Numerical Methods for Partial Differential Equations

• Some theoretical background on linear elliptic equations •

Lecture notes by

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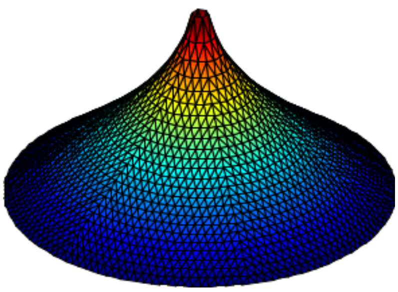

Graph of the function $\log(\log(1/|x|))$ on $B_{1/e}(0)$.

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Contents

1 Function spaces

$\Omega \subset \mathbb{R}^d$ ($d \geq 1$) will always denote a bounded *domain* (i.e., an open and connected set).

1.1 Hölder spaces

The *(vector) space of continuous functions* is defined by

$$\mathcal{C}^0(\overline{\Omega}) \equiv \mathcal{C}^0(\overline{\Omega}, \mathbb{R}) := \Big\{ v : \overline{\Omega} \to \mathbb{R} \,:\, v \text{ is continuous} \Big\}.$$

For mappings $v : \overline{\Omega} \to \mathbb{R}^m$ with $m > 1$, we define $\mathcal{C}^0(\overline{\Omega})^m \equiv \mathcal{C}^0(\overline{\Omega}, \mathbb{R}^m)$ componentwise. As a norm on $\mathcal{C}^0(\overline{\Omega})^m$ we define

$$\|v\|_{\mathcal{C}^0(\overline{\Omega})^m} := \sup_{x \in \overline{\Omega}} \big\{ |v(x)| \big\},$$

where $|\,.\,|$ is a suitable vector norm. For $k \geq 1$ let

$$\mathcal{C}^k(\overline{\Omega}) := \Big\{ v \in \mathcal{C}^0(\overline{\Omega}) \,: v \text{ is } k\text{-times continuously differentiable in } \Omega \\ \text{and all } k\text{-th derivatives can be continuously extended to } \overline{\Omega} \Big\}.$$

Then, for $k \geq 0$, the normed spaces

$$\Big(\mathcal{C}^k(\overline{\Omega}),\ \|\,.\,\|_{\mathcal{C}^k(\overline{\Omega})} := \max_{0 \leq l \leq k} \big\{ \|\boldsymbol{\nabla}^l(\,.\,)\|_{\mathcal{C}^0(\overline{\Omega})^{d^l}} \big\} \Big)$$

are *Banach*[1] *spaces* (complete normed spaces). For $\alpha \in (0,1]$ we define the semi-norm

$$\big[v\big]_{\alpha;\Omega} := \sup_{x,y \in \Omega \,:\, x \neq y} \left\{ \frac{|v(x) - v(y)|}{|x - y|^\alpha} \right\}$$

and the norm

$$\|v\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega})} := \|v\|_{\mathcal{C}^0(\overline{\Omega})} + \big[v\big]_{\alpha;\Omega}$$

and let

$$\mathcal{C}^{0,\alpha}(\overline{\Omega}) := \Big\{ v \in \mathcal{C}^0(\overline{\Omega}) \,:\, \|v\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega})} < \infty \Big\}.$$

For $k \geq 1$ we then define

$$\mathcal{C}^{k,\alpha}(\overline{\Omega}) := \Big\{ v \in \mathcal{C}^k(\overline{\Omega}) \,:\, \text{All partial derivatives of order } k \text{ are in } \mathcal{C}^{0,\alpha}(\overline{\Omega}) \Big\}.$$

The normed spaces

$$\Big(\mathcal{C}^{k,\alpha}(\overline{\Omega}),\ \|\,.\,\|_{\mathcal{C}^{k,\alpha}(\overline{\Omega})} := \|\,.\,\|_{\mathcal{C}^k(\overline{\Omega})} + \big[\boldsymbol{\nabla}^k(\,.\,)\big]_{\alpha;\Omega} \Big) \qquad (\textit{Hölder}^{2}\ \textit{space})$$

are Banach spaces. In case of infinitely many existing derivatives we define

$$\mathcal{C}^\infty(\Omega) := \Big\{ v : \Omega \to \mathbb{R} \,:\, v \text{ is infinitely often continuously differentiable in } \Omega \Big\}$$

and

$$\mathcal{C}^\infty(\overline{\Omega}) := \Big\{ v \in \mathcal{C}^\infty(\Omega) \,:\, v \text{ and all its derivatives can be continuously extended to } \overline{\Omega} \Big\}.$$

Both spaces are not normed [Rud78, Ch. 1.46].

[1] Stefan Banach (1892–1945), Polish mathematician.
[2] Otto Hölder (1859–1937), German mathematician.

Examples

(i) $\alpha = 1$: Functions in $\mathcal{C}^{0,1}(\overline{\Omega})$ are called *Lipschitz*[3] *continuous*. The function $x \mapsto |x|$ is in $\mathcal{C}^{0,1}([-1,1])$ but not in $\mathcal{C}^{1}([-1,1])$.
(ii) $d = 1$, $\Omega := (0,1)$, $v(x) := \sqrt{x}$: $v \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$ for $0 < \alpha \leq 1/2$.

Remark

For $\alpha \in (0,1)$ we have the inclusions

$$\mathcal{C}^{1}(\overline{\Omega}) \subsetneq \mathcal{C}^{0,1}(\overline{\Omega}) \subsetneq \mathcal{C}^{0,\alpha}(\overline{\Omega}) \subsetneq \mathcal{C}^{0}(\overline{\Omega}).$$

$\mathcal{C}^{0,\alpha}(\overline{\Omega})$ is *compactly embedded* in $\mathcal{C}^{0}(\overline{\Omega})$ for $\alpha \in (0,1]$ by the Theorem of Arzelà–Ascoli[4] [DL00, Ch. II, §4], [Heu03, Satz 106.2] (i.e., the *embedding* $Id : \mathcal{C}^{0,\alpha}(\overline{\Omega}) \to \mathcal{C}^{0}(\overline{\Omega})$ is a compact mapping).

1.2 Lebesgue and Sobolev spaces

Here, we further assume that $\partial\Omega$ is Lipschitz continuous (or of class $\mathcal{C}^{0,1}$), i.e., $\partial\Omega$ can be locally represented as a graph of a Lipschitz continuous mapping $D \subset \mathbb{R}^{d-1} \to \partial\Omega$.

1.2.1 Lebesgue spaces

For $p \in [1,\infty]$ we define normed (vector) spaces by

$$\mathcal{L}^p(\Omega) := \Big\{ v : \Omega \to \mathbb{R} \,:\, v \text{ measurable and } \|v\|_{\mathcal{L}^p(\Omega)} < \infty \Big\},$$

where

$$\|v\|_{\mathcal{L}^p(\Omega)} := \begin{cases} \Big(\int_\Omega |v|^p \Big)^{1/p} & \text{for } p \in [1,\infty), \\ \operatorname*{ess\,sup}_{x\in\Omega} \big\{ |v(x)| \big\} & \text{for } p = \infty. \end{cases}$$

Functions in $\mathcal{L}^p(\Omega)$ are defined *pointwise a.e.* only. We have

$$v = w \text{ in } \mathcal{L}^p(\Omega) \quad :\Longleftrightarrow \quad (v-w)(x) = 0 \text{ pointwise a.e. in } \Omega.$$

The normed spaces

$$\Big(\mathcal{L}^p(\Omega), \|\,.\,\|_{\mathcal{L}^p(\Omega)} \Big) \qquad (\textit{Lebesgue}^{5} \textit{ space})$$

are Banach spaces. For $p = 2$ we can define the scalar product (inner product)

$$\big(v, w\big)_{\mathcal{L}^2(\Omega)} := \int_\Omega vw$$

Thus $(v,v)_{\mathcal{L}^2(\Omega)} = \|v\|^2_{\mathcal{L}^2(\Omega)}$ and

$$\Big(\mathcal{L}^2(\Omega), \big(\,.\,,\,.\,\big)_{\mathcal{L}^2(\Omega)} \Big)$$

is a *Hilbert*[6] *space* (complete inner product space).

[3] Rudolf Otto Sigismund Lipschitz (1832–1903), German mathematician.
[4] Giulio Ascoli (1843–1896), Cesare Arzelà (1847–1912), Italian mathematicians.
[5] Henri Léon Lebesgue (1875–1941), French mathematician.
[6] David Hilbert (1862–1943), German mathematician.

Example

Let $\Omega := B_1(0) \subset \mathbb{R}^d$ and $v(x) := |x|^\alpha$. Then

$$v \in \mathcal{L}^p(\Omega) \iff \int_\Omega |v|^p < \infty \iff \int_0^1 r^{\alpha p} r^{d-1}\,\mathrm{d}r < \infty \iff \int_0^1 r^{\alpha p+d-1}\,\mathrm{d}r < \infty$$
$$\iff \alpha p + d - 1 > -1 \iff \alpha > -d/p.$$

Remarks

(i) A function $f \in \mathcal{L}^p(\Omega)$, $p \in [1,\infty]$, does not have well defined values on any smooth lower dimensional submanifold $S \subset \Omega$, since such a set is a zero set for the d-dimensional Lebesgue measure.

(ii) For bounded Ω we have $\mathcal{L}^p(\Omega) \subsetneq \mathcal{L}^q(\Omega)$ for $1 \le q < p \le \infty$. This follows from the estimate

$$\|f\|^q_{\mathcal{L}^q(\Omega)} = \int_\Omega |f|^q \le \Big(\int_\Omega 1\Big)^{1-1/s}\Big(\int_\Omega |f|^{qs}\Big)^{1/s}$$

for $s \ge 1$ (using Hölder's inequality) since taking $s = p/q$ yields

$$\|f\|_{\mathcal{L}^q(\Omega)} \le |\Omega|^{1/q-1/p}\|f\|_{\mathcal{L}^p(\Omega)}.$$

To show strict inclusions one uses counterexamples of the form $v(x) := |x|^\alpha$, see the example above.
For unbounded Ω, there is no inclusion between $\mathcal{L}^p(\Omega)$ and $\mathcal{L}^q(\Omega)$ for $p \ne q$. However, it holds $\mathcal{L}^p(\Omega)\cap\mathcal{L}^\infty(\Omega) \subsetneq \mathcal{L}^q(\Omega)\cap\mathcal{L}^\infty(\Omega)$ for $1 \le p < q \le \infty$ (opposite to the above!). This is seen by proving $\|f\|_{\mathcal{L}^q(\Omega)} \le \|f\|^{1-p/q}_{\mathcal{L}^\infty(\Omega)}\|f\|^{p/q}_{\mathcal{L}^p(\Omega)}$ (show this for $g := f/\|f\|_{\mathcal{L}^\infty(\Omega)}$ exploiting $|g| \le 1$).

1.2.2 Weak differentiability

We weaken the concept of differentiability. Let

$$\mathcal{L}^1_{\mathrm{loc}}(\Omega) := \Big\{ v : \Omega \to \mathbb{R} \,:\, v \in \mathcal{L}^1(K) \text{ for all } K \subset\subset \Omega \Big\}$$

and

$$\mathcal{D}(\Omega) \equiv \mathcal{C}^\infty_{\mathrm{c}}(\Omega) := \Big\{ v \in \mathcal{C}^\infty(\overline{\Omega}) \,:\, \mathrm{supp}(v) \subset \Omega \Big\} \qquad (\textit{test function space}).$$

$v \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$ is called *weakly differentiable*, if there exists a function $g \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)^d$ such that

$$\int_\Omega v \cdot \boldsymbol{\nabla} w = -\int_\Omega g w \qquad \text{for all } w \in \mathcal{D}(\Omega).$$

g is then uniquely defined and for $v \in \mathcal{C}^1(K)$ one obtains $g = \boldsymbol{\nabla} v$ on K for all open $K \subset \Omega$. We call g the *weak derivative* of v and write $g = \boldsymbol{\nabla} v$ as in the case of the classical derivative. Higher order weak derivatives are defined analogously.

Examples

(i) $d = 1$, $\Omega := (-1,1)$, $v(x) := |x|$. What is v'? We claim:

$$v'(x) = \mathrm{sign}(x) := \begin{cases} -1 & \text{for } x < 0, \\ 0 & \text{for } x = 0, \\ 1 & \text{for } x > 0. \end{cases}$$

Indeed, applying the definition for arbitrary $w \in \mathcal{D}(\Omega)$ yields after integration by parts

$$\int_\Omega vw' = \int_{-1}^1 |x|w' = -\int_{-1}^0 xw' + \int_0^1 xw' = \int_{-1}^0 w - \int_0^1 w$$
$$= -\Big(\int_{-1}^0 (-1)w + \int_0^1 (+1)w\Big) = -\int_0^1 \operatorname{sign}(x)w = -\int_\Omega v'w.$$

(ii) Discontinuous functions on $\mathbb{R}$ do not have weak derivatives! Let $\Omega := (-1,1)$ and consider the function

$$\chi_{(0,\infty)}(x) := \begin{cases} 0 & \text{for } x \le 0, \\ 1 & \text{for } x > 0. \end{cases}$$

We show that $\chi_{(0,\infty)}$ does not have a weak derivative. If a weak derivative $g \in \mathcal{L}^1_{\text{loc}}(-1,1)$ would exist, it has to vanish a.e. and thus

$$0 = \int_{-1}^1 gw \overset{!}{=} -\int_{-1}^1 \chi_{(0,\infty)} w' = -\int_0^1 w' = w(0).$$

However, $w(0) = 0$ does not hold for *arbitrary* $w \in \mathcal{D}(-1,1)$.

Remark

The derivative of $\chi_{(0,\infty)}$ can be defined as a member of $\mathcal{C}^0(\overline{\Omega})'$, the *dual space* of $\mathcal{C}^0(\overline{\Omega})$. Thus it is a linear continuous mapping $\chi'_{(0,\infty)} : \mathcal{C}^0(\overline{\Omega}) \to \mathbb{R}$ satisfying

$$\chi'_{(0,\infty)}[v] = v(0) \qquad \text{for all } v \in \mathcal{C}^0(\overline{\Omega})$$

and therefore obeys the bound

$$\big|\chi'_{(0,\infty)}[v]\big| \le \|v\|_{\mathcal{C}^0(\Omega)}.$$

The mapping $\delta_0 := \chi'_{(0,\infty)}$ is called *Dirac*[7] *distribution in* 0.

1.2.3 Sobolev spaces

For $p \in [1,\infty]$ we define

$$\mathcal{W}^{1,p}(\Omega) := \Big\{ v \in \mathcal{L}^p(\Omega) \,:\, v \text{ is weakly differentiable and } |\boldsymbol{\nabla} v| \in \mathcal{L}^p(\Omega) \Big\}.$$

For $m > 1$ correspondingly

$$\mathcal{W}^{m,p}(\Omega) := \Big\{ v \in \mathcal{W}^{m-1,p}(\Omega) \,:\, \text{All partial derivatives of order } m-1 \text{ are weakly differentiable and } |\boldsymbol{\nabla}^m v| \in \mathcal{L}^p(\Omega) \Big\}.$$

For convenience we let $\mathcal{W}^{0,p}(\Omega) := \mathcal{L}^p(\Omega)$. With

$$\|v\|_{\mathcal{W}^{m,p}(\Omega)} := \Big| \big[\|\boldsymbol{\nabla}^l v\|_{\mathcal{L}^p(\Omega)^{d^l}} \big]_l \Big|_{\ell^p}$$

the normed space

$$\Big(\mathcal{W}^{m,p}(\Omega), \|\,.\,\|_{\mathcal{W}^{m,p}(\Omega)} \Big) \qquad (\textit{Sobolev}^{8} \textit{ space})$$

[7] Paul Dirac (1902–1984), British physicist.

is a Banach space. For $p = 2$ one can define the scalar product

$$(v, w)_{\mathcal{W}^{m,2}(\Omega)} := \sum_{l=0}^{m} (\nabla^l v, \nabla^l w)_{\mathcal{L}^2(\Omega)^{d^l}}$$

and obtain the Hilbert space

$$\Big(\mathcal{H}^m(\Omega), (\,.\,,\,.\,)_{\mathcal{H}^m(\Omega)}\Big) := \Big(\mathcal{W}^{m,2}(\Omega), (\,.\,,\,.\,)_{\mathcal{W}^{m,2}(\Omega)}\Big).$$

Clearly, $\mathcal{C}^m(\overline{\Omega}) \subset \mathcal{W}^{m,p}(\Omega)$ and $\mathcal{W}^{m,p}(\Omega) \subset \mathcal{W}^{n,p}(\Omega)$ for all $m > n$ and all $p \in [1, \infty]$. $\mathcal{W}^{m,p}(\Omega)$ can also be characterized by

$$\mathcal{W}^{m,p}(\Omega) = \overline{\mathcal{C}^\infty(\overline{\Omega})}^{\|\,.\,\|_{\mathcal{W}^{m,p}(\Omega)}}.$$

Especially, this means that $\mathcal{C}^\infty(\overline{\Omega})$ *is dense in* $\mathcal{W}^{m,p}(\Omega)$: For each $w \in \mathcal{W}^{m,p}(\Omega)$ there is a sequence $\{w_j\}_{j\in\mathbb{N}} \subset \mathcal{C}^\infty(\overline{\Omega})$ such that $w_j \to w$ with respect to $\|\,.\,\|_{\mathcal{W}^{m,p}(\Omega)}$ for $j \to \infty$. This approximation argument is very often used when dealing with Sobolev functions.

Example

Let $\Omega := B_1(0) \subset \mathbb{R}^d$ and $v(x) := |x|^\alpha$. For which α holds $v \in \mathcal{W}^{m,p}(\Omega)$? By definition we have

$$\|v\|_{\mathcal{W}^{m,p}(\Omega)} < \infty \quad \iff \quad \|\nabla^l v\|_{\mathcal{L}^p(\Omega)} < \infty \text{ for } l = 0, \dots, m.$$

Thus we obtain

$$\|\nabla^l v\|_{\mathcal{L}^p(\Omega)} \sim \int_0^1 |r^{\alpha-l}|^p r^{d-1}\,\mathrm{d}r = \int_0^1 r^{(\alpha-l)p+d-1}\,\mathrm{d}r \le \infty$$
$$\iff \quad (\alpha - l)p + d - 1 > -1 \quad \iff \quad \alpha > l - d/p.$$

In summary, $x \mapsto |x|^\alpha \in \mathcal{W}^{m,p}(\Omega) \Leftrightarrow \alpha > m - d/p$.

1.2.4 Fractional Sobolev spaces

For $s \in \mathbb{R}_{>0}$ let $\lfloor s \rfloor := \max\{m \in \mathbb{N} : m \le s\}$ and define the *fractional Sobolev spaces* (or *Sobolev–Slobodeckij*[9] *spaces*) by

$$\mathcal{W}^{s,p}(\Omega) := \{w : \Omega \to \mathbb{R} \,:\, \|w\|_{\mathcal{W}^{s,p}(\Omega)} < \infty\},$$

where, with $m := \lfloor s \rfloor$,

$$\|w\|^p_{\mathcal{W}^{s,p}(\Omega)} := \|w\|^p_{\mathcal{W}^{m,p}(\Omega)} + \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x-y|^{(s-m+d/p)p}}\,\mathrm{d}x\,\mathrm{d}y.$$

This is, for $s \ge 0$ and $p \in (1, \infty)$, a separable and reflexive Banach space or a Hilbert space, if $p = 2$. $\mathcal{C}^\infty(\overline{\Omega})$ is dense in $\mathcal{W}^{s,p}(\Omega)$. $\mathcal{W}_0^{s,p}(\Omega)$ is defined to be the closure of $\mathcal{D}(\Omega)$ with respect to $\|\,.\,\|_{\mathcal{W}^{s,p}(\Omega)}$ and the vector valued function space is defined componentwise.

An alternative norm equivalent construction for $p = 2$ is as follows: let E be a continuous *extension operator* $E : f : \Omega \to \mathbb{R} \;\mapsto\; Ef : \mathbb{R}^d \to \mathbb{R}$ (see e.g. [BS94, Ch. 14] [EG04, Ch. B.3.2]) and let

$$\|f\|^2_{\mathcal{H}^s(\Omega)} := \int_{\mathbb{R}^d} |\widehat{E}f(\xi)|^2 \big(1 + |\xi|^2\big)^s\,\mathrm{d}\xi.$$

Details can be found in [Wlo82, §3] (or its English translation [Wlo87]). Such spaces are also defined for example in [BS94, Ch. 14] and [EG04, Ch. B.3.1].

[8] Sergei Lwowitsch Sobolev (1908–1989), Russian mathematician.
[9] Lev Naumovich Slobodeckij (1914–1976), Russian mathematician. Publication from 1964.

Remark

The vector spaces presented in Sections 1.1 and 1.2 are of infinite dimension. We consider for example $\mathcal{L}^2(0,1)$. Then the set $\mathcal{B} := \{f_j\}_{j\in\mathbb{N}}$ defined by $f_j(x) := \cos(j\pi x)$ satisfies

$$\big(f_i, f_j\big)_{\mathcal{L}^2(\Omega)} = \frac{1}{2}\delta_{ij} \qquad \text{for all } i,j \in \mathbb{N}.$$

Thus, each finite subset of the unbounded set $\mathcal{B} \subset \mathcal{L}^2(0,1)$ is linearly independent.

1.3 Properties of Sobolev functions

In the following we assume that $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is a bounded domain with $\partial\Omega$ assumed to be Lipschitz continuous.

1.3.1 Embedding theorems

(i) Let $m, l \in \mathbb{N}$ and $p \in (1,\infty)$, $q \in (1,\infty]$. If $m \geq l$ and $m - d/p > l - d/q$, then $\mathcal{W}^{m,p}(\Omega)$ in *continuously embedded* in $\mathcal{W}^{l,q}(\Omega)$, hence

$$\|v\|_{\mathcal{W}^{l,q}(\Omega)} \leq C\|v\|_{\mathcal{W}^{m,p}(\Omega)} \qquad \text{for all } v \in \mathcal{W}^{m,p}(\Omega)$$

with a constant C independent of v. The embedding is compact if the first inequality is strict, i.e. $m > l$. $m - d/p$ is called the *Sobolev number* of $\mathcal{W}^{m,p}(\Omega)$ (see the example in Sect. 1.2.3).

(ii) If $m - d/p > k + \alpha$, then $\mathcal{W}^{m,p}(\Omega)$ is continuously (even compactly) embedded in $\mathcal{C}^{k,\alpha}(\overline{\Omega})$.

Example

Let $v \in \mathcal{W}^{m,p}(\Omega)$. Is v a bounded function, i.e., holds $v \in \mathcal{L}^\infty(\Omega)$? By (i), we have to verify $m - d/p > 0$. Let for example $m = 1$ and $p = 2$.

- $d = 1$: $m > 1/2$ is satisfied, hence $\|v\|_{\mathcal{L}^\infty(\Omega)} \leq C\|v\|_{\mathcal{W}^{1,2}(\Omega)}$.
- $d = 2$: $m > 1$ is not satisfied, hence there are functions in $\mathcal{W}^{1,2}(\Omega)$ that are not bounded! Indeed, let $\Omega := B_{1/e}(0) \subset \mathbb{R}^2$ and $v(x) := \log\big(\log(1/|x|)\big)$. Then $v \in \mathcal{W}_0^{1,2}(\Omega)$, but v is unbounded.
- $d = 3$: $m > 3/2$ is not satisfied, an unbounded function in $\mathcal{W}^{1,2}(\Omega)$ is $x \mapsto |x|^\alpha$ for $0 > \alpha > 1 - 3/2 = -1/2$.

1.3.2 Traces

Functions in $\mathcal{W}^{m,p}(\Omega)$ have "traces" on lower dimensional manifolds in the following sense: If $S \subset \Omega$ is a $d-r$–dimensional Lipschitz continuous hypersurface and $m > l$, $m - d/p > l - (d-r)/q$, then there exists a continuous linear mapping

$$\gamma_S : \mathcal{W}^{m,p}(\Omega) \to \mathcal{W}^{l,q}(S) \qquad (\textit{trace operator})$$

with $\gamma_S[v] = v\lfloor_S$ for all $v \in \mathcal{C}^\infty(\overline{\Omega})$ and

$$\|\gamma_S[v]\|_{\mathcal{W}^{l,q}(S)} \leq C\|v\|_{\mathcal{W}^{m,p}(\Omega)} \qquad \text{for all } v \in \mathcal{W}^{m,p}(\Omega)$$

with some constant C that is independent of v. In this sense, we will in the following use the notation $\gamma_S[v] = v\lfloor_S$ ("in trace sense") for all $v \in \mathcal{W}^{m,p}(\Omega)$. Note also, that $\gamma_{\partial\Omega}$ is surjective in this case: for each $g \in \mathcal{W}^{r,q}(\partial\Omega)$ there is a $v \in \mathcal{W}^{m,p}(\Omega)$ such that $\gamma_{\partial\Omega}v = g$.

Example

Let $v \in \mathcal{W}^{1,2}(\Omega)$. Then $v\lfloor_{\partial\Omega} \in \mathcal{L}^2(\partial\Omega)$ (in trace sense), since $1 - d/2 = 1/2 + (1-d)/2 > -(d-1)/2$.

1.3.3 Poincaré–Friedrichs inequality

Since functions in $\mathcal{W}^{1,p}(\Omega)$ admit boundary values in trace sense (see Sect. 1.3.2), we obtain a closed subspace of $\mathcal{W}^{1,p}(\Omega)$ by

$$\mathcal{W}_0^{1,p}(\Omega) := \ker(\gamma\lfloor_{\partial\Omega}) = \{v \in \mathcal{W}^{1,p}(\Omega) : v\lfloor_{\partial\Omega} = 0\}.$$

It can be equivalently characterised (on bounded domains) by

$$\mathcal{W}_0^{1,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{\|\,.\,\|_{\mathcal{W}^{1,p}(\Omega)}}.$$

For $p = 2$ we use also the notation $\mathcal{H}_0^1(\Omega) = \mathcal{W}_0^{1,2}(\Omega)$. The *Poincaré–Friedrichs*[10] *inequality* states that

$$\|v\|_{\mathcal{L}^p(\Omega)} \le d_\Omega \|\nabla v\|_{\mathcal{L}^p(\Omega)} \qquad \text{for all } v \in \mathcal{W}_0^{1,p}(\Omega),$$

with d_Ω being the smallest diameter of Ω. As a result, $\|\nabla\,.\,\|_{\mathcal{L}^p(\Omega)}$ is a norm equivalent to $\|\,.\,\|_{\mathcal{W}^{1,p}(\Omega)}$, i.e., there is a constant $c > 0$ such that

$$c\|v\|_{\mathcal{W}^{1,p}(\Omega)} \le \|\nabla v\|_{\mathcal{L}^p(\Omega)} \le \|v\|_{\mathcal{W}^{1,p}(\Omega)} \qquad \text{for all } v \in \mathcal{W}_0^{1,p}(\Omega).$$

Obviously, we can take $c = 1/(1 + d_\Omega)$ here.

Remark

(i) Let $p = 2$. The optimal constant in the Poincaré–Friedrichs inequality is given by $1/C_\mathrm{P}$, where

$$C_\mathrm{P} := \inf_{v \in \mathcal{H}_0^1(\Omega)} \left\{ \frac{\|\nabla v\|_{\mathcal{L}^2(\Omega)}}{\|v\|_{\mathcal{L}^2(\Omega)}} \right\} \qquad (\textit{Poincaré constant}).$$

One can show that this minimum is attained and that C_P is the *smallest eigenvalue* of $-\Delta u$ in Ω, i.e., the smallest (positive) number Λ such that the equation $-\Delta u = \Lambda u$ in Ω admits a non–trivial solution u with $u\lfloor_{\partial\Omega} = 0$.

(ii) Estimates of the same form as the Poincaré–Friedrichs inequality hold if $v\lfloor_S = 0$ is imposed for some measurable set $S \subset \partial\Omega$ with $\mathrm{meas}_{d-1}(S) \neq 0$. The constant, however, is then different and depends on S and Ω [Alt85, Ch. 5.15].

1.3.4 Remark

For $m > 0$ and $p \in (1, \infty)$, the dual space of $\mathcal{W}^{m,p}(\Omega)$ is $\big(\mathcal{W}^{m,p}(\Omega)\big)' = \mathcal{W}^{-m,p'}(\Omega)$ with $p' = p/(p-1)$. We thus have Sobolev spaces $\mathcal{W}^{s,q}(\Omega)$ for $s \in \mathbb{R}$ and $q \in [1, \infty]$.

Bibliographical notes. Definitions and properties of all the mentioned function spaces are contained in many modern books on theory and numerics of partial differential equations, for example [Alt85], [GT98], [BS94], [EG04], [Eva10], [GR05].

[10] Jules Henri Poincaré (1854–1912), French mathematician. Kurt Otto Friedrichs (1901–1982), German–American mathematician.

2 Linear elliptic equations

2.1 Physical motivation

2.1.1 Flux balance

Let $\Omega \subset \mathbb{R}^d$, for $d \geq 1$, be a bounded domain. Inside Ω there is a *flux* $q : \Omega \to \mathbb{R}^d$ of a certain quantity, e.g., energy or mass. The total flow of q through a volume $V \subset \Omega$ is given by $\int_{\partial V} q \cdot n_{\partial V}$, where $n_{\partial V}$ is the outer normal field on ∂V. Inside V there might be a source or a sink for the flowing quantity of strength f. The total production (if positive) or loss (if negative) is $\int_V f$. Assuming conservation of the flowing quantity we require

$$\int_{\partial V} q \cdot n_{\partial V} = \int_V f$$

and by Gauß'[11] theorem

$$\int_V \boldsymbol{\nabla} \cdot q = \int_V f.$$

The quantity $\boldsymbol{\nabla} \cdot q \equiv \operatorname{div}(q) := \sum_{i=1}^d \partial_i q_i$ is called "divergence". Since $V \subset \Omega$ is an arbitrary open set, we conclude

$$\boldsymbol{\nabla} \cdot q = f \qquad \text{in } \Omega.$$

In case the flux q is driven by spatial differences between the flowing quantity u, one often assumes *Fourier's law*

$$q = -a \boldsymbol{\nabla} u \qquad \text{in } \Omega.$$

Here, $a : \Omega \to \mathbb{R}_+$ is a material property describing the mobility of the flow. Thus we arrive at the *flux balance equation* or *diffusion equation*

$$-\boldsymbol{\nabla} \cdot \big(a \boldsymbol{\nabla} u\big) = f \qquad \text{in } \Omega. \tag{1}$$

Generalisations:

- Include transport by a flow field $b : \Omega \to \mathbb{R}^d$ with $\boldsymbol{\nabla} \cdot b = 0$ (in many cases), hence $q = \boldsymbol{\nabla} \cdot (bu) = b \cdot \boldsymbol{\nabla} u$;

- Include a linear reaction term $f = -cu$ with $c : \Omega \to \mathbb{R}$ and $c > 0$ describing consumption of u;

- a may be a matrix field: $a : \Omega \to \mathbb{R}^{d,d}$ with $a(x)z \cdot z > 0$ for all $z \in \mathbb{R}^d$ and all $x \in \Omega$.

The general linear balance equation has then the form

$$-\boldsymbol{\nabla} \cdot \big(a \boldsymbol{\nabla} u\big) + b \cdot \boldsymbol{\nabla} u + cu = f. \tag{1'}$$

[11] Johann Carl Friedrich Gauß (1777–1855), German mathematician, astronomer, geodesist, and physicist.

2.1.2 Boundary conditions

The equation (1) or (1') needs to be accompanied by conditions on the boundary $\partial\Omega$. One might impose values for u on $\partial\Omega$ by a given function $g^{\mathrm{D}} : \partial\Omega \to \mathbb{R}$, hence

$$u = g^{\mathrm{D}} \qquad \text{on } \partial\Omega. \tag{2}$$

This is called *Dirichlet*[12] *boundary condition*. In case of a thermally isolated boundary, there will no flux through $\partial\Omega$, that is

$$n_{\partial\Omega} \cdot (a\nabla u) = 0 \qquad \text{on } \partial\Omega.$$

More generally, we may require

$$n_{\partial\Omega} \cdot (a\nabla u) = g^{\mathrm{N}} \qquad \text{on } \partial\Omega \tag{2'}$$

for a given function $g^{\mathrm{N}} : \partial\Omega \to \mathbb{R}$. (2') is called *Neumann*[13] *boundary condition*. Both conditions may appear at the same time on a *disjoint decomposition* of $\partial\Omega$,

$$\partial\Omega = \Gamma_{\mathrm{D}} \dot{\cup} \Gamma_{\mathrm{N}},$$

where we impose

$$\begin{aligned} u &= g^{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}} \quad (\textit{Dirichlet boundary}),\\ n_{\partial\Omega} \cdot (a\nabla u) &= g^{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}} \quad (\textit{Neumann boundary}). \end{aligned}$$

A pointwise mixture of these two conditions is called *Robin*[14] *boundary condition*

$$n_{\partial\Omega} \cdot (a\nabla u) + \widetilde{c}u = g^{\mathrm{R}} \qquad \text{on } \Gamma_{\mathrm{R}} \quad (\textit{Robin boundary})$$

with $\widetilde{c}, g^{\mathrm{R}} : \partial\Omega \to \mathbb{R}$.

2.2 Classical theory

2.2.1 Existence and regularity for the classical Dirichlet problem

We consider, for given $\Omega, a, b, c, f, g^{\mathrm{D}}$, the boundary value problem (1'), (2). We require that

(i) Regularity assumptions. Assume that for some $\alpha \in (0,1)$

- Ω is bounded and $\partial\Omega$ is a local $\mathcal{C}^{2,\alpha}$-graph,
- $a \in \mathcal{C}^{1,\alpha}(\overline{\Omega})^{d,d}$, $b \in \mathcal{C}^{0,\alpha}(\overline{\Omega})^d$, $c, f \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$,
- $g^{\mathrm{D}} = \widetilde{g}^{\mathrm{D}}\lfloor_{\partial\Omega}$ for some $\widetilde{g}^{\mathrm{D}} \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$.

(ii) Structural conditions.

- *Uniform ellipticity* $a : \Omega \to \mathbb{R}^{d,d}$ is symmetric and uniformly positive definit

  $$z \cdot a(x)z \ge a_0|z|^2 \qquad \text{for all } z \in \mathbb{R}^d \text{ and for all } x \in \Omega$$

  with some constant $a_0 > 0$ (a is called *uniformly elliptic*),
- $c \ge 0$.

Then there is a unique solution $u \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$ of the boundary value problem (1'), (2) and this solution depends continuously on the data in the respective topology [GT98, Thm. 6.14]. Assuming, for $k \ge 0$, $\partial\Omega \in \mathcal{C}^{2+k,\alpha}$, $a \in \mathcal{C}^{1+k,\alpha}(\overline{\Omega})^{d,d}$, $b \in \mathcal{C}^{k,\alpha}(\overline{\Omega})^d$, $c, f \in \mathcal{C}^{k,\alpha}(\overline{\Omega})$, and $\widetilde{g}^{\mathrm{D}} \in \mathcal{C}^{2+k,\alpha}(\overline{\Omega})$ in the regularity requirements yields a solution $u \in \mathcal{C}^{2+k,\alpha}(\overline{\Omega})$ [GT98, Thm. 6.19].

[12] Johann Peter Gustav Lejeune Dirichlet (1805–1859), German mathematician.
[13] Carl Gottfried Neumann (1832–1925), German mathematician.
[14] Victor Gustave Robin (1855–1897), French mathematician.

2.2.2 Some counterexamples

(i) If $\partial\Omega$ is not differentiable, then u may not be in $\mathcal{C}^2(\overline{\Omega})$. Let $\Omega := (0,1)^2$, $f := 0$ and $g^{\mathrm{D}}(x_1, x_2) := x_1^2$. Then, necessarily, $\partial_1^2 u(x_1, 0) = 2$ and $\partial_2^2 u(0, x_2) = 0$ and therefore the second derivatives cannot be continuous in $(0, 0)$.

(ii) Let $\Omega \subset \mathbb{R}^2$ be a domain with an *obtuse interior angle*, i.e., an interior angle of more than π. For example, let Ω be a segment of the unit disk with interior angle $\beta\pi$ for some $\beta \in (1, 2)$ and choose $f := 0$ and $\widetilde{g}^{\mathrm{D}}(x) := r^{1/\beta} \sin(\phi/\beta)$ in polar coordinates (r, ϕ). The unique solution of this problem is $u(x) = \widetilde{g}^{\mathrm{D}}(x)$ for $x \in \Omega$. However, $\boldsymbol{\nabla} u$ is unbounded in 0. It is said that u has a *corner singularity*. In three space dimensions there will be in addition also edge singularities. There exists a detailed theory that describes the behaviour of solutions near exceptional parts of $\partial\Omega$ for two and three space dimensions [Gri85] [KS87] [Dau88].

(iii) If Ω is unbounded, then a condition at "∞" is needed. A typical condition is $u(x) \to 0$ as $|x| \to \infty$. The existence and uniqueness theory is much more complicated than for bounded domains [MS60] [McO81], see Sect. 2.3.6.

(iv) If $f \in \mathcal{C}^0(\overline{\Omega})$, then u needs not to be in $\mathcal{C}^2(\overline{\Omega})$ and if $f \in \mathcal{C}^1(\overline{\Omega})$, then u needs not to be in $\mathcal{C}^{2,1}(\overline{\Omega})$ [GT98, Exercise 4.9].

(v) If a has zeros, there will be in general no solution at all or no solution in $\mathcal{W}_0^{1,2}(\Omega)$. As an example one may consider the equation $-(au')' = f$ with homogeneous boundary conditions for $a(x) := 0$ or $a(x) := x^m$ (for $m \in \mathbb{N}$) and $\Omega = (-1, 1)$. For examples in two space dimensions see [GT98, Ch. 6.6].

(vi) In case of $c < 0$ uniqueness may not hold. Consider for example the *eigenvalue problem*

$$\begin{aligned} -\Delta u - \lambda u &= 0 \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

There is a monotone increasing sequence of positive numbers $\{\lambda_j\}_{j\in\mathbb{N}}$ (with $\lambda_j \to \infty$ for $j \to \infty$) for which non-zero solutions u exist (see Sect. 2.6). For each such u we have the additional solutions su for all $s \in \mathbb{R}$.

(vii) If a is discontinuous, the problem cannot be stated as in (1'). This issue is treated in the next section.

2.3 Weak solutions

Low regularity, as in the examples of Sect. 2.2.2, is a typical problem in applications. For example, a may be discontinuous or $\partial\Omega$ may have corners and edges (i.e., $\partial\Omega \notin \mathcal{C}^{2,\alpha}$). We thus need a concept to formulate the boundary value problem (1'), (2) for much weaker regularity conditions on the data. Of course, this will lead to a weaker solution concept.

2.3.1 Weak formulation of the Dirichlet problem

We multiply the equation (1') by an arbitrary $v \in \mathcal{D}(\Omega)$ and integrate by parts. Hence the solution u of (1'), (2) satisfies

$$\int_\Omega \Big\{ a\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v + (b \cdot \boldsymbol{\nabla} u)v + cuv \Big\} = \int_\Omega fv. \tag{3}$$

We now use (3) as a definition for a solution. With the notation

$$\begin{aligned} A[u, v] &:= \int_\Omega \Big\{ a\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v + (b \cdot \boldsymbol{\nabla} u)v + cuv \Big\}, \\ F[v] &:= \int_\Omega fv, \end{aligned}$$

the *weak formulation* of (1'), (2) reads: Seek $u \in \mathcal{W}^{1,2}(\Omega)$ with $u\lfloor_{\partial\Omega} = g^{\mathrm{D}}$ and

$$A[u, v] = F[v] \qquad \text{for all } v \in \mathcal{C}_0^\infty(\Omega). \tag{4}$$

This formulation allows for discontinuous a and for $\partial\Omega$ that is only Lipschitz continuous. The required regularity for u only needs one derivative in $\mathcal{L}^2(\Omega)$. Note that these functions may not even be continuous (see Sect. 2.2.2). The problem (1') can be solved in a very general setting.

2.3.2 The Lax–Milgram Theorem

(Lax, Milgram[15] 1954) Let $\mathcal{H}$ be a Hilbert space with norm $\|\,.\,\|_{\mathcal{H}}$ and let $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a *coercive* and *continuous* bilinear form, that is, there are strictly positive constants α_0, α_1 such that

$$\begin{aligned} B[v, v] &\geq \alpha_0 \|v\|_{\mathcal{H}}^2, \\ \big|B[v, w]\big| &\leq \alpha_1 \|v\|_{\mathcal{H}} \|w\|_{\mathcal{H}} \qquad \text{for all } v, w \in \mathcal{H}, \end{aligned}$$

and let $G : \mathcal{H} \to \mathbb{R}$ be a *continuous linear functional* ($:\Leftrightarrow G \in \mathcal{H}'$), i.e.,

$$\big|G[v]\big| \leq \|G\|_{\mathcal{H}'} \|v\| \qquad \text{for all } v \in \mathcal{H}.$$

Then there exists a unique solution $u \in \mathcal{H}$ of the *variational equality*

$$B[u, v] = G[v] \qquad \text{for all } v \in \mathcal{H},$$

that obeys the bound (*a priori estimate*)

$$\|u\|_{\mathcal{H}} \leq \frac{1}{\alpha_0} \|G\|_{\mathcal{H}'}.$$

Proof. This theorem is a standard result from functional analysis, see for example [Alt85, Ch. 4.9]. The estimate follows easily from

$$\|u\|_{\mathcal{H}}^2 \leq \frac{1}{\alpha_0} B[u, u] = \frac{1}{\alpha_0} G[u] \leq \frac{1}{\alpha_0} \|G\|_{\mathcal{H}'} \|u\|_{\mathcal{H}}.$$

□

2.3.3 Existence and regularity for the weak Dirichlet problem

(i) Let $a \in \mathcal{L}^\infty(\Omega)^{d,d}$, $b \in \mathcal{W}^{1,\infty}(\Omega)^d$, $c \in \mathcal{L}^\infty(\Omega)$, $\Omega \subset \mathbb{R}^d$ a bounded domain with Lipschitz continuous boundary, and $F[v] := \int_\Omega fv$ for some $f \in \mathcal{L}^2(\Omega)$. Let g^{D} be the trace of a function $\widetilde{g}^{\mathrm{D}} \in \mathcal{W}^{1,2}(\Omega)$. Assume that a is symmetric positive definit a.e. as required in Sect. 2.2.1 und that $c - 1/2\,\boldsymbol{\nabla}\cdot b \geq 0$ a.e. in Ω. Then there exists a unique solution $u \in \mathcal{W}^{1,2}(\Omega)$ of the variational problem (3) und $\|u\|_{\mathcal{W}^{1,2}(\Omega)} \leq C$ with C depending on the data.

(ii) Let, in addition to the requirements stated in (i), $a \in \mathcal{C}^{0,1}(\Omega)^{d,d}$ and $\widetilde{g}^{\mathrm{D}} \in \mathcal{W}^{2,2}(\Omega)$. Then $u \in \mathcal{W}^{2,2}(D)$ for every $D \subset\subset \Omega$. In case $D \subset \Omega$ with $S := D \cap \partial\Omega \neq \emptyset$, this holds true if S is of class $\mathcal{C}^{2,\alpha}$. If $\partial\Omega$ is $\mathcal{C}^2$-regular as a whole or Ω is convex, then $u \in \mathcal{W}^{2,2}(\Omega)$. Assuming, for $k \geq 0$, $\partial\Omega \in \mathcal{C}^{2+k}$, $a \in \mathcal{W}^{k,\infty}(\Omega)^{d,d}$, $b \in \mathcal{W}^{1+k,\infty}(\Omega)^d$, $c, f \in \mathcal{W}^{k,\infty}(\Omega)$, and $\widetilde{g}^{\mathrm{D}} \in \mathcal{W}^{2+k,2}(\Omega)$ in the regularity requirements yields a solution $u \in \mathcal{W}^{2+k,2}(\Omega)$ [GT98, Thm. 8.12, 8.13].

[15] Peter David Lax (1926–), Hungarian/American mathematician. Wolf Prize in Mathematics 1987, Abel Prize 2005. Arthur Norton Milgram (1912–1961), American mathematician.

Proof. Proof of (i): We want to apply Theorem 2.3.2. First split $u = u_0 + \widetilde{g}^{\mathrm{D}}$ with $u_0 \in \mathcal{W}_0^{1,2}(\Omega)$. u_0 will then be the solution of

$$A[u_0, v] = \widetilde{F}[v] := F[v] - A[\widetilde{g}^{\mathrm{D}}, v] \qquad \text{for all } v \in \mathcal{D}(\Omega).$$

By density of $\mathcal{D}(\Omega)$ in $\mathcal{W}_0^{1,2}(\Omega)$, we can state the problem equivalently as: Seek $u_0 \in \mathcal{W}_0^{1,2}(\Omega)$ such that $A[u_0, v] = \widetilde{F}[v]$ for all $v \in \mathcal{W}_0^{1,2}(\Omega)$. Now let $\mathcal{H} := \mathcal{W}_0^{1,2}(\Omega)$ and $\|\,.\,\|_{\mathcal{H}} := \|\boldsymbol{\nabla}\,.\,\|_{\mathcal{L}^2(\Omega)}$. Then using the Cauchy–Bunjakovski–Schwarz[16] inequality and the Poincaré–Friedrichs inequality we have for all $v, w \in \mathcal{H}$

$$\begin{aligned} \big|A[v,w]\big| &\le \|a\|_{\mathcal{L}^\infty(\Omega)}\|v\|_{\mathcal{H}}\|w\|_{\mathcal{H}} + \|b\|_{\mathcal{L}^\infty(\Omega)}\|v\|_{\mathcal{H}}\|w\|_{\mathcal{L}^2(\Omega)} + \|c\|_{\mathcal{L}^\infty(\Omega)}\|v\|_{\mathcal{L}^2(\Omega)}\|w\|_{\mathcal{L}^2(\Omega)} \\ &\le \Big(\|a\|_{\mathcal{L}^\infty(\Omega)} + d_\Omega\|b\|_{\mathcal{L}^\infty(\Omega)} + d_\Omega^2\|c\|_{\mathcal{L}^\infty(\Omega)}\Big)\|v\|_{\mathcal{H}}\|w\|_{\mathcal{H}} \\ &\le C(A,b,c,\Omega)\,\|v\|_{\mathcal{H}}\|w\|_{\mathcal{H}} \end{aligned}$$

and

$$\begin{aligned} A[v,v] &= \int_\Omega \Big\{a\boldsymbol{\nabla}v\cdot\boldsymbol{\nabla}v + (b\cdot\boldsymbol{\nabla}v)v + c|v|^2\Big\} \\ &\ge \int_\Omega \Big\{a_0|\boldsymbol{\nabla}v|^2 + (c - \frac{1}{2}\boldsymbol{\nabla}\cdot b)|v|^2\Big\} \ge a_0\int_\Omega |\boldsymbol{\nabla}v|^2 = a_0\|v\|_{\mathcal{H}}^2. \end{aligned}$$

For $\widetilde{F}$ we immediately get

$$\widetilde{F}[v] \le \Big(\|F\|_{\mathcal{W}^{-1,2}(\Omega)} + C(A,b,c,\Omega)\|g^{\mathrm{D}}\|_{\mathcal{W}^{1,2}(\Omega)}\Big)\|v\|_{\mathcal{H}}.$$

By the Lax–Milgram[17] theorem there exists a unique $u_0 \in \mathcal{H}$ such that $A[u_0, v] = \widetilde{F}[v]$ for all $v \in \mathcal{H}$ and the bound for u_0 follows with the previous estimates. The result for u is then immediate. □

2.3.4 Examples

(i) The theoretical result allows the right hand side F to be a functional. For $\Omega := (-1,1) \subset \mathbb{R}$ one may take for example δ_0, the Dirac distribution from Sect. 1.2.2, since such F is continuous by Sect. 1.3.1: $\big|F[v]\big| = \big|\delta_0[v]\big| = |v(0)| \le \|v\|_{\mathcal{L}^\infty(\Omega)} \le C\|v\|_{\mathcal{W}_0^{1,2}(\Omega)}$. This is not true for $d \ge 2$ [ibd]. Examples on $\mathbb{R}^d$ could be $F[v] := \int_\Omega f\cdot\boldsymbol{\nabla}v$ for some $f \in \mathcal{L}^2(\Omega)^d$, or, for a smooth $d-1$ dimensional manifold S in Ω, $F[v] := \int_S v$. The continuity of F in the second example follows from the trace theorem 1.3.2.

(ii) This solution concept works for solutions with singularities as in Sect. 2.2.2. They are all in $\mathcal{W}^{1,2}(\Omega)$, but not in $\mathcal{W}^{2,2}(\Omega)$. However, they are in $\mathcal{W}^{2,p}(\Omega)$ for some $p \in (1,2)$. Let for example $\Omega := (-1,1)^2 \subset \mathbb{R}^2$ and $a = 1$ in $\{[x_1,x_2] \in \Omega : x_1x_2 > 0\}$, $a = 100$ in $\{[x_1,x_2] \in \Omega : x_1x_2 < 0\}$, $b = 0$, $c = 0$, $f = 1$, $g^{\mathrm{D}} = 0$. A unique solution exists in $\mathcal{W}^{1,2}(\Omega)$ and it can be shown that u behaves like r^β with $\beta \approx 0.1$ for $r \to 0$. Thus $u \in \mathcal{W}^{2,p}(\Omega)$ for $p \in (1, 2/(2-\beta))$ or $u \in \mathcal{W}^{1+\beta,2}(\Omega)$ only [Kel75] [MNS00, Sect. 5.3].

[17] Peter David Lax (1926–??), Hungarian/American mathematician. Wolf Prize 1987, Abel Prize 2005. Arthur Norton Milgram (1912–1961), American mathematician.

[17] Augustin-Louis Cauchy (1789–1857), French mathematician. Viktor Jakovlevitsh Bunjakovski (1804–1889), Russian mathematician. Hermann Amandus Schwarz (1843–1921), German mathematician.

2.3.5 The Neumann problem

Let for simplicity $b = c = 0$. Assume that $\partial\Omega = \Gamma_{\mathrm{D}} \dot\cup \Gamma_{\mathrm{N}}$ (disjoint decomposition) and that the boundary condition are as follows

$$\begin{aligned} u &= g^{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}}, \\ n_{\partial\Omega} \cdot \big(a\boldsymbol{\nabla} u\big) &= g^{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}}. \end{aligned}$$

By subtracting a global function $\widetilde{g}^{\mathrm{D}}$, as in the proof of Sect. 2.3.3, we may assume that $g^{\mathrm{D}} = 0$. We now multiply (1) by an arbitrary function $v \in \mathcal{C}^\infty(\overline{\Omega})$ with $v(x) = 0$ for all $x \in \Gamma_{\mathrm{D}}$ and obtain

$$\begin{aligned} A[u,v] := \int_\Omega a\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v &= \int_\Omega fv + \int_{\partial\Omega} n_{\partial\Omega} \cdot (a\boldsymbol{\nabla} u)v \\ &= \int_\Omega fv + \int_{\Gamma_{\mathrm{N}}} g^{\mathrm{N}} v =: F[v]. \end{aligned}$$

Using this as a definition of a weak solution $u \in \mathcal{H} := \big\{ w \in \mathcal{W}^{1,2}(\Omega) : w\lfloor_{\Gamma_{\mathrm{D}}} = 0 \big\}$ with $\|\,.\,\|_{\mathcal{H}} := \|\boldsymbol{\nabla}\,.\,\|_{\mathcal{L}^2(\Omega)}$, we can proceed as in Sect. 2.3.3.(i). Assuming $g^{\mathrm{N}} \in \mathcal{L}^2(\Gamma_{\mathrm{N}})$ we obtain with Sect. 1.3.2

$$\begin{aligned} A[v,v] &\geq \alpha_0 \|v\|_{\mathcal{H}}^2, \\ \big|F[v]\big| = \Big| \int_\Omega fv + \int_{\Gamma_{\mathrm{N}}} g^{\mathrm{N}} v \Big| &\leq \|f\|_{\mathcal{L}^2(\Omega)} \|v\|_{\mathcal{W}^{1,2}(\Omega)} + \|g^{\mathrm{N}}\|_{\mathcal{L}^2(\Gamma_{\mathrm{N}})} \|v\|_{\mathcal{L}^2(\partial\Omega)} \\ &\leq C(f, g^{\mathrm{N}})\, \|v\|_{\mathcal{W}^{1,2}(\Omega)}. \end{aligned}$$

In case $\mathrm{meas}_{d-1}(\Gamma_{\mathrm{D}}) \neq 0$, the Poincaré–Friedrichs inequality in Sect. 1.3.3 still holds and F is proved to be continuous on $\mathcal{H}$. However, in case $\mathrm{meas}_{d-1}(\Gamma_{\mathrm{D}}) = 0$ this is not true, and in fact, the problem might not be solvable: if u is a solution, then $u + k$ is also a solution for any $k \in \mathbb{R}$. However, if we impose the necessary condition $\int_\Omega f + \int_{\Gamma_{\mathrm{N}}} g^{\mathrm{N}} = 0$ and require the additional constraint $\int_\Omega u = 0$ (to fix one specific k) one can prove again existence and uniqueness. To this end let

$$\mathcal{H} := \Big\{ v \in \mathcal{W}^{1,2}(\Omega) \,:\, \int_\Omega v = 0 \Big\}$$

and let $\|\cdot\|_{\mathcal{H}} := \|\boldsymbol{\nabla}\,.\,\|_{\mathcal{L}^2(\Omega)}$. One can prove the Poincaré-type inequality

$$\|v\|_{\mathcal{L}^2(\Omega)} \leq C(\Omega)\|v\|_{\mathcal{H}} \qquad \text{for all } v \in \mathcal{H}$$

and from this the norm equivalence between $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{W}^{1,2}(\Omega)}$ as in Sect. 1.3.3. Now we define the $\mathcal{L}^2$-orthogonal decomposition

$$\begin{aligned} \mathcal{W}^{1,2}(\Omega) &= \mathcal{H} \oplus \mathbb{R} \\ v &= w + m(v) \end{aligned}$$

with

$$\begin{aligned} m(v) &:= \frac{1}{|\Omega|} \int_\Omega v \;\in \mathbb{R}, \\ w &:= v - m(v) \in \mathcal{H}. \end{aligned}$$

The weak equation can now be formulated in $\mathcal{H}$. For all $v \in \mathcal{W}^{1,2}(\Omega)$ one has

$$\begin{aligned}\int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v &= \int_\Omega fv + \int_{\Gamma_\mathrm{N}} g^\mathrm{N} v \\ &= \int_\Omega fw + \int_{\Gamma_\mathrm{N}} g^\mathrm{N} w + \Big(\int_\Omega f + \int_{\Gamma_\mathrm{N}} g^\mathrm{N}\Big) m(v) \\ &= \int_\Omega fw + \int_{\Gamma_\mathrm{N}} g^\mathrm{N} w.\end{aligned}$$

Since $w \in \mathcal{H}$ is also arbitrary, we will seek $u \in \mathcal{H}$ such that

$$\int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} w = \int_\Omega fw + \int_{\Gamma_\mathrm{N}} g^\mathrm{N} w \qquad \text{for all } w \in \mathcal{H}.$$

Existence and uniqueness for this problem is now proved as in Sect. 2.3.3.

Remark

If we would have defined $\|\,.\,\|_\mathcal{H} := \|\,.\,\|_{\mathcal{W}^{1,2}(\Omega)}$, then F would immediately be continuous but then we would need the same requirements as above to show that A is coercive on $\mathcal{H}$.

2.3.6 The Dirichlet problem on an exterior domain

Let $G \subset \mathbb{R}^d$ be a bounded and simply connected domain with Lipschitz continuous boundary and define $\Omega := \mathbb{R}^d \backslash G$. Such an Ω is called *exterior domain*. We further assume $a = 1$, $b = 0$, and $c = 0$ and assume homogeneous boundary conditions on ∂G and for $|x| \to \infty$. To this end we seek the solution in (compare Sect. 1.3.3)

$$\mathcal{W}_0^{1,p}(\Omega) := \overline{\mathcal{D}(\Omega)}^{\|\boldsymbol{\nabla}\,\cdot\,\|_{\mathcal{L}^2(\Omega)}}.$$

On an exterior domain the Poincaré–Friedrichs inequality does not hold so we need different arguments to prove continuity of $f \mapsto \int_\Omega fv$. The only continuous embedding $\mathcal{W}_0^{1,2}(\Omega) \to \mathcal{L}^p(\Omega)$ holds in case of equality of the Sobolev numbers, $1 - d/2 \overset{!}{=} -d/p \Leftrightarrow d > 2$ and $p = 1/(1/2 - 1/d)$ [BF96]. If we fix $d = 3$, then $p = 6$ and we find (with Hölder's inequality)

$$\Big|\int_\Omega fv\Big| \le \Big(\int_\Omega |f|^{6/5}\Big)^{5/6}\Big(\int_\Omega |v|^6\Big)^{1/6} \le \|f\|_{\mathcal{L}^{6/5}(\Omega)} \|\boldsymbol{\nabla} v\|_{\mathcal{L}^2(\Omega)}.$$

Alternatively, one may exploit *Hardy's*[18] *inequality* [BGH90] [SS96] (assuming that without loss of generality $0 \in G$)

$$\int_\Omega \frac{|v|^2}{|x|^2} \le 4 \int_\Omega |\boldsymbol{\nabla} v|^2.$$

We then get

$$\Big|\int_\Omega fv\Big| \le \int_\Omega |x| f \frac{|v|}{|x|} \le \Big(\int_\Omega |x|^2 |f|^2\Big)^{1/2}\Big(\int_\Omega \frac{|v|^2}{|x|^2}\Big)^{1/2} \le 2\|\,|x| f\|_{\mathcal{L}^2(\Omega)} \|\boldsymbol{\nabla} v\|_{\mathcal{L}^2(\Omega)}.$$

Thus we need $f \in \mathcal{L}^{6/5}(\Omega)$ or $|x| f \in \mathcal{L}^2(\Omega)$. Note that $u \in \mathcal{L}^2(\Omega)$ does in general not hold. More general results on this topic can be found in the monograph [SS96].

[18] Godfrey Harold Hardy (1877–1947), British mathematician.

2.3.7 Dirichlet principle

Let a, c, f, Ω as in Sect. 2.3.3.(i) but with $b = 0$. Then u is a solution (3) if and only if u is a minimiser of

$$I : \mathcal{W}_0^{1,2}(\Omega) \to \mathbb{R}$$
$$I(w) := \int_\Omega \Big\{ \frac{1}{2} a \nabla w \cdot \nabla w + \frac{1}{2} c|w|^2 - fw \Big\}.$$

Proof. It is easy to see that $I \in \mathcal{C}^2\big(\mathcal{W}_0^{1,2}(\Omega)\big)$ is strictly convex and bounded from below. Hence, if u is a minimiser of I, it is completely characterised by $I'(u)[v] = 0$ for all $v \in \mathcal{W}_0^{1,2}(\Omega)$ and this is (3). □

2.4 Maximum principles

2.4.1 The classical weak maximum principle

Let $L_0 v := -\nabla \cdot (a \nabla v) + b \cdot \nabla v$. Assume that $a \in \mathcal{C}^1(\overline{\Omega})^{d,d}$, $b \in \mathcal{C}^0(\overline{\Omega})^d$, and that a fulfills the structural conditions in Sect. 2.2.1.(ii). Then, for $v \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$,

$$L_0 v \le 0 \quad \text{in } \Omega \qquad \Longrightarrow \qquad \max_\Omega\{v\} = \max_{\partial\Omega}\{v\}.$$

Especially,

$$L_0 v = 0 \quad \text{in } \Omega \qquad \Longrightarrow \qquad \min_{\partial\Omega}\{v\} = \min_\Omega\{v\} \le \max_\Omega\{v\} = \max_{\partial\Omega}\{v\}.$$

Proof. [GT98, Thm. 3.1] Assume first that we have $L_0 v < 0$ in Ω. If v has an interior maximum at $x_0 \in \Omega$, then

$$L_0 v(x_0) = \underbrace{-a(x_0) : \underbrace{\nabla^2 v(x_0)}_{\le 0}}_{\ge 0} + \underbrace{\big(b(x_0) - \nabla \cdot a(x_0)\big)}_{=:\widetilde{b}(x_0)} \cdot \underbrace{\nabla v(x_0)}_{=0} \ge 0,$$

contradicting $L_0 v(x_0) \le 0$. For $\epsilon > 0$ consider now $v_\epsilon(x) := v(x) + \epsilon e^{\gamma x_1}$ for v as in the assumption. Then, for sufficiently large $\gamma > 0$

$$L_0 v_\epsilon(x) = L_0 v(x) + \epsilon L_0(e^{\gamma x_1}) = L_0 v(x) + \epsilon\big(-\gamma^2 a_{11}(x) + \gamma \widetilde{b}_1(x)\big) e^{\gamma x_1} < 0$$

and with the first part of the proof we conclude

$$\max_\Omega\{v + \epsilon e^{\gamma x_a}\} = \max_{\partial\Omega}\{v + \epsilon e^{\gamma x_1}\}.$$

With $\epsilon \to 0$ we obtain the first result. The proof of the second result follows since we can apply the first result to $-v$ also. □

2.4.2 A comparison result

Let $Lv := -\nabla \cdot (a \nabla v) + b \cdot \nabla v + cv$. Assume that $a \in \mathcal{C}^1(\overline{\Omega})^{d,d}$, $b \in \mathcal{C}^0(\overline{\Omega})^d$, $c \in \mathcal{C}^0(\overline{\Omega})$, and that a and c fulfill the structural conditions in Sect. 2.2.1.(ii). If $v, w \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ are such that

$$\begin{aligned} Lv &\le Lw \qquad \text{a.e. in } \Omega, \\ v &\le w \qquad \text{a.e. on } \partial\Omega, \end{aligned}$$

then $v \le w$ in Ω.

Proof. [GT98, Thm. 3.3] Replacing v by $v - w$, it remains to show: $Lv \le 0$ in Ω and $v \le 0$ on $\partial\Omega$ implies $v \le 0$ in Ω. For such v we define the open set $\Omega^+ := \{x \in \Omega : v(x) > 0\}$. On Ω^+ we have (with L_0 from Sect. 2.4.1) $L_0 v = Lv - cv \le 0$. Hence $\max_{x \in \Omega^+}\{v\} = \max_{x \in \partial\Omega^+}\{v\} = 0$ since $\partial\Omega \cap \partial\Omega^+ = \emptyset$ by assumption and therefore $\Omega^+ = \emptyset$, which is $v \le 0$ in Ω. □

2.4.3 Applications

(i) *Uniqueness:* For L as in Sect. 2.4.2 let u_1, u_2 be two solutions of $Lu = f$ in Ω and $u = g^{\mathrm{D}}$ on $\partial\Omega$. Then $L(u_1) \leq L(u_2)$ in Ω and $u_1 \leq u_2$ on $\partial\Omega$ and the same holds if we interchange u_1, u_2. We conclude by Sect. 2.4.2 that $u_1 \leq u_2$ and $u_1 \leq u_2$ in Ω, hence $u_1 = u_2$ in Ω.

(ii) *A priori estimates in* $\|\,.\,\|_{\mathcal{L}^\infty(\Omega)}$: For L as in Sect. 2.4.2 let u be the solution of $Lu = f$ in Ω and $u = g^{\mathrm{D}}$ in $\partial\Omega$. Then

$$\|u\|_{\mathcal{L}^\infty(\Omega)} \leq \|g^{\mathrm{D}}\|_{\mathcal{L}^\infty(\partial\Omega)} + C\|f\|_{\mathcal{L}^\infty(\Omega)},$$

with C depending on Ω, a, b, c.

Proof. [GT98, Thm. 3.7] Let

$$v(x) := \|g^{\mathrm{D}}\|_{\mathcal{L}^\infty(\partial\Omega)} + \frac{1}{\alpha}\|f\|_{\mathcal{L}^\infty(\Omega)}\big(e^{\gamma d_\Omega} - e^{\gamma x_1}\big),$$

where it is assumed that $x \in \Omega \Rightarrow 0 \leq x_1 \leq d_\Omega$ (w.l.o.g., this can be achieved by orthogonal rotations and translations). Then

$$\begin{aligned} Lv &= \frac{1}{\alpha}\Big(\gamma^2 a_{11} - (b_1 - \boldsymbol{\nabla}\cdot a_1)\gamma + c(e^{\gamma(d_\Omega - x_1)} - 1)\Big)\|f\|_{\mathcal{L}^\infty(\Omega)} e^{\gamma x_1} \\ &\geq \|f\|_{\mathcal{L}^\infty(\Omega)} \end{aligned}$$

for sufficiently large γ, depending on a, b, c. This yields

$$\begin{aligned} L(u - v) \leq f - \|f\|_{\mathcal{L}^\infty(\Omega)} &\leq 0 && \text{in } \Omega, \\ u - v \leq u - \|g^{\mathrm{D}}\|_{\mathcal{L}^\infty(\partial\Omega)} &\leq 0 && \text{on } \partial\Omega \end{aligned}$$

and hence $u \leq v$ in Ω by Sect. 2.4.2. Now we proceed by considering $-u$ instead of u. □

2.4.4 Remark

There is a large variety of results on maximum principles. One direction is to derive strong principles that state that, under appropriate conditions, $v(x) < \max_{\partial\Omega}\{v\}$ holds for all $x \in \Omega$ [GT98, Thm. 3.5]. Another direction generalises these results to the case of the weak solution concept [GT98, Ch. 8.1].

2.5 Fundamental solutions and Green's function

Under the assumptions of Sect. 2.2.1, there is a *fundamental solution* to the operator L on $\Omega := \mathbb{R}^d$, that is a function $\Gamma \in \mathcal{L}^1(\mathbb{R}^d \times \mathbb{R}^d)$, such that

$$u(x) := \int_{\mathbb{R}^d} \Gamma(x, y) f(y)\,\mathrm{d}y$$

is a solution of $Lu = f$ for any given $f \in \mathcal{C}^{0,1}_{\mathrm{c}}(\mathbb{R}^d) := \{v \in \mathcal{C}^{0,1}(\mathbb{R}^d) : \operatorname{supp}(v) \subset \mathbb{R}^d\}$. Γ is singular on „the diagonal“ $\triangle := \{[x, y] \in \mathbb{R}^d \times \mathbb{R}^d : x = y\}$ and $\mathcal{C}^\infty$ otherwise with

$$|\boldsymbol{\nabla}^l_x \Gamma(x, y)| + |\boldsymbol{\nabla}^l_y \Gamma(x, y)| \leq C_0 |x - y|^{2-d-l}$$

for all $l \in \mathbb{N}$. In the case $b = 0$, one has the special form $\Gamma(x, y) = \Gamma_0(x - y)$. Note that $\Gamma \notin \mathcal{C}^2(\Omega \times \Omega)$ but $\Gamma \in \mathcal{C}^2(\Omega \times \Omega \setminus \triangle)$. However, in $\mathcal{C}^0(\Omega \times \Omega)'$ we can write $L_x \Gamma(x, y) = \delta_\triangle$. For a bounded domain $\Omega \subset \mathbb{R}^d$ with smooth boundary, there is a smooth function $\widetilde{G}$, such that the solution u of the problem (1') and (2) is explicitly represented by

$$u(x) = \int_{\partial\Omega} \partial_{n(y)} G(x,y) u_D(y)\,\mathrm{d}y + \int_{\Omega} G(x,y) f(y)\,\mathrm{d}y,$$

with $G(x,y) = \Gamma(x,y) + \widetilde{G}(x,y)$. G is called *Green's*[19] *function* of the boundary value problem.

This representation is mostly of theoretical interest for proving special properties of u or if Ω is an unbounded domain. Practically, it is explicitly used in boundary element methods. However, G is explicitly known only in a view special cases.

Example

We study the case of $L = -\Delta$ in space dimensions $d = 1, 2, 3$:

$d = 1$: $\Gamma_0(z) = |z|$. Green's function for the Dirichlet problem on $[0,1]$ is

$$G(x,y) = \begin{cases} (1-x)y & \text{for } 0 \le y \le x \le 1, \\ x(1-y) & \text{for } 0 \le x \le y \le 1. \end{cases}$$

$d = 2$: $\Gamma_0(z) = -\frac{1}{2\pi}\log(|z|)$ for $z \in \mathbb{R}^2 \setminus \{0\}$. Green's function for the Dirichlet problem on $B_1(0)$ is

$$G(x,y) = \begin{cases} \Gamma_0\big(|x-y|\big) - \Gamma_0\big(|y|\,\big|x - \frac{y}{|y|^2}\big|\big) & \text{for } y \neq 0, \\ \Gamma_0(|x|) - \Gamma_0(1) & \text{for } y = 0. \end{cases}$$

$d > 2$: $\Gamma_0(z) = \frac{1}{n(2-n)\omega_n}|z|^{2-n}$ for $z \in \mathbb{R}^d \setminus \{0\}$ with $\omega_n = \mathrm{meas}_{n-1}(S^{n-1})$ (the measure of the sphere, e.g., $\omega_1 = 2$, $\omega_2 = 2\pi$, $\omega_3 = 4\pi$). Then G for $\Omega = B_1(0)$ follows from Γ_0 as for $d = 2$.

2.6 The elliptic eigenvalue problem

We let $Lv := -\boldsymbol{\nabla}\cdot(a\boldsymbol{\nabla} v) + cv$ under the assumptions of Sect. 2.2.1.

2.6.1 Eigenvalues and eigenfunctions

The eigenvalue problem for L is to find eigenvalues $\lambda \in \mathbb{C}$ and corresponding eigenfunctions $z_\lambda \neq 0$ with $z_\lambda\lfloor_{\partial\Omega} = 0$, $\|z_\lambda\|_{\mathcal{L}^2(\Omega)} = 1$, and

$$Lz_\lambda = \lambda z_\lambda.$$

In the weak form, this problem reads

$$\int_\Omega \big\{a\boldsymbol{\nabla} z_\lambda \cdot \boldsymbol{\nabla} v + c z_\lambda v\big\} = \lambda \int_\Omega z_\lambda v \quad \text{for all } v \in \mathcal{W}_0^{1,2}(\Omega),$$

which we will equivalently write as $((\,.\,,\,.\,)_0 := (\,.\,,\,.\,)_{\mathcal{L}^2(\Omega)})$

$$A(z_\lambda, v) = \lambda\big(z_\lambda, v\big)_0 \quad \text{for all } v \in \mathcal{W}_0^{1,2}(\Omega).$$

By symmetry and coercivity of A we obtain the following properties: All eigenvalues of L are positive and satisfy the following orthogonality relations

$$A(z_\lambda, z_\mu) = \lambda\big(z_\lambda, z_\mu\big)_0 = \lambda\delta_{\lambda\mu}.$$

[19] George Green (1793–1841), British mathematical physicist.

Proof. By definition and with the Poincaré–Friedrichs inequality from Sect. 1.3.3

$$0 < \frac{\alpha}{d_\Omega} = \frac{\alpha}{d_\Omega}\|z_\lambda\|^2_{\mathcal{L}^2(\Omega)} \le \alpha\|\boldsymbol{\nabla} z_\lambda\|^2_{\mathcal{L}^2(\Omega)} \le A(z_\lambda, z_\lambda) = \lambda,$$

hence $\lambda \in \mathbb{R}_{>0}$. For arbitrary eigenvalues λ, μ it holds that $\lambda(z_\lambda, z_\mu)_0 = A(z_\lambda, z_\mu) = A(z_\mu, z_\lambda) = \mu(z_\mu, z_\lambda)_0$, hence

$$(\lambda - \mu)\big(z_\lambda, z_\mu\big)_0 = 0.$$

$\lambda \neq \mu$ thus enforces $(z_\lambda, z_\mu)_0 = 0$ and hence $A(z_\lambda, z_\mu) = 0$. Otherwise $\|z_\lambda\|_{\mathcal{L}^2(\Omega)} = 1$ and $A(z_\lambda, z_\lambda) = \lambda$. □

2.6.2 Existence of eigenfunctions

Eigenvalues are extremal points of the *Rayleigh*[20] *quotient*

$$R(v) := \frac{A(v,v)}{\big(v,v\big)_0},$$

which is easily verified by differentiation of $v \mapsto R(v)$. The smallest (or principal) eigenvalue $\lambda_{\min}$ is characterized by

$$\lambda_{\min} = \inf_{v \in \mathcal{W}_0^{1,2}(\Omega)} \big\{R(v)\big\}$$

and we know from above $\lambda_{\min} > 0$. Henceforth, we use the notation $\lambda_{\min} = \lambda_1$ and z_1 for the corresponding eigenfunction with $\|z_1\|_{\mathcal{L}^2(\Omega)} = 1$. The eigenspace of λ_1 is one-dimensional and the function z_1 is of one sign. Furthermore, there is a sequence of eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ with $\lambda_1 < \lambda_2 \le \ldots$, $\lim_{j\to\infty}\lambda_j = \infty$, and eigenfunctions $\{z_j\}_{j\in\mathbb{N}}$ that are orthogonal as shown in Sect. 2.6.1. All eigenvalues are of finite multiplicity.

Proof. $R(v)$ is bounded from below, so there exists a sequence $\{v_k\}_{k\in\mathbb{N}}$ such that (w.l.o.g) $R(v_k) \searrow \lambda_1$. By homogeneity of R we can assume that $\|v_k\|_{\mathcal{L}^2(\Omega)} = 1$, hence $R(v_k) = A(v_k, v_k) \to \lambda_1$ for $k \to \infty$. By coercivity of A we get $\|\boldsymbol{\nabla} v_k\|^2_{\mathcal{L}^2(\Omega)} \le R(v_1)/\alpha$. By Rellich's[21] compactness theorem (bounded sets in $\mathcal{W}_0^{1,2}(\Omega)$ are precompact in $\mathcal{L}^2(\Omega)$; see also Sect. 1.3.1) we obtain a subsequence and $z_1 \in \mathcal{W}_0^{1,2}(\Omega)$ such that $v_k \to z_1$ in $\mathcal{L}^2(\Omega)$ (implying $\|z_1\|_{\mathcal{L}^2(\Omega)} = 1$) and by weak compactness in Hilbert spaces we get $R(z_1) = A(z_1, z_1) \le \liminf_{k\to\infty} A(v_k, v_k)$. Thus

$$\lambda_1 = \lim_{k\to\infty} R(v_k) \ge R(z_1) \ge \lambda_1$$

and therefore $R(z_1) = \lambda_1$. We now show that each eigenfunction to λ_1 has to be of one sign. This will prove that there cannot be a second one, since two functions of equal sign cannot be $\mathcal{L}^2$–orthogonal. We first split

$$z_1 = \max\{z_1, 0\} + \min\{z_1, 0\} \equiv z_1^+ + z_1^-$$

and note that $z_1^{+/-} \in \mathcal{W}_0^{1,2}(\Omega)$ and that

$$A(z_1^+, z_1^-) = 0 \quad \text{and} \quad (z_1^+, z_1^-)_0 = 0.$$

[20] John William Strutt “Lord Rayleigh” (1842–1919), English mathematician and physicist.
[21] Franz Rellich (1906–1955), Austrian–Italian mathematician

Hence

$$\begin{aligned}\lambda_1 = A(z_1, z_1) &= A(z_1^+, z_1^+) + A(z_1^-, z_1^-)\\ &\geq \lambda_1\big(\|z_1^+\|^2_{\mathcal{L}^2(\Omega)} + \|z_1^-\|^2_{\mathcal{L}^2(\Omega)}\big) = \lambda_1\|z_1\|^2_{\mathcal{L}^2(\Omega)} = \lambda_1\end{aligned}$$

and therefore $R(z_1^+) = R(z_1^-) = \lambda_1$. This implies $Lz_1^{+/-} = \lambda_1 z_1^{+/-}$. Especially $Lz_1^+ = \lambda_1 z_1^+ \geq 0$ in Ω and $z_1^+ \geq 0$ on $\partial\Omega$. Thus $z_1^+ > 0$ in Ω or $z_1^+ = 0$ in Ω by the strong maximum principle. We conclude $z_1 > 0$ or $z_1 < 0$ in Ω and this was to be proven. Now define $V_1 = \mathcal{W}_0^{1,2}(\Omega)$ and $V_2 = \{v \in V_1 : (v, z_1)_0 = 0\}$. λ_2 is characterised by $\lambda_2 = \inf_{v\in V_2}\{R(v)\}$. We repeat the previous arguments to establish existence of z_2, but this time the multiplicity may be larger than 1 (the previous proof used the absolute minimum property on $\mathcal{W}_0^{1,2}(\Omega)$). Repeating this argument gives a nondecreasing sequence of eigenvalues and corresponding eigenfunctions. This sequence cannot be bounded: if $\lambda_j \leq C$, then $\|\boldsymbol{\nabla} z_j\|_{(\Omega)} \leq A(z_j, z_j)/\alpha = \lambda_j/\alpha \leq C/\alpha$ and $\{z_j\}_{j\in\mathbb{N}}$ would have a converging subsequence in $\mathcal{L}^2(\Omega)$ (again by compactness). But it is not a Cauchy sequence since by orthogonality $\|z_j - z_k\|^2_{\mathcal{L}^2(\Omega)} = \|z_j\|^2_{\mathcal{L}^2(\Omega)} + \|z_k\|^2_{\mathcal{L}^2(\Omega)} = 2$ for $j \neq k$. Hence $\lim_{j\to\infty}\lambda_j = \infty$ and with this all multiplicities can only be finite [GT98, Ch. 8.12] [LT03, Ch. 6.1]. □

2.6.3 Eigenfunction expansion

The eigenfunctions of L form an orthogonal basis of $\mathcal{L}^2(\Omega)$. Each function $v \in \mathcal{L}^2(\Omega)$ can therefore be written as

$$v = \sum_{j=1}^{\infty} \big(v, z_j\big)_0 z_j$$

with $\sum_{j=1}^{\infty} |(v, z_j)|^2 = \|v\|^2_{\mathcal{L}^2(\Omega)}$.

Proof. It is sufficient to prove this on the dense (in $\mathcal{L}^2(\Omega)$) set $\mathcal{W}_0^{1,2}(\Omega)$. For $v \in \mathcal{W}_0^{1,2}(\Omega)$ we define

$$v_N := \sum_{j=1}^{N} \big(v, z_j\big)_0 z_j.$$

We have to prove $v_N \to v$ in $\mathcal{L}^2(\Omega)$. For $r_N := v - v_N$ we have the orthogonality relations

$$\big(r_N, z_j\big)_0 = 0 \quad \text{for all } j = 1, \ldots, N.$$

Hence, by construction of the eigenfunctions, see Sect. 2.6.2,

$$R(r_N) \geq \lambda_{N+1}$$

and thus

$$\|r_N\|^2_{\mathcal{L}^2(\Omega)} \leq \frac{1}{\lambda_{N+1}}\|\boldsymbol{\nabla} r_N\|^2_{\mathcal{L}^2(\Omega)}.$$

Since $\|\boldsymbol{\nabla} v_N\|_{\mathcal{L}^2(\Omega)} \leq \|\boldsymbol{\nabla} v\|_{\mathcal{L}^2(\Omega)}$ (Bessel's[22] inequality) we obtain $\|\boldsymbol{\nabla} r_N\|_{\mathcal{L}^2(\Omega)} \leq 2\|\boldsymbol{\nabla} v\|_{\mathcal{L}^2(\Omega)}$ and therefore $r_N \to 0$ since $\lambda_N \to \infty$ for $N \to \infty$ as shown above. □

[22] Friedrich Wilhelm Bessel (1784–1846), German mathematician and astronomer.

3 Applications

3.1 Heat conduction

What is the loss of energy of a house due to heat diffusion through the wall? For this we model the wall by a rectangular domain in $\mathbb{R}^2$ that is intended to describe a „typical“ cut through the wall. The whole wall is assumed to consist of many periodic copies of Ω along the periodic boundary $\Gamma^+_{\text{per}} \cup \Gamma^-_{\text{per}}$. At the inner ($\Gamma_{\text{in}}$) and outer ($\Gamma_{\text{out}}$) boundary, constant values of the temperature (u_{in} and u_{out}) are prescribed. The boundary value problem for the stationary temperature distribution u in Ω is given by (see Sect. 2.1)

$$\begin{aligned} -\nabla\cdot(a\nabla u) &= 0 && \text{in } \Omega, \\ u &= u_\flat && \text{at } \Gamma_\flat,\ \flat \in \{\text{in, out}\}, \\ u\lfloor_{\Gamma^+_{\text{per}}} &= u\lfloor_{\Gamma^-_{\text{per}}}. \end{aligned}$$

The quantity of interest is the amount of heat that flows through the inner wall Γ_{in},

$$-\int_{\Gamma_{\text{in}}} q \cdot n.$$

n is the outer normal with respect to Ω.

3.2 Electrostatics

A charge density $\rho : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}$ gives rise to an electric field $E : \Omega \to \mathbb{R}^3$. By *Maxwell's*[23] *equation* it satisfies $-\nabla\cdot(aE) = \rho$. $a : \Omega \to \mathbb{R}_{>0}$ is the *permittivity* of the material. By another Maxwell equation we have $\nabla \times E = 0$, so that $E = \nabla\Phi$ for some "potential" $\Phi : \Omega \to \mathbb{R}$. Again, we end up with the differential equation

$$-\nabla\cdot\big(a\nabla\Phi\big) = \rho \qquad \text{in } \Omega.$$

Prescribed potential values on $\partial\Omega$ lead to the Dirichlet boundary condition

$$\Phi = \Phi^{\text{D}} \qquad \text{in } \partial\Omega,$$

while a prescribed current density yields the Neumann boundary condition

$$a\partial_n\Phi = \Phi^{\text{N}} \qquad \text{on } \partial\Omega.$$

A quantity of interest (for the Dirichlet problem) could be the total current on $\partial\Omega$, $\int_{\partial\Omega} n \cdot (a\nabla\Phi)$.

3.3 Flow through porous media

We consider the stationary flow of (e.g.) water through soil inside a bounded domain Ω. Let the flow be described by the vector field $q : \Omega \to \mathbb{R}^d$. It may be driven by a pressure difference ∇p and by gravity $g\vec{\imath}_3$. The soil may have a strong anisotropic permeability κ. An experimentally verified model is

$$q = -\kappa(\nabla p + \rho g\vec{\imath}_3) \qquad (\textit{Darcy's}^{24}\ \textit{law}),$$

where $\kappa : \Omega \to \mathbb{R}^{d,d}$ is a pointwise a.e. symmetric positive matrix field (*permeability tensor*) and ρ is the constant density of the water. Let $f : \Omega \to \mathbb{R}$ describe the sinks and sources of the flow field. Then the pressure p satisfies the equation

$$-\nabla\cdot\big(\kappa(\nabla p + \rho g\vec{\imath}_3)\big) = f \qquad \text{in } \Omega$$

[23] James Clerk [Maxwell] (1831–1879), Scottish physicist.

To simplify this equation, one introduces the *piezometric height* $u(x) := p(x)/(\rho g) + x_3$ to get

$$-\boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}u) = f \qquad \text{in } \Omega.$$

This equation may be accomplished by boundary conditions for pressure (Dirichlet boundary conditions) or flow rates (Neumann boundary condition).

Actually, the permeability will depend on the saturation of the water itself. A model for flow in a saturated porous media is for example

$$-\boldsymbol{\nabla}\cdot\big(\kappa(\boldsymbol{\nabla}u)\boldsymbol{\nabla}u\big) = f \qquad \text{in } \Omega,$$

with $\kappa(s) := |s|^m$ for some $m > 0$. This leads to a nonuniform nonlinear elliptic boundary value problem since $\kappa(\boldsymbol{\nabla}u)$ will not be strictly positive (although nonnegative) in general. For further modeling see [KA02, Ch. 0].

3.4 Stationary waves

We consider electromagnetic waves (the potential Φ of an electric field E in a dielectric medium) or mechanical waves (the displacement of an oscillating membrane, e.g., a drum). Then $\Phi : \mathbb{R}_{\geq 0} \times \bar{\Omega} \to \mathbb{R}$, $(t, x) \mapsto \Phi(t, x)$, satisfies the *wave equation*

$$\begin{aligned} \partial_t^2\Phi - \boldsymbol{\nabla}\cdot(c^2\boldsymbol{\nabla}\Phi) &= 0 \qquad \text{in } \mathbb{R}_{>0} \times \Omega, \\ \Phi &= 0 \qquad \text{on } \mathbb{R}_{>0} \times \partial\Omega. \end{aligned}$$

This time dependent problem requires in addition initial data for $\Phi(t = 0, .)$ and $\partial_t\Phi(t = 0, .)$. Here, we look for *harmonically oscillating functions*, i.e., we try the ansatz

$$\Phi(t, x) := e^{\mathrm{i}\omega t}u(x).$$

Then $\partial_t^2\Phi(t, x) = -\omega^2\Phi(t, x)$ and we obtain for u the *Helmholtz*[25] *equation*

$$\begin{aligned} -\Delta u - \Big(\frac{\omega}{c}\Big)^2 u &= 0 \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

(in case c is constant). This only has a nontrivial solution if $(\omega/c)^2$ is an eigenvalue of $-\Delta$ on Ω with homogenous boundary conditions. Hence only a discrete set of values for ω lead to nontrivial solutions (see Sect. 2.6).

3.5 Reactor equation

We study the temperature u in a chemical reactor. Exothermic processes start after a certain critical temperature is reached. A simplified model for this is the stationary reactor equation

$$\begin{aligned} -\Delta u &= e^{-\lambda/u} \qquad \text{in } \Omega, \\ u &= u_0 > 0 \qquad \text{on } \partial\Omega \end{aligned}$$

for given $u_0 : \partial\Omega \to \mathbb{R}_{>0}$ and some $\lambda \in \mathbb{R}_{>0}$. This equation is called *semi-linear*, since the differential equation is nonlinear only in the zero order term (with respect to derivatives). For certain values of λ more than one solution may exist [BE89].

[24] Henry Philibert Gaspard Darcy (1803–1858), French engineer.
[25] Hermann Ludwig Ferdinand von Helmholtz (1821–1894), German physiologist and physicist.

3.6 Minimal surface equation

Let $\Omega \subset \mathbb{R}$ and $u : \Omega \to \mathbb{R}$ be a differentiable function. The graph F of u is defined by

$$\begin{aligned} F \,:\, \Omega &\longrightarrow \mathbb{R}^3 \\ (x_1, x_2) &\mapsto [x_1, x_2, u(x_1, x_2)]. \end{aligned}$$

The area of the graph is defined to be

$$\mathcal{A}(u) := \int_\Omega |\partial_1 F \times \partial_2 F| = \int_\Omega \sqrt{1 + |\boldsymbol{\nabla} u|^2}.$$

We assume that the boundary is prescribed, i.e.,

$$u\lfloor_{\partial\Omega} = g$$

for some given function $g : \Omega \to \mathbb{R}$. We seek the graph with smallest area amongst all surfaces of graph-type with this boundary conditions. A necessary condition is that for all $\xi \in \mathcal{D}(\Omega)$

$$\begin{aligned} 0 = \frac{d}{dt}\lfloor_{t=0} \mathcal{A}(u + t\xi) &= \int_\Omega \frac{1}{\sqrt{1 + |\boldsymbol{\nabla} u|^2}} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} \xi \\ &= \int_\Omega -\boldsymbol{\nabla}\cdot \Big(\frac{1}{\sqrt{1 + |\boldsymbol{\nabla} u|^2}} \boldsymbol{\nabla} u \Big) \xi. \end{aligned}$$

Thus u is a solution of the boundary value problem

$$\begin{aligned} -\boldsymbol{\nabla}\cdot \Big(\frac{\boldsymbol{\nabla} u}{\sqrt{1 + |\boldsymbol{\nabla} u|^2}} \Big) = 0 \qquad & \text{in } \Omega, \\ u = g \qquad & \text{on } \partial\Omega. \end{aligned}$$

This equation is again (cf. Section 3.3) of the form $-\boldsymbol{\nabla} \cdot \big(a(\boldsymbol{\nabla} u) \boldsymbol{\nabla} u \big) = 0$. Note that $a(u) > 0$, but there is no a priori lower bound, since it could be that $|\boldsymbol{\nabla} u|$ becomes infinite (in fact, this happens for the upper half sphere in every boundary point (case of a constant right hand side)!).

3.7 The stationary incompressible Navier–Stokes equation

We consider the stationary flow of an incompressible fluid in a flow domain $\Omega \subset \mathbb{R}^d$. The relevant quantities are the velocity $u : \Omega \to \mathbb{R}^d$ and the pressure $p : \Omega \to \mathbb{R}$. There might be a volume force (e.g., gravity) $f : \Omega \to \mathbb{R}^d$ and boundary data $u = g^{\mathrm{D}}$ on $S_{\text{in}} \cup S_{\text{out}}$ and $u = 0$ on Γ (*no-slip condition*). The flow obeys the *stationary Navier–Stokes*[26] *equation*

$$\begin{aligned} -\nu \Delta u + u \cdot \boldsymbol{\nabla} u + \boldsymbol{\nabla} p = f \qquad & \text{in } \Omega, \\ \boldsymbol{\nabla}\cdot u = 0 \qquad & \text{in } \Omega, \\ u = g^{\mathrm{D}} \qquad & \text{on } S_{\text{in}} \cup S_{\text{out}}, \\ u = 0 \qquad & \text{on } \Gamma, \end{aligned}$$

with $\nu > 0$ (viscosity). Clearly, p is only determined up to a constant, that is usually fixed by $\int_\Omega p = 0$ and g^{D} is required to satisfy $\int_{S_{\text{in}} \cup S_{\text{out}}} g^{\mathrm{D}} \cdot n = \int_{\partial\Omega} u \cdot n = \int_\Omega \boldsymbol{\nabla}\cdot u = 0$. There are $d + 1$ equations for the $d + 1$ unknowns u, p. For small ν the problem is very complicated! Examples for nonuniqueness are known. Physically the solutions will lose then stationarity and nonstationary turbulence will set on. Many problems are still open for $d = 3$.

[26] Claude Louis Marie Henri Navier (1785–1836), French mathematician and physicist. Sir George Gabriel Stokes (1819–1903), Irish mathematician and physicist.

3.8 Obstacle problem

Consider a one-dimensional string, described as a graph of a function $u : \Omega := (0,1) \to \mathbb{R}$, that is deformed by a force density f. This force will be in equilibrium with the curvature if the deformation gradient is small. In this case, u is the solution of

$$\begin{aligned} -u'' &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Now we consider an obstacle that constraints the string to $u \geq \Psi$ in Ω for some differentiable function Ψ. Clearly, we should require $\Psi(0), \Psi(1) < 0$. We derive an equation for u. Define an energy

$$\mathcal{E}(v) := \frac{1}{2} \int_\Omega |v'|^2$$

on

$$\mathcal{M} := \Big\{ v : \Omega \to \mathbb{R} \,:\, v\lfloor_{\partial\Omega} = 0,\, v \geq \Psi \text{ a.e. in } \Omega,\, \mathcal{E}(v) < \infty \Big\} \subset \mathcal{W}_0^{1,2}(\Omega).$$

Now seek $u \in \mathcal{M}$ such that

$$\mathcal{E}(u) = \min_{v \in \mathcal{M}} \big\{ \mathcal{E}(v) \big\}.$$

Our aim is now to characterize u. Observe that $\mathcal{M}$ is convex, hence we have $u + \alpha(v-u) \in \mathcal{M}$ for all $v \in \mathcal{M}$ and for all $\alpha \in [0,1]$. Therefore

$$\begin{aligned} & \mathcal{E}(u) \leq \mathcal{E}(u + \alpha(v-u)) \\ \iff \quad & \frac{1}{2} \int_\Omega |u'|^2 \leq \frac{1}{2} \int_\Omega \Big\{ |u'|^2 + 2\alpha u'(v-u)' + \alpha^2 |(v-u)'|^2 \Big\} \\ \iff \quad & 0 \leq \int_\Omega \Big\{ u'(v-u)' + \frac{1}{2}\alpha |(v-u)'|^2 \Big\}. \end{aligned}$$

After $\alpha \searrow 0$, we end up with

$$\int_\Omega u'(v-u)' \geq 0 \qquad \text{for all } v \in \mathcal{M} \qquad (\textit{variational inequality}).$$

If $u \in \mathcal{W}^{2,2}(\Omega)$, then

$$\int_\Omega -u''(v-u) \geq 0 \qquad \text{for all } v \in \mathcal{M}.$$

From this we can derive that u fulfills

$$\begin{aligned} -u'' &\geq 0 \qquad \text{in } \Omega, \\ -u'' &= 0 \qquad \text{in } \Omega \cap \{u > \Psi\}, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Regularity theory gives $u \in \mathcal{W}^{2,\infty}(\Omega)$ under some mild assumptions, thus u is continuously differentiable [Fri82].

Proof. We may assume $u \in \mathcal{C}^0(\Omega)$. If $x_0 \in \{u > \Psi\}$, then for arbitrary $\zeta \in \mathcal{D}(B_\delta(x_0))$ (for sufficiently small δ) one has $v := u + \zeta \in \mathcal{M}$. This shows $\int_{B_\delta(x_0)} -u''\zeta \geq 0$ from which $-u''(x_0) = 0$ is derived. If $x_0 \in \{u = \Psi\}$, we take $v = \Psi + \zeta$ for arbitrary $\zeta \in \mathcal{D}(B_\delta(x_0))$ with $\zeta \geq 0$. This shows $\int_{B_\delta(x_0)} -u''\zeta \geq 0$ and this proves $-u''(x_0) \geq 0$. □

Remarks

(i) Note that the difference to Sect. 2.3.7 is that the minimisation is performed over a convex subset of $\mathcal{W}_0^{1,2}(\Omega)$ instead of the whole of $\mathcal{W}_0^{1,2}(\Omega)$.

(ii) The *coincidence set* $\{u = \Psi\}$ and the *noncoincidence set* $\{u > \Psi\}$ are not a priori known. The points of detachment $\partial\{u > \Psi\}$ are called the *free boundary.* This is more apparent in the case of two space dimensions, where the free boundary is a curve, what is of course to be proved. For theory and applications of free boundary value problems see [Fri82].

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