In the following let \( \Omega \subseteq \mathbb{R}^d \) be a domain with boundary \( \Gamma \).

**Exercise 4** (Variational Formulation and \( V \)-ellipticity)
Consider the boundary value problem for a function \( u : \Omega \rightarrow \mathbb{R} \) given by

\[
-\Delta u + \sum_{k=1}^{d} \partial_{x_k} u + u = f \quad \text{in } \Omega \\
u = 0 \quad \text{on } \Gamma
\]  

(1)

with \( f \in L^2(\Omega) \).

a) Derive the variational formulation of (1) such that
\[
a(u, v) = \ell(v),
\]
with a bilinear form \( a : V \times V \rightarrow \mathbb{R} \) and a functional \( \ell : V \rightarrow \mathbb{R} \) for a suitable space \( V \).

Define \( V, a \) and \( \ell \) explicitly.

b) Show that \( a \) is coercive and bounded, i.e. show that there are constants \( 0 < \alpha, M < \infty \) s.t.
\[
a(u, u) \geq \alpha \|u\|_{H^1}^2 \quad \text{for all } u \in V \\
a(u, v) \leq M \|u\|_{H^1} \cdot \|v\|_{H^1} \quad \text{for all } u, v \in V.
\]

Hint: Show that for all \( k = 1, \ldots, d \) we have
\[
\langle \partial_{x_k} u, u \rangle_{L^2} = 0
\]
for all \( u \in V \) and use the results of exercise 5 d).

**Exercise 5** (On weak derivatives and Sobolev spaces)

a) Show that for \( u \in C^1(\Omega) \cap C(\overline{\Omega}) \) the weak derivative \( \partial_{x_i} u, i = 1, \ldots, d \) coincides with the corresponding classical derivative.

b) Let the function \( u : (0, 4) \rightarrow \mathbb{R} \) be defined as follows:
\[
u(x) = \begin{cases} 
0, & x \in (0, 1) \cup (3, 4) \\
x - 1, & x \in [1, 2) \\
3 - x, & x \in [2, 3]
\end{cases}
\]

Make a guess about the weak derivative of \( u \) and show that your guess is actually the weak derivative.

c) Let \( \Omega = (-1, 1) \), \( u(x) = |x| \). Then \( u \in H^1(\Omega) \) but \( u \notin \text{C}^{\infty,1}(\Omega) \), where
\[
\text{C}^{\infty,1}(\Omega) = \{ v \in C^\infty(\Omega) | \int_{\Omega} |\partial_x v(x)|^2 \, dx < \infty \}
\]

Show that for the sequence \( (v_n)_n \subset \text{C}^{\infty,1}(\Omega) \), \( v_n(x) = \sqrt{x^2 + \frac{1}{n^2}} \) we have
\[
\lim_{n \to \infty} \|v_n - u\|_{H^1} = 0.
\]

Remark: This exercise is an example for the density of \( \text{C}^{\infty,1} \) in \( H^1 \).

d) Show that the seminorm \( |\cdot|_{H^1} \) is a norm on \( H^1_0(\Omega) \) and furthermore show that on \( H^1_0(\Omega) \) the norms \( |\cdot|_{H^1} \) and \( \|\cdot\|_{H^1} \) are equivalent, i.e. that there are constants \( c, C > 0 \) such that for \( u \in H^1_0(\Omega) \)
\[
c \|u\|_{H^1} \leq \|u\|_{H^1} \leq C \|u\|_{H^1}.
\]

Remark: The second inequality is called Poincaré-Friedrichs inequality.

Discussion in the problem class Monday 11:30, in room 3.061 in building 20.30.