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Numerical Integrators for Nonlinear Dispersive Equations — Exercise Sheet 03 December 17, 2018

The aim of this exercise sheet is to construct and analyze a first order exponential type integration scheme for the cubic Schrödinger equation

$$i\partial_t u = -\Delta u + f(u), \quad u(0,x) = u^0(x) \in \mathbb{C}, \quad (t,x) \in [0,T] \times \mathbb{T}^d, \quad f(u) = \lambda |u|^2 u, \quad \lambda \in \mathbb{R}, \tag{NLS}$$

equipped with periodic boundary conditions, i.e., for example for $d = 1$ we have

$$u(t,-\pi) = u(t,+\pi), \quad \partial_x^m u(t,-\pi) = \partial_x^m u(t,+\pi), \quad \text{for } m \in \mathbb{N} \qquad \text{for all } t \in [0,T].$$

We look for solutions $u(t,\cdot)$ of (NLS) in Sobolev spaces $H^r(\mathbb{T}^d)$, $r > d/2$.

Moreover, we introduce the so-called φ_1 function, defined as

$$\varphi_1(z) = \frac{e^z - 1}{z}, \quad z \in \mathbb{C}, \quad \text{where in particular } \varphi_1(0) = 1, \text{ see Exercise 5 b).}$$

Exercise 5: (Exponential Integrator)

(a) Write down Duhamel's formula for the solution u of (NLS). Thereby, consider the time step from t to $t+\tau$, for $t, \tau \in [0,T]$ such that $t + \tau \in [0,T]$.

(b) Show that $|\varphi_1(ix)| \leq 1$ for all $x \in \mathbb{R}$ and that in particular $\varphi_1(0) = 1$.

(c) Use part b) in order to show that

$$\left\| e^{is\Delta} w - w \right\|_r \leq |s| \, \|w\|_{r+2}, \quad \text{for all } w \in H^{r+2}.$$

(d) Show that for $r > d/2$ (cf. Exercise 3 d))

$$\|u(t+s) - u(t)\|_r \leq |s| \left(\|u(t)\|_{r+2} + \sup_{\xi \in [0,s]} \|u(t+\xi)\|_r^3 \right),$$

and that

$$\|f(u(t+s)) - f(u(t))\|_r \leq K_u \, |s| \left(\|u(t)\|_{r+2} + \sup_{\xi \in [0,s]} \|u(t+\xi)\|_r^3 \right),$$

where K_u depends on $\sup_{\xi \in [0,s]} \|u(t+\xi)\|_r$.
Hint: *Exploit part c) and use that $e^{i\tau\Delta}$ is an isometry in H^r.*

(e) Show that

$$\varphi_1(-i\tau\Delta) w = e^{-i\tau\Delta} \varphi_1(i\tau\Delta) w, \quad \text{for all } w \in H^r.$$

(f) Based on Duhamel's formula from part a) use the previous results in order to construct an exponential type integrator (cf. Chapter 3 from the lecture)

$$u_{\mathrm{EXP}}^{n+1} = \Phi_{\exp}^{\tau}(u^n), \quad u^0 = u(0)$$

for the numerical approximation of the exact solution $u(t_{n+1})$ of (NLS), where $t_n = n\tau$, $n = 0, 1, 2, \ldots$.

(g) Carry out a global convergence analysis for $\Phi_{\exp}^{\tau}$ using classical Lady-Windermere's fan arguments (local error and stability analysis).

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Programming Exercise 4: (Implementation of the Exponential Integrator)

In this programming exercise, we implement an exponential type numerical integrator for solving the nonlinear Schrödinger equation with cubic nonlinearity, i.e.,

$$i\partial_t u(t,x) = -\Delta u(t,x) + \lambda |u(t,x)|^2 u(t,x), \quad u(0,x) = u^0(x), \quad \lambda \in \mathbb{R} \tag{1}$$

on the torus $\mathbb{T} = [-\pi, \pi]$ (i.e., $d = 1$ and periodic boundary conditions) and for $t \in [0, T]$.

In the following let $T = 1$ and let

$$u^0(x) := \cos(x).$$

For practical implementation issues, we consider the discretization of space $x_j = jh,\ j = -\frac{N_x}{2} + 1, \ldots \frac{N_x}{2}$ with $N_x = 64$, i.e. $h = \frac{2\pi}{N_x}$, and of time $t_n = n\tau,\ n = 0,1,2,\ldots \lfloor T/\tau \rfloor =: N_T$ for time step sizes

$$\tau = \tau_m = \frac{T}{N_T^m}, \quad \text{where} \quad N_T^m, m = 1,2,\ldots,m_{\max}$$

is the m-th divisor of the number $N_T^{m_{\max}} = 120$. All divisors of a number $N \in \mathbb{N}$ can be found in MATLAB (since version R2014b) via the function `divisors()`.

Same as in Programming Exercise 3, we use a Strang splitting scheme

$$u^{n+1} = \Phi^\tau_{\text{Strang}}(u^n) = \left(\varphi_T^{\tau/2} \circ \varphi_P^\tau \circ \varphi_T^{\tau/2}\right)(u^n).$$

for computing a reference approximation to the exact solution of (1) (see Exercise Sheet 2) with time step $\tau_{\text{ref}} := \frac{\min_m \tau_m}{M_{\text{ref}}} \ll \tau_m$ for $M_{\text{ref}} = 200$.

In the following, fix $\lambda = 0.5$.

(a) Write a MATLAB function `phi1(Z)` which takes an array `Z` and computes the pointwise evaluation $\varphi_1(Z_j)$, where Z_j are the entries of `Z`. Thereby, take into account that $\varphi_1(0) = 1$.

(b) Building up on the code from Programming Exercise 3 implement the exponential integrator Φ^τ_{exp} from Exercise 5 together with the Lie and Strang splitting schemes for approximations u^n_{LIE} and u^n_{STRANG} from the previous Exercise Sheet 2.

(c) Analogously to Programming Exercise 3 b) Animate the numerical solutions u^n_{EXP} and u^n_{LIE} over times t_n with time step size $\tau = \tau_{m_{\max}} = \frac{1}{120}$ in one subplot. In another subplot, animate the corresponding Fourier actions $|\left(\widehat{u^n_{\text{EXP}}}\right)_k|^2$ and $|\left(\widehat{u^n_{\text{LIE}}}\right)_k|^2$, respectively, for each $t_n = n\tau$.

(d) In a semi-logarithmic subplot (MATLAB command `semilogy()`) for each method Φ^τ_{Lie}, $\Phi^\tau_{\text{Strang}}$ and Φ^τ_{exp} plot the time evolution of the actions $|\left(\widehat{u^n_{\text{LIE}}}\right)_k|^2$, $|\left(\widehat{u^n_{\text{STRANG}}}\right)_k|^2$ and $|\left(\widehat{u^n_{\text{EXP}}}\right)_k|^2$, for each $k = -\frac{N_x}{2} + 1, \ldots \frac{N_x}{2}$. Each subplot should now contain $N_x = 64$ lines, where each line illustrates the behaviour of the k-th action $|\left(\widehat{u^n_{\text{X}}}\right)_k|^2$ over the time interval $[0, T]$.

(e) Proceed as in Programming Exercise 3 d) in order to create order plots for each of the schemes Φ^τ_{Lie}, $\Phi^\tau_{\text{Strang}}$ and Φ^τ_{exp}, using the time step sizes τ_m from above. We measure the maximal H^1 error of the respective numerical soultions over the time interval $[0, T]$, i.e., corresponding to the Lie splitting scheme with step size τ_m we measure (cf. Exercise Sheet 2)

$$\text{err}^m_{\text{Lie}} = \frac{\sqrt{2\pi}}{N_x} \max_{t_n \in [0,T]} \left\| (1 + |k|)^1 \cdot \texttt{fft}(u_{\text{ref}}(t_n) - u^n_{\text{LIE}}) \right\|_{\ell^2}.$$

The reference approximation $u_{\text{ref}}(t_n)$ is computed with the Strang splitting scheme $\Phi^{\tau_{\text{ref}}}_{\text{Strang}}$ with time step τ_{ref}.

(f) Repeat parts c)-e) with initial data

$$u^0(x) = e^{ix}, \qquad u^0(x) = 1 + e^{ix}, \qquad u^0(x) = \frac{1 - \sin(2x)}{2 - \cos(x)} \qquad \text{and} \qquad u^0(x) = \frac{1 - \sin(x)}{2 - \cos(2x)}.$$

What do you observe in the different behaviour of the actions and in the order plots?

Discussion in the problem class monday 8:00 am, in room 3.061 in the Kollegiengebäude Mathematik 20.30.