In the following let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with boundary $\Gamma$.

**Exercise 6** (Maximum principle)

We consider the Poisson problem

$$
\begin{cases}
-\Delta u = f & \text{in } \Omega \\
u = g & \text{on } \Gamma 
\end{cases}
$$

with $f \in C(\Omega)$, $g \in C(\Gamma)$.

a) Let $f(x,y) \leq 0$ for all $(x,y) \in \Omega$ and let $u : \Omega \rightarrow \mathbb{R}$ be a classical solution of $(P)$.

Show that $u$ has its maximum on the boundary, i.e. show that

$$
\max_{(x,y) \in \Omega} u(x,y) = \max_{(x,y) \in \Gamma} g(x,y).
$$

Furthermore explain, why the maximum is not necessarily unique.

b) Now consider the Poisson problem with perturbed data $\tilde{g} \in C(\Gamma)$ on the boundary, i.e.

$$
\begin{cases}
-\Delta \tilde{u} = f & \text{in } \Omega \\
\tilde{u} = \tilde{g} & \text{on } \Gamma 
\end{cases}
$$

and assume that $u, \tilde{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ solve $(P)$ and $(\tilde{P})$ respectively.

Show that the perturbation of $u$ is bounded by the perturbation of $g$, i.e. show that

$$
\sup_{(x,y) \in \Omega} |\tilde{u}(x,y) - u(x,y)| \leq \sup_{(x,y) \in \Gamma} |\tilde{g}(x,y) - g(x,y)|.
$$

**Exercise 7** (Edge singularities)

Consider the domain

$$
\Omega := \{(x,y) \in \mathbb{R}^2 | \ x^2 + y^2 < 1, \ x < 0 \text{ or } y > 0 \}
$$

and identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$.

Then $w : \Omega \rightarrow \mathbb{C}, \ w(z) := z^{\frac{1}{2}}$ is analytic in $\Omega$.

Now consider the boundary value problem

$$
\begin{cases}
-\Delta u = 0 & \text{in } \Omega \\
u(e^{i\varphi}) = \sin(\frac{2}{3}\varphi), & \text{on } \Gamma \text{ for } \varphi \in [0, \pi] \\
u = 0, & \text{on } \Gamma \text{ otherwise.}
\end{cases}
$$

(BVP)

a) Show that $u(z) := \text{Im}(w(z))$ solves the boundary value problem (BVP).

**Hint:** Polar coordinates. You can use that $\Delta u(r, \varphi) = \left(\frac{\partial^2}{\partial r^2} + \frac{\varphi}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}\right)u(r, \varphi)$.

b) Show that for $z \rightarrow 0$ the first derivatives of $u$ are unbounded.

**Hint:** It is enough to show that for $z = r \cdot e^{i\varphi} \rightarrow 0$ the derivative $\partial_r u(r \cdot e^{i\varphi})$ is unbounded.

Discussion in the problem class Monday 11:30, in room 3.061 in building 20.30.