-\gamma N^\beta})$, $\beta \in \{1/5, 1/4\}$. However, it is not yet clear whether these results are optimal.

A posteriori meshes It is possible to define an a posteriori controlled algorithm that chooses among possibilities to perform local h or p refinement. The convergence in sense of error decay has been proved for 1D–3D Poisson problems [DH07] [BD11]. In the 1D situation the optimal solution mentioned above could be recovered. An optimality result for an adaptive algorithm in 1D is recently published in [CNV13]. The cases 2D and 3D are open.

4.8 References

This section is based on [Sch98, Sect. 3.3].

5 Numerical examples

We compute the solution of a model problem with $a = 1$, $b = c = 0$. f is defined such that $u(x) = x(1-x)\exp(6x)$. We run the problem on different meshes from $n = 10$ to $n = 160$ and $p = 1, \ldots, 6$. Figure 3 shows N vs the H^1-error in a double-logarithmic plot. The lines have increasing slope from 1 to 6.

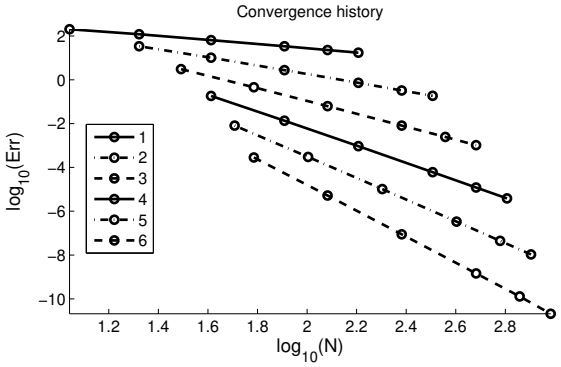

Figure 3: N vs the H^1-error in a double-logarithmic plot.

If we run the *hp*-adaptive code for the case $u(x) = x^\alpha$, $\alpha = 0.7$ (see Section 4.7), we get the optimal polynomial distribution in Figure 4.

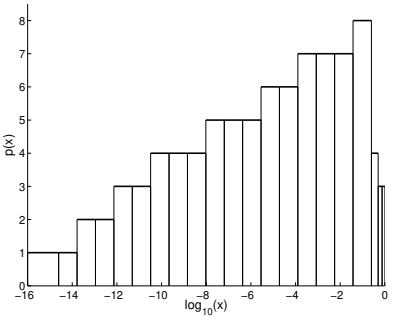

Figure 4: Distance from 0 (logarithmic) vs the polynomial degree.

References

[Alt99] H. W. Alt. *Lineare Funktionalanalysis. Eine anwendungsorientierte Einführung (3. überarb. und erw. Auflage).* Springer, Berlin, 1999.

[Alt16] H. W. Alt. *Linear Functional Analysis. An Application-Oriented Introduction.* Springer, Berlin, 2016.

[BD11] M. Bürg and W. Dörfler. Convergence of an adaptive *hp* finite element strategy in higher space dimensions. *Appl. Numer. Math.*, 61(11):1132–1146, 2011.

[BG96] I. Babuška and B. Q. Guo. Approximation properties of the *h-p* version of the finite element method. *Comput. Methods Appl. Mech. Engrg.*, 133:319–346, 1996.

[BS94] S. C. Brenner and L. R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, 1994.

[CNV13] C. Canuto, R. H. Nochetto, and M. Verani. Contraction and optimality properties of adaptive Legendre-Galerkin methods: The 1-dimensional case, 2013. arXiv:1206.5524.

[Dav75] P. J. Davis. *Interpolation and approximation.* Dover Books on Advanced Mathematics. Dover Publications, New York, 2 edition, 1975.

[DH07] W. Dörfler and V. Heuveline. Convergence of an adaptive *hp* finite element strategy in one space dimension. *Appl. Numer. Math.*, 57(10):1108–1124, 2007.

[GR05] C. Großmann and H.-G. Roos. *Numerik partieller Differentialgleichungen.* B. G. Teubner, Wiesbaden, 3 edition, 2005.

[KA02] P. Knabner and L. Angermann. *Numerical methods for elliptic and parabolic partial differential equations*, volume 44 of *Texts in Applied Mathematics*. Springer, New York, 2002.

[LT03] S. Larsson and V. Thomée. *Partial differential equations with numerical methods*, volume 45 of *Texts in Applied Mathematics*. Springer, Berlin, 2003.

[Sch98] C. Schwab. *p- and hp-finite element methods. Theory and applications in solid and fluid mechanics.* Clarendon Press, Oxford, 1998.

[SM03] E. Süli and D. Mayers. *An Introduction to Numerical Analysis.* Cambridge University Press, Cambridge, 2003.